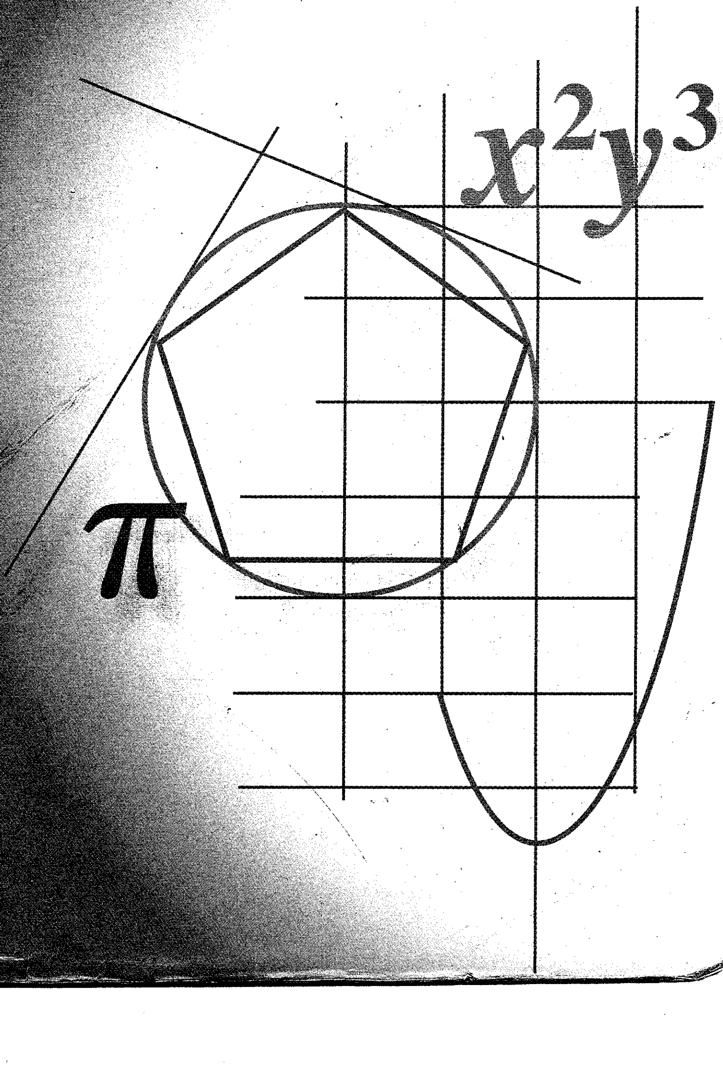
Second Edition

MATHΣMATICS
A COMPLETE COURSE
WITH CXC QUESTIONS

√olume One

Raymond Toolsie BSc, MACP



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MATHEMATICS

A COMPLETE COURSE
WITH C.X.C. QUESTIONS

VOLUME 1

Raymond Toolsie BSc, MACP
Mathematics Department
Couva Government Secondary School.
Formerly Head of Mathematics Department
Holy Faith Convent
Penal.

Caribbean Educational Publishers Ltd.

MATHEMATICS: A COMPLETE COURSE WITH C.X.C. QUESTIONS

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FOR

MY

SON

YURI ALBERT RAMAN

AND

MY DAUGHTER

CHRISTINE ANNA MARIA

CARIBBEAN EDUCATIONAL PUBLISHERS LTD.,

Teddy's Shopping Centre
Gulf View Luk Road
Gulf View, La Revenue
The Trinidad. West Indies
email: mbscep@jstt.net.tc.
Telephone: 868 657 9613 Fax: 868 652 5620

INTRODUCTION

This book is written not only for use by students in schools writing the C.X.C. Examination in Mathematics, but also by G.C.E./G.C.S.E. students, repeaters and adults who can use it as a teach yourself course.

One of the aims is to teach the student to understand basic mathematical concepts and principles, and to comprehend what they are learning or trying to learn. It places great emphasis on problem solving and tries to develop and nurture these skills.

Teachers will find this book very helpful, in that they can use it to supplement or highlight their strategies in teaching this subject.

This textbook endeavours to assist students to overcome any basic or inherent weaknesses they may have to bridge, or narrow any gaps in their knowledge and to develop their self confidence. Examples used are from everyday life situations so students can see and appreciate the value and the place of Mathematics in our world.

Mr. Toolsie, in his book, also exposes students to the language and vocabulary of Mathematics, and how to translate, or relate events from our daily lives into a mathematical framework using this language.

Graded worked examples of actual and typical examination questions (with alternative solutions) are dealt with in each chapter, and further exercises with answers are given at the end of each chapter and also at the end of the book.

Readers will find this book very stimulating and certainly Mr. Toolsie must be commended for producing this work.

S.C.Ramkeerat BSc, Dip Ed
Head-Mathematics Department
Titchfleld High School
Jamaica.

Formerly Deputy Headmaster
Head Mathematics Department
Richard Ishmael Secondary School
Guyana.
Examiner - Mathematics C.X.C.

PREFACE

Mathematics: A Complete Course with C.X.C. Questions was written specifically for students sitting the C.X.C. Mathematics Examinations - at both the Basic and General Proficiency levels.

Numerous exercises with graded questions are included in each chapter in order to give students enough practice to master the topics taught. At the end of each topic is an exercise with many graded questions-starting from the simplest type and proceeding to the more challenging problems. The questions in the exercises were tested at the *Holy Faith Convent, Penal* and the *Couva Government Secondary School*, over a twelve-year period.

Many different types of examples are worked out so that students can acquire the necessary skills in order to solve the problems in the given exercises.

In the solution to examples, diagrams are utilized where appropriate so that students can have a visual feeling and understanding of the problems and their solutions. Sometimes, alternative methods are used to solve a problem in order to satisfy students at different levels of mathematical abilities and maturity.

At the end of each chapter, from chapter 3 to chapter 24, different types of C.X.C. Past Papers Questions were given so that students can get a feel, appreciation and understanding of the language and solution, to such problems as expected of them in the actual examination. A total of 155 C.X.C. Questions are included. The author would like to thank the Caribbean Examinations Council for its kind permission to use the C.X.C. Past Paper Questions.

This work is divided into two volumes which is further subdivided into three parts:

Volume 1 Part 1 - consists of the C.X.C. Basic Proficiency Syllabus **(Core Syllabus)** which is the **Foundation Course**. The first 11 chapters give students the necessary grounding to move forward to the more challenging aspects of the syllabus. At the end of this Basic Course there are six C.X.C. Model Examinations-both Paper 1 (**Multiple Choice Questions**) and Paper 2 (**Essay Type Questions**).

Volume 2 Part 2 - consists of the C.X.C. General Proficiency syllabus (**Compulsory Objectives**). Chapters 12 to 20 take students through the process of preparing for the ever more challenging-questions that are set in Paper 2 Section 1 of the C.X.C. General Proficiency Mathematics Examination which are all compulsory questions.

Volume 2 Part 3 - consists of the C.X.C. General Proficiency syllabus (**Optional Objectives**). Chapters 21 to 24 groom students for the most challenging of all questions, that are set in Paper 2 Section 2 of the C.X.C. General Proficiency Mathematics Examination. At the end of this General Course there are six C.X.C. Model Examinations - both Paper 1 (**Multiple Choice Questions**) and Paper 2 (**Essay Type Questions**).

The calculation of standard deviation and the use of an assumed mean have been included in chapters dealing with statistics, although that knowledge is not presently required by the C.X.C. Mathematics syllabus. The topics have been included in the text because knowledge of them are required for more advanced studies in mathematics.

It is hoped that with a strong foundation, C.X.C. Mathematics students will be able to move on to more advanced work, which they can also master and hence achieve success in their examination. If students are able to achieve such standards, then my labour, and the labour of your teachers would not be in vain.

Raymond Toolsie.

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PART ONE
THE C.X.C. BASIC PROFICIENCY SYLLABUS (CORE SYLLABUS)

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PREREQUISITES FOR CHAPTER 1

The **set of natural numbers, N** = **{1, 2, 3, …}**

The **set of whole numbers, W** = **{0, 1, 2, 3, …}**

The **set of integers, Z** = **{…, −3, −2, −1, 0, 1, 2, 3, …}**

The **set of square numbers (perfect squares)** = **{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, …}**

The **set of prime numbers,** = **{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, …}**

The **set of even numbers,** = **{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, …}**

The **set of odd numbers,** = **{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, …}**

The **factors of a number** are those **numbers, including 1 and itself,** which can **divide exactly** into the **number**. The factors of 12 are 1, 2, 3, 4, 6 and 12.

The **multiples of a number** is k **times** the **number,** where k is a **natural number** (or a **counting number**). The multiples of 3 are 3, 6, 9, 12, 15, 18, 21, …, $3k$.

PART 1

The CXC Basic Proficiency Syllabus

(Core Syllabus)

CHAPTER 1

Sets 1

This chapter will teach you how to

- define a set, elements of a set, finite and infinite sets, a null set, a universal set, a subset, equal and equivalent sets, one-to-one correspondence and disjoint sets.
- determine the elements in the complement, union and intersection of two sets using Venn diagrams.
- calculate the number of elements in a given subset of two intersecting sets.

Defining a Set

A *set* is a collection of items usually of the same kind. Each set can be denoted by a *capital letter*. A set may be defined by *listing the members* or *describing the members*. A *set* is said to be *well-defined* when *all* its *members* can be *listed*.

(a) a flock of sheep
(b) a pack of cards } *Non-well-defined sets*
(c) the vowels in the English alphabet
(d) even numbers less than 13
(e) prime numbers between 5 and 17. } *Well-defined sets*

Example 1

(a) A = the *set* of vowels in the English alphabet
= {vowels in the English alphabet}
= {a, e, i, o, u}

(b) B = the *set* of even numbers between 1 and 9
= {even numbers between 1 and 9}
= {2, 4, 6, 8}

The curly brackets or braces { } means '*the set of*'.

Exercise 1a

List the members in each of the following sets:

1. A = {even numbers less than 13}.

2. B = {prime numbers between 15 and 30}.

3. C = {multiples of 5 between 12 and 47}.

4. X = {whole numbers greater than 10 but less than 20}.

5. Y = {letters used in the word 'mathematics'}.

6. Z = {prime numbers less than 21}.

7. D = {odd numbers less than 21}.

8. E = {even numbers from 4 to 16 inclusive}.

9. F = {odd numbers from 3 to 15 exclusive}.

10. H = {the first five letters in the English alphabet}.

Describe in words, each of the following sets:

11. $P = \{2, 3, 5, 7, 11, 13, 17\}$.

12. $Q = \{25, 30, 35, 40, 45\}$.

13. $R = \{w, x, y, z\}$.

14. $S = \{1, 4, 9, 16, 25, 36, 49\}$.

15. $T = \{15, 17, 19, 21, 23, 25\}$.

Element—Number of Elements in a Set

Each item in a set is called a *member* (or *element*).

Example 2

Let $B = \{2, 4, 6, 8\}$.

Then 2, 4, 6, and 8 are *elements* of set B.

We write $2 \in B$, $4 \in B$, $6 \in B$ and $8 \in B$.

The symbol $\in$ means '*is an element of*'.

The *number of elements* in set B is 4.

This can be written as $n(B) = 4$.

The notation $n(B)$ means '*the number of elements in set B*'.

Since 5 is *not an element* of set B, we write $5 \notin B$.

The symbol $\notin$ means '*is not an element of*'.

Exercise 1b

Re-write each of the following word statements using set notation:

1. Turtle is a member of the set of living things.

2. Brazil is not an Asian country.

3. Orange is a member of the set of fruits.

4. Electricity is not a member of the set of living things.

5. Mathematics is a member of the set of school subjects.

6. Curry is not a member of the set of cars.

7. A carite is a fish.

8. Three members who belong to {calypso singers}.

9. Grape is not a member of the set of animals.

10. Zero is not a natural number.

State the meaning of each of the following statements written in set notation:

11. Physics $\in$ {science subjects}.

12. French $\notin$ {science subjects}.

13. Cricket $\in$ {team games}.

14. Albert $\notin$ {girls' names}.

15. $1 \in$ {odd numbers}, $3 \in$ {odd numbers} and $5 \in$ {odd numbers}.

16. $2 \notin$ {odd numbers}, $4 \notin$ {odd numbers} and $6 \notin$ {odd numbers}.

State the number of elements in each of the following sets using the notation $n(?) = ?$:

17. A = {even numbers less than 15}.

18. B = {even numbers less than 16 inclusive}.

19. C = {even numbers less than 14 exclusive}.

20. P = {odd numbers less than 14}.

21. Q = {odd numbers less than 15 inclusive}.

22. R = {odd numbers less than 13 exclusive}.

23. X = {prime numbers less than 12}.

24. Y = {prime numbers less than 13 inclusive}.

25. Z = {prime numbers less than 17 exclusive}.

Finite and Infinite Sets

In a *finite set* it is possible to count and name all the elements in the set. In an *infinite set* it is not possible to count or name all the elements in the set.

Example 3

(a) Let X = {months of the year}.
Then X = {January, February, March,

April, May, June, July, August,
September, October, November,
December}.

So $n(X) = 12$.

We say that set X is *finite*, since all its elements can be counted and named.

(b) Let $Y = \{x: x \geqslant 0, x \in W\}$.
Then $Y = \{0, 1, 2, 3, \ldots\}$.
So $n(Y) =$ unknown.

We say that set Y is *infinite*, since the series is continuous indefinitely.

The notation $\{x: \ldots\}$ means '*the set of all x such that*'. It is a part of the *set builder notation*.

Null Set (or Empty Set)

The *null set* (or *empty set*) contains no elements and it is denoted by the symbols { } or ø.

Example 4

(a) Let $Z =$ {human beings on earth who are older than 300 years}.
Then Z is an *empty set*.
That is $Z = \{\}$.
Or $Z = ø$.

(b) Let $C =$ {Soviet cosmonauts who walked on Mars}.
Then C is an *empty set*.
That is $C = \{\}$.

Exercise 1c

State whether or not each of the following sets is finite, infinite or null:

1. $X =$ {even numbers less than 100}.
2. $Y =$ {even numbers}.
3. $Z =$ {people with six legs}.
4. $P = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$.
5. $Q = \{x: x \geqslant 5, x \in R\}$.
6. $R = \{x: 0 < x \leqslant 4, x \in W\}$.
7. $L =$ {children who have swam across the Caribbean Sea}.
8. The set of odd numbers which can be exactly divided by 2.
9. $L = \{y: y \leqslant -1.5 \text{ and } y \geqslant 7.5, y \in R\}$.
10. $M = \{r: r > 0 \text{ and } r < 8, r \in N\}$.

Universal Set

For any particular problem, the *universal set* is the set from which all the elements are taken. The *universal set* is denoted by the symbol U.

Example 5

(a) If $P = \{x: x \geqslant 0, x \in Z\}$, then the *universal set*, $U = W$, the set of whole numbers.

(b) If $R = \{x: x > 0, x \in Z\}$, then the *universal set*, $U = N$, the set of natural numbers.

Exercise 1d

Suggest a suitable universal set for each of the following subsets:

1. $A = \{12, 16, 20, 21, 23\}$. W
2. $B =$ {protractor, ruler, set square, compass, divider}. GI
3. $X = \{-3, -2, -1, 0, 1, 2, 3, 4\}$. I
4. $Y = \{2, 3, 5, 7, 11, 13, 19, 23\}$. PN
5. $P = \{0, 1, 2, 3, 4, 5\}$. WN
6. Give examples of a few empty sets.

Subset

If A and B are any two sets, and all the elements of A are *contained* in B, then we say that A is a *subset* of B. We write $A \subset B$.

Example 6

Let $A = \{9, 11, 13\}$, $B = \{7, 9, 11, 13, 15\}$ and $C = \{1, 2, 3\}$.

Then A is a *subset* of B, $A \subset B$,
since $\{9, 11, 13\} \subset \{7, 9, 11, 13, 15\}$.

The symbol $\subset$ means '*is contained in*' or '*is a subset of*'.

Also C is *not a subset* of A, $C \not\subset A$
and C is *not a subset* of B, $C \not\subset B$,
since $\{1, 2, 3\} \not\subset \{9, 11, 13\}$
and $\{1, 2, 3\} \not\subset \{7, 9, 11, 13, 15\}$.

The symbol $\not\subset$ means '*is not contained in*' or '*is not a subset of*'.

Number of Subsets

Let $R = \{a, b, c\}$.
Then the *subsets* of R are as follows:
$\{a\}, \{b\}, \{c\}$,
$\{a, b\}, \{a, c\}, \{b, c\}$,
$\{\ \}, \{a, b, c\}$.

This indicates that the *empty set* is a *subset* of all sets, and *each set* is a *subset* of itself. That is, for any set A, $\emptyset \subset A$ and $A \subset A$.

The first six *subsets* of R are called *proper subsets*.

The last two *subsets* of R are called *improper subsets* (or non-proper subsets).

From above, the number of elements in set R, $n(R) = 3$.

And the *number of subsets*, $S = 8$.

Now $S = 2^n$ is a *formula* that can be used to calculate the *number of subsets* of a *particular set*,
where $S =$ the *number of subsets*
and $n =$ the *number of elements*.

For the given example, the *number of subsets*,

$$\begin{aligned} S &= 2^n \\ &= 2^3 \\ &= 2 \times 2 \times 2 \\ &= 8 \end{aligned}$$

Exercise 1e

1. If $A = \{3, 6, 9, 12, 15, 18, 21\}$, $B = \{3, 6, 9\}$, $C = \{9, 12\}$ and $D = \{3, 12, 21\}$, then complete each of the following statements:

(a) $B \subset A \Rightarrow \{\quad\} \subset \{\quad\}$.
(b) $C \subset A \Rightarrow \{\quad\} \subset \{\quad\}$.
(c) $D \subset A \Rightarrow \{\quad\} \subset \{\quad\}$.
(d) $C \not\subset B \Rightarrow \{\quad\} \not\subset \{\quad\}$.
(e) $C \not\subset D \Rightarrow \{\quad\} \not\subset \{\quad\}$.
(f) $B \not\subset D \Rightarrow \{\quad\} \not\subset \{\quad\}$.

2. If $A = \{2, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{2, 4, 6, 8, 10, 12, 14\}$, state whether the statement $A \subset B \subset C$ is true of false. T

3. If $A = \{3, 5\}$, $B = \{3, 7, 9\}$ and $C = \{3, 7, 11, 13, 15\}$, state whether the statement $A \subset B \subset C$ is true or false. F

4. If $A = \{p, q, r, s\}$, list all the subsets of A. Indicate the proper subsets of A.

5. Given the set $\{1, 2, 4, 5, 7, 8, 10, 11, 19, 22, 35, 39, 41, 54\}$, state the set of numbers which is (are):

(a) prime (b) odd
(c) even (d) multiples of 2
(e) multiples of 3 (f) factors of 39.

6. (a) List the perfect squares in the set $\{2, 4, 8, 10, 16, 20, 25\}$.

(b) List the cubes in the set $\{8, 9, 27, 54, 64, 96\}$.

7. Given that the set $P = \{2, 4, 6, 8\}$, calculate the number of possible subsets of P.

8. Given that the set $T = \{5, 7, 8, 10, 15\}$, calculate the number of possible subsets of T.

9. If $P = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$, how many subsets can be formed from P?

State whether each of the following statements is true or false:

10. {squares} $\subset$ {rectangles}.

11. {rhombuses} $\subset$ {parallelograms}.

12. {squares} $\subset$ {rhombuses}.

13. {rectangles} $\subset$ {parallelograms}.

14. {kites} $\subset$ {rhombuses}.

15. {kites} $\subset$ {parallelograms}.

16. {trapeziums} $\subset$ {parallelograms}.

17. {kites} $\subset$ {trapeziums}.

18. {trapeziums} $\subset$ {kites}.

19. {kites} $\subset$ {squares}.

20. {trapeziums} $\subset$ {rectangles}.

Equal Sets

Two sets A and B are said to be *equal* if they both have the same elements, that is, every element which belongs to A also belongs to B, and every element which belongs to B also belongs to A.

That is, if $A \subset B$ and $B \subset A$, then $A = B$.

Example 7

Let $\quad A = \{15, 16, 17\}$ and $B = \{16, 17, 15\}$.
Then $\quad A = B = \{15, 16, 17\}$.

That is, the *order* in which the elements of a set are written does not matter.

Equivalent Sets—One-to-One Correspondence

Sometimes, it may be necessary to ask whether or not two sets have the *same number of elements*. When two sets, A and B, have the *same number of elements*, that is, $n(A) = n(B)$, we say that they are *equivalent*.

When two sets are *equivalent*, we say that there exists a *one-to-one correspondence* between the elements of the two sets.

We write $A \fallingdotseq B$.

Example 8

Let $\quad A = \{2, 3, 5, 7\}$ and $B = \{p, q, r, s\}$.
Then $\quad n(A) = n(B) = 4$.
So the sets A and B are *equivalent*.

Also the *elements* in set A can be *paired* off with the *elements* of set B; and the *elements* in set B can be *paired* off with the *elements* of set A. So the sets A and B have *one-to-one correspondence*.

Thus we write $A \fallingdotseq B$.

Let $\quad R = \{2, 3, 5\}$ and $S = \{2, 5\}$.
Then $\quad n(R) = 3$ and $n(S) = 2$.
So $\quad n(R) \neq n(S)$.

Hence the sets R and S are *not equivalent*. Also the sets R and S *do not have one-to-one correspondence*.

Equal sets P and Q always have *one-to-one correspondence*, since $n(P) = n(Q)$, and are *equivalent*. That is $P \fallingdotseq Q$.

Exercise 1f

Determine whether or not each of the following pairs of sets are equal:

1. $X = \{6, 8, 10, 12, 14, 16\}$ and
$Y = \{$even numbers from 6 to 16 inclusive$\}$.

2. $A = \{5, 7, 11, 13, 15\}$ and
$B = \{$odd numbers less than 16$\}$.

3. $E = \{$even numbers from 6 to 14 inclusive$\}$ and $F = \{6, 8, 10, 12, 14\}$.

4. $C = \{$vowels$\}$ and $D = \{a, e, i, o, u, f\}$.

5. $P = \{1, 3, 5, 7, 11, 13\}$ and
$Q = \{$prime numbers less than 14$\}$.

Place the correct symbol(s) ($=$, $\neq$ or $\fallingdotseq$) connecting each of the following pairs of sets:

6. $\{2, 4, 6, 8\} \fallingdotseq \{6, 8, 2, 4\}$.

7. $\{2, 4, 6, 8\} \fallingdotseq \{p, q, r, s\}$.

8. $\{2, 4, 6, 8\} \neq \{2, 4\}$.

9. $\{2, 4, 6, 8\} \neq \{6, 2, 4\}$.

10. $\{a, e, i, o, u\} = \{e, i, o, u, a\}$.

11. $\{a, e, i, o, u\} \fallingdotseq \{2, 3, 5, 7, 11\}$.

12. $\{a, e, i\} \neq \{2, 3, 5, 7, 11\}$.

13. $\{a, e, i, o, u\} \neq \{2, 3, 5, 7\}$.

Venn Diagrams

In drawing a *Venn diagram*, we use a *rectangle* to represent the *universal set*, and a *circle* inside the rectangle to represent a *subset*.

Complement

The *complement* of a set A, A', is the set of all elements in the universal set U, that are *not in set* A.

If $A \subset U$, then $A' \subset U$,

where $A' = \{x: x \in U \text{ but } x \notin A\}$.

The *Venn diagram* representing the above statement is shown below.

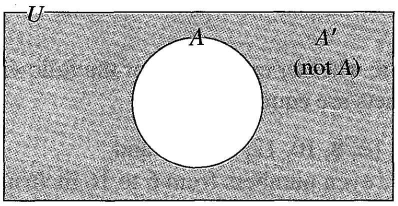


Fig. 1.1 Venn diagram

The shaded region represents *A complement*, A'.

Note that $(A')' = A$.

Example 9

Let $U = \{15, 16, 17, 18, 19\}$ and $P = \{16, 17\}$. Then the *complement* of P, $P' = \{15, 18, 19\}$.

The *Venn diagram* representing the above example can be seen below.

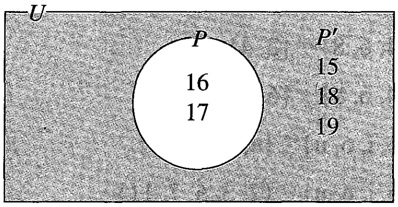


Fig. 1.2 Venn diagram

Intersection of Two Sets

If A and B are two sets, then the *intersection* of A and B is the set of all elements that are *common* to *both* A and B. That is,

$$A \cap B = \{x: x \in A \text{ and } x \in B\}.$$

The *Venn diagram* representing the above statement can be seen below.

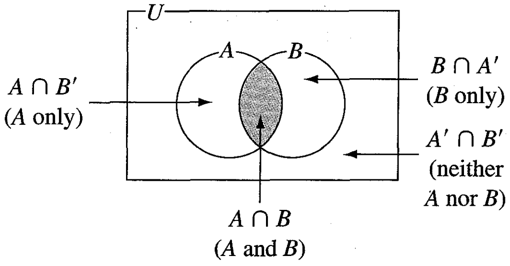


Fig. 1.3 Venn diagram

The *shaded region* represents *A intersection B*, $A \cap B$.

Example 10

Let $U = \{11, 12, 13, 14, 15, 16, 17, 18\}$,
$A = \{12, 15, 18\}$ and
$B = \{13, 15, 17\}$.

Then *A intersection B*, $A \cap B = \{15\}$,

A intersection B complement, $A \cap B' = \{12, 18\}$

and

B intersection A complement, $B \cap A' = \{13, 17\}$.

The *Venn diagram* representing the above example can be seen below.

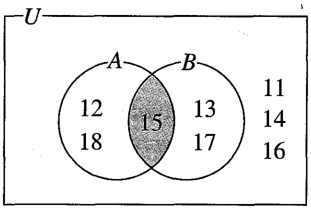


Fig. 1.4 Venn diagram

Union of Two Sets

The *union* of two sets A and B is the set of all elements that are in *either* A or B. That is,

$$A \cup B = \{x: x \in A \text{ or } x \in B \text{ or both}\}.$$

The *Venn diagram* representing the above statement can be seen below.

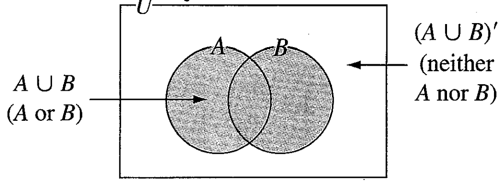


Fig. 1.5 Venn diagram

The *shaded region* represents A union B, $A \cup B$.

Example 11

Let $U = \{11, 12, 13, 14, 15, 16, 17, 18\}$,
$A = \{12, 15, 18\}$ and $B = \{13, 15, 17\}$.

Then A *union* B, $A \cup B = \{12, 13, 15, 17, 18\}$
and the *complement* of A *union* B,
$(A \cup B)' = \{11, 14, 16\}$.

Also $A' = \{11, 13, 14, 16, 17\}$
and $B' = \{11, 12, 14, 16, 18\}$.
Therefore $A' \cap B' = \{11, 14, 16\}$.

Hence $(A \cup B)' = A' \cap B'$.

Now A *intersection* B, $A \cap B = \{15\}$
and the *complement* of A *intersection* B,
$(A \cap B)' = \{11, 12, 13, 14, 16, 17, 18\}$.
Also $A' \cup B' = \{11, 12, 13, 14, 16, 17, 18\}$.

Hence $(A \cap B)' = A' \cup B'$.

The rules $(A \cup B)' = A' \cap B'$ and
$(A \cap B)' = A' \cup B'$

are called *De Morgan's Laws*.

The *Venn diagram* representing the above example can be seen below.

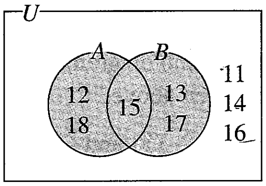


Fig. 1.6 Venn diagram

Subset

If A and B are two sets, and A is a *subset* of B, then we write $A \subset B$.

Further $A \subset B = \{x: x \in A \Rightarrow x \in B\}$

Also $A \cap B \neq \{\}$, $A \cap B = A$ and $A \cup B = B$.

So $(A \cup B)' = B'$.

The *Venn diagrams* representing the above statement can be seen below.

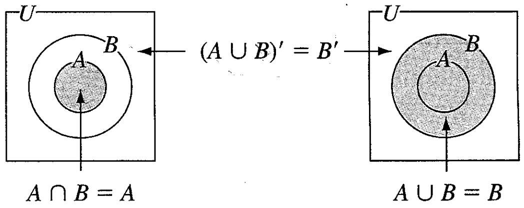


Fig. 1.7 Venn diagrams

Example 12

Let $U = \{2, 3, 5, 7, 11, 13, 17, 19\}$,
$A = \{5, 11, 17\}$ and
$B = \{2, 5, 7, 11, 17, 19\}$.

Then $A \cap B = \{5, 11, 17\} = A$
and $A \cup B = \{2, 5, 7, 11, 17, 19\} = B$.

Also $B' = \{3, 13\}$
and $(A \cup B)' = \{3, 13\} = B'$.

Hence A is a *subset* of B, $A \subset B$.

The *Venn diagrams* representing the above example can be seen below.

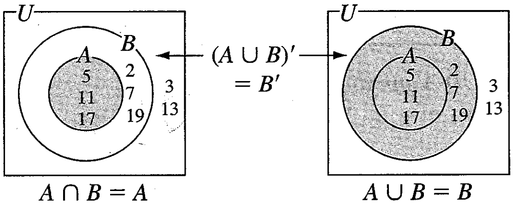


Fig. 1.8 Venn diagrams

Disjoint Sets

Two sets A and B are said to be *disjoint* if $A \cap B = \{\}$, when $A \neq \{\}$ and $B \neq \{\}$.
That is, $A \cap B = \{x: x \in A \Rightarrow x \notin B\}$.

The *Venn diagram* representing the above statement can be seen below.

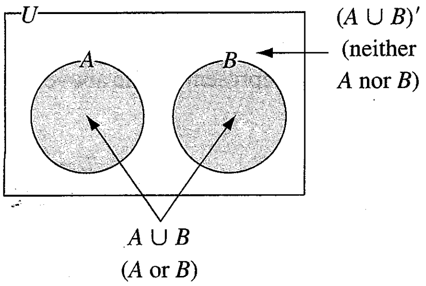


Fig. 1.9 Venn diagram

Example 13

Let $U = \{20, 22, 24, 26, 28, 30, 32, 34\}$,
$A = \{20, 24, 28\}$ and $B = \{22, 26\}$.

Then $A \cap B = \{\}$. So the sets A and B are *disjoint*.

Also $A \cup B = \{20, 22, 24, 26, 28\}$

and $(A \cup B)' = \{30, 32, 34\}$.

The *Venn diagram* representing the above example can be seen below.

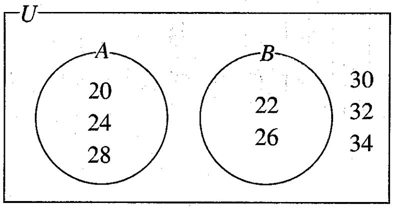


Fig. 1.10 Venn diagram

Exercise 1g

1. Given $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and
   $Y = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$,
   then the elements in:

   (a) $X \cup Y = \{\ \}$? (b) $X \cap Y = \{\ \}$?

   Draw a suitable Venn diagram to show the union of the two sets.

2. Given $A = \{3, 6, 9, 12, 15, 18, 21\}$ and
   $B = \{3, 5, 7, 9, 11, 13, 15, 17, 19, 21\}$,
   then the elements in $A \cap B = \{\ \}$?

   Draw a suitable Venn digram to show the intersection of the two sets.

3. Describe using set notation, the shaded area in each of the following Venn diagrams:

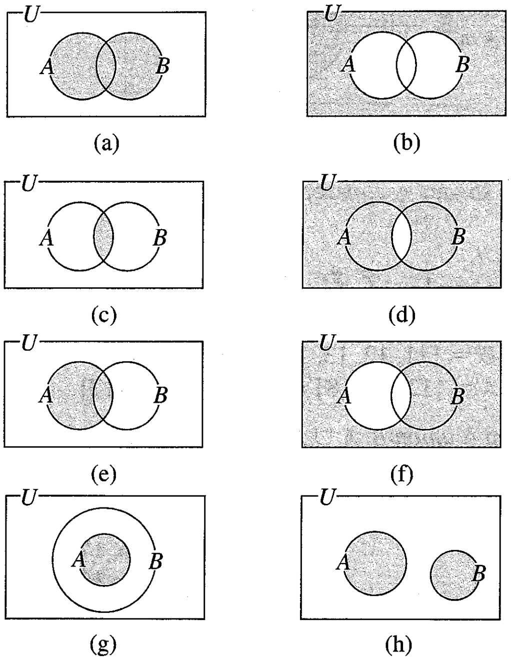


Fig. 1.11 Venn diagrams

4. $X = \{$prime numbers less than 20$\}$ and
   $Y = \{$odd numbers less than 16$\}$.
   (a) Draw a suitable Venn diagram to represent the information given above.
   State the elements in each of the following:
   (b) $X \cap Y$ (c) $X \cup Y$
   (d) $X \cap Y'$ (e) $Y \cap X'$.

5. Determine the elements in the union and intersection of the two given sets in each of the following:
   (a) $A = \{3, 6, 9, 12, 15\}$ and
   $B = \{6, 8, 10, 12, 14\}$.
   (b) $X = \{1, 3, 5, 7, 11, 13\}$ and $Y = \{1, 5, 11\}$.
   Draw suitable Venn diagrams to show the information given above.

6. If $P = \{3, 6, 9, 12, 15\}$ and $Q = \{2, 4, 6, 8, 10\}$, state the elements in $P \cap Q$. Draw a suitable Venn diagram to represent the information. Shade the region $P \cap Q$.

7. Determine the elements in the union of the two given sets in each of the following:
   (a) $X = \{3 ,6, 9, 12, 15\}$ and $Y = \{6, 12, 18, 24\}$.
   (b) $P = \{2, 3, 5, 7, 11\}$ and $Q = \{2, 5, 11\}$.
   Draw suitable Venn diagrams to show the union of the sets above.

8. Determine the elements in the intersection of the two given sets in each of the following:

(a) R = {odd numbers less than 15} and
S = {prime numbers less than 12}.

(b) L = {even numbers between 6 and 16 inclusive} and
$M = \{10, 12, 14\}$.

Draw suitable Venn diagrams to show the intersection of the sets above.

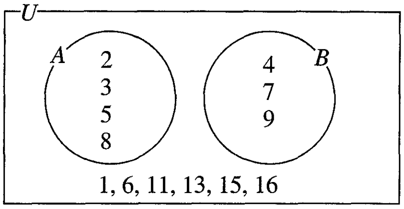


Fig. 1.12 Venn diagram

The Venn diagram above shows two sets A and B, which are subsets of the universal set, U. Determine the elements in each of the following:

9. $U = \{\ \}$.

10. $A = \{\ \}$.

11. $B = \{\ \}$.

12. $A \cap B = \{\ \}$.

13. $A \cup B = \{\ \}$.

14. $(A \cup B)' = \{\ \}$.

15. $A \cap B' = \{\ \}$.

16. Consider the following three statements:

(1) Some students play cricket.

(2) Short students are less than 2 meters in height.

(3) All cricket players are short students.

(a) Represent the statements in a suitable Venn diagram, showing and stating an appropriate universal set.

(b) Show on your Venn diagram that:

(i) Viv is 2.1 m tall.

(ii) Frank, who is 1.5 m tall, does not play cricket.

17. Consider the following three statements:

(1) Some students play basketball.

(2) Tall students are more than 2 meters in height.

(3) All basketball players are tall students.

(a) Represent the statements in a suitable Venn diagram, showing and stating an appropriate universal set.

(b) Show on your Venn diagram that:

(i) Samuel is 1.7 m tall.

(ii) Albert, who is 2.2 m tall, does not play basketball.

State whether the following statements are empty or not:

18. {rhombuses} ∩ {rectangles}.

19. {parallelograms} ∩ {squares}.

20. {squares} ∩ {rectangles}.

21. {rhombuses} ∩ {parallelograms}.

22. {kites} ∩ {trapeziums}.

23. {trapeziums} ∩ {parallelograms}.

24. If P = {whole numbers that divide exactly into 15} and
Q = {whole numbers that divide exactly into 18}, then $P \cap Q = \{\ \}$.

Draw a Venn diagram to show the intersection of the two sets.

25. If A = {factors of 12} and B = {factors of 16}, then $A \cup B = \{\ \}$.

Draw a Venn diagram to show the union of the two sets.

26. If X = {whole numbers less than 18} and
Y = {prime numbers less than 18},
state the elements in $X \cup Y$. Draw a suitable Venn diagram to represent the information. Shade the region $X \cup Y$.

27. If P = {multiples of 3 less than 19} and
Q = {multiples of 2 less than 13},
state the elements in $P \cap Q$. Draw a suitable Venn diagram to represent the information. Shade the region $P \cap Q$.

Number of Elements in Two Sets

Venn diagrams can be very useful in finding the *number of elements* in certain *subsets* of *two intersecting sets*.

Example 14

In a class of 30 students, 20 played cricket, 17 played football and 7 played both cricket and football. Each student played either cricket or football. Find the number of students who played:

(a) cricket only *(b) football only.*

Solution

Let $C = \{\text{students who played cricket}\}$

and $F = \{\text{students who played football}\}$.

Then $n(U) = n(C \cup F) = 30$ students,

$n(C) = 20$ students,

$n(F) = 17$ students

and $n(C \cap F) = 7$ students.

Then we have the following Venn diagram:

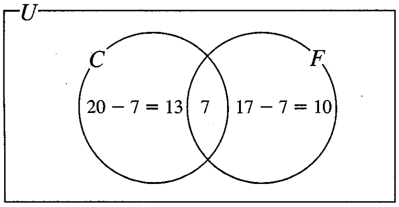


Fig. 1.13 Venn diagram

(a) The *number* of students who played *cricket only*,

$$\begin{aligned} n(C \cap F') &= n(C) - n(C \cap F) \\ &= (20 - 7) \text{ students} \\ &= 13 \text{ students} \end{aligned}$$

Hence 13 students played *cricket only.*

(b) The *number* of students who played *football only*,

$$\begin{aligned} n(F \cap C') &= n(F) - n(C \cap F) \\ &= (17 - 7) \text{ students} \\ &= 10 \text{ students} \end{aligned}$$

Hence 10 students played *football only.*

Example 15

In a class of 30 students, 21 like Mathematics, 12 like Physics and 6 like neither Mathematics nor Physics. Determine the number of students who like:

(a) both Mathematics and Physics

(b) only Mathematics *(c) only Physics.*

Solution

Let $M = \{\text{students who like Mathematics}\}$

and $P = \{\text{students who like Physics}\}$.

Then $n(U) = 30$ students,

$n(M) = 21$ students,

$n(P) = 12$ students

and $n(M \cup P)' = 6$ students.

Let the *number* of students who like *both Mathematics and Physics*, $n(M \cap P) = x$ students.

Then the *number* of students who like *Mathematics only*,

$$n(M \cap P') = (21 - x) \text{ students.}$$

And the *number* of students who like *Physics only*,

$$n(P \cap M') = (12 - x) \text{ students.}$$

Then we have the following Venn diagram:

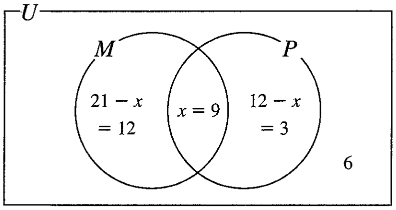


Fig. 1.14 Venn diagram

(a) Now $n(U) = (21 - x + x + 12 - x + 6)$ students

$= (21 + 12 + 6 - x + x - x)$ students

$= (39 - x)$ students

And $n(U) = 30$ students

Thus $30 = 39 - x$

i.e. $x = 39 - 30$

$= 9$

Hence 9 students like *both Mathematics and Physics.*

(b) The *number* of students who like *Mathematics only*,

$$\begin{aligned} n(M \cap P') &= (21 - x) \text{ students} \\ &= (21 - 9) \text{ students} \\ &= 12 \text{ students} \end{aligned}$$

Hence 12 students like *Mathematics only.*

(c) The *number* of students who like *Physics only*,

$$\begin{aligned} n(P \cap M') &= (12 - x) \text{ students} \\ &= (12 - 9) \text{ students} \\ &= 3 \text{ students} \end{aligned}$$

Hence 3 students like *Physics only.*

Exercise 1h

1.

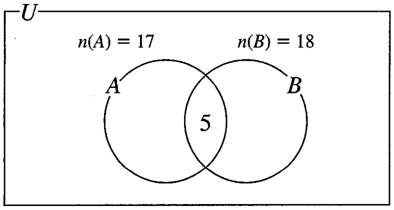


Fig. 1.15 Venn diagram

In the Venn diagram above, $n(A) = 17$, $n(B) = 18$ and $n(A \cap B) = 5$. Calculate

(a) $n(A \cap B')$ (b) $n(B \cap A')$.

2.

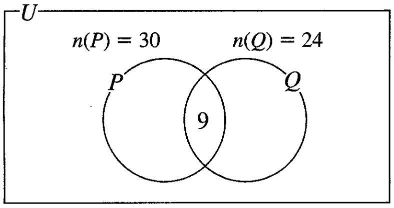


Fig. 1.16 Venn diagram

In the Venn diagram above, $n(P) = 30$, $n(Q) = 24$ and $n(P \cap Q) = 9$. Evaluate

(a) $n(P \cap Q')$ (b) $n(Q \cap P')$.

3. In a class of 35 students, 29 play draughts, 16 play chess and 10 play both draughts and chess. Each student plays either draughts or chess. Find the number of students who play:

(a) draughts only (b) chess only.

4. Of 26 students, 13 play the violin and 21 play the guitar. Each student plays the violin or guitar. If 8 students play both the violin and guitar, find how many students play:

(a) the violin only (b) the guitar only.

5. Of 45 students, 30 play badminton and 26 play tennis. Each student plays badminton or tennis. If 11 students play both badminton and tennis, determine how many students play:

(a) badminton only (b) tennis only.

6.

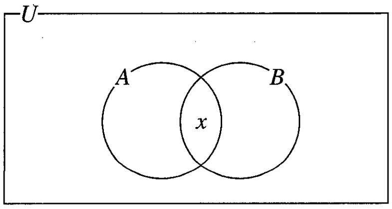


Fig. 1.17 Venn diagram

In the previous Venn diagram $n(U) = 45$, $n(A) = 20$, $n(B) = 18$, $n(A \cup B)' = 15$ and $n(A \cap B) = x$. Calculate:

(a) x (b) $n(A \cap B')$ (c) $n(B \cap A')$.

7.

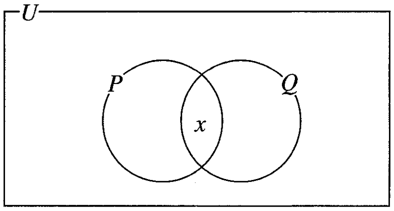


Fig. 1.18 Venn diagram

In the Venn diagram above, $n(U) = 50$, $n(P) = 27$, $n(Q) = 31$, $n(P \cup Q)' = 4$ and $n(P \cap Q) = x$.
Evaluate:

(a) x (b) $n(P \cap Q')$ (c) $n(Q \cap P')$.

8. In a group of 60 students, 31 speak French, 23 speak Spanish and 14 speak neither French nor Spanish. Determine the number of students who speak:

(a) both French and Spanish
(b) French only
(c) Spanish only.

9. Of 60 martial arts experts, 26 are Karatekas, 23 are Judokas and 16 are neither Karatekas nor Judokas. Calculate the number of martial arts experts who are:

(a) both Karatekas and Judokas
(b) only Karatekas
(c) only Judokas.

10. Of 100 athletes, 31 like to run, 65 like to walk, and 22 neither like to walk nor run. Determine the number of athletes who like:

(a) both to run and walk
(b) to run only
(c) to walk only.

CHAPTER 2

Number Theory

This chapter will teach you how to

- define natural numbers, whole numbers, integers, rational and irrational numbers, and real numbers.
- add, subtract, multiply and divide.
- use the identity and inverse for addition and multiplication; and multiply and divide by zero.
- define and use the law of closure, commutative law, associative law and distributive law.
- solve problems dealing with the powers of numbers; and use a defined arithmetic operation.
- obtain the factors, prime factors and multiples of a number; and determine the L.C.M. and H.C.F. of a set of numbers.
- define the sets of square numbers, rectangular numbers, prime numbers, composite numbers, even numbers and odd numbers.
- define a sequence and determine a term in the sequence.
- understand and use the binary system, quinary system, octal system and denary system.

Set of Natural Numbers

The *set of natural numbers* is another name given to the *set of counting numbers* and it is represented by the *symbol N*.

The *set of natural numbers*, $N = \{1, 2, 3, \ldots\}$.

Set of Whole Numbers

The *set of whole numbers* is the *set of natural numbers* or *counting numbers* and *zero*. It is represented by the *symbol W*.

It should be obvious from the previous statements that *zero* is *not* a *natural number*. *Zero* is represented by the *symbol* 0 (*nought*).

The *set of whole numbers*, $W = \{0, 1, 2, 3, \ldots\}$.

Set of Integers

The *set of integers* can be accepted as the *set of negative* and *positive natural numbers* and *zero*.

Alternatively, the *set of integers* can be regarded as the *set of negative* and *positive whole numbers* including *zero*. Note, however, that *zero* is *neither positive nor negative*. That is, $\pm 0 = 0$.

The *set of integers* is represented by the *symbol Z*.

The *set of integers*, $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

Set of Rational Numbers

The *set of rational numbers* is really the *set of numbers* that *can* be written as *fractions*. It is the *set of negative* and *positive fractions, including zero*.

For example: $-\frac{2}{3}, -\frac{1}{2}, \frac{3}{5}$ and $\frac{8}{9}$.

A *rational number* can always be *written* as a *decimal*, whether *terminating* or *recurring*.

For example: 0.8, 0.65, 0.3 and 0.6.

It should be obvious from the statements above that the *set of rational numbers contains* the *set of integers*, since all *whole numbers* can be *written* with 1 as their *denominator*.

For example: $-5 = \frac{-5}{1}, 6 = \frac{6}{1}$ and $0 = \frac{0}{1}$.

The *set of rational numbers* is represented by the *symbol Q*.

The *set of rational numbers*,

$$Q = \left\{\frac{n}{d} : n \in Z, d \in Z, d \neq 0, \text{ and, } n \text{ and } d \text{ have no common factor}\right\},$$

where n = the *numerator*,
d = the *denominator*

and $\frac{n}{d}$ = a *fraction* in its *simplest terms*.

From the statements above, it can be seen that:

(i) the *set of rational numbers contains* the *set of integers*;

(ii) the *set of integers contains* the *set of whole numbers*; and

(iii) the *set of whole numbers contains* the *set of natural numbers*.

So we can write:

$$Q \supset Z \supset W \supset N$$

or

$$N \subset W \subset Z \subset Q.$$

So we have the following Venn diagram representing the information stated previously.

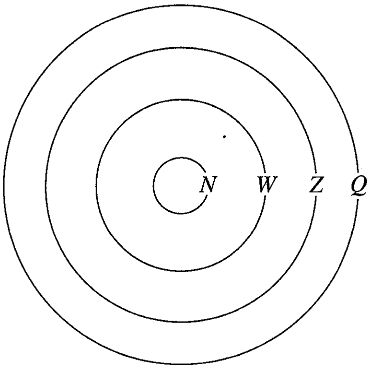


Fig. 2.1 Venn diagram

Set of Irrational Numbers

The *set of irrational numbers* is the *set of numbers* that *cannot* be written as *fractions*. For example:

$-\sqrt{3}, \sqrt{7}, -\sqrt{\frac{5}{4}}$ and $\sqrt{\frac{2}{9}}$.

Further, when *irrational numbers* are *written* as *decimals* they *do not terminate* or *recur*.
For example:

$\pi = 3.141\,592\,7\ldots$ (correct to 7 decimal places) and

$\sqrt{3} = 1.732\,050\,8\ldots$ (correct to 7 decimal places).

The *set of irrational numbers* is represented by either the *symbol Q'* or the *symbol I*.

The *set of irrational numbers*,

$$Q' \neq \left\{\frac{n}{d} : n \in Z, d \in Z, d \neq 0, \text{ and, } n \text{ and } d \text{ have no common factor}\right\}'.$$

Or the *set of irrational numbers*,

$$I = \left\{\frac{n}{d} : n \in Z, d \in Z, d \neq 0, \text{ and, } n \text{ and } d \text{ have no common factor}\right\}'.$$

Set of Real Numbers

The *set of real numbers* is the *union* of the *set of rational numbers* and the *set of irrational numbers*, and it is represented by the *symbol R*.

Thus $R = Q \cup Q' = Q \cup I$, when $U = R$.

So we have the following Venn diagram representing the *set of real numbers*.

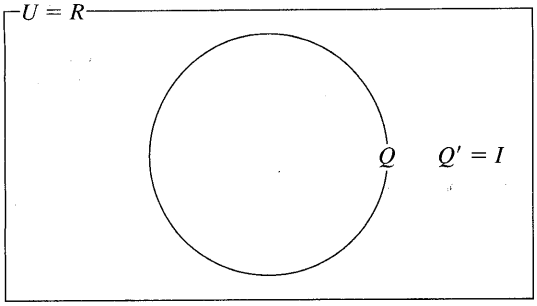


Fig. 2.2 Venn diagram

Alternatively, we have the more detailed Venn diagram representing the *set of real numbers*.

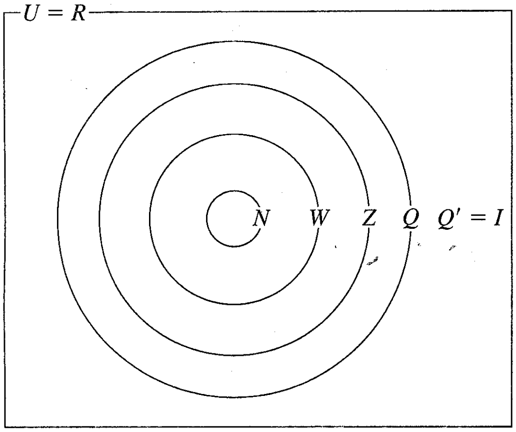


Fig. 2.3 Venn diagram

Thus $\quad N \subset W \subset Z \subset Q \subset R$
and $\quad Q' \subset R.$
So $\quad R = Q \cup Q' = Q \cup I.$

Basic Arithmetic Operations

The *four basic arithmetic operations* are:

(1) *Addition.*

(2) *Subtraction.*

(3) *Multiplication.*

(4) *Division.*

Thus:

(1) To *add* means to calculate a *sum*.
For example:
(a) Add the numbers 4 and 9.
The *sum* of the numbers $= 4 + 9$
$= 13.$

(2) To *subtract* means to *take away* or to calculate a *difference*.
For example:
(b) Subtract the number 4 from the number 9.
The *difference* of the numbers $= 9 - 4$
$= 5.$

(3) To *multiply* means to calculate a *product*.
For example:
(c) Multiply the number 3 and 5.
The *product* of the numbers $= 3 \times 5$
$= 15.$

(4) To *divide* means to calculate a *quotient*.
For example:
(d) Divide the number 8 by the number 2.
The *quotient* of the numbers $= 8 \div 2$
$= \frac{8}{2}$
$= 4.$

Some Meanings of Zero

Some *meanings* of *zero* are:

(a) *Zero* is used to *indicate* an *empty place value* in any number with *more* than *one digit*. For example: 74035 indicates that there are *zero hundreds* in the number seventy-four thousand and thirty-five.

(b) *Zero* is also the *number of elements* in the *empty* or *null set*. That is $n(\emptyset) = 0$.

(c) *Zero* is used to *represent* the *mid-point* on the *number line* between -1 and 1, -2 and 2, -3 and 3, et cetera.

This *fact* can be seen *illustrated* below.

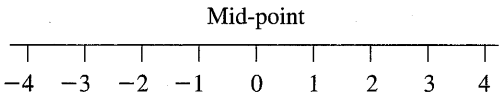


Fig. 2.4 Number line

(d) *Zero* can also be seen as the *identity* for the *addition of numbers*.

That is: $\quad 4 + 0 = 4$
$0 + 5 = 5$
$-4 + 0 = -4$
$0 + (-5) = -5$

Identity for Addition

The *identity* for an *operation* leaves the *original number unchanged* under the operation.

If *zero* is *added* to any *number*, then the *sum* is the *original number*.

Thus:
$$4 + 0 = 4$$
$$0 + 3 = 3$$
$$-4 + 0 = -4$$
$$0 + (-3) = -3$$

We say that *zero* is the *identity* for the *addition of numbers*.

Identity for Multiplication

If any *number* is *multiplied* by 1, then the *product* is the *original number*.

Thus:
$$8 \times 1 = 8$$
$$1 \times 9 = 9$$
$$-8 \times 1 = -8$$
$$1 \times (-9) = -9$$

We say that 1 is the *identity* for the *multiplication of numbers*.

Inverse for Numbers Under Addition

The *inverse of a number* for a given *operation combines* with the *number* under the *operation* to give the *identity*.

Thus:

The *inverse of* 5 *under addition* is −5,
since $5 + (-5) = 0$ (*identity*).
The *inverse of* −3 *under addition* is 3,
since $-3 + 3 = 0$ (*identity*).

Inverse for Numbers Under Multiplication

The *definition* for the *inverse of a number* was stated above.

Thus:

The *inverse of* 6 *under multiplication* is $\frac{1}{6}$,
since $6 \times \frac{1}{6} = 1$ (*identity*).

The *inverse of* −7 *under multiplication* is $-\frac{1}{7}$,
since $-7 \times \left(-\frac{1}{7}\right) = 1$ (*identity*).

Multiplication by Zero

If any *number* is *multiplied by zero*, then the *product* is always *zero*.

Thus:
$$8 \times 0 = 0$$
$$0 \times 7 = 0$$
$$-3 \times 0 = 0$$
$$0 \times (-1) = 0$$

Division by Zero

If any *number* is *divided by zero*, then we say that the *result* is *infinity*.

Thus:
$$\frac{3}{0} = +\infty$$
$$\frac{-4}{0} = -\infty$$

Sometimes it is *easier* to say that *division by zero* is a *meaningless operation*.

However, the *quotient of zero divided* by any *number other than zero* is always *zero*.

Thus:
$$\frac{0}{1} = \frac{0}{5} = \frac{0}{-3} = \frac{0}{-4} = 0$$

Law of Closure

The *law of closure* states that a *set of numbers* is *closed* under an *operation*, if when the operation is *performed* on any *two members of the set*, then the *result* is a *member of the set*.

Thus:

(a) $6 + 5 = 11$

(b) $3 \times 4 = 12$

So we say that the *set of whole numbers* is *closed* with respect to the *addition of numbers*, and the *multiplication of numbers*.

(c) $5 - 8 = -3$

(d) $-7 \div 2 = -3\frac{1}{2} = -3.5$

So we say that the *set of whole numbers* is *not closed* with respect to the *subtraction of numbers*, and the *division of numbers*.

Commutative Law

The *commutative law* for an *arithmetic operation* deals with the *order* in which the operation is *performed*.

Thus:

(a) $2 + 6 + 9 = 9 + 6 + 2 = 17$

(b) $2 \times 3 \times 5 = 5 \times 3 \times 2 = 30$

Hence the *addition of numbers*, and the *multiplication of numbers* are both *commutative*.

(c) $7 - 2 \neq 2 - 7$
i.e. $5 \neq -5$

(d) $8 \div 2 \neq 2 \div 8$ or $\frac{8}{2} \neq \frac{2}{8}$
i.e. $4 \neq \frac{1}{4}$

Hence the *subtraction of numbers*, and the *division of numbers* are both *non-commutative*.

Associative Law

The *associative law* for an *arithmetic operation* deals with *grouping* the numbers.

Thus:

(a) $3 + 4 + 7 = (3 + 4) + 7 = 3 + (4 + 7) = 14$

(b) $2 \times 4 \times 5 = (2 \times 4) \times 5 = 2 \times (4 \times 5) = 40$

Hence the *addition of numbers*, and the *multiplication of numbers* are both *associative*.

(c) $9 - 5 - 2 = (9 - 5) - 2 \neq 9 - (5 - 2)$
i.e. $2 = 2 \neq 6$

(d) $8 \div 4 \div 2 = (8 \div 4) \div 2 \neq 8 \div (4 \div 2)$
i.e. $1 = 1 \neq 4$

Hence the *subtraction of numbers*, and the *division of numbers* are both *non-associative*.

Distributive Law

The *distributive law* for an *arithmetic operation* deals with the *multiplication of numbers* in *brackets*.

(a) $3 \times (4 + 7) = 3 \times 4 + 3 \times 7 = 12 + 21$
$= 33$

(b) $4 \times (8 - 3) = 4 \times 8 + 4 \times (-3) = 32 - 12$
$= 20$

Hence we say that *multiplication is distributive* with respect to the *addition of numbers*, and the *subtraction of numbers*.

Powers of Numbers

The *number* $2 \times 2 \times 2 \times 2 \times 2$ can be written as 2^5, in index form; where 5 is called the *power* or *index*, and 2 is called the *base*. The *power* or *index* indicates how many *times* we are to *multiply* the *base*. For example: 3^4 means '3 *to the fourth power*'. The *index* 4 tells us that the *base* 3, appears *four times in the product*.

Thus: $3^4 = 3 \times 3 \times 3 \times 3 = 81.$

Note that $9^1 = 9$ and $1^0 = 2^0 = 3^0 = \cdots = 1$.

Hence any *number raised* to the *zero power* is equal to 1.

Example 1

(a) Simplify each of the following products, leaving your answers in index form:

(i) $2 \times 2 \times 3 \times 2 \times 3 \times 3 \times 2$
(ii) $4 \times 3 \times 5 \times 3 \times 4 \times 5 \times 5$
(iii) $7 \times 7 \times 8 \times 9 \times 8 \times 7$

(b) Calculate the value of each of the following:

(i) 2^8 *(ii)* 5^4 *(iii)* 7^3

(c) Determine the value of each of the following:

(i) $1^2 + 2^2 + 3^2 + 4^2$ *(ii)* $1^3 + 3^3 + 5^3$

(d) Simplify each of the following expressions:

(i) $8^5 \div 8^4$ *(ii)* $6^3 \div 6^3$

(iii) $9^5 \div 9^3$ *(iv)* $\dfrac{3^4 \times 5^3 \times 7^2}{3^2 \times 5^2 \times 7}$

(v) $(2^3)^4$

Solution

(a) (i) Now $2 \times 2 \times 3 \times 2 \times 3 \times 3 \times 2$

$= 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3$

$= 2^4 \times 3^3$

(ii) Now $4 \times 3 \times 5 \times 3 \times 4 \times 5 \times 5$

$= 3 \times 3 \times 4 \times 4 \times 5 \times 5 \times 5$

$= 3^2 \times 4^2 \times 5^3$

(iii) Now $7 \times 7 \times 8 \times 9 \times 8 \times 7$

$= 7 \times 7 \times 7 \times 8 \times 8 \times 9$

$= 7^3 \times 8^2 \times 9$

(b) (i) Now $2^8 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 256$

(ii) Now $5^4 = 5 \times 5 \times 5 \times 5 = 625$

(iii) Now $7^3 = 7 \times 7 \times 7 = 343$

(c) (i) Now $1^2 + 2^2 + 3^3 + 4^2$

$= 1 \times 1 + 2 \times 2 + 3 \times 3 + 4 \times 4$

$= 1 + 4 + 9 + 16$

$= 30$

(ii) Now $1^3 + 3^3 + 5^3$

$= 1 \times 1 \times 1 + 3 \times 3 \times 3 + 5 \times 5 \times 5$

$= 1 + 27 + 125$

$= 153$

(d) (i) Now $8^5 \div 8^4$

$$= \frac{\overset{1}{\cancel{8}} \times \overset{1}{\cancel{8}} \times \overset{1}{\cancel{8}} \times \overset{1}{\cancel{8}} \times 8}{\underset{1}{\cancel{8}} \times \underset{1}{\cancel{8}} \times \underset{1}{\cancel{8}} \times \underset{1}{\cancel{8}}} = \frac{8}{1} = 8$$

(ii) Now $6^3 \div 6^3$

$$= \frac{\overset{1}{\cancel{6}} \times \overset{1}{\cancel{6}} \times \overset{1}{\cancel{6}}}{\underset{1}{\cancel{6}} \times \underset{1}{\cancel{6}} \times \underset{1}{\cancel{6}}} = \frac{1}{1} = 1$$

(iii) Now $9^5 \div 9^3$

$$= \frac{\overset{1}{\cancel{9}} \times \overset{1}{\cancel{9}} \times \overset{1}{\cancel{9}} \times 9 \times 9}{\underset{1}{\cancel{9}} \times \underset{1}{\cancel{9}} \times \underset{1}{\cancel{9}}}$$

$= 9^2$ (in *index form*)

$= 81$ (as a *number*)

(iv) Now

$$\frac{3^4 \times 5^3 \times 7^2}{3^2 \times 5^2 \times 7}$$

$$= \frac{\overset{1}{\cancel{3}} \times \overset{1}{\cancel{3}} \times 3 \times 3 \times \overset{1}{\cancel{5}} \times \overset{1}{\cancel{5}} \times 5 \times \overset{1}{\cancel{7}} \times 7}{\underset{1}{\cancel{3}} \times \underset{1}{\cancel{3}} \times \underset{1}{\cancel{5}} \times \underset{1}{\cancel{5}} \times \underset{1}{\cancel{7}}}$$

$= 3^2 \times 5 \times 7$ (in *index form*)

$= 315$ (as a *number*)

(v) Now $(2^3)^4$

$= 2^3 \times 2^3 \times 2^3 \times 2^3$

$= 8 \times 8 \times 8 \times 8$

$= 8^4$ (in *index form*)

$= 4\,096$ (as a *number*)

Exercise 2a

1. Find the value of each of the following:

(a) 2^5 (b) 8^3 (c) 4^5

2. Calculate the value of each of the following:

(a) 3^4 (b) 10^1 (c) 10^5

3. Determine the value of each of the following:

(a) 6^3 (b) 7^3 (c) 9^4

4. Find the value of each of the following expressions:

(a) $2^3 \times 3^2$ (b) $1^4 + 3^4 + 5^4$

5. Calculate the value of each of the following expressions:

(a) $5^4 + 4^2$ (b) $2^3 \times 3^2 \times 4^2$

(c) $1^3 + 5^3 + 7^3$ (d) $2^3 \times 3^2 \times 7$

6. Determine the value of each of the following expressions:

(a) $2^3 \times 5^2$ (b) $2 \times 3^2 \times 7^2$

(c) $1^3 + 3^3 + 5^3 + 7^3$ (d) $1^4 + 2^4 + 3^4$

7. Calculate the exact value of $2^3 \times 3^2 \times 4$.

8. Calculate the value of each of the following expressions:

(a) $2^4 \times 3^2$ (b) $1^4 + 3^4 + 5^4 + 7^4$

9. Determine the value of $2^2 \times 3^2 \times 7^2$.

10. Find the value of $2 \times 3^2 \times 6^2$.

11. Simplify each of the following expressions:

(a) $2^3 \times 3^4 \times 2 \times 3$ (b) $\frac{5^2 \times 3^4 \times 2}{3^2 \times 5}$

(c) $(4^3)^2$ (d) $(8^2)^0$

12. Simplify each of the following expressions, leaving your answers in index form where possible:

(a) $8^3 \times 8^5 \times 8^2$ (b) $7^4 \div 7^4$
(c) $9^4 \div 9^5$ (d) $5^2 \times 5^4 \times 5^3$

13. Write the following product in index form:
$2 \times 3 \times 5 \times 2 \times 2 \times 3 \times 5 \times 5$.

14. Express the following product in index form:
$2 \times 3 \times 2 \times 5 \times 3 \times 2 \times 5$.

15. Write each of the following products in index form:

(a) $18 \times 18 \times 18 \times 18$
(b) $2 \times 2 \times 3 \times 5 \times 3 \times 3 \times 5 \times 5 \times 5 \times 2$

16. Express each of the following products in index form:

(a) $2 \times 3 \times 5 \times 2 \times 2 \times 5 \times 3$
(b) $6 \times 6 \times 7 \times 9 \times 9 \times 9 \times 7 \times 6 \times 3 \times 9$

17. Write each of the following numbers in index form:

(a) 27 (b) 64

18. State each of the following products in index form:

(a) $2 \times 2 \times 2 \times 2 \times 2$
(b) $2 \times 2 \times 3 \times 3 \times 2 \times 3 \times 5 \times 5 \times 2 \times 5$

19. Express each of the following numbers in index form:

(a) 16 (b) 81

20. Write as a single expression in index form:

(a) $3^6 \times 3^5$ (b) $10^5 \times 10^3 \times 10$

21. Express as a single expression in index form:

(a) $10^9 \div 10^5$ (b) $9^{13} \div 9^{12}$

22. Write each of the following products in index form:

(a) $18 \times 18 \times 18 \times 18 \times 18$
(b) $2 \times 2 \times 3 \times 5 \times 3 \times 3 \times 5 \times 5 \times 5$

23. State as a single expression in index form:

(a) $4^5 \times 4^3 \div 4^6$ (b) $(7^5 \div 7^2) \times 7$

24. Write as a single number in index form:

$$\frac{4^2 \times 4^6}{4^3}.$$

25. Simplify the following expressions:

(a) $5^3 \div 5^3$ (b) $6^4 \div 6^3$
(c) $7^5 \div 7^5$ (d) $8^3 \div 8^4$

26. Simplify the following expressions:

(a) $(2^3)^2$ (b) $(3^2)^3$

27. Simplify the following expressions:

(a) $(5^2)^4$ (b) $(8^3)^2$

28. Simplify the following expressions:

(a) $(10^3)^5$ (b) $(7^3)^2$

29. Find the value of $3^5 \times 3^2 \div 3^7$.

30. Simplify the following expressions:

$$\frac{5^2 \times 3^4 \times 2}{3^2 \times 5}.$$

Defined Arithmetic Operations

Apart from the *four basic arithmetic operations* we can *define* many more *operations in arithmetic*.

Example 2

(a) The operation † means subtract 3 from the first number, then add the result to the second number.
Use the defined operation † to work out the following:
(i) 8 † 2 (ii) 5 † 1

*(b) The operation * means multiply the first number by 5, then subtract the second number from the result.*
*Use the defined operation * to work out the following:*
*(i) 3 * 1 (ii) 4 * 7*

Solution

(a) (i) Now $8 \dagger 2 = (8 - 3) + 2 = 5 + 2 = 7$
(ii) Now $5 \dagger 1 = (5 - 3) + 1 = 2 + 1 = 3$

(b) (i) Now $3 * 1 = (3 \times 5) - 1 = 15 - 1 = 14$
(ii) Now $4 * 7 = (4 \times 5) - 7 = 20 - 7 = 13$

Exercise 2b

1. The operation † means add 4 to the first number, then add the result to the second number. Use the defined operation † to work out the following:

   (a) 3 † 2 (b) 16 † 5 (c) 210 † 17

2. The operation * means divide the first number by 3, then subtract the second number from the result. Use the defined operation * to work out the following:

   (a) 9 * 2 (b) 81 * 15 (c) 243 * 70

3. The operation □ means subtract 5 from the first number, then add the result to the second number. Use the defined operation □ to work out the following:

   (a) 9 □ 4 (b) 18 □ 11 (c) 125 □ 50

4. The operation ▭ means mutiply the first number by 4, then subtract the second number from the result. Use the defined operation ▭ to work out the following:

   (a) 7 ▭ 5 (b) 12 ▭ 11 (c) 15 ▭ 23

5. The operation ø means to multiply the first number by 10, then subtract twice the second number from the result. Hence solve the following:

   (a) 9 ø 4 (b) 12 ø 13 (c) 18 ø 17

6. The operation α means to divide the first number by 5, then add the result to twice the second number. Hence solve the following:

   (a) 25 α 3 (b) 120 α 15 (c) 125 α 18

7. The operation β means to add 9 to the first number and then subtract the second number from the result. Hence solve the following:

   (a) 11 β 5 (b) 21 β 18 (c) 25 β 19

8. The operation η means to subtract 8 from the first number and then add the result to thrice the second number. Hence solve the following:

   (a) 9 η 2 (b) 15 η 5 (c) 28 η 4

9. The operation γ means double the first number and then add the second number to the result. Hence solve the following:

   (a) 2 γ 14 (b) 4 γ 13 (c) 7 γ 19

10. The operation △ means square the first number and then subtract the second number from the result. Hence solve the following:

    (a) 3 △ 12 (b) 5 △ 15 (c) 7 △ 19

11. The operation μ means cube the first number and then add the result to twice the second number. Hence solve the following:

    (a) 1 μ 2 (b) 2 μ 3 (c) 3 μ 4

12. The operation ? means take the square root of the first number and then add the result to thrice the second number. Hence solve the following:

    (a) 9 ? 2 (b) 25 ? 4 (c) 49 ? 5

Factors of a Number

The *factors of a number* are those *numbers*, *including* 1 *and itself*, which can *divide exactly* into the *number*.

Example 3

(a) Find the factors of 40.

(b) State the set of factors of 40.

(c) List the pairs of factors of 40.

Solution

(a) Now $\frac{40}{1} = 40, \frac{40}{2} = 20, \frac{40}{4} = 10, \frac{40}{5} = 8,$

$\frac{40}{8} = 5, \frac{40}{10} = 4, \frac{40}{20} = 2$ and $\frac{40}{40} = 1.$

So the *factors* of 40 are 1, 2, 4, 5, 8, 10, 20 and 40.

(b) The *set of factors* of 40 is:
{1, 2, 4, 5, 8, 10, 20, 40}.

(c) Now
$$\begin{aligned} 40 &= 1 \times 40 \\ &= 2 \times 20 \\ &= 4 \times 10 \\ &= 5 \times 8 \end{aligned}$$

So the *pairs of factors* of 40 are:
1 × 40, 2 × 20, 4 × 10 and 5 × 8.

Set of Square Numbers

A *square number* is a number which can be *represented* by a *pattern of dots* in the *shape of a square.*

The *set of square numbers*
= {1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, …}.

Example 4

Represent the number 36 as a pattern of dots in the shape of a square.

Now $36 = 6 \times 6$.

So we have the following *pattern of dots* in the *shape of a square* representing the number 36:

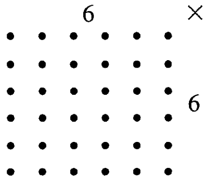


Fig. 2.5 Square

Set of Rectangular Numbers

A *rectangular number* is a number which can be *represented* by a *pattern of dots* in the *shape of a rectangle.*

The *set of rectangular numbers*
= {4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30, 32, 33, 34, 35, 36, 38, 39, 40, …}.

Note that the *number* 1 is a *square number* but *not* a *rectangle number.*

Example 5

Represent the number 36 as patterns of dots in the shape of a rectangle, giving all possibilities.

Now $36 = 2 \times 18 = 3 \times 12 = 4 \times 9 = 6 \times 6$.

So we have the following *patterns of dots* in the *shape of a rectangle* representing the number 36:

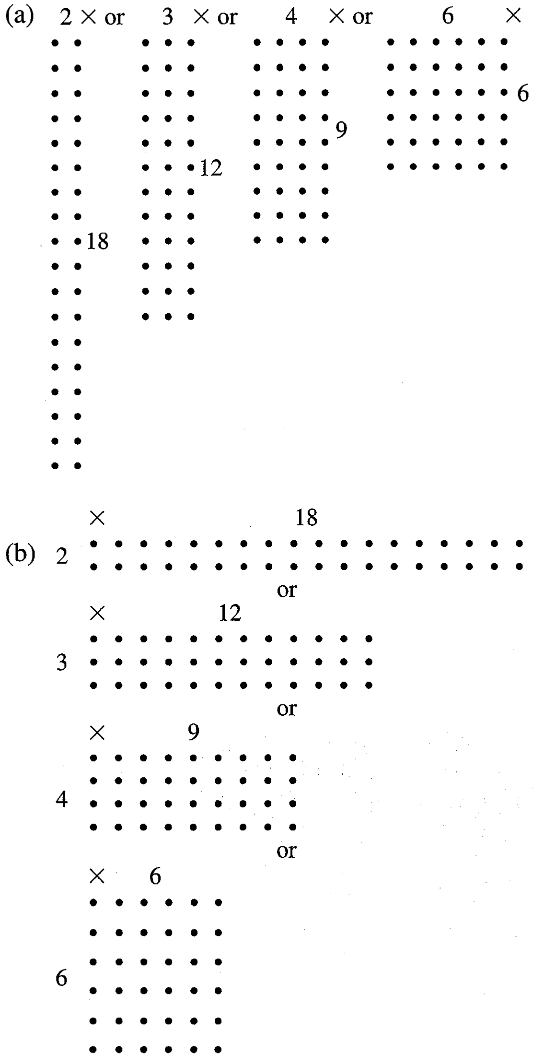


Fig. 2.6 Rectangles

Note that a *square* is a *rectangle* with *four equal sides. All square numbers, except* 1, are also *rectangular numbers.* The number 1 has only one factor.

Exercise 2c

1. State 88 as the product of two factors, giving all possibilities.
2. Express 18 as the product of two factors, giving all possibilities.

3. Write each of the following numbers as the product of two factors, giving all possibilities:

(i) 36 (ii) 100

4. List the set of factors of 88.

5. Write down the set of factors of 15.

6. State the pairs of factors of 28.

7. List the set of factors of 42.

8. State the set of factors of 55.

9. Represent the number 16 as a square array of dots.

10. Represent the number 49 as a pattern of dots in the shape of a square.

11. Represent the number 81 as a square array of dots.

12. Represent the number 144 as a pattern of dots in the shape of a square.

13. Represent the number 15 as a rectangular array of dots.

14. Represent the number 28 as a pattern of dots in the shape of a rectangle.

15. Represent the number 34 as a rectangular array of dots.

16. Represent the number 40 as a pattern of dots in the shape of a rectangle.

17. In the set {2, 4, 8, 16, 32, 64, 128, 516}, which of the members are perfect squares?

18. In the set {2, 4, 8, 16, 32, 64, 128, 516}, which of the members are perfect cubes?

Set of Prime Numbers

A *prime number* is a number which can only be *divided* exactly by *itself and* 1. That is, it has *itself and* 1 as the only *factors*. For example: $11 = 11 \times 1$, $23 = 23 \times 1$ and $37 = 37 \times 1$.

The *set of prime numbers*
= {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, ...}.

From above, it can be *seen* that:

> (i) A *prime number* is a number that is *not* a *rectangular number.*
>
> (ii) The *number* 1 is *neither* a *prime number, nor* a *rectangular number.*
>
> (iii) 2 is the *only prime number* that is also an *even number. All* other *prime numbers* are *odd numbers.*

Set of Composite Numbers

A *composite number* is a number which has *other factors* besides *itself* and 1.

The *set of composite numbers*
= {4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30, 32, 33, 34, 35, 36, 38, 39, 40, ...}.

Clearly it can be *seen* that, the *set of composite numbers* is *equal* to the *set of rectangular numbers.* That is, a *composite number* is a *rectangular number.*

Set of Either Prime or Composite Numbers

From above, it can be *seen* that:
The *set of either prime or composite numbers*
= {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, ...}

So {*prime numbers*} ∪ {*composite numbers*}

$= \{x: x \geqslant 2,\ x \in N\}$

$= \{x: x \geqslant 2,\ x \in W\}$

$= \{x: x \geqslant 2,\ x \in Z\}$.

Prime Factors of a Number

The *prime factors* of a number are *factors of the number* which are also *prime numbers.* We can write *any number* as a *product of prime factors.*

Example 6

(a) Find the set of factors of:
(i) 39 (ii) 40

(b) Hence state the set of prime factors of:
(i) 39 (ii) 40

(c) Write each of the following numbers as a product of prime factors:
(i) 28 (ii) 36 (iii) 420

Solution

(a) (i) Now $\frac{39}{1} = 39, \frac{39}{3} = 13, \frac{39}{13} = 3$ and $\frac{39}{39} = 1$

or $39 = 1 \times 39 = 3 \times 13.$

So the *set of factors* of 39 is $\{1, 3, 13, 39\}$.

(ii) Now $40 = 1 \times 40 = 2 \times 20 = 4 \times 10 = 5 \times 8.$

So the *set of factors* of 40 is $\{1, 2, 4, 5, 8, 10, 20, 40\}$.

(b) (i) The *set of prime factors* of 39 = $\{3, 13\}$

(ii) The *set of prime factors* of 40 = $\{2, 5\}$.

(c) (i) Now

$$\begin{array}{r|r} 2 & 28 \\ \hline 2 & 14 \\ \hline 7 & 7 \\ \hline & 1 \\ \hline \end{array}$$

So 28 as a *product of prime factors*

$= 2 \times 2 \times 7$

$= 2^2 \times 7.$

(ii) Now

$$\begin{array}{r|r} 2 & 36 \\ \hline 2 & 18 \\ \hline 3 & 9 \\ \hline 3 & 3 \\ \hline & 1 \\ \hline \end{array}$$

So 36 as a *product of prime factors*

$= 2 \times 2 \times 3 \times 3$

$= 2^2 \times 3^2.$

(iii) Now

$$\begin{array}{r|r} 2 & 420 \\ \hline 2 & 210 \\ \hline 3 & 105 \\ \hline 5 & 35 \\ \hline 7 & 7 \\ \hline & 1 \\ \hline \end{array}$$

So 420 as a *product of prime factors*

$= 2 \times 2 \times 3 \times 5 \times 7$

$= 2^2 \times 3 \times 5 \times 7.$

From above, it can be *seen* that:

(i) We *divide* each *number continuously* by the *smallest prime number*, until the number *cannot* be *divided* exactly again by that *particular factor*.

(ii) We then perform the *division*, if possible, for the *next larger prime number*.

(iii) We keep *dividing* the number in the above fashion until the *quotient is* 1.

Multiples of a Number

A *multiple of a number* is *k times* the *number*, where *k* is a *natural number* or a *counting number*. For example: the *multiples* of 5 between 4 and 36 are 5, 10, 15, 20, 25, 30 and 35. And the *set of multiples of* 9 = $\{9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99, \ldots, 9k\}$. This is so since, $1 \times 9 = 9, 2 \times 9 = 18, 3 \times 9 = 27, 4 \times 9 = 36, 5 \times 9 = 45, 6 \times 9 = 54, 7 \times 9 = 63, 8 \times 9 = 72, 9 \times 9 = 81, 10 \times 9 = 90, 11 \times 9 = 99$ and $k \times 9 = 9k$, where $k \geqslant 1$ and $k \in N$.

Example 7

(a) State the set of multiples of 3 between 5 and 29.

(b) State the set of multiples of 14 between 7 and 84 inclusive.

(c) State the set of multiples of 10 between 20 and 80 exclusive.

Solution

(a) {*multiples* of 3 *between* 5 and 29}
$= \{6, 9, 12, 15, 18, 21, 24, 27\}$.

(b) {*multiples* of 14 *between* 7 and 84 *inclusive*}
$= \{14, 28, 42, 56, 70, 84\}$.

(c) {*multiples* of 10 *between* 20 and 80 *exclusive*}
$= \{30, 40, 50, 60, 70\}$.

Exercise 2d

1. Write the prime numbers that are less than 12.
2. List the set of prime numbers between 0 and 18.
3. State the set of prime numbers less than 25.
4. Write the set of prime numbers less than 31 inclusive.
5. State the set of prime numbers between 31 and 59 inclusive.
6. List the set of prime numbers between 42 and 71 exclusive.
7. Determine the set of prime numbers less than 100.
8. Express 760 in prime factors.
9. State the prime factors of 720.
10. Express 342 in prime factors.
11. State the prime factors of 750.
12. Express 360 as a product of prime factors.
13. Write 540 as a product of prime factors.
14. Express 504 as a product of prime factors.
15. Write 768 as a product of prime factors.
16. State 315 as a product of prime factors.
17. Write 1 575 as a product of prime factors.
18. Express 4 725 as a product of prime factors.
19. Write the set of multiples of 5 between 12 and 47.
20. State the set of multiples of 13 between 11 and 99.
21. Write the set of multiples of 7 between 33 and 64.
22. List the set of multiples of 8 between 5 and 95.
23. State the set of multiples of 4 between 8 and 36 inclusive.
24. List the set of multiples of 6 between 36 and 72 exclusive.
25. Write the set of multiples of 9 less than 63.
26. State the set of multiples of 10 less than 80.
27. What is the set of multiples of 2 less than 26?
28. What is the set of multiples of 3 greater than 3 but less than 27?

Set of Even Numbers

The *set of even numbers* consists of *natural numbers* that *can* be *exactly divided by* 2. So the *set of even numbers* consists of numbers that are *multiples of* 2. Hence, *even numbers* are natural numbers *ending* with the *digits* 0, 2, 4, 6 or 8. For example: 30, 12, 24, 56 and 78.

The *set of even numbers*

$$= \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, \ldots, 2k\},$$

where $k \in N$.

Set of Odd Numbers

The *set of odd numbers* consists of *natural numbers* that *cannot* be *exactly divided by* 2. So the *set of odd numbers* consists of *counting numbers* that are *not even*. Hence, *odd numbers* are natural numbers *ending* with the *digits* 1, 3, 5, 7 or 9. For example: 21, 53, 65, 87 and 69.

The *set of odd numbers*

$$= \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, \ldots, 2k + 1\},$$

where $k \in W$.

Set of Either Odd or Even Numbers

From above, it can be *seen* that:
The *set of either odd or even numbers*

$$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, \ldots\}$$

So

$$\begin{aligned} & \{\textit{odd numbers}\} \cup \{\textit{even numbers}\} \\ &= \{\textit{natural numbers}\} \\ &= \{x\colon x \geqslant 1,\ x \in N\} \\ &= \{x\colon x > 0,\ x \in W\}. \end{aligned}$$

Obviously then, *zero* is *neither odd nor even.*

Example 8

Write the members of each of the following sets:

(a) {even numbers less than 14}.

(b) {odd numbers less than 15}.

(c) {even numbers from 8 to 20 inclusive}.

(d) {odd numbers from 9 to 21 inclusive}.

(e) {even numbers between 36 and 48 exclusive}.

(f) {odd numbers between 39 and 49 exclusive}.

(g) {even numbers less than 13}.

(h) {odd numbers less than 12}.

Solution

(a) {*even numbers less than* 14}
= {2, 4, 6, 8, 10, 12}.

(b) {*odd numbers less than* 15}
= {1, 3, 5, 7, 9, 11, 13}.

(c) {*even numbers* from 8 to 20 *inclusive*}
= {8, 10, 12, 14, 16, 18, 20}.

(d) {*odd numbers* from 9 to 21 *inclusive*}
= {9, 11, 13, 15, 17, 19, 21}.

(e) {*even numbers between* 36 and 48 *exclusive*}
= {38, 40, 42, 44, 46}.

(f) {*odd numbers between* 39 and 49 *exclusive*}
= {41, 43, 45, 47}.

(g) {*even numbers less than* 13}
= {2, 4, 6, 8, 10, 12}.

(h) {*odd numbers less than* 12}
= {1, 3, 5, 7, 9, 11}.

Exercise 2e

1. Write the members of the set of even numbers less than 18.
2. State the members of the set of even numbers from 12 to 34 inclusive.
3. List the members of the set of even numbers between 28 and 46 exclusive.
4. Determine the members of the set of even numbers less than 21.
5. Write the members of the set of odd numbers less than 19.
6. State the members of the set of odd numbers from 15 to 33 inclusive.
7. List the members of the set of odd numbers between 21 and 45 exclusive.
8. Determine the members of the set of odd numbers less than 18.
9. Write the set of even numbers greater than 18 but less than 36.
10. State the set of even numbers greater than 21 but less than 45.
11. List the set of odd numbers greater than 31 but less than 49.
12. Determine the set of odd numbers greater than 52 but less than 74.

Highest Common Factor (H.C.F.)

The *highest common factor* (abbreviated to *H.C.F.*) is the *greatest of the common factors* of two or more *positive integers*. For example:

{*factors* of 15} = {1, 3, 5, 15} and
{*factors* of 18} = {1, 2, 3, 6, 9, 18}.
So {*common factors* of 15 and 18} = {1, 3}.

Hence the *highest common factor* (*H.C.F.*) of the numbers 15 and 18 is 3.

Example 9

Find the highest common factor (H.C.F.) of the numbers 15, 18 and 21.

Solution

Now {*factors* of 15} = {1, 3, 5, 15},
{*factors* of 18} = {1, 2, 3, 6, 9, 18}
and {*factors* of 21} = {1, 3, 7, 21}.
So {*common factors* of 15, 18 and 21}
= {1, 3}.

Hence the *highest common factor* (*H.C.F.*) of the numbers 15, 18 and 21 is 3.

Alternative Method 1

Now 15 as a *product of prime factors* = 3×5,
18 as a *product of prime factors* = $2 \times 3 \times 3$
and 21 as a *product of prime factors* = 3×7.

So the *highest common factor* (*H.C.F.*) of the numbers 15, 18 and 21 is 3.

Note that the *factor* 3 is *common* to *each of the numbers* 15, 18 and 21.

Alternative Method 2

Now
$$\begin{array}{r|l} 3 & 15,\ 18,\ 21 \\ \hline & 5,\ 6,\ 7 \end{array}$$

3 is the *greatest factor* that can *divide exactly* into 15, 18 and 21 at the *same time*.
Hence the *highest common factor* (*H.C.F.*) of the numbers 15, 18 and 21 is 3.

Lowest Common Multiple (L.C.M.)

The *lowest common multiple* (abbreviated to *L.C.M.*) is the *smallest of the common multiples* of two or more *positive integers*. For example:

{*multiples* of 6} = {6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96, …}.

and

{*multiples* of 9} = {9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99, …}.

So {*common multiples* of 6 and 9} = {18, 36, 54, 72, 90, …}.

Hence the *lowest common multiple (L.C.M.)* of the numbers 6 and 9 is 18.

Example 10

Find the lowest common multiple (L.C.M.) of the numbers 6, 9 and 15.

Solution

Now {*multiples* of 6}
= {6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 84, 90, 96, 102, 108, 114, 120, 126, 132, 138, 144, 150, 156, 162, 168, 174, 180, 186, …},

{*multiples* of 9}
= {9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99, 108, 117, 126, 135, 144, 153, 162, 171, 180, 189, …}.

and {*multiples* of 15}
= {15, 30, 45, 60, 75, 90, 105, 120, 135, 150, 165, 180, 195, …}.

So {*common multiples* of 6, 9 and 15}
= {90, 180, …}.

Hence the *lowest common multiple* (*L.C.M.*) of the numbers 6, 9 and 15 is 90.

Alternative Method 1

Now 6 as a *product of prime factors* = 2 × 3,
9 as a *product of prime factors* = 3 × 3
and 15 as a *product of prime factors* = 3 × 5.

So the *lowest common multiple* (*L.C.M.*) of the numbers 6, 9 and 15 = 2 × 3 × 3 × 5 = 90.

From above, it can be *seen* that:

(i) The *multiple of* 3, 9 = 3 × 3, is the *largest multiple of* 3 of the numbers 6, 9 and 15.

(ii) The *multiple of* 3, 9 = 3 × 3, is *not common* to *each of the numbers* 6, 9 and 15.

Alternative Method 2

Now
$$\begin{array}{r|l} 2 & 6,\ 9,\ 15 \\ \hline 3 & 3,\ 9,\ 15 \\ \hline 3 & 1,\ 3,\ 5 \\ \hline 5 & 1,\ 1,\ 5 \\ \hline & 1,\ 1,\ 1 \end{array}$$

Hence the *lowest common multiple* (*L.C.M.*) of the numbers 6, 9 and 15 = 2 × 3 × 3 × 5 = 90.

Note that in this *method*:

(i) We *divide* the *numbers* by *prime numbers* until the *quotients* are all 1.

(ii) The *lowest common multiple* (*L.C.M.*) is then the *product* of the *prime numbers*.

Exercise 2f

1. Find the H.C.F. of 24, 60 and 96.
2. State the H.C.F. of 12, 18 and 24.
3. Determine the H.C.F. of 20, 25, 35 and 45.
4. Find the H.C.F. of 12, 48 and 60.
5. State the highest number which is a factor of both 25 and 30.
6. A room measures 450 cm by 250 cm. Determine the length of the largest square tile that can be used to tile the floor without cutting.

7. A bathroom measures 250 cm by 175 cm. Calculate the side of the largest square tile that can be used to tile the floor without cutting.

8. A living room measures 450 cm by 330 cm. Find the dimension of the largest square tile that can be used to tile the floor without cutting.

9. Determine the largest number which is a factor of the numbers 130, 169 and 195.

10. Find the L.C.M. of 24, 60 and 96.

11. State the L.C.M. of 2, 6 and 9.

12. Determine the L.C.M. of 20, 25, 35 and 45.

13. Find the L.C.M. of 12, 48 and 60.

14. State the lowest number that is a multiple of 4 and 5.

15. What is the least sum of money that can be made up of an exact number of 5 ¢ pieces or 25 ¢ pieces?

16. What is the least sum of money that can be made up of an exact number of 10 ¢ pieces or 25 ¢ pieces?

17. In a school, it is possible to divide the pupils into equal sized classes of either 24 or 30 or 36 pupils and have no pupils left over. Determine the least number of pupils that can make this possible. How many classes will there be if each class is to have 30 pupils?

18. What is the smallest number of sweets that can be shared exactly among 5, 10 or 15 students?

Sequence of Numbers

A *sequence of numbers* is a *set of numbers* that follows a *mathematical rule. Each number* in the *sequence* is called a *term* and is given a *value* according to its *position. Each term* is *represented* by the *symbol T.* For example:

Given the sequence of numbers:
$-6, -4, -2, 0, 2, \ldots$

Then the *first term*, $T_1 = -6$.
The *second term*, $T_2 = -6 + 2 = -4$.
And the *third term*, $T_3 = -4 + 2 = -2$.

Hence the *rule* is:
Add 2 to the *previous term* in order to *obtain* the *next term.*

Thus:
The *sixth term*, $T_6 = 2 + 2 = 4$.
And the *seventh term*, $T_7 = 4 + 2 = 6$.

Example 11

Given the sequence of numbers: 5, 2.5, 1.25, 0.625, …

(a) State the rule being used to obtain a term in the sequence of numbers.

(b) Determine the fifth and sixth terms of the sequence.

Solution

(a) Now the *first term*, $T_1 = 5$

The *second term*, $T_2 = \frac{5}{2} = 2.5$

And the *third term*, $T_3 = \frac{2.5}{2} = 1.25$

Hence the *rule* being used is:
Divide the *previous term* by 2 in order to *obtain* the *next term.*

(b) The *fifth term*, $T_5 = \frac{0.625}{2} = 0.3125$

And the *sixth term*, $T_6 = \frac{0.3125}{2} = 0.15625$

Exercise 2g

1. Write down the next two terms in the sequence 3, 15, 75, …

2. List the next two terms in the sequence 1, 3, 2, 4, 3, …

3. State the next two terms in the series 7, 6, 8, …

4. Find the next two terms in the sequence: 9, 8, 10, 9, 11, …

5. Write the next two terms in the sequence: 81, 27, 9, …

6. List the next two terms in the sequence of numbers: 1, 3, 5, 7, …

7. State the next two terms in the sequence of numbers: 3, 12, 48, …

8. Determine the next two terms in the sequence of numbers: 162, 54, 18, …

9. Write the next two terms in the sequence of numbers: 6, 5, 7, 6, 8, …

10. Find the next two terms in the series: 1, 8, 27, …

11. Determine the next two terms in the sequence: 1, 4, 9, 16, 25, 36, …

12. State the next two terms in the sequence: 1, 9, 25, 49, …

13. State the next two terms in the sequence: 4, 16, 36, 64, …

14. List the next two terms in the series: 6, 9, 8, 11, 10, 13, 12, …

15. Determine the next two terms in the sequence of numbers: $-9, -6, -3, 0, 3, \ldots$

16. State the next two terms in the sequence of numbers: $-8, -4, -2, -1, -\frac{1}{2}, \ldots$

Number Bases

In *counting* the *number* of things we always use *groups*.

The *base* of a *number* is the *size* of the *group* used.

Human beings normally have *ten fingers* and *ten toes*, so it is natural for us to *count* in *groups of ten*. Our *normal counting system* uses a *base of ten*, because the *group size* used is ten, and it is called the *denary system* or the *decimal system*.

In the *denary system* we use the *ten digits* 0, 1, 2, 3, 4, 5, 6, 7, 8 *and* 9, along with the principle of place value to write all of our numbers.

Each *digit* of a number in base 10 is linked to a *place value* which is a *power of* 10.

Thus:

The number $9\,734_{10}$ = The number 9 734

$= (9 \times 10^3) + (7 \times 10^2) + (3 \times 10^1) + (4 \times 10^0)$

$= (9 \times 1\,000) + (7 \times 100) + (3 \times 10) + (4 \times 1)$

We can also *count* in *other bases*. *Digital computers* store and process data using *base two*. Because the *group size* used is *two*, this system is called the *binary system* or the *bicimal system*.

In the *binary system* we use the *two digits* 0 *and* 1. Each *digit* of a number in base 2 is linked to a *place value* which is a *power of* 2.

Thus:

The number $1\,101_2$

$= (1 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0)$

$= (1 \times 2^3) + (1 \times 2^2) + (1 \times 2^0)$

$= (1 \times 8) + (1 \times 4) + (1 \times 1)$

The *suffix* 2 in the *number* $1\,101_2$ indicates the *base* being used. A suffix is used normally, in order to indicate the base of a number, except when the base is 10.

In *the quinary system*, the *group size* used is *five*, and therefore we use the *five digits* 0, 1, 2, 3 *and* 4.

Each *digit* of a number in base 5 is linked to a *place value* which is a *power of* 5.

Thus:

The number $3\,142_5$

$= (3 \times 5^3) + (1 \times 5^2) + (4 \times 5^1) + (2 \times 5^0)$

$= (3 \times 125) + (1 \times 25) + (4 \times 5) + (2 \times 1)$

In the octal system, the *group size* used is *eight*, and therefore we use the *eight digits* 0, 1, 2, 3, 4, 5, 6 *and* 7.

Each *digit* of a number in base 8 is linked to a *place value* which is a *power of* 8.

Thus:

The number $6\,713_8$

$= (6 \times 8^3) + (7 \times 8^2) + (1 \times 8^1) + (3 \times 8^0)$

$= (6 \times 512) + (7 \times 64) + (1 \times 8) + (3 \times 1)$

These are just a few of the *number bases* possible. There are *computers* that use *base* 16 which is called the *hexadecimal system.*

Decimal System

In the *decimal system* or *denary system*, we *count* in *base* 10 and use the *ten digits* 0 *to* 9. Since the *number base* or *scale* is 10, each *digit* of a *number* has a *place value* in terms of *powers of* 10.

Thus:

The number $983\,275_{10}$ can be *represented* as:

Table 2.1

Place name	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Units
Place value	100 000 $= 10^5$	10 000 $= 10^4$	1 000 $= 10^3$	100 $= 10^2$	10 $= 10^1$	1 $= 10^0$
Digit	9	8	3	2	7	5

$= 9 \times 10^5 + 8 \times 10^4 + 3 \times 10^3 + 2 \times 10^2$
$+ 7 \times 10^1 + 5 \times 10^0$

And the number 0.4601_{10} can be *represented* as:

Table 2.2

Place name	Tenths	Hundredths	Thousandths	Ten thousandths
Place value	$\frac{1}{10} = 0.1$ $= 10^{-1}$	$\frac{1}{100} = 0.01$ $= 10^{-2}$	$\frac{1}{1000} = 0.001$ $= 10^{-3}$	$\frac{1}{10000} = 0.0001$ $= 10^{-4}$
Digit	4	6	0	1

$$= 4 \times 10^{-1} + 6 \times 10^{-2} + 0 \times 10^{-3} + 1 \times 10^{-4}$$
$$= 4 \times 10^{-1} + 6 \times 10^{-2} + 1 \times 10^{-4}$$

Hence the number 983275.4601_{10} can be *represented* as:

$9 \times 10^5 + 8 \times 10^4 + 3 \times 10^3 + 2 \times 10^2 +$
$7 \times 10^1 + 5 \times 10^0 + 4 \times 10^{-1} + 6 \times 10^{-2} +$
$0 \times 10^{-3} + 1 \times 10^{-4}$

Note that for any number, *powers* keep *increasing* by *increments* of *one* moving to the *left*, from *digit to digit*: and *decreasing* by *increments* of *one*, moving to the *right*, from *digit to digit*. The decimal point separates the positive indices and zero index from the negative indices of the place values.

Binary System

In the *binary system* or *bicimal system*, we *count* in *base* 2 and use the *two digits* 0 *and* 1. Since the *number base* or *scale* is 2, each *digit* of a *number* has a *place value* in terms of *powers of* 2.

Thus:

The number 111001_2 can be *represented* as:

Table 2.3

Place value	$2^5 = 32$	$2^4 = 16$	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$
Digit	1	1	1	0	0	1

$$= 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$
$$= 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^0$$

And the number 0.1101_2 can be *represented* as:

Table 2.4

Place value	2^{-1} $= \frac{1}{2}$ $= 0.5$	2^{-2} $= \frac{1}{2^2} = \frac{1}{4}$ $= 0.25$	2^{-3} $= \frac{1}{2^3} = \frac{1}{8}$ $= 0.125$	2^{-4} $= \frac{1}{2^4} = \frac{1}{16}$ $= 0.0625$
Digit	1	1	0	1

$$= 1 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3} + 1 \times 2^{-4}$$
$$= 1 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-4}$$

Hence the number 111001.1101 can be *represented* as:

$1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 +$
$1 \times 2^0 + 1 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3} + 1 \times 2^{-4}$

Note that *powers* keep *increasing* by *increments* of *one* moving to the *left*, from *digit to digit*; and *decreasing* by *increments* of *one*, moving to the *right*, from *digit to digit*. The bicimal point separates the positive indices and zero index from the negative indices of the place values.

Exercise 2h

1. Represent each of the following binary numbers using place values which are powers of 2:

(a) 101_2 (b) 1011_2
(c) 11011_2 (d) 101101_2

2. Represent each of the following binary numbers using place values which are powers of 2:

(a) 0.11_2 (b) 0.101_2
(c) 0.111_2 (d) 0.1101_2

3. Represent each of the following binary numbers using place values which are powers of 2:

(a) 11.1_2 (b) 101.01_2
(c) 1101.11_2 (d) 10111.101_2

4. Write each of the following ternary (base 3) numbers using place values which are powers of 3:

(a) 21_3 (b) 121_3
(c) 2012_3 (d) 21012_3

5. Write each of the following ternary (base 3) numbers using place values which are powers of 3:

(a) 0.21_3 (b) 0.212_3
(c) 0.1201_3 (d) 0.12102_3

6. Write each of the following ternary (base 3) numbers using place values which are powers of 3:

(a) 21.01_3 (b) 121.11_3
(c) 2121.01_3 (d) 21201.102_3

7. State each of the following quaternary (base 4) numbers with place values which are powers of 4:

(a) 31_4 (b) 213_4
(c) 1032_4 (d) 31203_4

8. State each of the following quaternary (base 4) numbers with place values which are powers of 4:

(a) 0.31_4 (b) 0.132_4
(c) $0.312\,3_4$ (d) $0.021\,31_4$

9. State each of the following quaternary (base 4) numbers with place values which are powers of 4:

(a) 21.3_4 (b) 132.12_4
(c) $2\,031.312_4$ (d) $31\,021.213_4$

10. Express each of the following quinary (base 5) numbers with place values which are powers of 5:

(a) 41_5 (b) 314_5
(c) $2\,034_5$ (d) $13\,421_5$

11. Express each of the following quinary (base 5) numbers with place values which are powers of 5:

(a) 0.43_5 (b) 0.412_5
(c) $0.304\,1_5$ (d) $0.413\,02_5$

12. Express each of the following quinary (base 5) numbers with place values which are powers of 5:

(a) 42.01_5 (b) 104.32_5
(c) $2\,413.03_5$ (d) $13\,402.104_5$

13. Write each of the following senary (base 6) numbers using place values which are powers of 6:

(a) 54_6 (b) 451_6
(c) $3\,504_6$ (d) $20\,513_6$

14. Write each of the following senary (base 6) numbers using place values which are powers of 6:

(a) 0.51_6 (b) 0.415_6
(c) $0.014\,3_6$ (d) $0.34\,105_6$

15. Write each of the following senary (base 6) numbers using place values which are powers of 6:

(a) 53.2_6 (b) 451.32_6
(c) $3\,450.014_6$ (d) $40\,513.205_6$

16. State each of the following septenary (base 7) numbers with place values which are powers of 7:

(a) 65_7 (b) 506_7
(c) $4\,613_7$ (d) $63\,045_7$

17. State each of the following septenary (base 7) numbers with place values which are powers of 7:

(a) 0.65_7 (b) 0.145_7
(c) $0.460\,5_7$ (d) $0.516\,04_7$

18. State each of the following septenary (base 7) numbers with place values which are powers of 7:

(a) 6.14_7 (b) 50.603_7
(c) $462.301\,5_7$ (d) $6\,504.013_7$

19. Represent each of the following octonary (octal or base 8) numbers using place values which are powers of 8:

(a) 74_8 (b) 607_8
(c) $7\,650_8$ (d) $57\,632_8$

20. Represent each of the following octonary (octal or base 8) numbers using place values which are multiples of 8:

(a) 0.71_8 (b) 0.673_8
(c) $0.507\,2_8$ (d) $0.071\,54_8$

21. Represent each of the following octonary (octal or base 8) numbers using place values which are powers of 8:

(a) 7.61_8 (b) 65.701_8
(c) 570.62_8 (d) $4\,673.710\,4_8$

22. Express each of the following nonary (base 9) numbers with place values which are powers of 9:

(a) 84_9 (b) 768_9
(c) $5\,438_9$ (d) $7\,680_9$

23. Express each of the following nonary (base 9) numbers with place values which are powers of 9:

(a) 0.48_9 (b) 0.763_9
(c) $0.840\,1_9$ (d) $0.704\,58_9$

24. Express each of the following nonary (base 9) numbers with place values which are powers of 9:

(a) 76.8_9 (b) $5\,048.05_9$
(c) $8\,160.134_9$ (d) $76\,800.451\,3_9$

25. State each of the following denary numbers with place values which are powers of 10:

(a) 98_{10} (b) 987_{10}
(c) $8\,695_{10}$ (d) $78\,903_{10}$

26. State each of the following denary numbers with place values which are powers of 10:

(a) 0.96_{10} (b) 0.895_{10}
(c) 0.768_{10} (d) $0.907\,63_{10}$

27. State each of the following denary numbers with place values which are powers of 10:

(a) 9.05_{10} (b) 95.13_{10}
(c) 894.043_{10} (d) $7\,640.981\,3_{10}$

Converting from Decimal to Bicimal

The *denary number equivalent* in *base two* is obtained from the *remainders* under *division by* 2, taken in a *specific order defined* by the *arrows* in each *problem* worked below.

Example 12

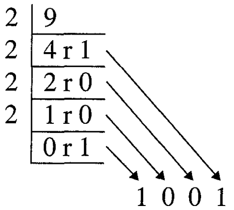
(a) Convert 9_{10} to a binary number.

(b) Convert 25_{10} to a binary number.

(c) Convert 147_{10} to a binary number.

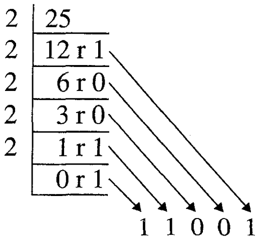
Solution

(a) Now



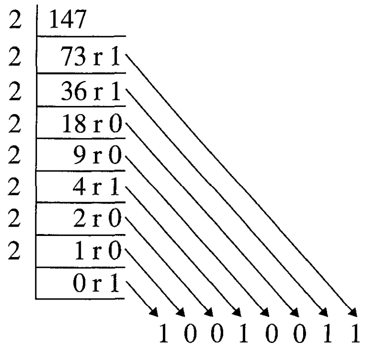
Thus $9_{10} = 1001_2$

(b) Now



Thus $25_{10} = 11001_2$

(c) Now



Thus $147_{10} = 10010011_2$

Alternative Method

In this *method*, we start by *dividing* the *denary number* by the *highest power of* 2 that can divide exactly into the number. The *remainder* is then exactly *divided* by the *highest possible power of* 2. We keep *dividing* in this manner until the *remainder* is *less than* 2.

The denary number is then written in terms of powers of 2, from which the *denary number equivalent* in *base* 2 can be obtained, as seen in the *problems* worked below.

(a) Now $\frac{9}{2^3} = \frac{9}{8} = 1 \text{ r } 1 \Rightarrow 1 \times 2^3$

And $\frac{1}{2^1} = \frac{1}{2} = 0 \text{ r } 1 \Rightarrow 0 \times 2^1 + 1 \times 2^0$

So $9_{10} = 1 \times 2^3 + 0 \times 2^1 + 1 \times 2^0$
$= 1001_2$

(b) Now $\frac{25}{2^4} = \frac{25}{16} = 1 \text{ r } 9 \Rightarrow 1 \times 2^4$

And $\frac{9}{2^3} = \frac{9}{8} = 1 \text{ r } 1 \Rightarrow 1 \times 2^3$

Also $\frac{1}{2^1} = \frac{1}{2} = 0 \text{ r } 1 \Rightarrow 0 \times 2^1 + 1 \times 2^0$

So $25_{10} = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^1 + 1 \times 2^0$
$= 11001_2$

(c) Now $\frac{147}{2^7} = \frac{147}{128} = 1 \text{ r } 19 \Rightarrow 1 \times 2^7$

And $\frac{19}{2^4} = \frac{19}{16} = 1 \text{ r } 3 \Rightarrow 1 \times 2^4$

Also $\frac{3}{2^1} = \frac{3}{2} = 1 \text{ r } 1 \Rightarrow 1 \times 2^1 + 1 \times 2^0$

So $147_{10} = 1 \times 2^7 + 1 \times 2^4 + 1 \times 2^1 + 1 \times 2^0$
$= 10010011_2$

Converting from Bicimal to Decimal

In *converting* from *binary numbers* to *denary numbers*, we use the *fact* that *each place value* is a *power of* 2.

Example 13

(a) Convert 1001_2 *to a decimal number.*

(b) Convert 11001_2 *to a decimal number.*

(c) Convert 10010011_2 *to a decimal number.*

(d) Convert 0.001_2 *to a decimal number.*

(e) Convert 0.0111_2 *to a decimal number.*

(f) Convert 0.110001_2 *to a decimal number.*

(g) Convert 11.01_2 *to a decimal number.*

(h) Convert 101.11_2 *to a decimal number.*

(i) Convert 1010.101_2 *to a decimal number.*

Solution

(a) Now

$$\begin{aligned}
&1001_2\\
&= 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0\\
&= 1 \times 8 + 0 \times 4 + 0 \times 2 + 1 \times 1\\
&= 8 + 0 + 0 + 1\\
&= 9_{10}
\end{aligned}$$

(b) Now

$$\begin{aligned}
&11001_2\\
&= 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0\\
&= 1 \times 16 + 1 \times 8 + 0 \times 4 + 0 \times 2 + 1 \times 1\\
&= 16 + 8 + 0 + 0 + 1\\
&= 25_{10}
\end{aligned}$$

(c) Now

$$\begin{aligned}
&10010011_2\\
&= 1 \times 2^7 + 0 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0\\
&= 1 \times 128 + 0 \times 64 + 0 \times 32 + 1 \times 16 + 0 \times 8 + 0 \times 4 + 1 \times 2 + 1 \times 1\\
&= 128 + 0 + 0 + 16 + 0 + 0 + 2 + 1\\
&= 147_{10}
\end{aligned}$$

(d) Now

$$\begin{aligned}
&0.001_2\\
&= 0 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3}\\
&= 0 + 0 + 0.125\\
&= 0.125_2
\end{aligned}$$

(e) Now

$$\begin{aligned}
&0.0111_2\\
&= 0 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3} + 1 \times 2^{-4}\\
&= 0 + 0.25 + 0.125 + 0.0625\\
&= 0.4375_{10}
\end{aligned}$$

(f) Now

$$\begin{aligned}
&0.110001_2\\
&= 1 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3} + 0 \times 2^{-4} + 1 \times 2^{-5}\\
&= 0.5 + 0.25 + 0 + 0 + 0.03125\\
&= 0.78125_{10}
\end{aligned}$$

(g) Now

$$\begin{aligned}
&11.01_2\\
&= 1 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2}\\
&= 1 \times 2 + 1 \times 1 + 0 \times \frac{1}{2} + 1 \times \frac{1}{4}\\
&= 2 + 1 + 0 + 0.25\\
&= 3.25_{10}
\end{aligned}$$

(h) Now

$$\begin{aligned}
&101.11_2\\
&= 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 1 \times 2^{-1} + 1 \times 2^{-2}\\
&= 1 \times 4 + 0 \times 2 + 1 \times 1 + 1 \times \frac{1}{2} + 1 \times \frac{1}{4}\\
&= 4 + 0 + 1 + 0.5 + 0.25\\
&= 5.75_{10}
\end{aligned}$$

(i) Now

$$\begin{aligned}
&1010.101_2\\
&= 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 + 1 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3}\\
&= 1 \times 8 + 0 \times 4 + 1 \times 2 + 0 \times 1 + 1 \times \frac{1}{2} + 0 \times \frac{1}{4} + 1 \times \frac{1}{8}\\
&= 8 + 0 + 2 + 0 + 0.5 + 0 + 0.125\\
&= 10.625_{10}
\end{aligned}$$

Adding Binary Numbers

The following *rules* apply when *adding binary numbers*:

(1) Now

$$\begin{array}{ccl} \underline{\text{Twos}} & \underline{\text{Units}} & \\ 0 & 1 & + \\ 0 & 0 & \\ \hline 0 & 1 & \\ \hline \end{array}$$

Thus $1 + 0 = 1$

(2) Now

$$\begin{array}{ccl} \underline{\text{Twos}} & \underline{\text{Units}} & \\ 0 & 0 & + \\ 0 & 1 & \\ \hline 0 & 1 & \\ \hline \end{array}$$

Thus $0 + 1 = 1$

(3) Now

$$\begin{array}{ccl} \underline{\text{Twos}} & \underline{\text{Units}} & \\ \overset{1}{0} & 1 & + \\ 0 & 1 & \\ \hline 1 & 0 & \text{(That is, 0 carry 1)} \\ \hline \end{array}$$

Thus $1 + 1 = 10$

Note that $1 + 1 = 10$ implies that the *sum* has *zero units* and one *group of two*.

Example 14

Add the following binary numbers:

(a) 111 and 1

(b) 1111 and 11

(c) 110101 and 10011

(d) 1110, 101 and 11011

Solution

(a) Now

$$\begin{array}{ccccl} & {\scriptstyle 1} & {\scriptstyle 1} & & \\ & 1 & 1 & 1 & + \\ & & & 1 & \\ \hline 1 & 0 & 0 & 0_2 & \\ \hline \end{array}$$

(b) Now

$$\begin{array}{cccccl} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & \\ & 1 & 1 & 1 & 1 & + \\ & & & 1 & 1 & \\ \hline 1 & 0 & 0 & 1 & 0_2 & \\ \hline \end{array}$$

(c) Now

$$\begin{array}{cccccccl} & {\scriptstyle 1} & & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & \\ & 1 & 1 & 0 & 1 & 0 & 1 & + \\ & & 1 & 0 & 0 & 1 & 1 & \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 0_2 & \\ \hline \end{array}$$

(d) Now

$$\begin{array}{ccccccl} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & \\ & & 1 & 1 & 1 & 0 & + \\ & & & 1 & 0 & 1 & \\ & 1 & 1 & 0 & 1 & 1 & \\ \hline 1 & 0 & 1 & 1 & 1 & 0_2 & \\ \hline \end{array}$$

Subtracting Binary Numbers

The following *rules* apply when *subtracting binary numbers*:

(1) Now

$$\begin{array}{ccl} \underline{\text{Twos}} & \underline{\text{Units}} & \\ 0 & 1 & - \\ 0 & 0 & \\ \hline 0 & 1 & \\ \hline \end{array}$$

Thus $1 - 0 = 1$

(2) Now

$$\begin{array}{ccl} \underline{\text{Twos}} & \underline{\text{Units}} & \\ 0 & 1 & - \\ 0 & 1 & \\ \hline 0 & 0 & \\ \hline \end{array}$$

Thus $1 - 1 = 0$

Example 15

Find the difference between each pair of the following binary numbers:

(a) 11101 − 101 *(b) 11111 − 1011*

(c) 10001 − 1011

Solution

(a) Now

$$\begin{array}{cccccl} 1 & 1 & 1 & 0 & 1 & - \\ & & 1 & 0 & 1 & \\ \hline 1 & 1 & 0 & 0 & 0_2 & \\ \hline \end{array}$$

(b) Now

$$\begin{array}{cccccl} 1 & 1 & 1 & 1 & 1 & - \\ & 1 & 0 & 1 & 1 & \\ \hline 1 & 0 & 1 & 0 & 0_2 & \\ \hline \end{array}$$

(c) Now

$$\begin{array}{cccccl} {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\ \cancel{1} & \cancel{0} & \cancel{0} & {}^{1}0 & 1 & - \\ & 1 & 0 & 1 & 1 & \\ \hline & & 1 & 1 & 0_2 & \\ \hline \end{array}$$

Multiplying Binary Numbers

The *multiplication table* for *binary numbers* is as follows:

Table 2.5

×	0	1
0	0	0
1	0	1

Thus $0 \times 0 = 0$

$0 \times 1 = 0$

$1 \times 0 = 0$

And $1 \times 1 = 1$

Example 16

Find the product of the following binary numbers:

(a) 111 × 10

(b) 1011 × 101

(c) 11101 × 1011

Solution

(a) Now

$$\begin{array}{rl} 111 & \times \\ 10 & \\ \hline 1110_2 & \\ \hline \end{array}$$

(b) Now

$$\begin{array}{rl} 1011 & \times \\ 101 & \\ \hline 1\overset{1}{0}1100 & + \\ 1011 & \\ \hline 110111_2 & \\ \hline \end{array}$$

(c) Now

$$\begin{array}{rl} 11101 & \times \\ 1011 & \\ \hline \overset{1}{1}\overset{1}{1}\overset{1}{1}\overset{1}{0}1000 & \\ 111010 & + \\ 11101 & \\ \hline 100111111_2 & \\ \hline \end{array}$$

Exercise 2i

1. Convert each of the following denary numbers to a binary number:
   (a) 5_{10} (b) 8_{10}
   (c) 10_{10} (d) 19_{10}
2. Convert each of the following denary numbers to a binary number:
   (a) 67_{10} (b) 78_{10}
   (c) 185_{10} (d) 341_{10}
3. Convert each of the following decimal numbers to a bicimal number:
   (a) 435_{10} (b) 487_{10}
   (c) 507_{10} (d) 510_{10}
4. Convert each of the following binary numbers to a denary number:
   (a) 101_2 (b) 1110_2
   (c) 10111_2 (d) 111011_2
5. Convert each of the following bicimal numbers to a decimal number:
   (a) 0.111_2 (b) 0.1110_2
   (c) 0.11101_2 (d) 0.11111_2
6. Convert each of the following bicimal numbers to a decimal number:
   (a) 11.01_2 (b) 101.11_2
   (c) 1111.01_2 (d) 111.11_2
7. Add the following binary numbers as indicated:
   (a) $1011_2 + 101_2$ (b) $1111_2 + 110_2$
   (c) $1001_2 + 111_2$ (d) $10011_2 + 110_2$
8. Sum the following binary numbers as indicated:
   (a) $11101_2 + 111_2$ (b) $11111_2 + 101_2$
   (c) $111101_2 + 1011_2$ (d) $11111_2 + 1111_2$
9. Add the following binary numbers as indicated:
   (a) $11101_2 + 1101_2$ (b) $11011_2 + 1111_2$
   (c) $11111_2 + 1011_2$ (d) $11001_2 + 1111_2$
10. Sum the following binary numbers as indicated:
    (a) $110101_2 + 110111_2$
    (b) $111111_2 + 101101_2$
    (c) $1011101_2 + 101101_2$
    (d) $1111011_2 + 110111_2$
11. Subtract the following binary numbers as indicated:
    (a) $111_2 - 101_2$ (b) $1111_2 - 1101_2$
    (c) $1110_2 - 1011_2$ (d) $1010_2 - 111_2$
12. Subtract the following binary numbers as indicated:
    (a) $1101_2 - 1011_2$ (b) $1011_2 - 1001_2$
    (c) $10111_2 - 1011_2$ (d) $1111_2 - 101_2$
13. Find the difference between each pair of the following binary numbers:
    (a) $110101_2 - 110011_2$
    (b) $110111_2 - 10111_2$
    (c) $100101_2 - 11011_2$
    (d) $111111_2 - 11011_2$
14. Multiply each pair of the following binary numbers:
    (a) $111_2 \times 10_2$ (b) $1011_2 \times 101_2$
    (c) $1111_2 \times 11_2$ (d) $1111_2 \times 111_2$

15. Find the product of the following binary numbers as indicated:

(a) $101_2 \times 11_2$ (b) $111_2 \times 101_2$
(c) $1011_2 \times 111_2$ (d) $1101_2 \times 111_2$

16. Find the product of each of the following pairs of binary numbers:

(a) $11111_2 \times 111_2$ (b) $10011_2 \times 101_2$
(c) $11101_2 \times 11_2$ (d) $10111_2 \times 101_2$

Numbers to Base Five

In writing *numbers* to *base* 5, we use the *digits* 0 *to* 4. Since the *number base* or *scale* is 5, each *digit* of a *number* has a *place value* in terms of *powers of* 5.

Thus:

The number 10324_5 can be *represented* as:

Table 2.6

Place value	5^4 = 625	5^3 = 125	5^2 = 25	5^1 = 5	5^0 = 1
Digit	1	0	3	2	4

$$= 1 \times 5^4 + 0 \times 5^3 + 3 \times 5^2 + 2 \times 5^1 + 4 \times 5^0$$
$$= 1 \times 5^4 + 3 \times 5^2 + 2 \times 5^1 + 4 \times 5^0$$

And the number 0.312_5 can be *represented* as:

Table 2.7

Place value	$5^{-1} = \frac{1}{5}$ = 0.2	$5^{-2} = \frac{1}{25}$ = 0.04	$5^{-3} = \frac{1}{125}$ = 0.008
Digit	3	1	2

$$= 3 \times 5^{-1} + 1 \times 5^{-2} + 2 \times 5^{-3}$$

Hence the number 10324.312_5 can be represented as:
$1 \times 5^4 + 0 \times 5^3 + 3 \times 5^2 + 2 \times 5^1 + 4 \times 5^0 +$
$3 \times 5^{-1} + 1 \times 5^{-2} + 2 \times 5^{-3}$

Converting from Decimal to Base Five

The *denary number equivalent* in *base five* is obtained from the *remainders* under *division by* 5, taken in a *specific order defined* by the *arrows* in each *problem* worked below.

Example 17

(a) Convert 89_{10} to a number in base 5.

(b) Convert 348_{10} to a number in base 5.

Solution

(a) Now

$$\begin{array}{r|l} 5 & 89 \\ \hline 5 & 17 \text{ r } 4 \\ \hline 5 & 3 \text{ r } 2 \\ \hline & 0 \text{ r } 3 \\ \hline \end{array}$$

3 2 4

Thus $89_{10} = 324_5$

(b) Now

$$\begin{array}{r|l} 5 & 348 \\ \hline 5 & 69 \text{ r } 3 \\ \hline 5 & 13 \text{ r } 4 \\ \hline 5 & 2 \text{ r } 3 \\ \hline & 0 \text{ r } 2 \\ \hline \end{array}$$

2 3 4 3

Thus $348_{10} = 2343_5$

Alternative Method

In this *method*, we start by *dividing* the *denary number* by the *highest power of* 5 that can divide exactly into the number. The *remainder* is then exactly *divided* by the *highest possible power of* 5. We keep *dividing* in this manner until the *remainder* is *less than* 5.

The denary number is then written in terms of powers of 5, from which the *denary number equivalent* in *base* 5 can be obtained, as seen in the *problems* worked below.

(a) Now $\frac{89}{5^2} = \frac{89}{25} = 3 \text{ r } 14 \Rightarrow 3 \times 5^2$

And $\frac{14}{5^1} = \frac{14}{5} = 2 \text{ r } 4 \Rightarrow 2 \times 5^1 + 4 \times 5^0$

So $89_{10} = 3 \times 5^2 + 2 \times 5^1 + 4 \times 5^0 = 324_5$

(b) Now $\frac{348}{5^3} = \frac{348}{125} = 2 \text{ r } 98 \Rightarrow 2 \times 5^3$

And $\frac{98}{5^2} = \frac{98}{25} = 3 \text{ r } 23 \Rightarrow 3 \times 5^2$

Also $\frac{23}{5^1} = \frac{23}{5} = 4 \text{ r } 3 \Rightarrow 4 \times 5^1 + 3 \times 5^0$

So $348_{10} = 2 \times 5^3 + 3 \times 5^2 + 4 \times 5^1 + 3 \times 5^0$
$= 2343_5$

Converting from Base Five to Decimal

In *converting* from *numbers written* in *base* 5 to *denary numbers*, we use the fact that *each place value* is a *power of* 5.

Example 18

(a) Convert 341_5 to a decimal number.
(b) Convert $40\,312_5$ to a decimal number.
(c) Convert 0.324_5 to a decimal number.
(d) Convert $0.430\,2_5$ to a decimal number.
(e) Convert 41.23_5 to a decimal number.
(f) Convert 124.03_5 to a decimal number.

Solution

(a) Now

$$\begin{aligned} & 341_5 \\ &= 3 \times 5^2 + 4 \times 5^1 + 1 \times 5^0 \\ &= 3 \times 25 + 4 \times 5 + 1 \times 1 \\ &= 75 + 20 + 1 \\ &= 96_{10} \end{aligned}$$

(b) Now

$$\begin{aligned} & 40\,312_5 \\ &= 4 \times 5^4 + 0 \times 5^3 + 3 \times 5^2 \\ &\quad + 1 \times 5^1 + 2 \times 5^0 \\ &= 4 \times 625 + 0 \times 125 + 3 \times 25 \\ &\quad + 1 \times 5 + 2 \times 1 \\ &= 2500 + 0 + 75 + 5 + 2 \\ &= 2\,582_{10} \end{aligned}$$

(c) Now

$$\begin{aligned} & 0.324_5 \\ &= 3 \times 5^{-1} + 2 \times 5^{-2} + 4 \times 5^{-3} \\ &= 3 \times \frac{1}{5} + 2 \times \frac{1}{25} + 4 \times \frac{1}{125} \\ &= 0.6 + 0.08 + 0.032 \\ &= 0.712_{10} \end{aligned}$$

(d) Now

$$\begin{aligned} & 0.430\,2_5 \\ &= 4 \times 5^{-1} + 3 \times 5^{-2} + 0 \times 5^{-3} \\ &\quad + 2 \times 5^{-4} \\ &= 4 \times \frac{1}{5} + 3 \times \frac{1}{25} + 0 \times \frac{1}{125} \\ &\quad + 2 \times \frac{1}{625} \\ &= 0.8 + 0.12 + 0 + 0.003\,2 \\ &= 0.923\,2_{10} \end{aligned}$$

(e) Now

$$\begin{aligned} & 41.23_5 \\ &= 4 \times 5^1 + 1 \times 5^0 + 2 \times 5^{-1} \\ &\quad + 3 \times 5^{-2} \\ &= 4 \times 5 + 1 \times 1 + 2 \times \frac{1}{5} \\ &\quad + 3 \times \frac{1}{25} \\ &= 20 + 1 + 0.4 + 0.12 \\ &= 21.52_{10} \end{aligned}$$

(f) Now

$$\begin{aligned} & 124.03_5 \\ &= 1 \times 5^2 + 2 \times 5^1 + 4 \times 5^0 \\ &\quad + 0 \times 5^{-1} + 3 \times 5^{-2} \\ &= 1 \times 25 + 2 \times 5 + 4 \times 1 \\ &\quad + 0 \times \frac{1}{5} + 3 \times \frac{1}{25} \\ &= 25 + 10 + 4 + 0 + 0.12 \\ &= 39.12_{10} \end{aligned}$$

Adding Base Five Numbers

The *addition table* for *base* 5 *numbers* can be seen below.

Table 2.8

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

Example 19

Add the following quinary numbers:
(a) 432_5 and 104_5 *(b) 301_5 and $2\,144_5$*

Solution

(a) Now

$$\begin{array}{r} \overset{1}{4}32 \\ 104 \\ \hline 1\,041_5 \\ \hline \end{array} +$$

(b) Now

$$\begin{array}{r} {\scriptstyle 1\,1\,1} \\ 301 \\ 2\,144 \\ \hline 3\,000_5 \\ \hline \end{array} +$$

Subtracting Base Five Numbers

In *subtracting base* 5 *numbers*, we use *similar rules* as those for the *addition of base* 5 *numbers*.

Example 20

Find the difference between each pair of the following quinary numbers:

(a) $3412_5 - 203_5$ *(b)* $4210_5 - 2401_5$

Solution

(a) Now
$$\begin{array}{r} 3\,4\,\overset{0}{\cancel{1}}\,{}^{1}2 \\ \underline{203} \\ \underline{3\,204_5} \end{array} -$$

(b) Now
$$\begin{array}{r} \overset{3}{\cancel{4}}\,{}^{1}2\,\overset{0}{\cancel{1}}\,{}^{1}0 \\ \underline{2\,401} \\ \underline{1\,304_5} \end{array} -$$

Multiplying Base Five Numbers

The *multiplication table* for *base* 5 *numbers* is as follows:

Table 2.9

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

Example 21

Find the product of the following quinary numbers:

(a) $431_5 \times 20_5$ *(b)* $3412_5 \times 103_5$

Solution

(a) Now
$$\begin{array}{r} \overset{1}{4}31 \\ \underline{20} \\ \underline{14\,120_5} \end{array} \times$$

(b) Now
$$\begin{array}{r} \overset{2}{3}\,4\,\overset{1}{1}\,2 \\ \underline{103} \\ \overset{1}{3}41\,200 \\ \underline{21\,241} \\ \underline{412\,441_2} \end{array} \begin{array}{l} \times \\ \\ + \\ \\ \\ \end{array}$$

Exercise 2j

1. Convert each of the following denary numbers to its base 5 equivalent:

(a) 45_{10} (b) 67_{10}
(c) 89_{10} (d) 103_{10}

2. Convert each of the following decimal numbers to its base 5 equivalent:

(a) 247_{10} (b) 268_{10}
(c) 349_{10} (d) 847_{10}

3. Convert each of the following base 5 numbers to its denary equivalent:

(a) 34_5 (b) 41_5
(c) 134_5 (d) 431_5

4. Convert each of the following base 5 numbers to its decimal equivalent:

(a) 0.143_5 (b) 0.342_5
(c) 0.412_5 (d) 0.2143_5

5. Convert each of the following quinary numbers to a decimal number:

(a) 43.21_5 (b) 34.12_5
(c) 124.102_5 (d) 324.241_5

6. Add the following quinary numbers as indicated:

(a) $43_5 + 34_5$ (b) $343_5 + 132_5$
(c) $241_5 + 344_5$ (d) $143_5 + 234_5$

7. Sum the following quinary numbers as indicated:

(a) $1034_5 + 2331_5$ (b) $2134_5 + 1032_5$
(c) $4321_5 + 3412_5$ (d) $3412_5 + 4113_5$

8. Subtract the following quinary numbers as indicated:

(a) $321_5 - 42_5$ (b) $423_5 - 234_5$
(c) $201_5 - 124_5$ (d) $104_5 - 34_5$

9. Find the difference between each pair of the following quinary numbers:

(a) $1034_5 - 432_5$ (b) $2341_5 - 1342_5$
(c) $3044_5 - 2431_5$ (d) $4132_5 - 3432_5$

10. Multiply each pair of the following quinary numbers:

(a) $43_5 \times 20_5$ (b) $124_5 \times 31_5$
(c) $234_5 \times 14_5$ (d) $312_5 \times 13_5$

11. Find the product of each of the following pairs of quinary numbers:

(a) $123_5 \times 21_5$ (b) $243_5 \times 42_5$
(c) $302_5 \times 132_5$ (d) $412_5 \times 103_5$

Octal Numbers

Octal numbers are numbers to *base* 8. We therefore use the *digits* 0 *to* 7. Since the *number base* or *scale* is 8, each *digit* of a *number* has a *place value* in terms of *powers of* 8.

Octal numbers are used by *computers* as a *shorthand* for *binary numbers.*

Thus:

The number 76401_8 can be *represented* as:

Table 2.10

Place value	8^4 = 4096	8^3 = 512	8^2 = 64	8^1 = 8	8^0 = 1
Digit	7	6	4	0	1

$$= 7 \times 8^4 + 6 \times 8^3 + 4 \times 8^2 + 0 \times 8^1 + 1 \times 8^0$$
$$= 7 \times 8^4 + 6 \times 8^3 + 4 \times 8^2 + 1 \times 8^0$$

And the number 0.43_8 can be *represented* as:

Table 2.11

Place value	$8^{-1} = \frac{1}{8} = 0.125$	$8^{-2} = \frac{1}{64} = 0.015625$
Digit	4	3

$$= 4 \times 8^{-1} + 3 \times 8^{-2}$$

Hence the number 76401.43_8 can be *represented* as:
$7 \times 8^4 + 6 \times 8^3 + 4 \times 8^2 + 0 \times 8^1 + 1 \times 8^0 + 4 \times 8^{-1} + 3 \times 8^{-2}$

Converting from Decimal to Octal

The *denary number equivalent* in *base* 8 is obtained from the *remainders* under *division by* 8, taken in a *specific order defined* by the *arrows* in each *problem* worked below.

Example 22

(a) Convert 98_{10} to a number in base 8.
(b) Convert 985_{10} to a number in base 8.

Solution

(a) Now

$$\begin{array}{r|l} 8 & \overset{1}{9}8 \\ \hline 8 & 12 \text{ r } 2 \\ \hline 8 & 1 \text{ r } 4 \\ \hline & 0 \text{ r } 1 \\ \hline \end{array} \quad \searrow \; 1\;4\;2$$

Thus $98_{10} = 142_8$

(b) Now

$$\begin{array}{r|l} 8 & \overset{1\,2}{985} \\ \hline 8 & 1\overset{4}{2}3 \text{ r } 1 \\ \hline 8 & 15 \text{ r } 3 \\ \hline 8 & 1 \text{ r } 7 \\ \hline & 0 \text{ r } 1 \\ \hline \end{array} \quad \searrow \; 1\;7\;3\;1$$

Thus $985_{10} = 1731_8$

Alternative Method

In this *method*, we start by *dividing* the *denary number* by the *highest power of* 8 that can divide exactly into the number. The *remainder* is then exactly *divided* by the *highest possible power of* 8. We keep *dividing* in this manner until the *remainder* is *less than* 8.

The denary number is then written in terms of powers of 8, from which the *denary number equivalent* in *base* 8, can be obtained as seen in the *problems* worked below.

(a) Now $\frac{98}{8^2} = \frac{98}{64} = 1 \text{ r } 34 \Rightarrow 1 \times 8^2$

And $\frac{34}{8^1} = \frac{34}{8} = 4 \text{ r } 2 \Rightarrow 4 \times 8^1 + 2 \times 8^0$

So $98_{10} = 1 \times 8^2 + 4 \times 8^1 + 2 \times 8^0$
$= 142_8$

(b) Now $\frac{985}{8^3} = \frac{985}{512} = 1 \text{ r } 473 \Rightarrow 1 \times 8^3$

And $\frac{473}{8^2} = \frac{473}{64} = 7 \text{ r } 25 \Rightarrow 7 \times 8^2$

Also $\frac{25}{8^1} = \frac{25}{8} = 3 \text{ r } 1 \Rightarrow 3 \times 8^1 + 1 \times 8^0$

Thus $985_{10} = 1 \times 8^3 + 7 \times 8^2 + 3 \times 8^1 + 1 \times 8^0$
$= 1731_8$

Converting from Octal to Decimal

In *converting* from *octal numbers* to *denary numbers*, we use the fact that *each place value* is a *power of* 8.

Example 23

(a) Convert 743_8 to a decimal number.

(b) Convert $2\,405_8$ to a decimal number.

(c) Convert 0.74_8 to a decimal number.

(d) Convert 0.214_8 to a decimal number.

(e) Convert 74.3_8 to a decimal number.

(f) Convert 641.04_8 to a decimal number.

Solution

(a) Now 743_8

$= 7 \times 8^2 + 4 \times 8^1 + 3 \times 8^0$

$= 7 \times 64 + 4 \times 8 + 3 \times 1$

$= 448 + 32 + 3$

$= 483_{10}$

(b) Now $2\,405_8$

$= 2 \times 8^3 + 4 \times 8^2 + 0 \times 8^1 + 5 \times 8^0$

$= 2 \times 512 + 4 \times 64 + 0 \times 8 + 5 \times 1$

$= 1\,024 + 256 + 0 + 5$

$= 1\,285_{10}$

(c) Now 0.74_8

$= 7 \times 8^{-1} + 4 \times 8^{-2}$

$= 7 \times \frac{1}{8} + 4 \times \frac{1}{64}$

$= 0.875 + 0.0625$

$= 0.937\,5_{10}$

$= 0.94$ (*correct* to 2 *d.p.*)

(d) Now 0.214_8

$= 2 \times 8^{-1} + 1 \times 8^{-2} + 4 \times 8^{-3}$

$= 2 \times \frac{1}{8} + 1 \times \frac{1}{64} + 4 \times \frac{1}{512}$

$= 0.25 + 0.015\,625 + 0.007\,812\,5$

$= 0.273\,437\,5_{10}$

$= 0.273_{10}$ (*correct* to 3 *d.p.*)

(e) Now 74.3_8

$= 7 \times 8^1 + 4 \times 8^0 + 3 \times 8^{-1}$

$= 7 \times 8 + 4 \times 1 + 3 \times \frac{1}{8}$

$= 56 + 4 + 0.375$

$= 60.375_{10}$

$= 60.4_{10}$ (*correct* to 1 *d.p.*)

(f) Now 641.04_8

$= 6 \times 8^2 + 4 \times 8^1 + 1 \times 8^0 + 0 \times 8^{-1} + 4 \times 8^{-2}$

$= 6 \times 64 + 4 \times 8 + 1 \times 1 + 0 \times \frac{1}{8} + 4 \times \frac{1}{64}$

$= 384 + 32 + 1 + 0 + 0.0625$

$= 417.062\,5_{10}$

$= 417.06$ (*correct* to 2 *d.p.*)

Adding Octal Numbers

The *addition table* for *octal numbers* can be seen below:

Table 2.12

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

Example 24

Add the following octal numbers:

(a) 675_8 and 204_8

(b) $4\,763_8$ and 215_8

Solution

(a) Now
$$\begin{array}{rl} {\scriptstyle 1\,1} & \\ 675 & + \\ 204 & \\ \hline 1\,101_8 & \\ \hline \end{array}$$

(b) Now
$$\begin{array}{rl} {\scriptstyle 1\,1\,1} & \\ 4\,763 & + \\ 215 & \\ \hline 5\,200_8 & \\ \hline \end{array}$$

Subtracting Octal Numbers

In *subtracting octal numbers*, we use *similar rules* as those for the *addition of octal numbers*.

Example 25

Find the difference between each pair of the following octal numbers:

(a) $7632_8 - 475_8$ *(b)* $6701_8 - 5043_8$

Solution

(a) Now
$$\begin{array}{rl} \overset{1}{5}\,2\;\; & \\ 1\; & \\ 7\cancel{6}\cancel{3}2 & - \\ 475 & \\ \hline 7135_8 & \\ \hline \end{array}$$

(b) Now
$$\begin{array}{rl} 6\,7\;\; & \\ 1\,1\; & \\ 6\cancel{7}\cancel{0}1 & - \\ 5043 & \\ \hline 1636_8 & \\ \hline \end{array}$$

Multiplying Octal Numbers

The *multiplication table* for *octal numbers* is as follows:

Table 2.13

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	10	12	14	16
3	0	3	6	11	14	17	22	25
4	0	4	10	14	20	24	30	34
5	0	5	12	17	24	31	36	43
6	0	6	14	22	30	36	44	52
7	0	7	16	25	34	43	52	61

Example 26

Find the product of the following octal numbers:

(a) $761_8 \times 30_8$ *(b)* $6017_8 \times 472_8$

Solution

(a)
$$\begin{array}{rl} 2\;\;\, & \\ 761 & \times \\ 30 & \\ \hline 27230_8 & \\ \end{array}$$

(b)
$$\begin{array}{rl} 1\;\;\;\;1\; & \\ 5\;\;\;16\; & \\ 3\;\;\;\;\;\;\;3\;\; & \\ 6017 & \times \\ 472 & \\ \hline \overset{11}{3007}400 & \\ 521510 & + \\ 14036 & \\ \hline 3545146_8 & \\ \hline \end{array}$$

Exercise 2k

1. Convert each of the following denary numbers to an octal number:
(a) 84_{10} (b) 93_{10}
(c) 104_{10} (d) 137_{10}

2. Convert each of the following decimal numbers to its base 8 equivalent:
(a) 247_{10} (b) 384_{10}
(c) 841_{10} (d) 968_{10}

3. Convert each of the following octal numbers to a denary number:
(a) 47_8 (b) 135_8
(c) 436_8 (d) 647_8

4. Convert each of the following numbers in base 8 to a decimal number:
(a) 0.74_8 (b) 0.32_8
(c) 0.54_8 (d) 0.76_8

5. Convert each of the following numbers in base 8 to its decimal equivalent:
(a) 34.31_8 (b) 47.62_8
(c) 105.42_8 (d) 237.76_8

6. Add the following octal numbers as indicated:
(a) $47_8 + 36_8$ (b) $64_8 + 32_8$
(c) $57_8 + 346_8$ (d) $124_8 + 431_8$

7. Sum the following octal numbers as indicated:
(a) $1204_8 + 347_8$ (b) $2476_8 + 1463_8$
(c) $3471_8 + 436_8$ (d) $6741_8 + 3471_8$

8. Subtract the following octal numbers as indicated:
(a) $47_8 - 35_8$ (b) $65_8 - 43_8$
(c) $104_8 - 76_8$ (d) $243_8 - 106_8$

9. Find the difference between each pair of the following octal numbers:

(a) $1045_8 - 247_8$ (b) $4341_8 - 745_8$
(c) $5436_8 - 4716_8$ (d) $6471_8 - 5432_8$

10. Multiply each pair of the following octal numbers:

(a) $35_8 \times 40_8$ (b) $63_8 \times 34_8$
(c) $107_8 \times 24_8$ (d) $245_8 \times 63_8$

11. Find the product of each of the following pairs of octal numbers:

(a) $6431_8 \times 105_8$ (b) $4732_8 \times 215_8$
(c) $6134_8 \times 324_8$ (d) $5342_8 \times 407_8$

Other Number Bases

Having fully understood the *methods* explained for the *addition*, *subtraction* and *multiplication* of *number bases* 2, 5 and 8, students should now be able to *extend* their *knowledge* and *add*, *subtract* and *multiply numbers in any given number base.*

Exercise 2I

1. Carry out each of the following additions in base 3:

(a) $\begin{array}{r} 21_3 \\ 10_3 \end{array} +$ (b) $\begin{array}{r} 120_3 \\ 201_3 \end{array} +$ (c) $\begin{array}{r} 2120_3 \\ 1202_3 \end{array} +$

2. Carry out each of the following subtractions in base 3:

(a) $\begin{array}{r} 21_3 \\ 11_3 \end{array} -$ (b) $\begin{array}{r} 212_3 \\ 112_3 \end{array} -$ (c) $\begin{array}{r} 1221_3 \\ 1212_3 \end{array} -$

3. Carry out each of the following multiplications in base 3:

(a) $\begin{array}{r} 21_3 \\ 20_3 \end{array} \times$ (b) $\begin{array}{r} 212_3 \\ 21_3 \end{array} \times$ (c) $\begin{array}{r} 2211_3 \\ 201_3 \end{array} \times$

4. Perform each of the following additions in base 4:

(a) $\begin{array}{r} 31_4 \\ 12_4 \end{array} +$ (b) $\begin{array}{r} 231_4 \\ 123_4 \end{array} +$ (c) $\begin{array}{r} 2132_4 \\ 1313_4 \end{array} +$

5. Perform each of the following subtractions in base 4:

(a) $\begin{array}{r} 31_4 \\ 21_4 \end{array} -$ (b) $\begin{array}{r} 312_4 \\ 213_4 \end{array} -$ (c) $\begin{array}{r} 2312_4 \\ 1231_4 \end{array} -$

6. Perform each of the following multiplications in base 4:

(a) $\begin{array}{r} 32_4 \\ 20_4 \end{array} \times$ (b) $\begin{array}{r} 312_4 \\ 23_4 \end{array} \times$ (c) $\begin{array}{r} 2312_4 \\ 132_4 \end{array} \times$

7. Evaluate:

(a) $\begin{array}{r} 23_5 \\ 12_5 \end{array} +$ (b) $\begin{array}{r} 132_5 \\ 113_5 \end{array} +$ (c) $\begin{array}{r} 3132_5 \\ 1301_5 \end{array} +$

8. Evaluate:

(a) $\begin{array}{r} 31_5 \\ 21_5 \end{array} -$ (b) $\begin{array}{r} 213_5 \\ 132_5 \end{array} -$ (c) $\begin{array}{r} 3203_5 \\ 3112_5 \end{array} -$

9. Evaluate:

(a) $\begin{array}{r} 32_5 \\ 30_5 \end{array} \times$ (b) $\begin{array}{r} 321_5 \\ 32_5 \end{array} \times$ (c) $\begin{array}{r} 2321_5 \\ 102_5 \end{array} \times$

10. Add the following base 6 numbers:

(a) $\begin{array}{r} 53_6 \\ 12_6 \end{array} +$ (b) $\begin{array}{r} 345_6 \\ 124_6 \end{array} +$ (c) $\begin{array}{r} 4532_6 \\ 2354_6 \end{array} +$

11. Subtract the following base 6 numbers:

(a) $\begin{array}{r} 45_6 \\ 13_6 \end{array} -$ (b) $\begin{array}{r} 514_6 \\ 241_6 \end{array} -$ (c) $\begin{array}{r} 4351_6 \\ 3142_6 \end{array} -$

12. Multiply the following base 6 numbers:

(a) $\begin{array}{r} 53_6 \\ 30_6 \end{array} \times$ (b) $\begin{array}{r} 534_6 \\ 23_6 \end{array} \times$ (c) $\begin{array}{r} 3152_6 \\ 124_6 \end{array} \times$

13. Calculate:

(a) $\begin{array}{r} 61_7 \\ 34_7 \end{array} +$ (b) $\begin{array}{r} 456_7 \\ 134_7 \end{array} +$ (c) $\begin{array}{r} 1645_7 \\ 2643_7 \end{array} +$

14. Calculate:

(a) $\begin{array}{r} 64_7 \\ 32_7 \end{array} -$ (b) $\begin{array}{r} 456_7 \\ 341_7 \end{array} -$ (c) $\begin{array}{r} 645_7 \\ 364_7 \end{array} -$

15. Calculate:

(a) $\begin{array}{r} 56_7 \\ 40_7 \end{array} \times$ (b) $\begin{array}{r} 625_7 \\ 43_7 \end{array} \times$ (c) $\begin{array}{r} 562_7 \\ 124_7 \end{array} \times$

16. Express the solution to each of the following pairs of numbers in base 8:

(a) $\begin{array}{r} 75_8 \\ 43_8 \end{array} +$ (b) $\begin{array}{r} 573_8 \\ 432_8 \end{array} +$ (c) $\begin{array}{r} 3457_8 \\ 2341_8 \end{array} +$

17. Express the solution to each of the following pairs of numbers in base 8:

(a) $\begin{array}{r} 75_8 \\ 43_8 \end{array} -$ (b) $\begin{array}{r} 673_8 \\ 532_8 \end{array} -$ (c) $\begin{array}{r} 4573_8 \\ 3261_8 \end{array} -$

18. Express the solution to each of the following pairs of numbers in base 8:

(a) $\begin{array}{r} 73_8 \\ 50_8 \end{array} \times$ (b) $\begin{array}{r} 367_8 \\ 42_8 \end{array} \times$ (c) $\begin{array}{r} 7\,432_8 \\ 134_8 \end{array} \times$

19. Simplify each of the following operations, leaving your answers in base 9:

(a) $\begin{array}{r} 84_9 \\ 73_9 \end{array} +$ (b) $\begin{array}{r} 748_9 \\ 382_9 \end{array} +$ (c) $\begin{array}{r} 6\,417_9 \\ 5\,342_9 \end{array} +$

20. Simplify each of the following operations, leaving your answers in base 9:

(a) $\begin{array}{r} 85_9 \\ 74_9 \end{array} -$ (b) $\begin{array}{r} 748_9 \\ 485_9 \end{array} -$ (c) $\begin{array}{r} 8\,763_9 \\ 5\,371_9 \end{array} -$

21. Simplify each of the following operations, leaving your answers in base 9:

(a) $\begin{array}{r} 84_9 \\ 30_9 \end{array} \times$ (b) $\begin{array}{r} 507_9 \\ 45_9 \end{array} \times$ (c) $\begin{array}{r} 7\,680_9 \\ 312_9 \end{array} \times$

CHAPTER 3

Computation

This chapter will teach you how to

- carry out arithmetic operations in order.
- add, subtract, divide, multiply and simplify whole numbers, fractions, mixed numbers and decimals.
- solve word problems using a logical sequence.
- approximate a decimal number to its nearest whole number, to the nearest power of ten, to a number of decimal places and to a number of significant figures.
- write a number in standard form (or scientific notation).
- construct the range in which the exact value of a computation must lie.
- define and use ratio, proportional parts, direct proportion, ready reckoner and inverse proportion.
- calculate percentages and the arithmetic mean (or average).
- find the square, square root and reciprocal of a number using a calculator and using three figure-mathematical tables.

Order of Arithmetic Operations

The *order* in which *arithmetic operations* are carried out is *defined* below.

BODMAS

This *rule* tells us that in *solving problems* dealing with *arithmetic operations*, we work out *brackets* first, then *of*, *division* or *multiplication*, *addition* or *subtraction*, in that *order*.

It is always better to *divide*, that is, to *cancel* first when possible, then *multiply*.

Operations with Whole Numbers

Now $W = \{\textit{whole numbers}\} = \{0, 1, 2, 3, \ldots\}$

Example 1

(a) Add the numbers 475 and 329.

(b) Subtract 381 from 745.

(c) Divide 8 350 by 25.

(d) Multiply 431 by 247.

(e) Simplify $8 \div 2 \times 6 + (10 - 3)$.

Solution

(a) Now

$$\begin{array}{r} {}^{1\,1} \\ +\;475 \\ 329 \\ \hline 804 \end{array} \leftarrow \boxed{1\cancel{4}\;\big|\;1\cancel{0}\;\big|\;\cancel{8}}$$

So the *sum* is 804.

(b) Now

$$\begin{array}{r} -\;\overset{6}{\cancel{7}}{}^{1}45 \\ 381 \\ \hline 364 \end{array}$$

So the *difference* is 364.

The *example* below *shows* how to perform *long division.*

(c) Now

$$\begin{array}{r l} & 0\,334 \\ \cline{2-2} 25 & \!\!\big)\,\overset{7}{\cancel{8}}\overset{1}{3}50 \\ & -75\downarrow \\ \cline{2-2} & 085 \\ & -75\downarrow \\ \cline{2-2} & 100 \\ & -100 \\ \cline{2-2} & 000 \end{array} \qquad \boxed{\begin{array}{r} 25\times \\ 3 \\ \hline 75 \end{array}} \qquad \boxed{\begin{array}{r} 25\times \\ 4 \\ \hline 100 \end{array}}$$

So the *quotient* is 334.

The *example* below *shows* how to perform *long multiplication.*

(d) Now

$$\begin{array}{r} \times\;431 \\ 247 \\ \hline 86\,200 \\ +17\,240 \\ 3\,017 \\ \hline 106\,457 \end{array} \qquad \boxed{\begin{array}{r} 431\times \\ 2 \\ \hline 862 \end{array}} \quad \boxed{\begin{array}{r} 431\times \\ 4 \\ \hline 1724 \end{array}} \quad \boxed{\begin{array}{r} 431\times \\ 7 \\ \hline 3017 \end{array}}$$

So the *product* is 106 457.

In *solving problems* dealing with *mixed operations*, we need to follow the *order of arithmetic operations* defined by *BODMAS.*

(e) Using *BODMAS*

$$\begin{aligned} & 8 \div 2 \times 6 + (10 - 3) \\ &= \frac{8}{2} \times 6 + (7) \\ &= 4 \times 6 + 7 \\ &= 24 + 7 \\ &= 31 \end{aligned}$$

Alternatively,

$$\begin{aligned} & 8 \div 2 \times 6 + (10 - 3) \\ &= \frac{8}{2} \times 6 + (7) \\ &= \frac{48}{2} + 7 \\ &= 24 + 7 \\ &= 31 \end{aligned}$$

Exercise 3a

1. Evaluate 583×97.
2. Calculate $10\,325 \div 413$.
3. Determine the value of $15 \div 5 + 20 \div (3 + 2)$.
4. Simplify $(3 \times 2 - 1) + (44 \div 11 + 3)$.
5. Evaluate 147×230.
6. Calculate $1\,704 \div 24$.
7. Determine the value of $15 \div 5 + 7 \times 2$.
8. Simplify $(8 + 3) \times 2 + 10 \div (6 - 1)$.
9. Find the value for each of the following expressions:
   (a) $11 - 12 \div 4 + 3(6 - 2)$ (b) $0 \div 21$
10. Determine the value for each of the following expressions:
    (a) $17 - 2(5 - 3)$ (b) $762 \div 0$
11. Calculate:
    (a) $3 \times 4 \times 5$ (b) $5 + 3 \times 2$
12. Evaluate:
    (a) $4 \times 5 + 6$ (b) $16 \div 4 + 2$
13. Simplify:
    (a) $9 - 12 \div 3$ (b) $8 + 14 \div 7$
14. Evaluate $(5 - 3) \div 4$ of \$500.
15. Calculate:
    $(7 - 4) \div 9$ of $(8 \div 2 \times 3)$.

Word Problems— Whole Numbers

In a *word problem*, we have to *translate* the *English sentences* into a *problem* dealing with the stated *arithmetic operations.* We then *solve* the *problem* using a *logical sequence.*

Example 2

(a) The cost of a comic book is 620 ¢ and the cost of a pack of crayons is 1 140 ¢. Calculate the total cost of the purchases.

(b) A boy bought a video game for \$85. Calculate his change if he paid with a \$100 note.

(c) The cost of a magazine is $12. Calculate the cost of 15 such magazines.

(d) The cost of 85 kg of flour is 1 445 ¢. Calculate the cost of 1 kg of flour.

Solution

(a) The cost of one comic book = 620 ¢
The cost of a pack of crayons = + 1 140 ¢
∴ the *total cost* = 1 760 ¢

So the *total cost* of the purchases was 1 760 ¢.

(b) The value of the note used = $100
The cost of the video game = − 85
∴ the *change received* = $15

So the *change received* was $15.

(c) The cost per magazine = $12
∴ the *cost of* 15 magazines = $12 × 15
= $180

So the *cost* of 15 magazines was $180.

(d) The cost of 85 kg of flour = 1 445 ¢
∴ the *cost of* 1 kg of flour $= \dfrac{1\,445\text{ ¢}}{85} = 17\text{ ¢}$

So the *cost* of 1 kg of flour was 17 ¢.

Exercise 3b

1. A girl bought a comic book costing 625 ¢, a pen costing 570 ¢ and a chocolate bar costing 375 ¢. She paid her bill with a $20 note. How much change, in cents, did she receive?

2. I had a piece of string 300 cm in length. I cut off three pieces, one of length 97 cm, one of length 53 cm and one of length 112 cm. What is the length of the piece of string that I had left?

3. On Friday 2 000 patties were cooked in a school cafeteria. At the first meal 347 patties were served. At the second meal 652 patties were served. At the third meal 432 patties were served. How many patties were left after the three meals?

4. A light bulb was being tested so it was left on continuously. It failed after 29 days exactly. How many hours did it work altogether?

5. A girl saves the same amount each week. After 12 weeks she had $288. How many dollars did she save per week?

6. A car travelling from Town A to Town B at 50 km/h took 4 h. How far is Town A from Town B.

7. A girl can walk up a flight of stairs at the rate of 36 steps per minute. If it takes her 3 minutes to reach the top, determine how many steps there are.

8. A farmer has to pack 6 000 oranges in new boxes. Each box can hold 75 oranges. How many boxes are needed to hold the 6 000 oranges?

9. A girl bought a book costing $95 and a folder costing $17. She paid her bill with six 20-dollar notes. How many dollars change did she receive?

10. A piggy bank has fifteen 10 ¢ pieces and six 25 ¢ pieces in it. Another piggy bank has twenty 5 ¢ pieces and eight 50 ¢ pieces in it. What is the total sum of money, in cents, in the piggy banks?

11. At the newspaper stand I bought three comics costing 625 ¢ each, a magazine costing 3 540 ¢ and two newspapers costing 300 ¢ each. How much change, in cents, did I received from a $100 note?

12. A grocer bought 57 cases of sweet drinks. Each case contains 24 cans. How many cans were bought altogether?

13. A school day is 8 hours. How many minutes are there in a school day?

14. If 12 sweets cost 204 ¢, determine the cost of one sweet.

15. A woman saves the same amount each month. After 9 months she had $1 575. What amount of money did she save each month?

16. A girl's total marks in her end of term test was 480. She got 62 in Mathematics, 81 in English, 75 in Science, 76 in Spanish, 89 in Social Studies and the remainder was her Music marks. How many marks did she make in Music?

17. A boy had 50 marbles when he went to school on Tuesday morning. At break time he lost 18 marbles and at lunch time he won 11 marbles. After school he lost an additional 13 marbles. How many marbles did he go home with?

43. An escalator can move up at the rate of 25 steps per minute. It takes 4 minutes to reach the top from the bottom. How many steps are there?

44. A man is paid $525 for working a 5-day week. What amount does he get paid per day?

45. A girl bought a comic book costing 368 ¢ and a pencil costing 125 ¢. She paid with a $10 note. What amount of change did she receive?

46. The contents of a tin of sweets had a mass of 5 000 grams. The sweets are divided into packets each of mass 250 g. How many packets of sweets can be made?

47. One money box has six 5 ¢ pieces and seven 10 ¢ pieces in it. Another money box has twelve 25 ¢ pieces and nine 50 ¢ pieces in it. What is the total sum of money in the two boxes?

48. On Monday 1 500 hamburgers were cooked in a school canteen. At the first sitting 357 hamburgers were served. At the second sitting 655 hamburgers were served. At the third sitting 421 hamburgers were served. How many hamburgers were left after the three sittings?

49. A boy had 60 marbles when he arrived at school on Friday morning. At break time he lost 17 marbles. At lunch time he won 9 marbles. After school he lost 21 marbles. How many marbles did he take home that afternoon?

50. Anisa received 950 ¢ pocket money on Sunday. On Monday she spent 542 ¢ in school. On Tuesday her mom gave her 325 ¢ for doing a special chore at home. On Thursday she spent 152 ¢. On Friday she spent 403 ¢. How much money has she got left to spend at her church bazaar on Saturday?

51. I bought 9 oranges costing 75 ¢ each and 6 apples costing 290 ¢ each. What amount of money did I spend?

52. A car travelling 70 kilometres an hour took 3 hours to travel from Town A to Town B. How many kilometres did the car travel altogether, if it returned to Town A?

53. A boy can walk up a flight of stairs at a constant rate of 23 steps per minute. It takes him 4 minutes to reach the top. How many steps are there altogether?

54. A total of 9 000 oranges were packed into boxes, each holding 75 oranges. How many boxes were filled with oranges?

55. A man is paid $520 for a 5-day week. What amount does he get paid per working day?

56. A vase has seven 5 ¢ pieces and twenty 10 ¢ pieces in it. Another vase has eleven 25 ¢ pieces and eight 50 ¢ pieces in it. What is the total sum of money in the two vases?

57. Three friends, Sonia, Anu and Kelly went to shop. Sonia bought 7 sweets costing 12 ¢ each, Anu bought 9 sweets costing 8 ¢ each and Kelly bought 11 sweets costing 14 ¢ each. How much money, in cents, did they spend altogether in the shop?

Operations with Common Fractions

A *fraction* is part of a whole. It is a measure of how the whole is to be divided (or shared).

A *common fraction* (or *vulgar fraction*) is a number *written* in the form $\frac{n}{d}$, where n is a term called the *numerator*, d is a term called the *denominator*, $n \in N$ and $d \in N$.

CASE 1: PROPER FRACTIONS

A *proper fraction* is a *common fraction* in which the numerator is less than the denominator. A *proper fraction* is less than a whole, that is, it is less than one.

For example:

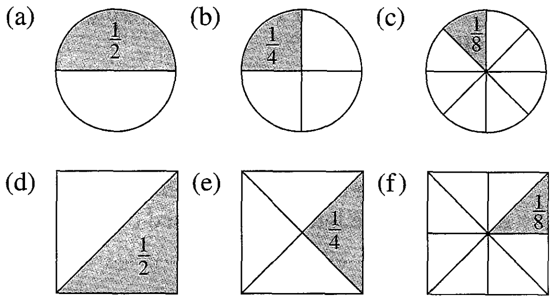


Fig. 3.1 Proper fractions

18. Multiply five thousand, seven hundred and one by twenty-three.

19. On a school outing 6 maxi taxis were used, each taking 31 students. How many students went on the outing?

20. A fuse was tested by being left on continuously. It failed after 45 days exactly. For how many hours was it working?

21. How many mangoes costing 50 ¢ each can I buy with $8.

22. If a maxi taxi holds 30 children, how many maxi taxis are needed to take 480 children?

23. A money box has nine 5 ¢ pieces and five 10 ¢ pieces in it. Another money box has seven 10 ¢ pieces and ten 25 ¢ pieces in it. What is the total sum of money, in cents, in the two money boxes?

24. An elevator can move up at the rate of 60 steps a minute. It takes 4 minutes to reach the top from the bottom. How many steps are there altogether?

25. I bought 7 mangoes costing 125 ¢ each and 5 apples costing 225 ¢ each. What amount of money did I spend?

26. Calculate the total cost of a tin of baked beans at 458 ¢, a cake at 75 ¢ and a soft drink at 150 ¢.

27. Determine the total cost of a washing machine at $2 341, a fridge at $3 642 and a gas cooker at $1 975.

28. In a school there are 597 children. There are 321 boys. How many girls are there?

29. The Middle Peak of the Blue Mountains in Jamaica is 2 270 m in height and Kaieteur Falls in Guyana is 256 m in height. How much higher than Kaieteur Falls is the Middle Peak of the Blue Mountains?

30. A mini mart had 39 kg of carrots when it opened on Monday morning. During the day the shop received a delivery of 63 kg of carrots and sold 27 kg of carrots. How many kilograms of carrots were left when it closed on Monday evening?

31. Nicole received 59 ¢ pocket money on Saturday. On Monday she spent 34 ¢. On Tuesday she was given 20 ¢ for doing a special job at home. On Thursday she spent 25 ¢. What amount of money was Nicole left with?

32. The office I work in has 83 computers. The office my friend works in has 37 computers. How many more computers are there in my office than in my friend's office?

33. When Judy went to school on Monday morning it took her 7 minutes to walk to the bus stop. She waited 11 minutes for a bus and the bus journey lasted 23 minutes. She then had an 8 minutes walk to school. How long, in minutes, did it take Judy to reach her school?

34. Sian received 940 ¢ pocket money on Sunday. On Monday she spent 341 ¢. On Tuesday her dad gave her 225 ¢ for doing a special chore at home. On Thursday she spent 142 ¢. On Friday she spent 402 ¢. What amount of money did she have left to spend during the weekend?

35. I have three pieces of rope. One piece is 24 cm in length, another piece is 47 cm in length and the third piece is 35 cm in length. What is the total length, in cm, of rope that I have?

36. On Monday morning Renee took 6 minutes to walk to the bus stop. She had to wait 9 minutes for the bus. The bus journey took 45 minutes. She then had a 3 minutes walk to her school. How long, in minutes, did it take Renee to reach her school?

37. Cindy's club dues for last week was 255 ¢. Cindy paid with a $5 note. How much change did she receive?

38. The arcade that I go to has 82 video games. The arcade that my friend goes to has 63 video games. How many more video games are there in my arcade than in my friend's arcade?

39. Kelly bought a notebook costing 255 ¢ and a pen costing 625 ¢. She paid with a $10 note. What amount of change did she receive?

40. Calculate the difference between three thousand, five hundred and forty-eight; and eight hundred and twenty-five. Add two hundred and three to the difference. What is the sum?

41. I have a piece of wire 300 cm in length. I cut off three pieces, one of length 53 cm, the second of length 24 cm and the third of length 85 cm. How long, in cm, is the piece of wire left?

42. A car travelling at 40 km/h took 3 hours to arrive at Toco. How many kilometres did the car travel?

(e) Now $\frac{1}{3}$ of 9

$$= \frac{1}{3} \times 9$$

$$= \frac{1}{\overset{}{\underset{1}{\cancel{3}}}} \times \overset{3}{\cancel{9}}$$

$$= \frac{1 \times 3}{1}$$

$$= 3$$

So $\frac{1}{3}$ of 9 is 3.

Exercise 3c

1. Evaluate:

(a) $\frac{21}{100} + \frac{19}{100} + \frac{15}{100}$ (b) $\frac{7}{15} + \frac{2}{15}$

2. Calculate:

(a) $\frac{4}{11} + \frac{7}{22}$ (b) $\frac{9}{17} - \frac{3}{17}$

3. Simplify:

(a) $\frac{13}{15} - \frac{2}{5}$ (b) $\frac{3}{5} + \frac{4}{7}$

4. Calculate:

(a) $\frac{5}{12} - \frac{1}{3}$ (b) $\frac{9}{11} - \frac{2}{3}$

5. Determine the value of:

(a) $\frac{5}{11}$ of 132 metres (b) $\frac{5}{8}$ of 40 metres.

6. Calculate:

(a) $\frac{3}{5}$ of 1 non-leap year, (state your answer in days)

(b) $\frac{5}{12}$ of 1 day, (state your answer in hours).

7. Calculate:

(a) $\frac{39}{50} \div \frac{13}{20}$

(b) $\frac{1}{2}$ of 1 month; September (state your answer in days).

8. State the fraction that is shaded in each of the following sketches:

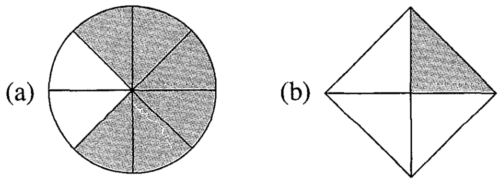


Fig. 3.3 Proper fractions

9. Write the fraction that is shaded in each of the following diagrams:

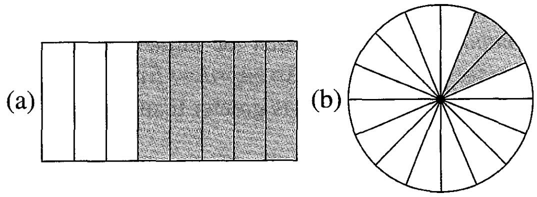


Fig. 3.4 Proper fractions

10. State the fraction that is shaded in each of the following diagrams:

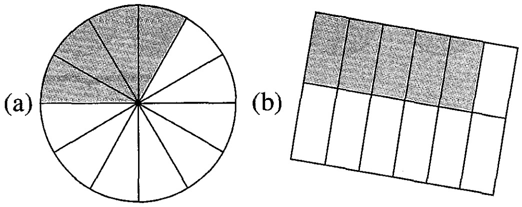


Fig. 3.5 Proper fractions

11. Evaluate:

(a) $\frac{5}{12}$ of 144 litres

(b) $\frac{3}{8}$ of 1 day, (state your answer in hours).

12. Calculate:

(a) $40 \div \frac{8}{11}$ (b) $\frac{9}{28} \div \frac{3}{14}$

13. Calculate:

(a) $\frac{5}{12} + \frac{1}{6} + \frac{2}{3}$ (b) $\frac{7}{9} - \frac{1}{3} + \frac{5}{6}$

14. Simplify each of the following expressions:

(a) $\frac{3}{4} + \frac{1}{3} - \frac{5}{8}$ (b) $\frac{4}{9} \times \frac{5}{8} \times \frac{9}{25}$

15. Calculate the value of each of the following expressions:

(a) $\frac{3}{4} + \frac{1}{12} + \frac{2}{3}$ (b) $\frac{2}{3} - \frac{5}{6} + \frac{3}{5}$

16. Evaluate each of the following expressions:

(a) $\frac{7}{16} \times \frac{10}{11} \times \frac{8}{35}$ (b) $\frac{3}{8} \times \frac{5}{9} \times \frac{16}{15}$

17. Determine the value of each of the following expressions:

(a) $\frac{5}{9} \times \frac{21}{25} \times \frac{5}{4}$ (b) $\frac{9}{14} \times \frac{21}{25} \times \frac{5}{18}$

CASE 2: IMPROPER FRACTIONS

An *improper fraction* (or *top heavy fraction*) is a *common fraction* in which the numerator is *greater than the denominator.* An *improper fraction* is greater than a whole, that is, it is greater than one.

For example:

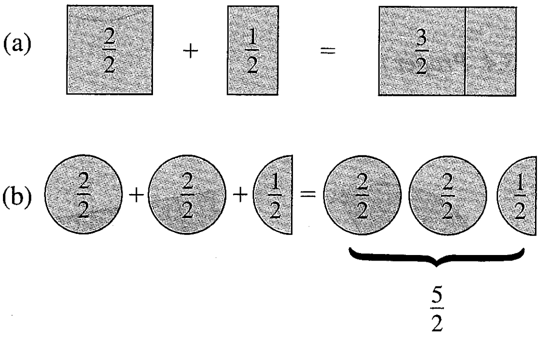


Fig. 3.2 Improper fractions

In *simplifying fractions*, it is necessary to find the *lowest common denominator, L. C. D.* (i.e. the *lowest common multiple, L. C. M.* of the *denominators*).

Example 3

Calculate each of the following expressions:

(a) $\frac{3}{4}+\frac{1}{5}$

(b) $\frac{4}{7}-\frac{1}{3}$

(c) $\frac{8}{9}\times\frac{3}{4}$

(d) $\frac{4}{9}\div\frac{2}{3}$

(e) $\frac{1}{3}$ *of 9*

Solution

(a) Now $\frac{3}{4}+\frac{1}{5}$

The *L.C.M.* of the *denominators* 4 and 5 is 20.

$$= \frac{3\times5+1\times4}{20} \leftarrow \boxed{\frac{20}{4}=5 \;\Big|\; \frac{20}{5}=4}$$

$$= \frac{15+4}{20}$$

$$= \frac{19}{20}$$

So the *sum* is $\frac{19}{20}$.

(b) Now $\frac{4}{7}-\frac{1}{3}$

The *L.C.M.* of the *denominators* 3 and 7 is 21.

$$= \frac{4\times3-1\times7}{21} \leftarrow \boxed{\frac{21}{7}=3 \;\Big|\; \frac{21}{3}=7}$$

$$= \frac{12-7}{21}$$

$$= \frac{5}{21}$$

So the *difference* is $\frac{5}{21}$.

(c) Now $\frac{8}{9}\times\frac{3}{4}$

$$= \frac{\overset{2}{\cancel{8}}}{\underset{3}{\cancel{9}}}\times\frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{4}}}$$

$$= \frac{2\times1}{3\times1}$$

$$= \frac{2}{3}$$

So the *product* is $\frac{2}{3}$.

When we are *dividing* by a *fraction*, we *invert* (i.e. *upturn*) the *fraction* which is the *divisor* and *multiply* instead.

(d) Now $\frac{4}{9}\div\frac{2}{3}$

$$= \frac{4}{9}\times\frac{3}{2}$$

(*Inverting* the *fraction* which is the *divisor* and *multiplying* instead)

$$= \frac{\overset{2}{\cancel{4}}}{\underset{3}{\cancel{9}}}\times\frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{2}}}$$

$$= \frac{2\times1}{3\times1}$$

$$= \frac{2}{3}$$

So the *quotient* is $\frac{2}{3}$.

The *operation* '*of*' means that we *multiply.*

(c) The mass of a bottle $= \frac{3}{7}\text{ g}$

∴ the *mass of* 14 such bottles

$$= \frac{3}{7}\text{ g} \times 14$$

$$= \frac{3}{\underset{1}{\cancel{7}}}\text{ g} \times \frac{\overset{2}{\cancel{14}}}{1}$$

$$= \frac{3 \times 2}{1 \times 1}\text{ g}$$

$$= 6\text{ g}$$

So the *mass* of 14 bottles is 6 g.

(d) $\frac{3}{11}$ of 121 cm $= \frac{3}{11} \times 121$ cm

$$= \frac{3}{\underset{1}{\cancel{11}}} \times \overset{11}{\cancel{121}}\text{ cm}$$

$$= \frac{3 \times 11}{1}$$

$$= 33\text{ cm}$$

So $\frac{3}{11}$ of 121 cm is 33 cm.

Exercise 3d

1. How many $\frac{3}{7}$s are there in 6?

2. Divide $\frac{1}{2}$ by $\frac{1}{4}$.

3. How many times does $\frac{2}{3}$ go into 12?

4. How many $\frac{1}{4}$s are there in 9?

5. A girl spent $\frac{1}{6}$ of her pocket money on sweets and $\frac{2}{5}$ on toys. What fraction of her pocket money did she spend? What fraction of her pocket money did she have remaining?

6. Write the first quantity as a fraction of the second quantity:

(a) 5 days; 1 non-leap year

(b) 42 minutes; 2 hours.

7. In a class of 35 students, 10 take Spanish and 9 take Geography. What fraction of the children in the class take:

(a) Spanish (b) Geography.

8. What fraction of an hour is 25 minutes?

9. In a class of 28 children, 12 take Spanish, 24 take Mathematics and 20 take English. What fraction of the children in the class take Spanish?

10. A boy spent $\frac{2}{5}$ of his money on sweets and $\frac{1}{3}$ on records.

(a) What fraction of his money did he spend?

(b) What fraction of his money did he have remaining?

11. In a class of 32 children, 28 like football and 20 like cricket. What fraction of the children:

(a) like football

(b) do not like football

(c) like cricket

(d) do not like cricket?

12. My school bag contain 9 books, each of mass $\frac{2}{9}$ kg and 5 folders, each of mass $\frac{7}{25}$ kg. What is the total mass of the books and folders in my bag? What fraction of the total mass is books?

13. How many $\frac{1}{2}$s are there in 8?

14. How many times does $\frac{2}{3}$ go into 6?

15. How many cents is $\frac{7}{10}$ of $1.

16. How many $\frac{3}{5}$ s are there in 27?

17. How many times does $\frac{2}{3}$ go into 10?

18. What fraction of an hour is 35 minutes?

19. Write the first quantity as a fraction of the second quantity:

53 days; 1 leap year.

20. In a class of 42 children, 10 take Spanish, 8 take French and 22 take Mathematics. What fraction of the children in the class take:

(a) Spanish (b) French (c) Mathematics.

21. At a school $\frac{1}{8}$ of the time is spent in Mathematics classes, $\frac{3}{16}$ of the time in English

18. Simplify each of the following expressions:

(a) $\frac{7}{12} + \frac{1}{6} + \frac{2}{3}$ (b) $\frac{12}{25} \times \frac{5}{9} \div \frac{5}{18}$

19. Evaluate each of the following expressions:

(a) $\frac{2}{5} \times \left(\frac{2}{3} + \frac{3}{4}\right)$ (b) $\frac{1}{3} + \frac{2}{9} + \frac{5}{6}$

20. Simplify each of the following expressions:

(a) $\frac{11}{12} + \frac{5}{6} - \frac{2}{3}$ (b) $\frac{2}{5} - \frac{3}{4} + \frac{5}{12}$

21. Evaluate:

$\left(\frac{16}{25} \times \frac{5}{7}\right) \div \frac{8}{25}$

Word Problems—Fractions

In a *word problem*, we have to *translate* the *English sentences* into a *problem* dealing with the stated *arithmetic operations.* We then *solve* the *problem* using a *logical sequence.*

Example 4

(a) Anu spent $\frac{1}{3}$ of her allowance on chocolate and $\frac{2}{5}$ on magazines.

(i) What fraction of her allowance did she spend?

(ii) What fraction has she left?

(b) Christine bought $\frac{4}{5}$ kg of sweets. If she divides it equally amongst 5 children, what mass of sweets does each child receive?

(c) A bottle has a mass of $\frac{3}{7}$ g. What is the mass of 14 such bottles?

(d) Calculate $\frac{3}{11}$ of 121 cm.

Solution

(a) (i) The fraction spent on chocolates $= \frac{1}{3}$

The fraction spent on magazines $= \frac{2}{5}$

∴ the *fraction spent*

$$= \frac{1}{3} + \frac{2}{5}$$
$$= \frac{1 \times 5 + 2 \times 3}{15}$$
$$= \frac{5 + 6}{15}$$
$$= \frac{11}{15}$$

So Anu *spent* $\frac{11}{15}$ of her allowance.

(ii) The *fraction* of her allowance *left*

$$= 1 - \frac{11}{15}$$
$$= \frac{15}{15} - \frac{11}{15}$$
$$= \frac{15 - 11}{15}$$
$$= \frac{4}{15}$$

So Anu has $\frac{4}{15}$ of her allowance *left.*

(b) The mass of sweets bought $= \frac{4}{5}$ kg

The number of children it is divided amongst $= 5$

Then the *mass* of sweets *each child receives*

$$= \frac{4}{5} \text{ kg} \div 5$$
$$= \frac{4}{5} \text{ kg} \div \frac{5}{1}$$

(*inverting* the *fraction* which is the *divisor*)

$$= \frac{4}{5} \text{ kg} \times \frac{1}{5}$$
$$= \frac{4 \times 1}{5 \times 5} \text{ kg}$$
$$= \frac{4}{25} \text{ kg}$$

So each child *received* $\frac{4}{25}$ kg of sweets.

(b) Now $5\frac{1}{3} - 2\frac{3}{8}$

The *L.C.M.* of the *denominators* 3 and 8 is 24.

$$= 5 - 2 + \frac{1}{3} - \frac{3}{8}$$

$$= 3 + \frac{1 \times 8 - 3 \times 3}{24} \leftarrow \boxed{\frac{24}{3} = 8 \;\Big|\; \frac{24}{8} = 3}$$

$$= 3 + \frac{8 - 9}{24}$$

$$= 3 - \frac{1}{24}$$

$$= 2 + 1 - \frac{1}{24}$$

$$= 2 + \frac{24 - 1}{24}$$

$$= 2 + \frac{23}{24}$$

$$= 2\frac{23}{24}$$

So the *difference* is $2\frac{23}{24}$.

Alternative Method

(b) Now $5\frac{1}{3} - 2\frac{3}{8}$

The *L.C.M.* of the *denominators* 3 and 8 is 24.

$$= 5 - 2 + \frac{1}{3} - \frac{3}{8}$$

$$= 3 + \frac{1 \times 8 - 3 \times 3}{24} \leftarrow \boxed{\frac{24}{3} = 8 \;\Big|\; \frac{24}{8} = 3}$$

$$= 3 + \frac{8 - 9}{24}$$

$$= 2 + \frac{24 + 8 - 9}{24}$$

$$= 2 + \frac{32 - 9}{24}$$

$$= 2 + \frac{23}{24}$$

$$= 2\frac{23}{24}$$

So the *difference* is $2\frac{23}{24}$.

(c) Now $2\frac{1}{3} \times \frac{5}{14}$

$$= \frac{7}{3} \times \frac{5}{14} \leftarrow \boxed{2 \times 3 + 1 = 6 + 1 = 7}$$

$$= \frac{\overset{1}{\cancel{7}}}{3} \times \frac{5}{\underset{2}{\cancel{14}}}$$

$$= \frac{1 \times 5}{3 \times 2}$$

$$= \frac{5}{6}$$

So the *product* is $\frac{5}{6}$.

(d) Now $2\frac{5}{8} \div 1\frac{3}{4}$

$$= \frac{21}{8} \div \frac{7}{4} \leftarrow \boxed{\begin{array}{l} 2 \times 8 + 5 = 16 + 5 = 21 \\ 1 \times 4 + 3 = \;\; 4 + 3 = 7 \end{array}}$$

$$= \frac{21}{8} \times \frac{4}{7}$$

(*Inverting* the *fraction* which is the *divisor* and *multiplying* instead)

$$= \frac{\overset{3}{\cancel{21}}}{\underset{2}{\cancel{8}}} \times \frac{\overset{1}{\cancel{4}}}{\underset{1}{\cancel{7}}}$$

$$= \frac{3 \times 1}{2 \times 1}$$

$$= \frac{3}{2}$$

$$= 1\frac{1}{2}$$

So the *quotient* is $1\frac{1}{2}$.

(e) Now $2\frac{1}{3}$ of 6

$$= \frac{7}{3} \times 6 \leftarrow \boxed{2 \times 3 + 1 = 6 + 1 = 7}$$

$$= \frac{7}{\underset{1}{\cancel{3}}} \times \overset{2}{\cancel{6}}$$

$$= \frac{7 \times 2}{1}$$

$$= 14$$

So $2\frac{1}{3}$ of 6 is 14.

Mixed Operations—Fractions

In *solving problems* dealing with *mixed operations*, we need to follow the *order of arithmetic operations* defined by *BODMAS.*

Example 6

(a) Simplify each of the following expressions:

(i) $8\frac{1}{3} + 2\frac{5}{6} - 3\frac{4}{9}$ *(ii)* $9\frac{3}{7} - 5\frac{1}{14} - 2\frac{5}{21}$

classes and $\frac{1}{16}$ on Sports. What fraction of the time is spent on:

(a) English and Mathematics together
(b) Mathematics and Sports
(c) All lessons except sports

22. A brick layer takes $1\frac{1}{5}$ minutes to lay one brick. How many minutes will it take him to lay 300 bricks?

23. Express the ratio of 35 ¢ to 225 ¢ as a fraction in its lowest terms.

24. Express the ratio $6:30 ¢ as a fraction in its lowest terms.

25. Write the fraction $\frac{3}{5}$ in equivalent form with denominator 45.

26. How many lengths of $1\frac{1}{2}$ m may be cut from a length of 36 m?

27. How many thirds are there is 5?

28. What is three-fifths of 6?

29. Write the first quantity as a fraction of the second quantity:

(a) 7 hours:1 day
(b) 9 months:1 year.

Operations with Mixed Numbers

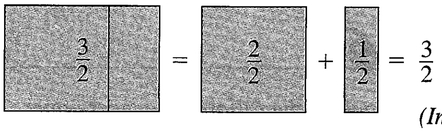
All *improper fractions* can be *written* as *mixed numbers*. A *mixed number* consists of a *whole number* and a *proper fraction.*

Thus,

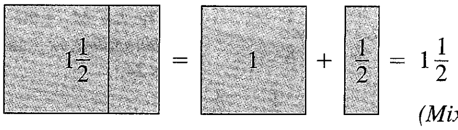
Mixed number (or *improper fraction*)	=	*whole number*	+	*proper fraction*

For example:

(a)



(Improper fraction)

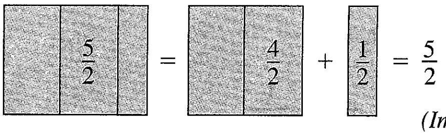


(Mixed number)

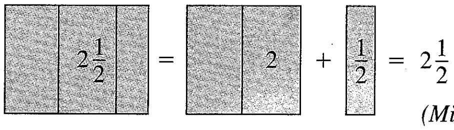
Fig. 3.6 Mixed number and improper fraction

Hence $1\frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2}$.

(b)



(Improper fraction)



(Mixed number)

Fig. 3.7 Mixed number and improper fraction

Hence $2\frac{1}{2} = 2 + \frac{1}{2} = \frac{5}{2}$.

Example 5

Simplify each of the following expressions:

(a) $3\frac{2}{5} + 1\frac{3}{7}$ *(b)* $5\frac{1}{3} - 2\frac{3}{8}$

(c) $2\frac{1}{3} \times \frac{5}{14}$ *(d)* $2\frac{5}{8} \div 1\frac{3}{4}$

(e) $2\frac{1}{3}$ *of 6*

Solution

(a) Now $3\frac{2}{5} + 1\frac{3}{7}$

The *L.C.M.* of the *denominators* 5 and 7 is 35.

$$= 3 + 1 + \frac{2}{5} + \frac{3}{7}$$

$$= 4 + \frac{2 \times 7 + 3 \times 5}{35}$$

$$\boxed{\frac{35}{5} = 7 \quad \frac{35}{7} = 5}$$

$$= 4 + \frac{14 + 15}{35}$$

$$= 4 + \frac{29}{35}$$

$$= 4\frac{29}{35}$$

So the *sum* is $4\frac{29}{35}$.

5. Calculate each of the following:

(a) $8\frac{7}{9} + 4\frac{1}{3} + 5\frac{11}{18}$ (b) $1\frac{1}{13} \times 3\frac{9}{10} \times 1\frac{2}{3}$

6. Simplify each of the following expressions:

(a) $3\frac{1}{2} \times 1\frac{3}{14} \times 2\frac{1}{34}$ (b) $3\frac{5}{12} \times 1\frac{7}{24} \times 2\frac{6}{41}$

7. Evaluate each of the following expressions:

(a) $2\frac{3}{5} \times 1\frac{1}{9} \div 8\frac{2}{3}$

(b) $\frac{8}{11} \times \left(\frac{5}{9} - \frac{1}{6}\right) \div 1\frac{5}{9}$

8. Calculate:

(a) $\frac{5}{9} \div \left(1\frac{1}{3} + \frac{5}{9}\right) + \frac{3}{8}$ (b) $2\frac{3}{4} \times \frac{4}{9} \div \frac{11}{12}$

9. Evaluate:

(a) $4\frac{4}{5} + 8\frac{4}{15} + 1\frac{2}{3}$ (b) $3\frac{1}{5} \times 1\frac{1}{8} \times 2\frac{1}{2}$

10. Calculate:

(a) $8\frac{7}{9} + 5\frac{1}{3} + 4\frac{7}{8}$ (b) $7\frac{1}{3} - 2\frac{4}{5} + 1\frac{7}{15}$

11. Determine:

(a) $7 \times \left(2\frac{1}{3} + 1\frac{1}{2}\right)$ (b) $2\frac{1}{4} \times 1\frac{1}{2} \times \frac{2}{3}$

12. Simplify the following:

(a) $\dfrac{6\frac{1}{2}}{9\frac{1}{2} - 5\frac{1}{4}}$ (b) $\dfrac{5\frac{4}{9} - 3\frac{2}{9}}{4\frac{1}{4} + 1\frac{2}{3}}$

13. Find the exact value of:

(a) $4\frac{3}{5} + 2\frac{1}{3} - 3\frac{1}{2}$ (b) $\dfrac{8\frac{1}{3} \times 3\frac{3}{5}}{1\frac{1}{2} \times 2\frac{2}{3}}$

14. Simplify the following:

(a) $3\frac{4}{9} + \left(1\frac{5}{36} \times 3\frac{3}{5}\right)$

(b) $8\frac{1}{3} \div \left(4\frac{1}{2} - 1\frac{1}{4}\right)$

15. Evaluate the following:

(a) $\dfrac{2\frac{1}{5} + 1\frac{2}{3}}{1\frac{1}{10} - \frac{2}{5}}$ (b) $\dfrac{\frac{3}{5} - \frac{7}{10}}{1 - \frac{3}{7} \times \frac{2}{5}}$

16. Determine the exact value of:

(a) $5\frac{3}{4} - 3\frac{1}{8}$ (b) $5\frac{2}{3} - 2\frac{1}{4}$

17. Calculate the exact value of:

(a) $\dfrac{6\frac{3}{8} + 3\frac{5}{8}}{5 - 3\frac{4}{7}}$ (b) $\left(5\frac{3}{7} - 3\frac{5}{14}\right) \div 2\frac{3}{14}$

18. Calculate the exact value of:

(a) $\left(3\frac{2}{5} - 1\frac{3}{10}\right) \div 1\frac{2}{5}$ (b) $\dfrac{3\frac{1}{4} + \frac{1}{3}}{2\frac{1}{4}}$

19. Find the exact value of:

(a) $\frac{8}{11} \times \left(\frac{5}{9} - \frac{1}{6}\right) \div 1\frac{5}{9}$

(b) $6\frac{3}{4} - 2\frac{1}{12} - 3\frac{1}{2}$

20. Simplify:

(a) $8\frac{5}{9} + 3\frac{5}{27} + 2\frac{1}{3}$ (b) $3\frac{1}{2} - 2\frac{1}{4} + 1\frac{3}{8}$

21. Evaluate:

(a) $5\frac{1}{3} - 4\frac{1}{2} + 1\frac{2}{3}$ (b) $2\frac{3}{5} \div 3\frac{3}{26}$

22. Determine the exact value of each of the following expressions:

(a) $\left(5\frac{3}{5} \div 1\frac{5}{9}\right) - \left(1\frac{1}{5}\right)^2$

(b) $\dfrac{5\frac{1}{4} - 2\frac{1}{3}}{2\frac{1}{2}}$

23. Calculate the exact value of each of the following expressions:

(a) $\left(4\frac{5}{6} - 1\frac{2}{3}\right) \div 1\frac{1}{3}$ (b) $\dfrac{5\frac{3}{5} + 2\frac{2}{5}}{4 - 1\frac{1}{5}}$

24. Evaluate each of the following expressions:

(a) $2\frac{1}{2} + 3\frac{1}{4} - 4\frac{3}{8}$

(b) $2\frac{9}{25} - 3\frac{4}{5} - 2\frac{7}{10} - \frac{3}{10}$

25. Simplify each of the following expressions:

(a) $2\frac{1}{2} \times 2\frac{2}{3}$ (b) $2\frac{3}{10} \div \frac{3}{5}$

26. Calculate the exact value of:

(a) $\dfrac{5\frac{3}{5} - 3\frac{1}{2} \times \frac{2}{3}}{2\frac{1}{3}}$ (b) $\dfrac{\frac{5}{9} - \frac{7}{15}}{1 - \frac{5}{9} \times \frac{7}{15}}$

(iii) $5\frac{1}{16} \times \frac{8}{9} \div 2\frac{3}{4}$

(b) Arrange the fractions:

$$\frac{3}{8}, \frac{4}{5}, \frac{2}{3}, \frac{1}{2} \text{ in}$$

(i) ascending order *(ii) descending order.*

Solution

(a) (i) Now $8\frac{1}{3} + 2\frac{5}{6} - 3\frac{4}{9}$

The *L.C.M.* of the *denominators* 3, 6 and 9 is 18.

$= 8 + 2 - 3 + \frac{1}{3} + \frac{5}{6} - \frac{4}{9}$

$= 10 - 3 + \frac{1 \times 6 + 5 \times 3 - 4 \times 2}{18}$

$= 7 + \frac{6 + 15 - 8}{18}$

$\frac{18}{3} = 6$	$\frac{18}{6} = 3$
$\frac{18}{9} = 2$	

$= 7 + \frac{21 - 8}{18}$

$= 7 + \frac{13}{18}$

$= 7\frac{13}{18}$

(ii) Now $9\frac{3}{7} - 5\frac{1}{14} - 2\frac{5}{21}$

The *L.C.M.* of the *denominators* 7, 14 and 21 is 42.

$= 9 - 5 - 2 + \frac{3}{7} - \frac{1}{14} - \frac{5}{21}$

$= 9 - 7 + \frac{3 \times 6 - 1 \times 3 - 5 \times 2}{42}$

$= 2 + \frac{18 - 3 - 10}{42}$

$\frac{42}{7} = 6$	$\frac{42}{14} = 3$
$\frac{42}{21} = 2$	

$= 2 + \frac{18 - 13}{42}$

$= 2 + \frac{5}{42}$

$= 2\frac{5}{42}$

(iii) Now $5\frac{1}{16} \times \frac{8}{9} \div 2\frac{3}{4}$

$= \frac{81}{16} \times \frac{8}{9} \div \frac{11}{4}$

$= \frac{81}{16} \times \frac{8}{9} \times \frac{4}{11}$

$= \frac{\overset{9}{\cancel{81}}}{\underset{\underset{1}{\cancel{2}}}{\cancel{16}}} \times \frac{\overset{1}{\cancel{8}}}{\underset{1}{\cancel{9}}} \times \frac{\overset{2}{\cancel{4}}}{11}$

$= \frac{9 \times 1 \times 2}{1 \times 1 \times 11}$

(*Inverting* the *fraction* which is the *divisor* and *multiplying* instead)

$= \frac{18}{11}$

5 × 16 + 1 = 80 + 1 = 81
2 × 4 + 3 = 8 + 3 = 11

$= 1\frac{7}{11}$

(b) Now $\frac{3}{8}, \frac{4}{5}, \frac{2}{3}, \frac{1}{2}$

The *L.C.M.* of the *denominators* 2, 3, 5 and 8 is 120.

$= \frac{3 \times 15, 4 \times 24, 2 \times 40, 1 \times 60}{120}$

$= \frac{45, 96, 80, 60}{120}$

$\frac{120}{8} = 15$	$\frac{120}{5} = 24$
$\frac{120}{3} = 40$	$\frac{120}{2} = 60$

(i) Since $\frac{45}{120} < \frac{60}{120} < \frac{80}{120} < \frac{96}{120}$

Then $\frac{3}{8} < \frac{1}{2} < \frac{2}{3} < \frac{4}{5}$.

So the *fractions* arranged in *ascending order* is:

$$\frac{3}{8}, \frac{1}{2}, \frac{2}{3}, \frac{4}{5}.$$

(ii) Since $\frac{96}{120} > \frac{80}{120} > \frac{60}{120} > \frac{45}{120}$

Then $\frac{4}{5} > \frac{2}{3} > \frac{1}{2} > \frac{3}{8}$.

So the *fractions* arranged in *descending order* is:

$$\frac{4}{5}, \frac{2}{3}, \frac{1}{2}, \frac{3}{8}.$$

Exercise 3e

1. Simplify:

(a) $8\frac{3}{4} - 5\frac{1}{3}$ (b) $5\frac{4}{9} \div 4\frac{2}{3}$

2. Evaluate:

(a) $8\frac{1}{5} - 3\frac{2}{9}$ (b) $10\frac{6}{7} - 5\frac{3}{4}$

3. Calculate:

(a) $7\frac{5}{8} - 4\frac{7}{16}$ (b) $4\frac{1}{3} - 1\frac{1}{4} + 2\frac{1}{6}$

4. Simplify each of the following:

(a) $3\frac{5}{9} + \left(\frac{1}{6} - \frac{3}{4} \div 4\frac{1}{2}\right)$

(b) $2\frac{1}{10} \times \frac{23}{24} \div \left(\frac{1}{6} + \frac{3}{5}\right)$

indicate that the values to its right is a decimal fraction. For example:

$$.5 = 0.5 = \frac{5}{10} = \frac{5}{10^1} = 5 \times 10^{-1}$$

$$.53 = 0.53 = \frac{53}{100} = \frac{53}{10^2} = 53 \times 10^{-2}$$

$$.531 = 0.531 = \frac{531}{1000} = \frac{531}{10^3} = 531 \times 10^{-3}$$

$$.5317 = 0.5317 = \frac{5\,317}{10\,000} = \frac{5317}{10^4} = 5\,317 \times 10^{-4}$$

$$.53179 = 0.53179 = \frac{53\,179}{100\,000} = \frac{53\,179}{10^5}$$

$$= 53\,179 \times 10^{-5}$$

From the above *examples* it can be *seen* that:

(i) *Zero* is placed to the *left* of the *decimal point* in order to *indicate* that the *whole number* is *zero*.

(ii) For a given *number of decimal places*, the *numerator* is *divided by* 10^n, where $n \in N$ and *n is equal* to the *number of decimal places*.

(iii) For a given *number of decimal places*, the *numerator* is *multiplied by* 10^{-n}, where $n \in N$ and *n is equal* to the *number of decimal places*.

(iv) The *digits* are *grouped* in *threes* from the *decimal point*, with a *space* left between each *group of three digits*.

(v) We can *convert* from a *decimal fraction* to a *common fraction* and *vice versa* (i.e. *convert* from a *common fraction* to a *decimal fraction* also).

Example 7

(a) Simplify the following:

(i) 0.45 × 10 *(ii) 0.354 × 100*

(iii) 0.0479 × 1000 *(iv) 0.13047 × 10000*

(v) 0.871 × 100000

(b) Simplify the following:

(i) 0.7 ÷ 10 *(ii) 0.81 ÷ 100*

(iii) 0.9857 ÷ 1000 *(iv) 0.147635 ÷ 10000*

(v) 0.893 ÷ 100000

Solution

(a) (i) Now $0.45 \times 10 = 4.5$

(ii) Now $0.354 \times 100 = 35.4$

(iii) Now $0.0479 \times 1\,000 = 47.9$

(iv) Now $0.13047 \times 10\,000 = 1\,304.7$

(v) Now $0.871 \times 100\,000 = 87\,100$

From the above *examples* it can be *seen* that:

(i) For *each zero* in the *power of* 10, we *shift* the *decimal point one place* to the *right* when we are *multiplying*.

(ii) We sometimes have to *add zeros* to a *number*, in order to *keep* the *place values of digits* in the *number*. This *fact* can be seen *illustrated* in *part* (*v*) above.

(b) (i) Now $0.7 \div 10 = \frac{0.7}{10} = 0.7 \times 10^{-1} = 0.07$

(ii) Now $0.81 \div 100 = \frac{0.81}{100} = 0.81 \times 10^{-2}$
$= 0.0081$

(iii) Now $0.9857 \div 1\,000 = \frac{0.9857}{1\,000}$
$= 0.9857 \times 10^{-3}$
$= 0.0009857$

(iv) Now $0.147\,635 \div 10\,000 = \frac{0.147\,635}{10\,000}$
$= 0.147\,635 \times 10^{-4}$
$= 0.0000147635$

(v) Now $0.893 \div 100\,000 = \frac{0.893}{100\,000}$
$= 0.893 \times 10^{-5}$
$= 0.00000893$

From the previous *examples* it can be *seen* that:

(i) For *each zero* in the *power of* 10, we *shift* the *decimal point one place* to the *left*, when we are *dividing*.

(ii) If the *whole number* part of the *decimal number* is *zero*, then we need to *add zeros*, in order to *keep* the *place values of digits* in the *number*, when we are dividing.

Exercise 3f

1. Simplify:
(a) 0.56396 × 10000 (b) 0.61345 ÷ 10000

2. Determine the value of:
(a) 37.58 × 1000 (b) 54.2 ÷ 100

27. Given the fractions $\frac{2}{3}, \frac{1}{2}, \frac{4}{5}$ and $\frac{7}{30}$.

(a) Write the fractions in ascending order.

(b) Write the fractions in descending order.

28. Arrange the following fractions in ascending order:

$$\frac{7}{11}, \frac{1}{2}, \frac{13}{22}, \frac{27}{44}$$

29. Arrange the following fractions in ascending order:

$$\frac{1}{2}, \frac{5}{6}, \frac{3}{4}, \frac{7}{12}$$

30. Arrange the following fractions:

$$\frac{1}{2}, \frac{2}{3}, \frac{7}{10}, \frac{3}{5}$$

(a) in ascending order

(b) in descending order.

31. Write the following fractions in ascending order:

$$\frac{3}{4}, \frac{5}{8}, \frac{7}{9}, \frac{3}{7}$$

32. Divide $8\frac{1}{4}$ by $\frac{9}{16}$.

33. Divide $6\frac{1}{2}$ by $2\frac{1}{4}$.

34. Divide $22\frac{2}{3}$ by $1\frac{8}{9}$.

35. If you read 81 pages of a book in $1\frac{4}{5}$ hours, how many minutes does it take to read one page?

36. Divide $7\frac{2}{3}$ by $5\frac{1}{9}$.

37. A bag of sugar has a mass of $4\frac{1}{2}$ kg. What is the mass of 30 bags?

38. Anna read 60 pages of a book in $1\frac{1}{2}$ hours. How many minutes did it take her to read one page?

39. Write the first quantity as a fraction of the second quantity

8 hours:1 day

40. Write the first quantity as a fraction of the second quantity

10 months:1 year

41. It takes $1\frac{3}{8}$ minutes to wrap a parcel and half a minute to address it. How many minutes does it take to wrap and address a dozen similar parcels?

42. A sheet of plywood has a thickness of $1\frac{1}{2}$ cm. How many sheets of plywood are there in a heap of thickness 105 cm?

43. A medicine bottle contains 60 tablets each of mass $\frac{1}{4}$ of a gram. The empty bottle has a mass of $125\frac{1}{2}$ grams. What is the total mass?

44. A pharmacist counts 45 capsules and puts them in a bottle. Each capsule has a mass of $\frac{7}{9}$ of a gram and the mass of the empty bottle is $152\frac{1}{2}$ grams. What is the total mass?

45. Put either > or < between the fractions:

(a) $\frac{3}{10} \quad \frac{1}{4}$ (b) $\frac{8}{11} \quad \frac{9}{10}$

46. Put either < or > between the fractions:

$$\frac{4}{15} \quad \frac{1}{5}$$

47. A bottle contains 70 vitamin tablets each of mass $\frac{1}{4}$ of a gram. The empty bottle has a mass of $122\frac{1}{2}$ grams. What is the total mass?

48. A girl spent $\frac{1}{5}$ of her pocket money on CDs and $\frac{2}{3}$ on clothes. What fraction of her money did she spend? What fraction of her money has she left?

49. Write either > or < between the following pairs of fractions:

(a) $\frac{7}{9} \quad \frac{5}{8}$ (b) $\frac{2}{3} \quad \frac{5}{6}$

Operations with Decimals

A *decimal fraction* is a way of expressing a proper fraction as a number in base 10, using the place value system.

It is a *fraction* whose *unwritten denominator* is a *power of* 10, and it is indicated by a *decimal point before the numerator*. The *decimal point* is used to

(b) Now

$$\begin{array}{r} 39.\overset{3}{\cancel{4}}\overset{^{1}7}{\cancel{8}}\overset{1}{0} \\ -\ \ 7.395 \\ \hline 32.085 \\ \hline \end{array}$$

So the *difference* is 32.085.

The *example* below *shows* how to perform *long division.*

(c) Now 130.158 ÷ 3.15
= 13 015.8 ÷ 315

And

$$\begin{array}{r} 00041.32 \\ 315\,\overline{)\,1\overset{2}{\cancel{3}}\overset{1}{0}15.80} \\ -\ 1260\downarrow \leftarrow \\ \hline 00415 \\ -\ 315\downarrow \\ \hline \overset{9}{\cancel{1}}\overset{1}{0}08 \\ -\ 945\downarrow \leftarrow \\ \hline 0630 \\ -\ 0630 \leftarrow \\ \hline 0000 \\ \hline \end{array} \qquad \boxed{\begin{array}{r} 315 \\ 4 \times \\ \hline 1260 \end{array}} \quad \boxed{\begin{array}{r} 315 \\ 3 \times \\ \hline 945 \end{array}} \quad \boxed{\begin{array}{r} 315 \\ 2 \times \\ \hline 630 \end{array}}$$

So the *quotient* is 41.32.

The *example* below *shows* how to perform *long multiplication.*

(d) Now

$$\begin{array}{r} 573.12 \\ 4.63 \times \\ \hline 22\,924\,800 \leftarrow \\ 3\,438\,720 + \leftarrow \\ 171\,936 \leftarrow \\ \hline 2\,653.5456 \\ \hline \end{array} \qquad \boxed{\begin{array}{r} 57\,312 \\ 4 \times \\ \hline 229\,248 \end{array}} \quad \boxed{\begin{array}{r} 57\,312 \\ 6 \times \\ \hline 343\,872 \end{array}} \quad \boxed{\begin{array}{r} 57\,312 \\ 3 \times \\ \hline 171\,396 \end{array}}$$

So the *product* is 2 653.545 6.

From the above *examples* it can be *seen* that:

(i) In *adding decimal numbers*, each *digit* with the *same place value* must be placed in the *same column.*

(ii) In *subtracting decimal numbers*, each *digit* with the *same place value* must be placed in the *same column.*
A *zero* is used to *indicate* the *absence* of a *natural* number in a *particular place value*, for the purpose of *subtracting.*

(iii) Before we *divide*, we must *always* make the *divisor* a *whole number.*
The *decimal point* in the *dividend* (i.e. the *number* being *divided*) gives the *decimal point* in the quotient.

(iv) In *multiplying decimal numbers*, the *number* of *decimal places* in the *product* is the *sum* of the *decimal places* of the *two numbers* being *multiplied.*

Thus: 573.12 (2 *decimal places*),
4.63 (2 *decimal places*) and
2 653.545 6 (4 *decimal places*).

So, 2 *decimal places* + 2 *decimal places* = 4 *decimal places*

Exercise 3g

1. Write down the value of:
   (a) 36.34 + 2.71 + 0.041
   (b) 4.317 − 0.015
2. Divide 1.45 by 5
3. Complete 9.2 − 1.82
4. Multiply 3.2 by 1.5
5. Determine the sum of 9.2, 5.6 and 1.3
6. Add 0.58 to 3.5
7. Evaluate 9.5 + 0.86 + 3.7
8. Take 18.3 from 75.6
9. Evaluate 8.62 − 0.51
10. Subtract 1.8 from 10.3
11. Determine the value of:
    (a) 56.8 ÷ 0.4 (b) 0.255 6 ÷ 15
12. Share 15.3 kg equally between two people.
13. Divide 97.8 into 8 equal parts.
14. Determine the value of each of the following expressions:
    (a) 27.418 + 0.967 + 25 + 1.467
    (b) 5.48 − 0.069 1
15. Calculate the exact value of:
    (a) 3.45 × 4.3 (b) 6.2 ÷ 1.24
16. Evaluate the exact value of:
    (a) 2.35 × 6.7 (b) 6.9 ÷ 1.15
17. Determine the exact value of:
    (a) 8.05 + 5.23 − 6.38 (b) 8.21 × 0.05
18. Find the exact value of 6.04 × 3.4

3. Divide 8.24 by 1000

4. Multiply 0.034 by (a) 10 (b) 100 (c) 1000

5. Divide 15.31 by (a) 10 (b) 100 (c) 1000

6. Complete the operation:

(a) 7.5×10^3 (b) 5.071×10^2
(c) 4.73×10^1 (d) 6.87×10^3

7. Complete the operation:

(a) 3.971×10^4 (b) 4.36×10^{-3}

8. Complete the operation:

(a) 4.971×10^2 (b) 1.032×10^{-2}

9. Simplify:

(a) $627.428 \div 10000$ (b) $0.943 \div 10000$

10. Simplify:

(a) 0.0847×100000 (b) $0.4531 \div 100000$

11. Express $\frac{3}{25}$ as a decimal.

12. Express 0.0085 as a common fraction in its lowest terms.

13. Express $\frac{3}{8}$ as a decimal.

14. Express 0.07 as a common fraction.

15. Multiply 0.029 by 10000.

16. Evaluate $3.15 \div 3$

17. Express 0.08 as a common fraction in its lowest terms.

18. What is the difference between 0.48162 and $\frac{7}{32}$?

19. Work out 0.09×0.05

20. Divide 0.0432 by 0.6

21. Express $\frac{4}{25}$ as a decimal.

22. Determine the value of $0.0085 - 0.0003$

23. Express $\frac{7}{8}$ as a decimal.

24. State $\frac{9}{100}$ as a decimal.

25. Write 0.8 as a common fraction in its lowest terms.

26. Express 0.95 as a common fraction in its lowest terms.

27. Change $\frac{5}{8}$ to a decimal.

28. Change 0.85 to a vulgar fraction and simplify.

29. Write in ascending order: $\frac{7}{8}$, 0.95, $\frac{9}{10}$.

30. Write in ascending order: 0.6, $\frac{2}{3}$, $\frac{5}{8}$.

31. Complete the operation: 5.43×10^9.

32. Complete the operation: 1.754×10^{-8}.

33. Write each of the following decimals as fractions in their lowest terms. State each of your answers as a mixed number where appropriate:

(a) 0.0807 (b) 9.07
(c) 17.75 (d) 15.25

34. State each of the following decimals as a common fraction in its lowest terms:

(a) 0.375 (b) 0.72
(c) 0.0315 (d) 0.00016

Decimal Numbers

A *decimal number* consists of a *whole number* and a *decimal fraction. Decimal numbers* are *similar* to *mixed numbers.* Thus:

Decimal number = whole number + decimal fraction.

For example,

$$9.8 = 9 + 0.8$$
$$84.56 = 84 + 0.56$$
$$127.3 = 127 + 0.3$$

Example 8

Simplify each of the following expressions:

(a) $4.57 + 0.8316 + 37.29$ *(b) $39.48 - 7.395$*
(c) $130.158 \div 3.15$ *(d) 573.12×4.63*

Solution

(a) Now

$$\begin{array}{r} {}^{1\;1\;1}\quad\;\; \\ 4.57\;\; \\ +\ 0.8316 \\ 37.29\;\; \\ \hline 42.6916 \\ \hline \end{array} \xleftarrow{\quad} \boxed{1\cancel{9}}\boxed{1\cancel{6}}\boxed{1\cancel{2}}$$

So the *sum* is 42.6916.

(c) The *perimeter* of the *regular hexagon*,

$$P = 67.5 \text{ cm}$$

So the *length* of *one side*, $l = \frac{P}{6}$

$$= \frac{67.5 \text{ cm}}{6}$$

$$= 11.25 \text{ cm}$$

Hence the *length* of *one side* of the *regular hexagon* is 11.25 cm.

(d) The *cost of* 1 m of cloth = \$8.95

∴ the *cost of*
7.5 m of cloth = \$8.95 × 7.5
= \$67.125
= \$67.13 (*correct* to the *nearest cent*)

Hence the *cost* of the cloth is \$67.13.

Exercise 3h

1. In a hardware store I bought 5 screws costing 20 ¢ each and 3 light bulbs costing \$3.99 each. If I paid with two \$10.00 notes, how much change did I receive?
2. On Sunday Anna received her \$25.00 weekly allowance. She spent \$3.75 in school on Monday. On Wednesday she collected \$6.95 as payment for a special chore from her dad. On Friday she pays \$5.87 for a chocolate bar. How much money does she have left to spend on Saturday?
3. A boy bought a comic book costing \$5.25 and a pencil costing \$1.55. He paid with a \$10 note. What amount of change he receive?
4. Sonia gets \$6.50 pocket money on Saturday. On Monday she spends \$2.71. On Tuesday she is given \$3.20 for a special chore at home. On Thursday she spends \$1.54. How much money has she got left?
5. Renald gets \$8.75 pocket money on Saturday. On Monday he spends \$5.65. On Tuesday he is given \$4.30 for a special chore at home. On Thursday he spends \$6.10. How much money has he got left?
6.

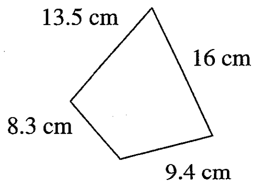


Fig. 3.10 Quadrilateral

Determine the perimeter of the quadrilateral.

7. The bill for two books is \$59.84. One book costs \$23.47. What is the cost of the other book?
8.

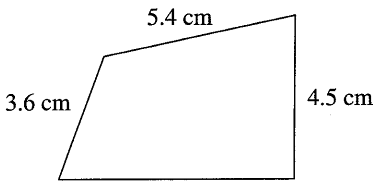


Fig. 3.11 Quadrilateral

The perimeter of the quadrilateral is 20 cm. What is the length of the fourth side?

9. I entered a shop with \$18.95 and bought two articles. One article costs \$6.37 and the other article costs \$9.47. What amount of money did I have left?
10.

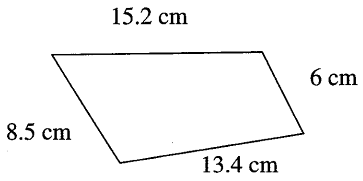


Fig. 3.12 Quadrilateral

Calculate the perimeter of the quadrilateral shown above.

11. The perimeter of an equilateral triangle is 17.4 cm. Determine the length of one side of the triangle.
12. What is the cost of 25 articles at \$2.35 each.
13. The perimeter of a regular nonagon (i.e. a polygon with 9 equal sides) is 31.23 cm. State the length of one of its sides?
14. Beef is sold for \$14.32 a kilogram. What is the cost of 0.36 kg of beef?
15. A book has a mass 0.65 kg. What is the mass of each page, if there are 125 pages and the book cover has a mass of 400 g?
16. Evaluate the cost of 4.5 m of ribbon at 97 ¢ a metre.
17. Add the following sums of money together: \$2.25, \$3.27, \$4.68, \$0.47
18. Subtract:
    (a) \$5.95 from \$11.68 (b) 57 ¢ from \$2.35

19. Add together 10.79, 8.43 and 1.52

20. Find the sum of 4.13, 8.4 and 12.5

21. Evaluate the sum of 19.47, 8.5 and 23.4

22. To 12.7 add 4.5 and 15.32

23. Take 19.5 from 84.3

24. Subtract 5.7 from 12.8

25. From 0.179 subtract 0.025

26. Evaluate 5.62 − 0.91

27. Divide 81.9 into 9 equal parts.

28. Share 25.3 kg of flour equally between two housewives.

29. Evaluate 25.3 ÷ 23

30. Determine the value of 79.8 ÷ 14

31. Evaluate 0.014 28 ÷ 12

32. Determine the value of 0.008 412 ÷ 24

33. Give 0.345 as a common fraction in its lowest terms.

34. Express $\frac{7}{8}$ as a decimal.

35. Evaluate 8.9 × 2.5

36. Add together 17.3, 6.15 and 9

37. Divide 3.5 by 25

38. Calculate each of the following products:

(a) 58.3 × 24 (b) 15.4 × 2.5

(c) 0.0568 × 0.57 (d) 7.73 × 30.6

39. Determine the cost of 15 articles at $24.30 each.

40. Divide 105.6 kg into 8 equal parts.

Word Problems—Decimals

In a *word problem*, we have to *translate* the *English sentences* into a *problem* dealing with the stated *arithmetic operations.* We then *solve* the *problem* using a *logical sequence.*

Example 9

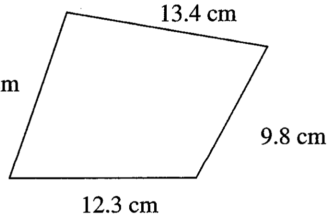


Fig. 3.8 Quadrilateral

(a) Determine the perimeter of the quadrilateral shown above.

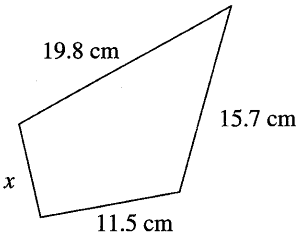


Fig. 3.9 Quadrilateral

(b) The perimeter of the quadrilateral shown above is 55.3 cm. Find the length of the fourth side which is represented by x.

(c) The perimeter of a regular hexagon (i.e. a polygon with 6 equal sides) is 67.5 cm. Calculate the length of one side of the hexagon.

(d) What is the cost of 7.5 metres of cloth at $8.95 per metre.

Solution

(a) The *perimeter* of the *quadrilateral*,

$P = (13.4 + 9.8 + 12.3 + 11.5)$ cm = 47 cm

Hence the *perimeter* of the *quadrilateral* is 47 cm.

(b) The *perimeter* of the *quadrilateral*,

$P = 55.3$ cm.

The *sum* of *three sides* of the *quadrilateral* $= (19.8 + 15.7 + 11.5)$ cm $= 47$ cm

So the *length* of the *fourth side*, $x = (55.3 - 47)$ cm $= 8.3$ cm

Hence the *length* of the *fourth side* of the *quadrilateral* is 8.3 cm.

Note that we *shifted* the *decimal points* in the *denominator* a *total of* 3 *places* to the *right* in order to make the numbers in the *denominator whole numbers*. We therefore need to *shift* the *decimal places* in the *numerator* a *total of* 3 *places* to the *right*. Of course, there are *many ways* of *doing* this. The choice above was mine.

(ii) Now
$$\begin{aligned} &\frac{47.6 + 34.3}{10.4 - 9.5} \\ &= \frac{81.9}{0.9} \\ &= \frac{819}{9} \\ &= 91 \end{aligned}$$

$$\begin{array}{r} \overset{1}{4}7.6 \\ +\ 34.3 \\ \hline 81.9 \end{array} \qquad \begin{array}{r} \overset{9}{\cancel{1}\cancel{0}}.{}^{1}4 \\ -\ \ 9.5 \\ \hline 0.9 \end{array}$$

Note that we *shifted* the *decimal point* in the *denominator* 1 *place* to the *right* in order to make the number in the *denominator* a *whole number*. We therefore need to *shift* the *decimal point* in the *numerator* 1 *place* to the *right* also.

(b) (i) Now
$$\begin{aligned} &\frac{5.7\,(8.9 + 3.5)}{17.3 - 4.9} \\ &= \frac{5.7 \times 12.4}{12.4} \\ &= 5.7 \times 1 \\ &= 5.7 \end{aligned}$$

$$\begin{array}{r} \overset{1}{8}.9 \\ +\ 3.5 \\ \hline 12.4 \end{array} \qquad \begin{array}{r} 1\overset{6}{\cancel{7}}.{}^{1}3 \\ -\ \ 4.9 \\ \hline 12.4 \end{array} \qquad \frac{\overset{1}{\cancel{12.4}}}{\underset{1}{\cancel{12.4}}} = 1$$

Note that a *number* can always *cancel* with an equal *value*. This is an *example* of a *case* where we *do not necessarily* have to make the *denominator a whole number*. The *rule* still *applies* however.

That is $\frac{124}{124} = 1$.

(ii) Now
$$\begin{aligned} &9.5 \times 4.3 - \frac{8.54}{0.7} \\ &= 40.85 - \frac{8.54}{0.7} \\ &= 40.85 - \frac{85.4}{7} \\ &= 40.85 - 12.2 \\ &= 28.65 \end{aligned}$$

$$\begin{array}{r} 9.5 \\ \times\ 4.3 \\ \hline 3\,800 \\ 285 \\ \hline 40.85 \end{array} \qquad \frac{8\overset{1}{5}.{}^{1}4}{7} = 12.2 \qquad \begin{array}{r} \overset{3}{\cancel{4}}{}^{1}0.85 \\ -\ 12.2 \\ \hline 28.65 \end{array}$$

Exercise 3i

1. Evaluate $\frac{35.05 \times 0.27}{0.03 \times 7.01}$
2. Without using tables, determine the exact value of $\frac{0.896 \times 0.01}{0.16}$
3. Without using tables and calculators, calculate the exact value of $\frac{4.5(6 - 2.85)}{3 \times 1.05}$
4. Determine the value of $\frac{0.3 \times 0.52}{0.6}$
5. Calculate the value of $\frac{6.9 \times 1.6}{4.0 \times 2.3}$
6. Without using tables and calculators, calculate the exact value of $\frac{45.37 - 24.16}{13.74 + 7.26}$
7. Without using tables, calculate $\frac{3.65 - 1.05}{1.25 + 0.05}$
8. Determine the exact value of $\frac{3.42 - (2.31 - 1.32)}{0.27 + (4.21 - 1.48)}$
9. Evaluate the exact value of $\frac{6.36 \times 2.5}{0.53}$
10. Without using tables and calculators, calculate the exact value of $\frac{0.45(7 - 3.85)}{1.05 \times 0.15}$
11. Calculators, slide rules and mathematical tables must NOT be used to answer this question. Show ALL steps clearly.
    (a) Calculate the exact value of
    $3.74 \times 5.2 - \frac{6.2}{1.55}$
    (b) Write your answer correct to 1 decimal place.
12. Without using tables, determine the exact value of $\frac{0.426 \times 0.03}{0.142}$
13. Without using tables, evaluate $\frac{7 \times 10.2}{34 \times 0.14}$
14. Without using tables, calculate the exact value of $\frac{0.996 \times 0.07}{0.012}$
15. Without using tables, evaluate $\frac{1.5 + 2.85}{4.66 - 3.21}$
16. Without using tables, determine the exact value of $\frac{6.5(7 - 2.75)}{1.3 \times 0.85}$

19.

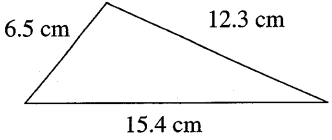


Fig. 3.13 Triangle

Determine the perimeter of the triangle.

20.

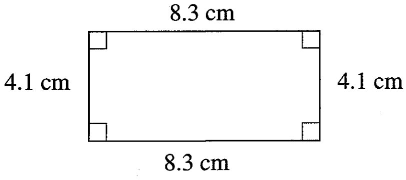


Fig. 3.14 Rectangle

Calculate the perimeter of the rectangle.

21.

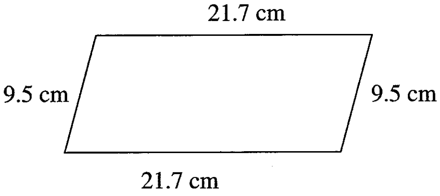


Fig. 3.15 Parallelogram

Determine the perimeter of the parallelogram.

22.

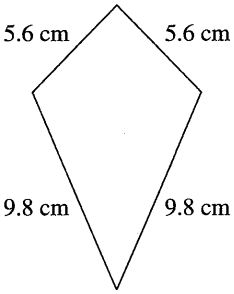


Fig. 3.16 Kite

Calculate the perimeter of the kite.

23.

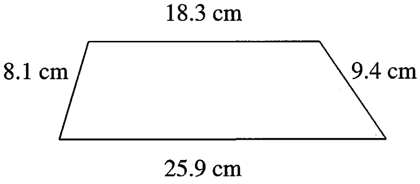


Fig. 3.17 Trapezium

Evaluate the perimeter of the trapezium.

24. The perimeter of a square is 16.4 cm. Calculate the length of a side?

25. The perimeter of an equilateral triangle is 27.9 cm. What is the value of the length of one side?

26. The perimeter of a regular pentagon (i.e. a polygon with 5 equal sides) is 86.5 cm. Calculate the length of one of its sides?

27. The length of a side of a regular nonagon is 7.86 cm. Evaluate the perimeter of the nonagon.

Mixed Operations—Decimals

In *solving problems* dealing with *mixed operations,* we need to follow the *order of arithmetic operations* defined by *BODMAS.*

It should also be noted that, we can only *divide* or *cancel* numbers when we have a *product* in the *numerator* and a *product* in the *denominator.* And it is always better to *divide* or *cancel* first then *multiply.*

Example 10

(a) Evaluate:

(i) $\frac{0.956 \times 10.5}{0.16 \times 0.7}$ *(ii)* $\frac{47.6 + 34.3}{10.4 - 9.5}$

(b) Find the exact value of:

(i) $\frac{5.7(8.9 + 3.5)}{17.3 - 4.9}$

(ii) $9.5 \times 4.3 - \frac{8.54}{0.7}$

Solution

(a) (i) Now $\frac{0.956 \times 10.5}{0.16 \times 0.7}$

$= \frac{95.6 \times 105}{16 \times 7}$

$= 5.975 \times 15$

$= 89.625$

$\frac{\overset{80}{\overset{120}{\overset{15}{95.6}}}}{16} = 5.975$

$\frac{10\overset{3}{5}}{7} = 15$

$$\begin{array}{r} 5.975 \\ 15 \times \\ \hline 59750 \\ 29875 \\ \hline 89.625 \end{array}$$

Example 12

(a) Write each of the following numbers correct to the nearest ten:

(i) 25 *(ii) 134*
(iii) 6598 *(iv) 80571*

(b) Write each of the following numbers as an approximate number of tens:

(i) 35 *(ii) 94*
(iii) 798 *(iv) 8473*

Solution

(a) (i) Now $25 = 2|5 \approx 20 + 10 = 30$
(*correct* to the *nearest ten*)

(ii) Now $134 = 13|4 = 130$
(*correct* to the *nearest ten*)

(iii) Now $6598 = 659|8 \approx 6590 + 10 = 6600$
(*correct* to the *nearest ten*)

(iv) Now $80571 = 8057|1 = 80570$
(*correct* to the *nearest ten*)

(b) (i) Now $35 = 3|5 \approx 30 + 10 = 40$
So $35 \approx 4$ tens.

(ii) Now $94 = 9|4 \approx 90$
So $94 \approx 9$ tens.

(iii) Now $798 = 79|8 \approx 790 + 10 = 800$
So $798 \approx 80$ tens.

(iv) Now $8473 = 847|3 \approx 8470$
So $8473 \approx 847$ tens.

CASE 2: CORRECT TO THE NEAREST HUNDRED

In *approximating numbers correct* to the *nearest hundred*, we have to look at the *digit value* of the *tens.* If the *digit value* of the *tens* is 5 *or more than* 5, then we have to *add* 1 to the *digit value* of the *hundreds.* Otherwise we *do not* have to *add* 1 to the *digit value* of the *hundreds.*

Example 13

(a) Write each of the following numbers correct to the nearest hundred:

(i) 453 *(ii) 768*
(iii) 8439 *(iv) 9827*

(b) Write each of the following numbers as an approximate number of hundreds:

(i) 453 *(ii) 8427*
(iii) 12981 *(iv) 728*

Solution

(a) (i) Now $453 = 4|53 \approx 400 + 100 = 500$
(*correct* to the *nearest hundred*)

(ii) Now $768 = 7|68 \approx 700 + 100 = 800$
(*correct* to the *nearest hundred*)

(iii) Now $8439 = 84|39 = 8400$
(*correct* to the *nearest hundred*)

(iv) Now $9827 = 98|27 = 9800$
(*correct* to the *nearest hundred*)

(b) (i) Now $453 = 4|53 \approx 400 + 100 = 500$
So $453 \approx 5$ hundreds.

(ii) Now $8427 = 84|27 \approx 8400$
So $8427 \approx 84$ hundreds.

(iii) Now $12981 = 129|81 \approx 12900 + 100 = 13000$
So $12981 \approx 130$ hundreds.

(iv) Now $728 = 7|28 \approx 700$
So $728 \approx 7$ hundreds.

The *method* of *approximating numbers correct* to the *nearest ten* or *correct* to the *nearest hundred* can be extended in order to *approximate numbers correct* to any *power of* 10.

Exercise 3j

1. Express each of the following decimals correct to the nearest whole number:

(a) 9.8 (b) 2.4
(c) 5.7 (d) 6.2

2. State each of the following decimal numbers correct to the nearest whole number:

(a) 15.7 (b) 24.1
(c) 39.5 (d) 75.2

3. Write each of the following decimals correct to the nearest whole number:

(a) 345.813 (b) 471.541
(c) 213.215 (d) 839.157

17. Calculate the exact value of $\frac{0.95(8 - 4.25)}{0.45 + 0.8}$

18. Without using tables and calculator, calculate the exact value of $\frac{45.35 + 13.15}{35.60 - 29.75}$

19. Without using calculator, evaluate the exact value of $\frac{0.508 \times 0.05}{0.127}$

20. (a) Determine the exact value of

$$3.45 \times 4.3 - \frac{6.2}{1.24}$$

(b) Write your answer correct to 1 decimal place.

21. Calculate the exact value of $\frac{9.6 \times 0.2}{1.2 \times 0.4}$

22. Add 5.2 and 0.7, then subtract the result from 8.1.

23. Calculate $\frac{0.6 \times 1.3}{0.026}$

24. Add 5.3 to 0.27, then subtract 1.5 from the result.

25. Evaluate $15.4 + 2.63 - 3.8$

26. Calculate $\frac{0.8 \times 0.3}{0.09}$

27. Evaluate $3.5^2 + 0.5 \times 3.5$

28. Calculate $8.75^2 - 4.75 \times 8.75$

29. Evaluate $5.839^2 - 3.161^2$

30. Determine the value of the following $\frac{0.7 \times 0.32}{0.14}$

31. Multiply 8.5 by 0.7 and divide the result by 0.05.

32. Calculate the value of each of the following expressions:

(a) $\frac{0.8 \times 0.14}{0.7}$ (b) $\frac{0.36}{0.2 \times 0.3}$

(c) $\frac{9.3 \times 0.04}{0.03 \times 2}$ (d) $\frac{0.85 \times 3}{0.6 \times 0.15}$

Approximation: Nearest Whole Number

An *approximation* is a stated value of a number that is close to, but not equal to, the exact value of the number. Several reasonable approximations are always possible for any number. The most suitable approximation of a number depends on the degree of accuracy required.

In *approximating* a *decimal number correct* to the *nearest whole number*, we have to look at the *digit value* of the *first decimal place.* If the *digit value* in the *first decimal place* is 5 *or more than* 5, then we have to *add* 1 to the *whole number part* in order to write the *decimal number correct* to the *nearest whole number.* If however, the *digit value* in the *first decimal place* is *less than* 5, then the *whole number part* is *equal* to the *decimal number correct* to the *nearest whole number.*

Note that the symbol ≈ means '*is approximately equal to*'.

Example 11

Write each of the following decimal numbers correct to the nearest whole number:

(a) 174.573 *(b) 247.48*

(c) 68.9 *(d) 59.3*

Solution

(a) Now $174.573 = 174.|573 \approx 174 + 1 = 175$
(*correct* to the *nearest whole number*)

(b) Now $247.48 = 247.|48 = 247$
(*correct* to the *nearest whole number*)

(c) Now $68.9 = 68.|9 \approx 68 + 1 = 69$
(*correct* to the *nearest whole number*)

(d) Now $59.3 = 59.|3 = 59$
(*correct* to the *nearest whole number*)

Approximation: Nearest Power of Ten

CASE 1: CORRECT TO THE NEAREST TEN

In *approximating numbers correct* to the *nearest ten*, we have to look at the *digit value* of the *units.* If the *digit value* of the *units* is 5 *or more than* 5, then we have to *add* 1 to the *digit value* of the *tens.* Otherwise, we *do not* have to *add* 1 to the *digit value* of the *tens.*

(ii) Now $0.0479 \div 0.00312 = 15.3525|641$

$= 15.3526$

(*correct* to 4 *d.p.*)

Recurring Decimals

Many *fractions cannot* be written as an *exact decimal fraction* because they *do not terminate.*

Such *fractions* are called *recurring decimals*, because they *do not terminate*, and one or more *decimal digit* keeps *recurring.* A *dot* is placed at the *top of the decimal digit* that *recurs.* If *more* than one *digit recurs,* then a *dot* is placed on the *first* and the *last digits* that *recur.*

Example 15

(a) *Write each of the following fractions as recurring decimals:*

(i) $\frac{1}{6}$ (ii) $\frac{1}{12}$ (iii) $\frac{3}{11}$

(iv) $\frac{5}{12}$ (v) $\frac{7}{11}$ (vi) $\frac{9}{11}$

(b) *Write each of the following fractions as decimals, correct to the number of decimal places given in brackets.*

(i) $\frac{2}{11}$ (5 *d.p.*) (ii) $\frac{5}{6}$ (4 *d.p.*)

(iii) $\frac{2}{3}$ (3 *d.p.*)

Solution

(a) (i) Now $\frac{1}{6} \approx 0.1666 = 0.1\dot{6}$

(*recurring decimal*)

(ii) Now $\frac{1}{12} \approx 0.0833 = 0.08\dot{3}$

(*recurring decimal*)

(iii) Now $\frac{3}{11} \approx 0.2727 = 0.\dot{2}\dot{7}$

(*recurring decimal*)

(iv) Now $\frac{5}{12} \approx 0.4166 = 0.41\dot{6}$

(*recurring decimal*)

(v) Now $\frac{7}{11} \approx 0.6363 = 0.\dot{6}\dot{3}$

(*recurring decimal*)

(vi) Now $\frac{9}{11} \approx 0.8181 = 0.\dot{8}\dot{1}$

(*recurring decimal*)

(b) (i) Now $\frac{2}{11} \approx 0.18181|8 = 0.18182$

(*correct* to 5 *d.p.*)

(ii) Now $\frac{5}{6} \approx 0.8333|3 = 0.8333$

(*correct* to 4 *d.p.*)

(iii) Now $\frac{2}{3} \approx 0.666|6 = 0.667$

(*correct* to 3 *d.p.*)

Exercise 3k

1. Write 25.0347 as a decimal correct to 3 decimal places.
2. Write 15.403 correct to two decimal places.
3. State each of the following numbers correct to 2 decimal places:
   (a) 5.126 (b) 0.085 (c) 3.999
4. State each of the following numbers correct to the number of decimal places given in brackets:
   (a) 5.05 (1 d.p.) (b) 286.598 (2 d.p.)
   (c) 0.003921 (4 d.p.) (d) 0.0088 (3 d.p.)
5. Write each of the following numbers correct to the number of decimal places given in brackets:
   (a) 5.06 (1 d.p.) (b) 289.597 (2 d.p.)
   (c) 0.003924 (4 d.p.) (d) 0.0085 (3 d.p.)
6. Give 9.7814 correct to:
   (a) the nearest whole number
   (b) 1 decimal place
   (c) 3 decimal places.
7. Calculate the value of $0.1394 \div 0.004$ correct to 2 decimal places.
8. State the value of $\frac{5}{7}$ as a recurring decimal.
9. Divide 41 by 15. Give your answer to 3 decimal places.
10. Divide 8.24 by 12. Give your answer as a recurring decimal.
11. Evaluate $14.8 \div 4.4$, stating your answer as a recurring decimal.
12. Evaluate $8.45 \div 0.7$ correct to 3 decimal places.

4. State each of the following decimals correct to the nearest whole number:

(a) 174.9 (b) 54.3
(c) 1895.8 (d) 347.1

5. Express each of the following decimal numbers correct to the nearest whole number:

(a) 799.5 (b) 39.1
(c) 349.8 (d) 47.3

6. Write each of the following numbers correct to the nearest ten:

(a) 58 (b) 54
(c) 59 (d) 53

7. State each of the following numbers as an approximate number of tens:

(a) 125 (b) 123
(c) 129 (d) 124

8. Write each of the following numbers to the nearest ten:

(a) 3542.25 (b) 4781.17
(c) 9435.08 (d) 2897.09

9. Express each of the following numbers correct to the nearest hundred:

(a) 4547 (b) 7952
(c) 3761 (d) 8419

10. State each of the following numbers as an approximate number of hundreds:

(a) 3145 (b) 3154
(c) 3137 (d) 3179

11. Express each of the following numbers correct to the nearest hundred:

(a) 71431.84 (b) 85749.32
(c) 97481.03 (d) 21752.09

12. Write 157508 correct to the nearest hundred.

13. State each of the following numbers correct to the nearest number of tens, and hence determine an approximate answer for each problem:

(a) $341 - 82$ (b) $267 + 109$
(c) $520 + 32 - 125$ (d) $947 - 839 + 341$

14. Write each of the following numbers correct to the nearest number of hundreds, and hence calculate an approximate answer for each problem:

(a) $345 + 178$ (b) $851 - 587$
(c) $751 + 349 - 463$ (d) $917 - 853 + 182$

15. Express each of the following numbers correct to the nearest number of hundreds, and hence state an approximate answer for each problem:

(a) $8471 + 6345$
(b) $3582 - 2954$
(c) $6453 + 1072 - 2371$
(d) $5164 - 3173 + 1045$

Approximation: Decimal Places

In *approximating* a *decimal number correct* to *n decimal places*, we have to look at the *digit value* of the $(n + 1)^{th}$ *decimal place.* If the *digit value* of the $(n + 1)^{th}$ *decimal place* is *greater than or equal to* 5, then we have to *add* 1 to the n^{th} *decimal place digit.* Otherwise, we *do not add* the 1.

Example 14

(a) Express each of the following numbers correct to the number of decimal places stated:

(i) 6.07 (1 d.p.) *(ii) 84.345 (2 d.p.)*
(iii) 124.09781 (3 d.p.)
(iv) 93.943973 (4 d.p.)

(b) Calculate the value of:

(i) 27.45×13.93 correct to 2 decimal places
(ii) $0.0479 \div 0.00312$ correct to 4 decimal places.

Solution

(a) (i) Now $6.07 = 6.0|7 \approx 6.0 + 0.1 = 6.1$
(*correct* to 1 *d.p.*)

(ii) Now $84.345 = 84.34|5 \approx 84.34 + 0.01$
$= 84.35$
(*correct* to 2 *d.p.*)

(iii) Now $124.09781 = 124.097|81$
$\approx 124.097 + 0.001$
$= 124.098$
(*correct* to 3 *d.p.*)

(iv) Now $93.943973 = 93.9439|73$
$\approx 93.9439 + 0.0001$
$= 93.9440$
(*correct* to 4 *d.p.*)

(b) (i) Now $27.45 \times 13.93 = 382.37|85$
$= 382.38$
(*correct* to 2 *d.p.*)

(iii) Now $0.005\,807\,016 = 0.005\,807|016$
$= 0.005\,807$
(*correct* to 4 *s.f.*)

(iv) Now $0.005\,807\,016 = 0.005\,80|7\,016$
$= 0.005\,81$
(*correct* to 3 *s.f.*)

(v) Now $0.005\,807\,016 = 0.005\,8|07\,016$
$= 0.005\,8$
(*correct* to 2 *s.f.*)

(vi) Now $0.005\,807\,016 = 0.005|807\,016$
$= 0.006$
(*correct* to 1 *s.f.*)

Exercise 3I

1. Write 19.407 correct to three significant figures.
2. Write 0.005 483 correct to two significant figures.
3. Give the following numbers correct to the number of significant figures indicated in the brackets:
   (a) 475.831 07 (4 s.f.) (b) 59.072 03 (3 s.f.)
   (c) 0.000 068 3 (2 s.f.) (d) 0.049 850 2 (1 s.f.)
4. Give the following numbers correct to the number of significant figures indicated in the brackets:
   (a) 478.831 06 (4 s.f.) (b) 58.073 02 (3 s.f.)
   (c) 0.000 068 2 (2 s.f.) (d) 0.048 851 (1 s.f.)
5. State 9 862 correct to one significant figure.
6. State 0.050 706 correct to two significant figures.
7. Determine the value of $(0.543)^2$ giving your answer correct to three significant figures.
8. Write each of the following numbers correct to the number of significant figures indicated in the brackets:
   (a) 46.931 06 (2 s.f.) (b) 4 537 (1 s.f.)
   (c) 0.067 34 (1 s.f.) (d) 37.856 72 (3 s.f.)
9. Express the number 816.095 4 correct to the number of significant figures stated below:
   (a) 6 s.f. (b) 5 s.f. (c) 4 s.f.
   (d) 3 s.f. (e) 2 s.f. (f) 1 s.f.
10. Express the number 0.007 836 152 correct to the number of significant figures stated below:
    (a) 6 s.f. (b) 5 s.f. (c) 4 s.f.
    (d) 3 s.f. (e) 2 s.f. (f) 1 s.f.
11. (a) Calculate the exact value of
    $$\left(5\frac{3}{5} \div 1\frac{5}{9}\right) - \left(1\frac{1}{5}\right)^2$$
    (b) Write your answer to part (a) as a decimal correct to three significant figures.
12. Determine the value of 26.32×15.4 correct to four significant figures.
13. State the value of 25.42×29.23 correct to five significant figures.
14. Determine the value of 0.043×0.032 correct to four significant figures.
15. State the value of $12.07 \div 0.008\,97$ giving your answer correct to three significant figures.
16. Calculate 8.65×0.105 giving your answer correct to four significant figures.

Standard Form (or Scientific Notation)

A *rational number* which is written in the *form* $a \times 10^n$, where $1 \leqslant a < 10$ and $n \in Z$ is said to be written in *standard form* (or *scientific notation*). This notation is widely used in science and engineering in order to represent very large and very small numbers.

Example 17

(a) Express each of the following numbers in standard form:
(i) 841 902 *(ii) 0.000 479 35*

(b) Express each of the following numbers in scientific notation:
(i) 749 543 (correct to 3 s.f.)
(ii) 0.000 578 49 (correct to 3 s.f.)

Solution

(a) (i) Now $841\,902 = 8.419\,02 \times 100\,000$
$= 8.419\,02 \times 10^5$
(in *standard form*)

(ii) Now $0.000\,479\,35 = 4.793\,5 \times \dfrac{1}{10\,000}$
$= 4.793\,5 \times 10^{-4}$
(in *standard form*)

13. Determine the value of:

(a) 25.42×29.23 correct to 2 decimal places
(b) 0.043×0.032 correct to 3 decimal places.

14. Determine the value of:

(a) $18.89 \div 14.2$ correct to 2 decimal places
(b) $0.1382 \div 0.0032$ correct to 1 decimal place.

15. Write $\frac{2}{3}$ as a recurring decimal.

16. Write each of the following fractions as a recurring decimals:

(a) $\frac{5}{11}$ (b) $\frac{10}{11}$

17. State each of the following fractions as decimals correct to two decimal places:

(a) $\frac{5}{11}$ (b) $\frac{10}{11}$

18. Write each of the following fractions as decimals correct to four decimal places:

(a) $\frac{8}{15}$ (b) $\frac{13}{15}$

19. State each of the following fractions as recurring decimals:

(a) $\frac{8}{15}$ (b) $\frac{13}{15}$

20. Write each of the following fractions as decimals correct to five decimal places:

(a) $\frac{7}{9}$ (b) $\frac{8}{9}$

21. Calculate, giving each of your answers correct to 2 decimal places:

(a) $27.85 \div 15$ (b) $54.9 \div 48$

22. Evaluate, giving each of your answers correct to 3 decimal places:

(a) $17.91 \div 0.8$ (b) $0.194 \div 2.3$

23. Express each of the following mixed numbers as recurring decimals:

(a) $3\frac{5}{9}$ (b) $4\frac{7}{11}$ (c) $2\frac{6}{7}$

24. Express each of the following sets of numbers as decimals correct to 2 decimal places and hence write them in ascending order:

(a) $\frac{1}{2}, 0.47, \frac{3}{5}$ (b) $\frac{3}{4}, 0.76, \frac{2}{5}$

(c) $0.\dot{6}, \frac{7}{10}, \frac{4}{5}$ (d) $0.\dot{3}, \frac{1}{4}, \frac{2}{5}$

Approximation: Significant Figures

In *approximating* a *number correct* to *n significant figures*, we have to look at the *digit value* of the $(n+1)^{th}$ *significant figure.* If the *digit value* of the $(n+1)^{th}$ *significant figure* is *greater than or equal to* 5, then we have to *add* 1 to the n^{th} *significant figure.* Otherwise, we *do not add* the 1. It should also be noted that the *first significant figure cannot* be *zero.* However, *zero* can be a *significant figure* otherwise. The first significant figure of a number is the first non-zero digit that occurs in the number, reading from left to right.

Example 16

(a) Express the number 105.8054 correct to the number of significant figures stated below:

(i) 6 s.f. (ii) 5 s.f. (iii) 4 s.f.
(iv) 3 s.f. (v) 2 s.f. (vi) 1 s.f.

(b) Express the number 0.005 807 016 correct to the number of significant figures stated below:

(i) 6 s.f. (ii) 5 s.f. (iii) 4 s.f.
(iv) 3 s.f. (v) 2 s.f. (vi) 1 s.f.

Solution

(a) (i) Now $105.8054 = 105.805|4 = 105.805$ (*correct* to 6 *s.f.*)

(ii) Now $105.8054 = 105.80|54 = 105.81$ (*correct* to 5 *s.f.*)

(iii) Now $105.8054 = 105.8|054 = 105.8$ (*correct* to 4 *s.f.*)

(iv) Now $105.8054 = 105.|8054 = 106$ (*correct* to 3 *s.f.*)

(v) Now $105.8054 = 10|5.8054 = 110$ (*correct* to 2 *s.f.*)

(vi) Now $105.8054 = 1|05.8054 = 100$ (*correct* to 1 *s.f.*)

(b) (i) Now $0.005\,807\,016 = 0.005\,807\,01|6 = 0.005\,807\,02$ (*correct* to 6 *s.f.*)

(ii) Now $0.005\,807\,016 = 0.005\,807\,0|16 = 0.005\,807\,0$ (*correct* to 5 *s.f.*)

Example 18

(a) A steel rod is measured to be 18.43 metres in length, correct to the nearest centimetre.
State:
  - *(i) the error interval involved in the measurement of the length*
  - *(ii) the greatest possible length of the steel rod*
  - *(iii) the least possible length of the steel rod*
  - *(iv) the range in which the exact length of the steel rod must lie in the form a ± b.*

(b) Two adjacent sides of a parallelogram are measured as 27.9 cm and 18.7 cm in length, correct to the nearest millimetre.
State:
  - *(i) the error interval involved in the measurement of the length of each adjacent side*
  - *(ii) the apparent semi-perimeter of the parallelogram*
  - *(iii) the greatest possible semi-perimeter of the parallelogram*
  - *(iv) the least possible semi-perimeter of the parallelogram*
  - *(v) the range in which the exact semi-perimeter of the parallelogram must lie in the form a ± b.*

(c) The length and breadth of a rectangle are measured as 15.6 cm and 9.4 cm correct to the nearest millimetre.
Determine:
  - *(i) the error interval involved in the measurement of the lengths*
  - *(ii) the apparent area of the rectangle*
  - *(iii) the greatest possible area of the rectangle*
  - *(iv) the least possible area of the rectangle*
  - *(v) the range in which the exact area of the rectangle must lie in the form a ± b.*

(d) Express the range in which the exact value of the difference of 8.6 g and 5.3 g must lie in the form a ± b, if both masses are correct to 2 significant figures.

(e) Express the range in which the exact value of the quotient $\frac{195}{130}$ *must lie in the form of a ± b, if both numbers are correct to three significant figures.*

Solution

(a) (i) The length of the steel rod is *measured* to be 18.43 metres *correct* to the *nearest centimetre.*
So the *smallest unit of measurement* $= 1 \text{ cm} = \frac{1}{100} \text{ m} = 0.01 \text{ m}.$
Then the *absolute error* $= \frac{0.01 \text{ m}}{2} = 0.005 \text{ m}.$
Hence the *error* interval involved in the measurement of the length is ± 0.005 m.

(ii) The *greatest possible length* of the steel rod.
$l_{max} = (18.43 + 0.005) \text{ m} = 18.435 \text{ m}$

(iii) The *least possible length* of the steel rod.
$l_{min} = (18.43 - 0.005) \text{ m} = 18.425 \text{ m}$

(iv) The *range* in which the *exact length* of the steel rod must lie in the *form*
$a \pm b = (18.43 \pm 0.005) \text{ m}.$

(b) (i) The two adjacent sides of a parallelogram are *measured* as 27.9 cm and 18.7 cm *correct* to the *nearest millimetre.* Therefore the *error* interval involved in the *measurement of the length of each adjacent side* is ± 0.05 cm.

(ii) The *apparent semi-perimeter* of the *parallelogram,*

$$s = l + b$$
$$= (27.9 + 18.7) \text{ cm} = 46.6 \text{ cm}$$

(iii) The *greatest possible length,*
$l_{max} = (27.9 + 0.05) \text{ cm} = 27.95 \text{ cm}$
The *greatest possible breadth,*
$b_{max} = (18.7 + 0.05) \text{ cm} = 18.75 \text{ cm}$
∴ the *greatest possible semi-perimeter* of the *parallelogram,*
$$s_{max} = l_{max} + b_{max} = (27.95 + 18.75) \text{ cm} = 46.7 \text{ cm.}$$

(iv) The *least possible length,*
$l_{min} = (27.9 - 0.05) \text{ cm} = 27.85 \text{ cm}$
The *least possible breadth,*
$b_{min} = (18.7 - 0.05) \text{ cm} = 18.65 \text{ cm}$
∴ the *least possible semi-perimeter* of the *parallelogram,*
$$s_{min} = l_{min} + b_{min} = (27.85 + 18.65) \text{ cm} = 46.5 \text{ cm}$$

(v) Now $+b = s_{max} - s = (46.7 - 46.6) \text{ cm} = +0.1 \text{ cm}$

And $-b = s_{min} - s = (46.5 - 46.6) \text{ cm} = -0.1 \text{ cm}$

Hence the *range* in which the *exact semi-perimeter* of the *parallelogram* must lie in the *form a ± b* = (46.6 ± 0.1) cm.

(b) (i) Now $749\,543 = 7.49|5\,43 \times 10^5$
$= 7.50 \times 10^5$
(*correct* to 3 *s.f.*)

(ii) Now $0.000\,578\,49 = 5.78|4\,9 \times 10^{-4}$
$= 5.78 \times 10^{-4}$
(*correct* to 3 *s.f.*)

Exercise 3m

1. Write each of the following numbers in standard form:
   (a) 7438 (b) 12 149
2. State each of the following numbers in scientific notation:
   (a) 0.004 79 (b) 0.094 31
3. Express each of the following numbers in standard form:
   (a) 15.78 (b) 224.09
4. Write each of the following numbers in scientific notation:
   (a) 847.08 (b) 12 436.3
5. Express each of the following numbers in scientific notation:
   (a) 0.047 98 (b) 0.000 034 5
6. (a) Write 0.009 352 in standard form.
   (b) State your answer to part (a) correct to:
      (i) three significant figures
      (ii) one decimal place.
7. Write in standard form:
   (a) 3 850 000 000 (b) 0.000 000 007 308
8. Express each of the following numbers in standard form:
   (a) 0.000 043 7 (*correct* to 2 s.f.)
   (b) 0.000 000 094 83 (*correct* to 3 s.f.)
   (c) 4 937 682 (*correct* to 3 s.f.)
   (d) 9 845 (*correct* to 2 s.f.)
9. State each of the following numbers in standard form:
   (a) 0.003 921 (*correct* to 2 s.f.)
   (b) 0.008 8 (*correct* to 2 s.f.)
10. Write each of the following numbers in standard form:
    (a) 0.000 042 7 (*correct* to 2 s.f.)
    (b) 0.000 000 095 3 (*correct* to 3 s.f.)
    (c) 4 927 683 (*correct* to 3 s.f.)
    (d) 9 835 (*correct* to 2 s.f.)
11. State 0.007 805 in standard form.
12. Write the following numbers in standard form (or scientific notation):
    (a) 736 000 (b) 43.5 (c) 0.004 37
13. Express each of the following numbers in standard form:
    (a) 630.21 (b) 62.79 (c) 0.080 5
14. Write 37 400 in standard form correct to two significant figures.
15. State 0.005 09 in standard form correct to one significant figure.

Constructing the Range in Which the Exact Value of a Computation must Lie

Everytime we perform the *process* of *measuring* there are *two inherent errors* occurring:

(i) An *error* due to the *inaccuracy* of the *instrument* being used.

We all know that *different watches*, for example give *different values* for the *same time* of the day. Some *watches* can *measure time correct* to the *nearest second*, while others can *measure time correct* to the *nearest minute* only.

(ii) An *error* due to the *judgement* of the *observer*. We all know that *two or more observers* can *measure* the *same length* of wood, for example, and state *two or more values* for the *length*. This is so because *different observers* can *measure* to *different degrees of accuracy*. Some *observers* can *measure correct* to the *nearest metre*, others can *measure correct* to the *nearest centimetre*, while others still can *measure correct* to the *nearest millimetre*.

(a) The *absolute error* involved in the *measurement* of a *quantity* is defined as *half* of the *smallest unit of measurement* (i.e. half of the smallest marked divisions on the instrument).

(b) The *error interval* is defined as ± *the absolute error*.

Determine:

(a) the error interval involved in the measurement of the length

(b) the greatest possible length of the gold string

(c) the least possible length of the gold string

(d) the range in which the exact length of the gold string must lie in the form $a \pm b$.

3. Write in the form $a \pm b$, the range within which the exact value of each of the following lengths must lie, if each is measured correct to the last stated digit.

(a) 125.6 cm (b) 18.53 cm (c) 9.237 cm

4. Express in the form $a \pm b$, the range within which the exact value of each of the following numbers must lie, if each is correct to the last given digit.

(a) 2347.8 (b) 145.94 (c) 768.149

5. Two adjacent sides of a rectangle are measured as 34.9 cm and 25.4 cm correct to the nearest millimetre.

State:

(a) the error interval involved in the measurement of each length

(b) the apparent semi-perimeter of the rectangle

(c) the greatest possible semi-perimeter of the rectangle

(d) the least possible semi-perimeter of the rectangle

(e) the range in which the exact semi-perimeter of the rectangle must lie in the form $a \pm b$.

6. Two unequal adjacent sides of a kite are measured as 127.82 cm and 85.83 cm in length correct to the nearest $\frac{1}{100}$ cm.

Determine:

(a) the error interval involved in the measurement of each length

(b) the apparent semi-perimeter of the kite

(c) the greatest possible semi-perimeter of the kite

(d) the least possible semi-perimeter of the kite

(e) the range in which the exact semi-perimeter of the kite must lie in the form $a \pm b$.

7. Write in the form $a \pm b$, the range within which the exact value of each of the following sum of lengths must lie, if each length is measured correct to the last stated digit.

(a) 9.8 cm + 7.4 cm

(b) 12.65 cm + 8.21 cm

(c) 72.134 cm + 89.768 cm

8. Express in the form $a \pm b$, the range within which the exact value of each of the following sum of numbers must lie, if each is correct to the last given digit.

(a) 12.4 + 8.2

(b) 93.71 + 84.93

(c) 124.134 + 97.847

9. The base and altitude of a parallelogram are measured as 27.8 cm and 12.7 cm correct to the nearest millimetre.

State:

(a) the error interval involved in the measurement of each length

(b) the apparent area of the parallelogram

(c) the greatest possible area of the parallelogram

(d) the least possible area of the parallelogram

(e) the range in which exact area of the parallelogram must lie in the form $a \pm b$.

10. The lengths of the diagonals of a kite are measured as 19.87 cm and 15.32 cm correct to the nearest $\frac{1}{100}$ cm.

Determine:

(a) the error interval involved in the measurement of each diagonal

(b) the apparent area of the kite

(c) the greatest possible area of the kite

(d) the least possible area of the kite

(e) the range in which exact area of the kite must lie in the form $a \pm b$.

11. Write in the form $a \pm b$, the range within which the exact value of each of the following product of lengths must lie, if each length is measured correct to the least stated digit.

(a) 7.5 cm × 8.9 cm

(b) 12.61 cm × 15.83 cm

(c) 124.763 cm × 29.872 cm

12. Express in the form $a \pm b$, the range within which the exact value of each of the following product of masses must lie, if each mass is correct to the last given digit.

(a) 9.7 g × 12.4 g

(b) 97.68 g × 24.91 g

(c) 127.83 g × 49.72 g

(c) (i) The length and breadth of the rectangle are *measured* as 15.6 cm and 9.4 cm correct to the *nearest millimetre*. Therefore the *theoretical error* involved in the *measurement of each length* is ±0.05 cm.

(ii) The *apparents area* of the *rectangle*,

$A = lb$

$= 15.6 \text{ cm} \times 9.4 \text{ cm}$

$= 146.64 \text{ cm}^2$

(iii) The *greatest possible length,*

$l_{max} = (15.6 + 0.05) \text{ cm} = 15.65 \text{ cm}$

The *greatest possible breadth,*

$b_{max} = (9.4 + 0.05) \text{ cm} = 9.45 \text{ cm}$

∴ the *greatest possible area* of the *rectangle,*

$A_{max} = l_{max} \times b_{max} = 15.65 \text{ cm} \times 9.45 \text{ cm}$

$= 147.89 \text{ cm}^2$

(*correct* to 2 *d.p.*)

(iv) The *least possible length,*

$l_{min} = (15.6 - 0.05) \text{ cm} = 15.55 \text{ cm}$

The *least possible breadth,*

$b_{min} = (9.4 - 0.05) \text{ cm} = 9.35 \text{ cm}$

∴ the *least possible area* of the *rectangle,*

$A_{min} = l_{min} \times b_{min} = 15.55 \text{ cm} \times 9.35 \text{ cm}$

$= 145.39 \text{ cm}^2$

(*correct* to 2 *d.p.*)

(v) Now $+b = A_{max} - A$

$= (147.89 - 146.64) \text{ cm}^2$

$= +1.25 \text{ cm}^2$

And $-b = A_{min} - A$

$= (145.39 - 146.64) \text{ cm}^2$

$= -1.25 \text{ cm}^2$

Hence the *range* in which the *exact area* of the *rectangle* must lie in the *form*

$a \pm b = (146.64 \pm 1.25) \text{ cm}^2$.

(d) The *apparent difference*, $a = (8.6 - 5.3) \text{ g}$

$= 3.3 \text{ g}$

The *greatest possible difference* $= (8.65 - 5.25) \text{ g}$

$= 3.4 \text{ g}$

The *least possible difference* $= (8.55 - 5.35) \text{ g}$

$= 3.2 \text{ g}$

So $+b = (3.4 - 3.3) \text{ g} = +0.1 \text{ g}$

And $-b = (3.2 - 3.3) \text{ g} = -0.1 \text{ g}$

Hence the *range* in which the *exact value* of the *difference* must lie in the *form*

$a \pm b = (3.3 \pm 0.1) \text{ g}$.

From above it can be *seen* that:

(i) The *greatest possible difference* is obtained by *subtracting* the *least possible value of the lesser mass* from the *greatest possible value* of the *greater mass.*

(ii) The *least possible difference* is obtained by *subtracting* the *greatest possible value* of the lesser mass from the *least possible value* of the *greater mass.*

(e) The *apparent quotient* $= \dfrac{195}{130} = 1.50$

The *greatest possible quotient* $= \dfrac{195.5}{129.5} = 1.51$ (*correct* to 3 *s.f.*)

The *least possible quotient* $= \dfrac{194.5}{130.5} = 1.49$ (*correct* to 3 *s.f.*)

So $+b = 1.51 - 1.50 = +0.01$

And $-b = 1.49 - 1.50 = -0.01$

Hence the *range* in which the *exact value* of the *quotient* must lie in the *form*

$a \pm b = 1.50 \pm 0.01$

From above it can be *seen* that:

(i) The *greatest possible quotient* is obtained by *dividing* the *greatest possible value* of the *numerator* by the *least possible value* of the *denominator.*

(ii) The *least possible quotient* is obtained by *dividing* the *least possible value* of the *numerator* by the *greatest possible value* of the *denominator.*

Exercise 3n

1. A brass rod is measured to be 15.6 m in length, correct to the nearest $\frac{1}{10}$ m.

State:

(a) the error interval involved in the measurement of the length

(b) the greatest possible length of the brass rod

(c) the least possible length of the brass rod

(d) the range in which the exact length of the brass rod must lie in the form *a* ± *b*.

2. A string of gold is measured to be 128.92 cm correct to the nearest $\frac{1}{100}$ cm.

Similar methods may be used to *multiply by* 0.25, 2.5, 250, 2 500, 25 000, et cetera.

(ii) *To multiply by* 125.

Since $125 = \frac{1\,000}{8}$, we *multiply* the *number* by 1 000 and then *divide* the *result* by 8.

Similar methods may be used to *multiply by* 0.125, 1.25, 12.5, 1 250, 12 500, 125 000, et cetera.

(iii) *To multiply by* 625.

Since $625 = \frac{10\,000}{16} = \frac{10\,000}{4 \times 4}$, we *multiply* the *number* by 10 000 and then *divide* the *result* by 4. This *result* we now *divide* by 4.

$$\text{Alternatively, } 625 = \frac{10\,000}{16} = \frac{10\,000}{8 \times 2} = \frac{10\,000}{2 \times 8}.$$

So we *multiply* the *number* by 10 000 and then *divide* the *result* by 8. This *result* we now *divide* by 2.

Or we *multiply* the *number* by 10 000 and then *divide* the *result* by 2. This *result* we now *divide* by 8.

Similar methods may be used to *multiply by* 0.625, 6.25, 62.5, 6 250, 62 500, 625 000, et cetera.

(iv) *To multiply by* 99.

Since $99 = 100 - 1$, we *multiply* the *number* by 100 and then *subtract* the *original number* from the *result.*

Similar methods may be used to *multiply by* 999, 98, 998, 97, 997, et cetera.

(v) *To multiply by* 101.

Since $101 = 100 + 1$, we *multiply* the *number* by 100 and then *add* the *original number* to the *result.*

Similar methods may be used to *multiply by* 1 001, 102, 1 002, 103, 1 003, et cetera.

SHORT CUTS IN DIVISION

The *reciprocal* (or *inverse*) of a *number* x is the number $\frac{1}{x}$. For example: the *reciprocal* of 2 is $\frac{1}{2}$, the *reciprocal* of $\frac{2}{3}$ is $\frac{3}{2}$, the *reciprocal* of $\frac{8}{3}$ is $\frac{3}{8}$, et cetera. Instead of *dividing* by a *number* we can *multiply* by its *inverse.*

For example:

$$\frac{3}{2} = 3 \times \frac{1}{2}, \frac{4}{3} = 4 \times \frac{1}{3}, \frac{7}{8} = 7 \times \frac{1}{8}, \text{ et cetera.}$$

So the *reciprocal* of a *number* is the *same* as the *multiplicative inverse* of the *number.*

(i) *To divide by* 25.

Since $\frac{1}{25} = \frac{4}{100}$, we *multiply* the *number* by 4 and then *divide* the *result* by 100.

Similar methods may be used to *divide* by 0.25, 2.5, 250, 2 500, 25 000, et cetera.

(ii) *To divide by* 125.

Since $\frac{1}{125} = \frac{8}{1\,000}$, we *multiply* the *number* by 8 and then *divide* the *result* by 1 000.

Similar methods may be used to *divide* by 0.125, 1.25, 12.5, 1 250, 12 500, 125 000, et cetera.

(iii) *To divide by* 625.

Since $\frac{1}{625} = \frac{16}{10\,000} = \frac{4 \times 4}{10\,000}$, we *multiply* the *number* by 4. We then *multiply* the *result* by 4 and *divide* this *result* by 10 000.

Alternatively,

$\frac{1}{625} = \frac{16}{10\,000} = \frac{8 \times 2}{10\,000} = \frac{2 \times 8}{10\,000}$, So we *multiply* the *number* by 8. We then *multiply* the *result* by 2 and *divide* the *result* by 10 000.

Or we *multiply* the *number* by 2 and then *multiply* the *result* by 8. This *result* we now *divide* by 10 000.

Example 19

(a) *Multiply the following numbers with as little working as possible:*

(i) *768 × 25* (ii) *768 × 0.25*
(iii) *1 439 × 625* (iv) *1 439 × 62.5*

(b) *Calculate each of the following products using short cuts in multiplication:*

(i) *863 × 99* (ii) *863 × 999*
(iii) *745 × 101* (iv) *745 × 1 001*

(c) *Evaluate each of the following quotients with as little working as possible:*

(i) *685 ÷ 25* (ii) *685 ÷ 2.5*
(iii) *1 478 ÷ 125* (iv) *1 478 ÷ 12.5*

Solution

13. Express the range in which the exact value of the difference of 9.4 kg and 3.7 kg must lie in the form $a \pm b$, if both masses are correct to two significant figures.

14. Express the range in which the exact value of the difference 47.3 kg $-$ 38.6 kg must lie in the form $a \pm b$, if both masses are correct to three significant figures.

15. Write in the form $a \pm b$, the range within which the exact value of the following differences must lie, if each measurement is correct to the last stated digit.

(a) 23 g $-$ 14 g

(b) 12.4 cm $-$ 7.3 cm

(c) 134.6 g $-$ 117.8 kg

(d) 75.93 mm $-$ 63.84 mm

16. Write in the form $a \pm b$, the range within which the exact value of each of the following differences must lie, if each measurement is correct to the last given digit.

(a) 9.6 mg $-$ 4.7 mg

(b) 5.46 kg $-$ 4.68 kg

(c) 124.7 cm $-$ 116.8 cm

(d) 25.64 km $-$ 19.52 km

17. Express the range in which the exact value of the quotient $\frac{81}{30}$ must lie, in the form $a \pm b$, if both numbers are correct to two significant figures.

18. Express the range in which the exact value of the quotient $\frac{625}{250}$ must lie, in the form $a \pm b$, if both numbers are correct to three significant figures.

19. Write in the form $a \pm b$, the range within which the exact value of each of the following quotients must lie, if each number is correct to two significant figures.

(a) $\frac{21}{70}$ (b) $\frac{45}{90}$ (c) $\frac{84}{16}$ (d) $\frac{98}{16}$

20. Express in the form $a \pm b$, the range within which the exact value of each of the following quotients must lie, if each number is correct to the last stated digit.

(a) $\frac{84}{96}$ (b) $\frac{768}{240}$ (c) $\frac{87}{15}$ (d) $\frac{750}{125}$

21. The lengths of three metal rods are 9.2 cm, 10.5 cm and 11.3 cm, to one decimal place.

(a) Write the greatest and the least possible values for the length of each rod to two decimal places.

(b) State the greatest and the least possible values for the total length of the three rods.

22. The edge-length of a wooden cube is recorded as 5 cm, correct to the nearest centimetre.

(a) What is the greatest and least possible length, for each edge?

(b) Calculate the volume of the cube, assuming each edge is:

(i) the greatest possible length

(ii) the least possible length.

(c) (i) what is the difference between the apparent volume and the greatest possible volume?

(ii) what is the difference between the apparent volume and the least possible volume?

23. The diameter of a human blood corpuscle is measured as $(7.5 \pm 0.05) \times 10^{-6}$ m.

(a) Calculate from this measurement

(i) the circumference of a blood corpuscle C metre in the form $k \times 10^{-6}$ m, where k is correct to two decimal places

(ii) the least possible value of the circumference.

(b) Write the possible circumference of a blood corpuscle in the form $(C \pm b) \times 10^{-6}$ m, where b is the error correct to two decimal places. (Take $\pi = 3.14$).

Short Cuts in Computation

As the *student* may know by now, it is often *easier* to *add* and *subtract*, *than* to *multiply* and *divide.*

There are some *methods* which we can use as *short cuts* in *multiplication* and *division.*

SHORT CUTS IN MULTIPLICATION

(i) *To multiply by* 25.

Since $25 = \frac{100}{4}$, we *multiply* the *number* by 100 and then *divide* the *result* by 4.

10. Evaluate each of the following quotients using short cuts in division:
   (a) $3472 \div 25$ (b) $3472 \div 250$
   (c) $3472 \div 2500$ (d) $3472 \div 25000$

11. Divide the following numbers with as little working as possible:
   (a) $8471 \div 125$ (b) $8471 \div 0.125$
   (c) $8471 \div 1.25$ (d) $8471 \div 12.5$

12. Divide the following numbers with as little working as possible:
   (a) $839847 \div 125$ (b) $839847 \div 1250$
   (c) $839847 \div 12500$ (d) $839847 \div 125000$

13. Calculate each of the following quotients using short cuts in division:
   (a) $642 \div 125$ (b) $642 \div 0.125$
   (c) $642 \div 1.25$ (d) $642 \div 12.5$

14. Calculate each of the following quotients using short cuts in division:
   (a) $209765 \div 125$ (b) $209765 \div 1250$
   (c) $209765 \div 12500$ (d) $209765 \div 125000$

Ratio

A *ratio* is a *relation* that is used to *compare the sizes of two* or *more quantities*. The *ratio* of one quantity *of measure n units* to another quantity of measure d units is written in the form $n{:}d$, where $n \in N$ *and* $d \in N$. The ratio of $n{:}d$ can be expressed as the fraction $\frac{n}{d}$. Hence *ratios* are *equal* when their *fractions* are *equivalent*, that is, $\frac{n}{d} = \frac{kn}{kd} \Rightarrow n{:}d = kn{:}kd$, where $k \in N$.

For example:

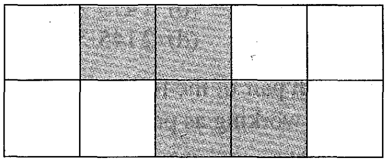


Fig. 3.18 Ratio

In the diagram above, the *ratio* of the *shaded region* to *unshaded region* is 4:6 = 2:3. The *ratio* of the *shaded region* to *total area* is 4:10 = 2:5. And the *ratio* of the *unshaded region* to *total area* is 6:10 = 3:5.

Note that the *symbol* : means '*to*'.

Example 20

(a) *In a cricket club there are 50 members of which 28 are females.*
Determine the ratio of:
   (i) *female members:total number of members*
   (ii) *male members:female members*
   (iii) *male members:total number of members.*

(b) *Express 3 m to 175 cm as a ratio in its lowest terms.*

Solution

(a) The total *number of members* = 50 members
The number of *female members* = 28 members
So the number of *male members* = (50 − 28) members
= 22 members
   (i) The *ratio* of *female members*:*total number of members*
   = 28 ~~members~~:50 ~~members~~ = 14:25
   (ii) The *ratio* of *male members*:*female members*
   = 22 ~~members~~:28 ~~members~~ = 11:14
   (iii) The *ratio* of *male members*:*total number of members*
   = 22 ~~members~~:50 ~~members~~ = 11:25

(b) Now 3 m:175 cm = 300 ~~cm~~:175 ~~cm~~ = 12:7
From the above *examples* it can be *seen* that:
   (i) We should always *state* a *ratio* in its *lowest terms*.
   (ii) When finding a *ratio*, we should always use the *same units* so that the units will *cancel* each other, since a *ratio* has no *units*.

Exercise 3p

1. Express \$6:30 ¢ as a ratio in its lowest terms.
2. Express 35 ¢ to \$2.25 as a ratio in its lowest terms.
3. I spent \$4.80 on groceries and \$1.20 on vegetables. What is the ratio of:
   (a) cost of groceries to cost of vegetables
   (b) cost of vegetables to cost of groceries

(a) (i) Now $768 \times 25 = 768 \times \frac{100}{4} = \frac{7{}^{3}6\,800}{4}$

$= 19\,200$

(ii) Now $768 \times 0.25 = 768 \times \frac{1}{4} = \frac{7{}^{3}6\,8}{4} = 192$

(iii) Now $1\,439 \times 625 = 1\,439 \times \frac{10\,000}{4 \times 4}$

$= \frac{14{}^{2}3{}^{3}9{}^{3}0{}^{2}000}{4 \times \underset{1}{\cancel{4}}}$

$= \frac{3\,{}^{3}5{}^{3}9{}^{1}7{}^{3}5{}^{2}00}{\underset{1}{\cancel{4}}}$

$= 899\,375$

(iv) Now $1\,439 \times 62.5 = 1\,439 \times \frac{1\,000}{4 \times 4}$

$= \frac{1\,4{}^{2}3{}^{3}9{}^{3}0{}^{2}00}{4 \times \underset{1}{\cancel{4}}}$

$= \frac{3{}^{3}5{}^{3}9{}^{1}7{}^{3}50{}^{20}}{\underset{1}{\cancel{4}}}$

$= 89\,937.5$

(b) (i) Now $863 \times 99 = 863 \times 100 - 863$
$= 86\,300 - 863$
$= 85\,437$

(ii) Now $863 \times 999 = 863 \times 1\,000 - 863$
$= 863\,000 - 863$
$= 862\,137$

(iii) Now $745 \times 101 = 745 \times 100 + 745$
$= 74\,500 + 745$
$= 75\,245$

(iv) Now $745 \times 1\,001 = 745 \times 1\,000 + 745$
$= 745\,000 + 745$
$= 745\,745$

(c) (i) Now $685 \div 25 = 685 \times \frac{4}{100} = \frac{274\cancel{0}}{10\cancel{0}}$

$= 27.4$

(ii) Now $685 \div 2.5 = 685 \times \frac{4}{10} = \frac{274\cancel{0}}{1\cancel{0}}$

$= 274$

(iii) Now $1\,478 \div 125 = 1\,478 \times \frac{8}{1\,000}$

$= \frac{11\,824}{1\,000}$

$= 11.824$

(iv) Now $1\,478 \div 12.5 = 1\,478 \times \frac{8}{100}$

$= \frac{11\,824}{100}$

$= 118.24$

Exercise 3o

1. Calculate each of the following products using short cuts in multiplication:
   (a) 847 × 25 (b) 847 × 0.25
   (c) 847 × 2.5 (d) 847 × 250
2. Evaluate each of the following products using short cuts in multiplication:
   (a) 384 × 25 (b) 384 × 250
   (c) 384 × 2 500 (d) 384 × 25 000
3. Multiply each of the following pairs of numbers with as little working as possible:
   (a) 2 371 × 125 (b) 2 371 × 0.125
   (c) 2 371 × 1.25 (d) 2 371 × 12.5
4. Multiply each of the following pairs of numbers with as little working as possible:
   (a) 984 × 125 (b) 984 × 1 250
   (c) 984 × 12 500 (d) 984 × 125 000
5. Determine each of the following products using short cuts in multiplication:
   (a) 347 × 625 (b) 347 × 0.625
   (c) 347 × 6.25 (d) 347 × 62.5
6. Calculate each of the following products using short cuts in multiplication:
   (a) 49 × 625 (b) 49 × 6 250
   (c) 49 × 62 500 (d) 49 × 625 000
7. Multiply each pair of the following numbers with as little working as possible:
   (a) 1 473 × 99 (b) 1 473 × 999
   (c) 2 145 × 97 (d) 2 145 × 997
8. Multiply each pair of the following numbers with as little working as possible:
   (a) 839 × 101 (b) 839 × 1 001
   (c) 573 × 102 (d) 573 × 1 002
9. Evaluate each of the following quotients using short cuts in division:
   (a) 584 ÷ 25 (b) 584 ÷ 0.25
   (c) 584 ÷ 2.5 (d) 584 ÷ 250

Alternative Method 1

(b) Given that 4 *proportional parts* $= \$420$

Then the *larger amount* $= \$420 \times \frac{11}{4}$

$= \$105 \times 11$

$= \$1\,155$

The *method illustrated* above is called the *fractional method.*

Alternative Method 2

(b) Let the larger amount $= \$x$

Then $4{:}11 = 420{:}x$

So $\frac{4}{11} = \frac{420}{x}$

i.e. $x = \frac{11 \times 420}{4}$

$= 11 \times 105$

$= 1\,155$

Hence the *larger amount* is \$1 155.

The *method illustrated* above is called the *ratio method.*

Exercise 3q

1. Two lengths are in the ratio 7:8. If the first length is 273 m, what amount is the second length?
2. Two amounts of money are in the ratio 8:3. If the second amount is \$4.05, what is the value of the first amount?
3. Two friends, Natasha and Tricia share a sum of money in the ratio 5:3 respectively. If Tricia's share was \$126.75, calculate the total sum of money shared.
4. A sum of money is divided between two friends in the ratio 4:9. If the smaller amount is \$560, determine the value of the larger amount.
5. A sum of money was divided between two friends, Karen and Natasha in the ratio 2:5. If Natasha received \$210 more than Karen, calculate the sum of money shared.
6. It costs \$112 to turf a lawn of area 56 m^2. How many dollars would it cost to turf a lawn of area 77 m^2?
7. At a constant speed a car uses five litres of petrol to travel 80 km. At the same speed how many litres of petrol is needed to travel
   (a) 120 km (b) 60 km
8. An alloy consist of 4 parts of gold and 9 parts of silver. What amount of gold should be mixed with 360 g of silver?
9. A photograph is enlarged in the ratio 10:3. Determine the length of a car in the enlargement, if its length was 4 cm in the original.
10. An express train is travelling at 80 km/h. How far does it go in:
    (a) 1 minute (b) 1 second?

Three quantities are said to be in *proportion* when *corresponding triples* of values are always in the *same ratio.* For example, 1:2:3 = 4:8:12 = 5:10:15, et cetera.

Example 22

(a) A sum of money was to be shared among three friends, Albert, Bruce and Christine in the ratio 3:7:10. If Christine received \$495 more than Bruce, determine the sum of money shared.

(b) A sum of money is to be divided among Yuri, Anna and Maria in the ratio 4:7:9. If Anna's share amounts to \$1 295, calculate:

(i) the total sum of money to be shared

(ii) Yuri's share

(iii) the percentage of the total amount that Maria receives.

Solution

(a) The *total number of proportional parts* $= (3 + 7 + 10)$ parts

$= 20$ parts

And *Christine's share – Bruce's share* $= (10 - 7)$ parts

$= 3$ parts

Then 3 *proportional parts* $= \$495$

So 1 *proportional part* $= \frac{495}{3} = \$165$

And 20 *proportional parts* $= \$165 \times 20$

$= \$3\,300$

Hence the *sum of money shared* was \$3 300.

(c) cost of vegetables to total cost
(d) cost of groceries to total cost?

4. Express in its simplest form, the ratio of 825 g to 1 kg.
5. Express the ratio 4:7 in the form n:1
6. Satate 7 m to 250 cm as a ratio in its lowest terms.
7. Write 9.85 km to 5 000 m as a ratio in its lowest terms.
8. Determine £2.15 to 125 p as a ratio in its lowest terms. (£1 = 100 p)
9. Evaluate 390 mm to 13 cm as a ratio in its lowest terms.
10. Evaluate 28 mg to 7 g as a ratio in its lowest terms.
11. Determine 750 centilitres to 1 litre as a ratio in its lowest terms.
12. Write 350 cm^3 to 1 litre as a ratio in its lowest terms.
13. Express the ratio of 11 cm^2 to 44 cm^2 as a ratio in its lowest terms.
14. State 12 cm^3 to 108 cm^3 as a ratio in its lowest terms.
15. Write 150 ml to 350 cm^3 as a ratio in its lowest terms.

Proportional Parts

Two quantities are said to be in *proportion* when *corresponding pairs* of values are always in the *same ratio*. For example: The number of pencils, n, bought in a bookshop at a cost of x dollars is shown in the table below:

Table 3.1

Number of pencils, n	1	2	3	4	5	10	20
Cost of pencils, in dollars x	1.25	2.50	3.75	5.00	6.25	12.50	25.00

Notice that 1:1.25 = 2:2.50 = 3:3.75 = 4:5.00 = 5:6.25 = 10:12.50 = 20:25.00. The ratios relating n to x are *all equal*.

Hence the *quantities* n and x are in *proportion*.

Example 21

(a) $25 000 is to be divided among two consultants in the ratio 2:3. What is the amount of the smaller share?

(b) A sum of money is divided among two friends in the ratio 4:11. If the smaller amount is $420, determine the larger amount.

Solution

(a) The *total number* of *proportional parts* = (2 + 3) parts = 5 parts

Then 5 *proportional parts* = $25 000

So 1 *proportional part* $= \dfrac{\$25\,000}{5} = \$5\,000$

And 2 *proportional parts* = 2 × $5 000 = $10 000

Hence the *amount* of the *smaller share* is $10 000.

The *method illustrated* above is called the *unitary method.*

Alternative Method

(a) The *total number* of *proportional parts* = (2 + 3) parts = 5 parts

So the *amount* of the *smaller share* $= \dfrac{2}{5}$ of \$25 000

$= \dfrac{2}{5} \times \$25\,000$

$= 2 \times \$5\,000$

$= \$10\,000$

The *method illustrated* above is called the *fractional method.*

(b) Given that 4 *proportional parts* = $420

Then 1 *proportional part* $= \dfrac{\$420}{4} = \$105$

So 11 *proportional parts* = 11 × $105

= $1 155

Hence the *larger amount* is $1 155.

The *method illustrated* above is called the *unitary method.*

8. A sum of money is divided among three girls, Anna, Barbara and Christy in the ratio 5:3:2. If Barbara received $400 less than Anna, calculate the amount of money each girl received.

9. Share the contents of a box containing 60 chocolates amongst Ann, Marie and James in the ratio 3:4:5. How many chocolates will each get?

10. A sum of money is to be divided among A, B and C in the ratio 2:3:5. The smallest share amounts to $600.
Calculate:
(a) the total sum of money to be shared
(b) C's share
(c) the percentage of the total amount that B receives.

11. A piece of ribbon of length 84 cm is divided into three pieces in the ratio 1:4:7. Calculate the length of the longest piece.

12. The sum of $4 500 is divided among Anesha, Sian and Joanne. Sian received half, Anesha received $1 050 and Joanne received the remainder.
Calculate:
(a) Sian's share
(b) Joanne's share
(c) the ratio in which the $4 500 was divided among the three persons
(d) the percentage of the total amount that Anesha received.

13. A sum of money is to be divided among three brothers A, B and C in the ratio 2:3:5. The largest share amounts to $1 500.
Calculate:
(a) the total sum of money to be shared
(b) B's share
(c) the percentage of the total amount that A receives.

14. The sum of money of $3 500 is divided among Adrian, Sean and James. Sean received half, Adrian received $850 and James received the remainder.
Calculate:
(a) Sean's share
(b) James' share
(c) the ratio in which the $3 500 was divided among the three persons
(d) the percentage of the total amount that Adrian received.

15. A sum of money is to be divided among Albert, Brian and Chrissy in the ratio 3:5:7. Chrissy's share amounts to $3 500.
Calculate:
(a) the total sum of money to be shared
(b) Brian's share
(c) the percentage of the total amount that Albert receives.

16. A sum of money was to be shared among three persons A, B and C in the ratio 3:2:5. If C received $420 more than B, determine the sum of money shared.

17. An alloy consists of steel, silver and copper in the ratio 6:5:9. If the smallest mass is 160 g, calculate the mass of the copper in the alloy.

Direct Proportion

Two quantities are said to be in *direct proportion* if they *increase* or *decrease* using a constant multiplier. That is, if *one quantity* is *doubled*, then the *other quantity* is *doubled* also. If we *halve* the *first quantity*, then the *second quantity* is also *halved*. For example: If the cost of 2 magazines is $15.00, then the cost of 4 magazines is $30.00. That is, $\frac{2}{4} = \frac{\$15.00}{\$30.00} = \frac{1}{2}$ or 2:4 = $15.00:$30.00 = 1:2

Example 23

(a) A car travels 180 km on 6 litres of petrol. How many litres of petrol will be needed to complete a journey of 540 km?

(b) An oxtail soup is made using the following ingredients:
1 onion
500 g eddoes
650 g yams
400 g plantains
4 dasheen leaves
750 g oxtail
2 tablespoons Chinese seasoning
salt and pepper to taste

Alternative Method 1

(a) The *total number* of *proportional parts* $= (3 + 7 + 10)$ parts

$= 20$ parts

Let the *sum* of money *shared* $= \$x$

Then *Christine's share* – *Bruce's share* $= \$\left(\frac{10}{20}x - \frac{7}{20}x\right)$

$= \$\frac{3}{20}x$

So $\frac{3}{20}x = 495$

And $x = 495 \times \frac{20}{3}$

$= 165 \times 20$

$= 3\,300$

Hence the *sum of money shared* was \$3 300.

Alternative Method 2

(a) Let the *sum* of money *received* by *Albert*, *Bruce* and *Christine* $= \$3x, \$7x$ and $\$10x$

Then the *total sum* of money *shared* $= \$(3x + 7x + 10x)$

$= \$20x$

And *Christine's share* – *Bruce's share* $= \$(10 - 7)x$

$= \$3x$

Thus $3x = 495$

So $x = \frac{495}{3} = 165$

i.e. $20x = 20 \times 165$

$= 3\,300$

Hence the *sum of money shared* was \$3 300.

(b) (i) The *total number* of *proportional parts* $= (4 + 7 + 9)$ parts

$= 20$ parts

And *Anna's share* $= 7$ parts

Then 7 *proportional parts* $= \$1\,295$

So 1 *proportional part* $= \frac{\$1\,295}{7}$

$= \$185$

And 20 *proportional parts* $= 20 \times \$185$

$= \$3\,700$

Hence the *total sum of money* to be shared is \$3 700.

(ii) Now *Yuri's share* $= 4$ parts

$= 4 \times \$185$

$= \$740$

(iii) Now *Maria's share* $= 9$ parts

$= 9 \times \$185$

$= \$1\,665$

And the *total amount* shared $= \$3\,700$

So the *percentage* of the *total amount* that Maria receives $= \frac{\$1\,665}{\$3\,7\cancel{00}} \times 1\cancel{00}\%$

$= 45\%$

Alternatively, the *required percentage* $= \frac{9\ \cancel{\text{parts}}}{2\cancel{0}\ \cancel{\text{parts}}} \times 10\cancel{0}\%$

$= 9 \times 5\%$

$= 45\%$

Exercise 3r

1. An estate valued at \$75 000 is divided among three daughters, Natasha, Natalie and Nadia in the ratio 5:8:2 respectively. Calculate the amount each receives.
2. A sum of money was to be shared among three friends, Albert, Michael and Moses, in the ratio 3:5:6. If Michael received \$196 more than Albert, find the sum of money shared.
3. An estate valued at \$45 000 is divided among three daughters, Anu, Betty and Chandra in the ratio 7:10:13 respectively. Calculate the amount each received.
4. A piece of string of length 85 cm, is divided into three pieces in the ratio 2:3:5. Calculate the length of the
   (a) shortest piece (b) longest piece.
5. An alloy consists of steel, gold and brass in the ratio 5:3:7. Determine the amount of each metal in 150 g of the alloy.
6. A sum of money was to be shared among three friends, Ann, Beryl and Candy, in the ratio 2:5:8. If Beryl received \$225 more than Ann, evaluate the sum of money shared.
7. An estate valued at \$60 000 is divided among three sons, Albert, Brian and Charles in the ratio 1:2:3 respectively. Calculate the amount each receives.

Table 3.2 Ready reckoner

X	$	*X*	$	*X*	$	*X*	$
15	3.45	51	11.73	201	46.23	501	115.23
16	3.68	52	11.96	202	46.46	502	115.46
17	3.91	53	12.19	203	46.69	503	115.69
18	4.14	54	12.42	204	46.92	504	115.92
19	4.37	55	12.65	205	47.15	505	116.15
20	4.60	56	12.88	206	47.38	506	116.38

Use the extract from the ready reckoner to determine:

(a) the cost of 19 buttons at 23 ¢ each

(b) the cost of 72 sweets at 23 ¢ each

(c) the cost of 254 dozens rubber bands at 23 ¢ per dozen

(d) the cost of 709 g of cherries at 23 ¢ per gram

(e) how many grams of cherries would cost $59.80.

Solution

From the *ready reckoner*:

(a) The *cost* of 19 *buttons* at 23 ¢ *each* = $4.37

(b) The *cost* of 56 *sweets* at 23 ¢ *each* = $12.88
The *cost* of 16 *sweets* at 23 ¢ *each* = $3.68
So the *cost* of 72 *sweets* at 23 ¢ *each* = $(12.88 + 3.68)
= $16.56

Or

The *cost* of 52 *sweets* at 23 ¢ *each* = $11.96
The *cost* of 20 *sweets* at 23 ¢ *each* = $4.60
So the *cost* of 72 *sweets* at 23 ¢ *each* = $(11.96 + 4.60)
= $16.56

Note that: 72 = 52 + 20 = 53 + 19 = 54 + 18 = 55 + 17 = 56 + 16.

(c) The *cost* of 201 *dozen* rubber bands at 23 ¢ *per dozen* = $46.23
The *cost* of 53 *dozen* rubber bands at 23 ¢ *per dozen* = $12.19
So the cost of 254 *dozen* rubber bands at 23 ¢ *per dozen* = $(46.23 + 12.19)
= $58.42

Note that: 254 = 201 + 53 = 202 + 52 = 203 + 51.

(d) The *cost* of 506 g of cherries at 23 ¢ *per gram* = $116.38
The *cost* of 203 g of cherries at 23 ¢ *per gram* = $46.69
So the *cost* of 709 g of cherries at 23 ¢ *per gram* = $(116.38 + 46.69)
= $163.07

Note that: 709 = 503 + 206 = 504 + 205 = 505 + 204 = 506 + 203.

(e) The *cost* of 206 g of cherries = $47.38
The *cost* of 54 g of cherries = $12.42
So the *cost* of 260 g of cherries = $(47.38 + 12.42)
= $59.80

Hence 260 g of cherries will cost $59.80.

Exercise 3s

1. If 26 articles cost $214.50, what amount does 1 article cost? What is the cost of 15 articles?

2. Eggs cost $5.40 per dozen. What amount will 25 eggs cost?

3. A train travels 252 km in 42 hours. What amount of time will it take to complete a journey of 350 km?

4. Calculate the cost of 15 articles at $1.95 each.

5. If 25 articles cost $43.75, how much does each cost?

6. A car travels 240 km on 20 litre of petrol. How many litres of petrol is needed to travel 600 km?

7. A train travels 300 km in 6 hours. How long will it take to complete a journey of 550 km?

8. A 5 kg bag of peas costs $17.91. At the same rate, what amount of money would a 9 kg bag of peas cost?

9. It costs $112 to turf a lawn of area 56 m^2. What amount would it cost to turf a lawn of area 99 m^2?

The above recipe is sufficient for 3 people. Calculate the ingredients necessary for 10 people.

(c) *Given that the cost of the recipe for 3 people is $19.50, what is the cost for the 10 people?*

Solution

The *unitary method* is illustrated below.

(a) The *volume of petrol* needed to *cover* 180 km $= 6\,l$

Then the *volume of petrol* needed to *cover* 1 km $= \frac{6}{180}\,l = \frac{1}{30}\,l$

So the *volume of petrol* needed to *cover* 540 km $= 54\cancel{0} \times \frac{1}{3\cancel{0}}\,l$

$= 18\,l$

Hence 18 litres of *petrol* is needed to *complete* a *journey of* 540 km.

Alternative Method

The *fractional method* is illustrated below.

(a) The *volume of petrol* needed to *cover* 180 km $= 6\,l$

Then the *volume of petrol* needed to *cover* 540 km $= 6\,l \times \frac{54\cancel{0}}{18\cancel{0}}$

$= 6\,l \times 3$

$= 18\,l$

Hence 18 litres of *petrol* is needed to *complete* a *journey of* 540 km.

(b) The *number of onions* needed $= 1 \text{ onion} \times \frac{10}{3}$

$= 3\frac{1}{3} \text{ onions}$

The mass *of eddoes* needed $= 500 \text{ g} \times \frac{10}{3}$

$= \frac{5\,000}{3} \text{ g}$

$= 1\,666\frac{2}{3} \text{ g}$

The mass *of yams* needed $= 650 \text{ g} \times \frac{10}{3}$

$= \frac{6\,500}{3} \text{ g}$

$= 2\,166\frac{2}{3} \text{ g}$

The mass *of plantains* needed $= 400 \text{ g} \times \frac{10}{3}$

$= \frac{4\,000}{3} \text{ g}$

$= 1\,333\frac{1}{3} \text{ g}$

The *number of dasheen leaves* needed $= 4 \text{ leaves} \times \frac{10}{3}$

$= \frac{40}{3} \text{ leaves}$

$= 13\frac{1}{3} \text{ leaves}$

The *mass of oxtail* needed $= 750 \text{ g} \times \frac{10}{3}$

$= \frac{7\,500}{3} \text{ g}$

$= 2\,500 \text{ g}$

The *number of tablespoons* of *chinese seasoning* needed $= 2 \text{ tbsp.} \times \frac{10}{3}$

$= \frac{20}{3} \text{ tbsp.}$

$= 6\frac{2}{3} \text{ tbsp.}$

(c) The *cost* of the *recipe* for 3 *people* $= \$19.50$

So the *cost* of the *recipe* for 10 *people* $= \$19.50 \times \frac{10}{3}$

$= \$6.50 \times 10$

$= \$65.00$

Ready Reckoner

The *ready reckoner* is a *table* that allows us to *multiply* sums of money *quickly*. It is used in business places that are not computerized or when easy and reliable access is required.

Example 24

The table below is an extract from a ready reckoner showing the cost of X articles at 23 ¢ each.

Table 3.3 Ready reckoner

g	$	g	$	g	$
500	1.50	1500	4.50	2500	7.50
550	1.65	1550	4.65	2550	7.65
600	1.80	1600	4.80	2600	7.80
650	1.95	1650	4.95	2650	7.95
700	2.10	1700	5.10	2700	8.10
750	2.25	1750	5.25	2750	8.25
800	2.40	1800	5.40	2800	8.40
850	2.55	1850	5.55	2850	8.55
900	2.70	1900	5.70	2900	8.70
950	2.85	1950	5.85	2950	8.85

Use the table to determine:

(a) the cost of 950 g of carrots

(b) the cost of 2 650 g of carrots

(c) the cost of 2 760 g of carrots

(d) how many grams of carrots would cost $6.15.

18. The table below is an extract from a ready reckoner showing the cost of pineapples at $1.80 per 500 g.

Table 3.4 Ready reckoner

g	$	g	$	g	$
500	1.80	1500	5.40	2500	9.00
550	1.98	1550	5.58	2550	9.18
660	2.16	1600	5.76	2600	9.36
650	2.34	1650	5.94	2650	9.54
700	2.52	1700	6.12	2700	9.72
750	2.70	1750	6.30	2750	9.90
800	2.88	1800	6.48	2800	10.08
850	3.06	1850	6.66	2850	10.26
900	3.24	1900	6.84	2900	10.44
950	3.42	1950	7.02	2950	10.62

Use the table of determine:

(a) the cost of 750 g of pineapples

(b) the cost of 2 350 g of pineapples

(c) the cost of 3 750 g of pineapples.

19. The table that follows is an extract from a ready reckoner giving the price of X articles at 29 ¢ each.

Table 3.5 Ready reckoner

X	$	X	$	X	$	X	$
21	6.09	31	8.99	201	58.29	500	145.00
22	6.38	32	9.28	202	58.58	525	152.25
23	6.67	33	9.57	203	58.87	550	159.50
24	6.96	34	9.86	204	59.16	575	166.75
25	7.25	35	10.15	205	59.45	600	174.00

Use the table above to find the cost of:

(a) 24 sweets at 29 ¢ each

(b) 584 plums at 29 ¢ each.

20. The table below is an extract from a ready reckoner showing the cost of a number of articles at 29 ¢ per article.

Table 3.6 Ready reckoner

No.	$	*No.*	$	*No.*	$	*No.*	$
5	1.45	15	4.35	51	14.79	101	29.29
6	1.74	16	4.64	52	15.08	102	29.58
7	2.03	17	4.93	53	15.37	103	29.87
8	2.32	18	5.22	54	15.66	104	30.16
9	2.61	19	5.51	55	15.95	105	30.45

Use the table above to determine:

(a) the cost of 17 biscuits at 29 ¢ each

(b) the cost of 175 sweets at 29 ¢ each

(c) the cost of 226 buttons at 29 ¢ each

(d) how many marbles at 29 ¢ each would cost $17.40.

Inverse Proportion

One *quantity* is said to be *inversely proportional* to another *quantity*, if when the *first quantity* is *doubled*, then the *second quantity* is *halved*. And if the *first quantity* is *halved*, then the *second quantity* is *doubled*. That is, one *quantity changes* by the *inverse ratio* (or *reciprocal ratio*) of the *other quantity*. For example, if 2 *men* can *weed* a compound in 6 *days*, then 4 *men* working at the *same rate* can *weed* the compound in 3 *days*.

Note that the *number of men* is *increased* in the *ratio* 4:2, that is, 2:1. And the *number of days* is *reduced* in the *ratio* 3:6, that is, 1:2.

Hence, when the *number of men* was *doubled*, then the *number of days* was *halved*.

10. At a constant speed a car used five litres of petrol to travel 80 km. At the same speed how many litres of petrol is needed to travel

(a) 120 km (b) 60 km

11. A car travels 240 km on 8 litre of petrol. How many litres will it take to complete a journey of 360 km?

12. The rates of currency exchange published in the newspapers on a certain day showed that 12 pounds could be exchanged for 96 dollars. How many dollars could be obtained for 102 pounds?

13. The ingredients to make Black Cake are as follows:

450 g raisins
450 g prunes
450 g currants
100 g mixed peel
$1\frac{1}{2}$ tablespoon ground cinnamon
225 g glace cherries
2 cups non-alcoholic red wine
2 cups cherry brandy
450 g granulated sugar
450 g butter
2 tablespoons baking powder
10 large eggs
225 g brown sugar for caramel colouring
$\frac{1}{2}$ cup of boiling water

The above recipe makes 2 Black Cakes of 25 cm diameter. What quantities are needed to make 5 Black Cakes of 25 cm diameter for a wedding?

14. A recipe for making Festive Light Fruit Cake uses the following quantities.

Ingredient	Cost
$1\frac{1}{2}$ kg cherries	$\frac{1}{2}$ kg costs \$4.80
1 kg raisins	$\frac{1}{2}$ kg costs \$3.50
1 kg pineapple	$\frac{1}{2}$ kg costs \$5.80
$\frac{1}{2}$ kg mixed fruit and peels	$\frac{1}{4}$ kg costs \$2.25
$\frac{1}{2}$ kg orange peel	$\frac{1}{4}$ kg costs \$1.75
1 kg walnuts	$\frac{1}{4}$ kg costs \$5.95
3 kg sifted sugar	1 kg costs \$9.95
2 tablespoons baking powder	1 tablespoon costs \$0.25
1 kg butter	$\frac{1}{4}$ kg costs \$1.50
1 kg sugar	$\frac{1}{4}$ kg costs \$4.35
12 eggs	4 eggs cost \$2.10
$\frac{1}{4}$ *l* corn syrup	$\frac{1}{2}$ *l* costs \$10.50
$\frac{1}{4}$ *l* orange juice	$\frac{1}{2}$ *l* costs \$5.20
$\frac{1}{4}$ *l* sherry	$\frac{1}{2}$ *l* costs \$10.20

Calculate the cost of making the Festive Light Fruit Cake correct to the nearest cent.

15. The ingredients to make Fruit Punch are as follows:

$1\frac{1}{2}$ bottles orange juice $\frac{1}{2}$ kg sugar
$\frac{1}{2}$ bottle tangerine juice $1\frac{1}{2}$ bottles grapefruit juice

If the recipe makes 4 bottles of Fruit Punch, determine the ingredients needed to make:

(a) 8 bottles of Fruit Punch

(b) 2 bottles of Fruit Punch.

16. A recipe for making non-alcoholic Ponche De Crème uses the following quantities:

Ingredient	Cost
1 dozen eggs	6 egg cost \$3.50
6 tins condensed milk	1 tin costs \$3.78
6 tins evaporated milk	1 tin costs \$2.25
1 green lime	1 lime costs 25 ¢
1 bottle non-alcoholic white wine	$\frac{1}{2}$ bottle costs \$14.90
bitters to taste	—

Determine the cost of making the non-alcholic Ponche De Crème to the nearest cent.

17. The table below is an extract from a ready reckoner showing the cost of carrots at \$1.50 per 500 g.

Example 26

(a) What is 30% of \$600?

(b) What is $12\frac{1}{2}$% of \$1 600?

(c) What is 0.45% of \$500?

(d) 25% of certain volume is 60 cm³. Calculate the total volume.

(e) Of a certain area $8\frac{1}{3}$% is 30 cm². Evaluate the total area.

(f) Of a certain length of string 12.5% is 43.5 cm. Determine the total length of the string?

Solution

(a) Now 30% of \$600 $= \frac{30}{1\cancel{00}} \times \$6\cancel{00}$

$= \$180$

(b) Now $12\frac{1}{2}$% of \$1 600 $= \frac{12\frac{1}{2}}{100} \times \$1\,600$

$= \frac{25}{2 \times 1\cancel{00}} \times \$1\,6\cancel{00}$

$= \$25 \times 8$

$= \$200$

(c) Now 0.45% of \$500 $= \frac{0.45}{1\cancel{00}} \times \$5\cancel{00}$

$= \$2.25$

(d) Now 25% of the volume $= 60 \text{ cm}^3$

Then 1% of the volume $= \frac{60}{25} \text{ cm}^3$

So 100% of the volume $= \frac{60}{25} \times 100 \text{ cm}^3$

$= 60 \times 4 \text{ cm}^3$

$= 240 \text{ cm}^3$

Hence the *total volume* is 240 cm³.

(e) Now $8\frac{1}{3}$% of the area $= 30 \text{ cm}^2$

Then 1% of the area $= \frac{30}{8\frac{1}{3}} \text{ cm}^2$

$= \frac{30}{\frac{25}{3}} \text{ cm}^2$

$= 30 \times \frac{3}{25} \text{ cm}^2$

$= \frac{90}{25} \text{ cm}^2$

So 100% of the *area* $= \frac{90}{25} \times 100 \text{ cm}^2$

$= 90 \times 4 \text{ cm}^2$

$= 360 \text{ cm}^2$

Hence the *total area* is 360 cm².

(f) Now 12.5% of the length of string $= 43.5$ cm

Then 1% of the length of string $= \frac{43.5}{12.5}$ cm

So 100% of the length of string $= \frac{43.5}{12.5} \times 100$ cm

$= 43.5 \times 8$ cm

$= 348.0$ cm

Hence the *total length* of the string is 348 cm.

Expressing One Quantity as a Percentage of Another

In order to *convert* a *fraction* or *decimal* to a *percentage*, we *multiply either value* by 100.

Example 27

(a) Express 975 cm³ as a percentage of 3 l.

(b) Express 4.5 cm² as a percentage of 20 cm².

(c) Copy and complete the following table:

Table 3.7

	Common fraction	*Percentage*	*Decimal fraction*
(i)	$\frac{1}{4}$		
(ii)		*50%*	
(iii)			*0.75*

Solution

(a) Now 975 cm³ *as a fraction* of 3 *l* $= \frac{975 \text{ cm}^3}{3\ l}$

$= \frac{975\ \cancel{\text{cm}^3}}{3\,000\ \cancel{\text{cm}^3}}.$

Example 25

(a) Two bicycle gear wheels mesh together. One gear has 35 teeth and other gear has 30 teeth. If the smaller wheel makes 70 revolutions per minute, how many revolutions per minute does the larger wheel make?

(b) If 12 men can sew 180 shirts in 5 days, how long will it take 15 men to sew the 180 shirts?

Solution

(a) Let the *number of revolutions per minute* made by the *larger wheel* = x rev/min

Then $\frac{x}{70} = \frac{30}{35}$

So $x = \frac{30}{35} \times 70 = 30 \times 2 = 60$ rev/min

Hence the *larger wheel* makes 60 *revolutions per minute.*

(b) Let the *time taken* by the 15 *men* = x days

Then $\frac{x}{5} = \frac{12}{15}$

So $x = \frac{12}{15} \times 5 = \frac{12}{3} = 4$

Hence the *time taken* by 15 *men* to sew the 180 shirts was 4 days.

Alternative Method

(b) The *time taken* by 12 *men* to sew the 150 shirts = 5 days

So the *time taken* by 1 *man* to sew the same 150 shirts = 5 days × 12 = 60 days

Hence the *time taken* by 15 *men* to sew the 150 shirts $= \frac{60}{15}$ days = 4 days

Note that the *number of shirts* sewn plays no *direct part* in the *calculations*. Usually the *volume of work* done plays no *direct part* in the *actual calculations* in *inverse proportion problems*, since it is *constant* (i.e. *fixed*).

Exercise 3t

1. Twelve men produce 700 watches in 9 working days. How long would it take 18 men to produce the 700 watches?
2. A rice farmer employs 15 men to harvest his crop. The men take 12 days to do the job. If he had employed 9 men, how many days would it have taken them?
3. Two gear wheels mesh together. One gear has 30 teeth and the other gear has 25 teeth. If the larger wheel makes 75 revolutions per minute, how many revolutions per minute does the smaller wheel make?
4. If 9 women can sew 375 dresses in 8 weeks, calculate the time it would take 4 women to perform the same task.
5. Nine taps fill a tank in 4 hours. How many hours would it take to fill the tank if only six taps are working?
6. A field of grass feeds 28 cows for 6 days. How many days would the same field feed 21 cows?
7. Nine taps can fill a tank in 3 hours. How many hours would it take to fill the tank if only three taps are working?
8. A field of grass feeds 36 cows for 4 days. How many days would the same field feed 20 cows?
9. A contractor decides that he can build a barn in 8 weeks using five men. If he employs three more men, how long will the job take? Assume that all the men work at the same rate.
10. A factory employs 18 women to sew 540 dresses. The women take 6 weeks to do the job. If 12 women had been employed instead, how many weeks would it have taken them to sew the 540 dresses?
11. Two gear wheels mesh together. One gear has 40 teeth and the other gear has 25 teeth. If the larger wheel makes 60 revolutions per minute, how many revolutions per minute does the smaller wheel make?

Percentage of a Quantity

A *percentage* is a *fraction* whose *denominator* is 100.

Thus: x per cent $= x\% = \frac{x}{100}$.

10. The price of a car that cost \$27 000 last year increased by 12.5% on 1st January this year. What amount is its present price?

11. A school employs 30 teachers. How many teachers will there be if there is a 10% reduction?

12. Evaluate $22\frac{1}{2}\%$ of 40 m.

13. Express 4 mm as a percentage of 3 cm.

14. There are 120 shops at a mall, of which 35% sell clothes. How many shops do not sell clothes?

15. Calculate the value of 15% of \$350.

16. Copy and complete the following table:

Table 3.10

	Common fraction	*Percentage*	*Decimal fraction*
(a)	$\frac{1}{8}$		
(b)		65%	
(c)			0.94

17. In an election, 38% of the electorate voted for Mrs. Khan, 45% for Mr. Sobers and the remainder voted for Miss Damme. What percentage voted for Miss Damme if there were only three candidates and 5% of the electorate did not vote?

18. Copy and complete the following table:

Table 3.11

	Common fraction	*Percentage*	*Decimal fraction*
(a)	$\frac{5}{8}$		
(b)		70%	
(c)			0.85

19. Fifteen percent of the persons taking a driving test fail to pass first time. What percentage passes first time?

20. A concert is attended by 2 500 people. If 47% are adult females and 32% are adult males, how many children attended?

21. There are 90 girls in the third year, 25 of whom study biology. What percentage of third year girls study biology?

22. If 42% of a crowd of 38 500 at a football match were females, how many females attended?

23. A mathematics book has 360 pages, of which 40% are on algebra, 35% on geometry and the remainder on arithmetic. How many pages on arithmetic are there in the book?

24. In a mathematics test, Mary scored 27 of a possible 60. What amount was her percentage score?

25. A science book has 428 pages, of which 47% are on biology, 28% on chemistry and the remainder on physics. How many pages on physics are there?

26. There are 150 shops at a mall, 56% of which sell toys. How many shops do not sell toys?

27. Determine $22\frac{1}{2}\%$ of 90 m.

28. Express 4 mm as a percentage of 9 cm.

29. There are 120 shops at a mall, of which 65% sell clothes. How many shops do not sell clothes?

30. In an English test, Crissy scored 49 of a possible 60. What amount was her percentage mark?

Arithmetic Mean (or Average)

The *arithmetic mean* is called the *average* by most non-mathematicians. The *arithmetic mean* (or *average*) is *one member*, or *value*, that *represents* a *whole group*. The *arithmetic mean* (or *average*) of a *set of n numbers* is the *quotient* of the *sum of the n numbers divided by n*.

Thus:

$$\text{The arithmetic mean (or average) of a set of quantities} = \frac{\text{The sum of the quantities}}{\text{The total number of quantities}}$$

$$\text{The sum of the quantities} = \text{The arithmetic mean (or average)} \times \text{The total number of quantities}$$

So 975 cm³ *as a percentage* of 3 *l* $= \frac{975}{3\,0\cancel{00}} \times 1\cancel{00}\%$

$= 32.5\%$

It can be *seen* from the *example* above that we need to *convert* the *units* in the *fraction*, if they are *not* the *same unit*, in *order* to *cancel* the *units*.

(b) Now 4.5 cm² *as a fraction* of 20 cm² $= \frac{4.5\ \cancel{cm^2}}{20\ \cancel{cm^2}}$

$= \frac{4.5}{20}$

So 4.5 cm² *as a percentage* of 20 cm² $= \frac{4.5}{2\cancel{0}} \times \$10\cancel{0}\%$

$= 4.5 \times 5\%$

$= 22.5\%$

(c) (i) Now $\frac{1}{4}$ *as a percentage* $= \frac{1}{4} \times 100\%$

$= 25\%$

And $\frac{1}{4}$ *as a decimal fraction* $= \frac{1}{4} = 0.25$

Or $\frac{1}{4}$ *as a decimal fraction* $= 25\% = \frac{25}{100}$

$= 0.25$

(ii) Now 50% *as a common fraction* $= \frac{5\cancel{0}}{10\cancel{0}} = \frac{1}{2}$ (in *lowest terms*)

And 50% *as a decimal fraction* $= \frac{5\cancel{0}}{10\cancel{0}} = 0.5$

Or 50% *as a decimal fraction* $= \frac{1}{2} = 0.5$

(iii) Now 0.75 *as a common fraction* $= \frac{75}{100} = \frac{3}{4}$ (in *lowest terms*)

And 0.75 *as a percentage* $= 0.75 \times 100\%$

$= 75\%$

Or 0.75 *as a percentage* $= \frac{3}{4} \times 100\%$

$= 3 \times 25\%$

$= 75\%$

Hence we have the following *completed table*:

Table 3.8

	Common fraction	*Percentage*	*Decimal fraction*
(i)	$\frac{1}{4}$	25%	0.25
(ii)	$\frac{1}{2}$	50%	0.5
(iii)	$\frac{3}{4}$	75%	0.75

From the *example* above it can be *seen* that we always *state* the *value* of a *common fraction* in its *lowest terms.*

Exercise 3u

1. Express $16\frac{1}{2}\%$ as a decimal fraction.

2. Express $\frac{17}{15}$ as a percentage.

3. Write 0.845 as a percentage.

4. Copy and complete the following table:

Table 3.9

	Common fraction	*Percentage*	*Decimal fraction*
(a)	$\frac{3}{5}$		
(b)		45%	
(c)			0.35

5. A football team won 67% of their matches and drew 24% of them. What percentage of the matches did they lose?

6. In a school, 33% of the pupils study Biology and 16% study Chemistry. If 9% study both sciences, what percentage does not study either subject?

7. Express 985 cm³ as a percentage of 1 *l*.

8. A mathematics book has 360 pages, of which 50% are on Algebra, 20% on Geometry and the remainder on Arithmetic. How many pages of Arithmetic are there in the book?

9. There are 530 students in my school and 30% are footballers. How many students are not footballers?

13. Albert bought six bottles of Cola each containing 75 cl, five bottles of Cola each containing 1 litre and four bottles of Cola each containing 2 litres. Determine, in centilitres, the mean amount in each of the bottles he bought.

14. The average daily 'takings' in the corner shop from Monday to Friday of a certain week was \$275.50, while the average daily 'takings' from Monday to Saturday of the same week was \$294.25.

(a) How many dollars was taken over the 5 days from Monday to Friday?
(b) What amount was taken over the 6 days from Monday to Saturday?
(c) How much money was taken on the Saturday?

15. In 11 completed innings, a batsman's average score was 59. After a further innings his average score increased to 60. How many runs did he score in his 12th innings?

16. Calculate the average age of five girls, given that three of them are each 14 years 6 months of age and the other two are each 16 years 2 months of age.

17. The average mark of 25 students in a Mathematics test was 48. Calculate what the average mark would have been, if a student who scored 84 marks had been absent.

Square of a Number

The *square of a number* is defined as the *number times itself*, that is, the *number multiplied by itself*. The *square of the number* 5 is therefore 5 × 5 and it is written as 5^2, where 5 is called the *base* and 2 is called the *power* (*or index*). The *power* 2 means '*the square of*'. Thus 5^2 is read as the square of 5 or 5 squared or 5 to the second power or 5 to the power 2 or 5 to the second power. And $5^2 = 5 \times 5 = 25$.

The *square of a number* can also be thought of as the *area of a square* whose *length* is *equal* to that of the *base*. Thus the *area of a square* whose *length* is 7 cm is *equal* to $(7\text{ cm})^2 = 7\text{ cm} \times 7\text{ cm} = 49\text{ cm}^2$.

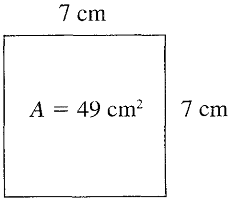


Fig. 3.19 Square

Example 29

Determine the square of each of the following numbers by multiplication:

(a) 7 *(b) 20* *(c) 0.3*
(d) 2.5 *(e) 0.09*

Solution

(a) The *square* of the number $7 = 7^2 = 7 \times 7 = 49$

(b) The *square* of the number $20 = 20^2 = 20 \times 20$
$= 400$

(c) The *square* of the number 0.3
$= 0.3^2 = 0.3 \times 0.3 = 0.09 \longleftarrow \boxed{\begin{array}{r} 3 \\ \underline{3} \times \\ \underline{9} \end{array}}$
1 *d.p.* + 1 *d.p.* = 2 *d.p.*

(d) The *square* of the number 2.5
$= 2.5^2 = 2.5 \times 2.5 = 6.25 \longleftarrow \boxed{\begin{array}{r} 25 \\ \underline{25} \times \\ \underline{625} \end{array}}$
1 *d.p.* + 1 *d.p.* = 2 *d.p.*

(e) The *square* of the number 0.09
$= 0.09^2 = 0.09 \times 0.09 = 0.0081 \longleftarrow \boxed{\begin{array}{r} 9 \\ \underline{9} \times \\ \underline{81} \end{array}}$
2 *d.p.* + 2 *d.p.* = 4 *d.p.*

From the above *examples*, it can be *seen* that the *sum of the decimal places* in the *product of the base* is *equal* to the *number of decimal places* in the *square of the number.*

Example 30

Calculate the area of a square of length 7.5 cm.

Solution

Example 28

(a) Lawrence's marks in nine consecutive tests in school are:

79, 84, 55, 49, 95, 64, 73, 97, 88.

(i) Calculate the total amount of marks he scored in the tests.

(ii) Hence determine his average mark per test.

(b) In nine completed innings, a batsman's average score was 47. After a further innings his average score increased to 51. How many runs did he score in his tenth innings?

Solution

(a) (i) Lawrence's *total amount of marks* in the 9 tests

$$= (79 + 84 + 55 + 49 + 95 + 64 + 73 + 97 + 88) \text{ marks}$$
$$= 684 \text{ marks}$$

(ii) Lawrence's *average mark per test*

$$= \frac{\textit{The total number of marks}}{\textit{The number of tests}}$$
$$= \frac{684 \text{ marks}}{9 \text{ tests}}$$
$$= 76 \text{ marks/test}$$

(b) *The total number of runs scored in n innings* $= \textit{The average score} \times n$

So the *total number of runs* scored in 10 *innings.*

$$= 51 \frac{\text{runs}}{\cancel{\text{innings}}} \times 10 \cancel{\text{innings}}$$
$$= 510 \text{ runs}$$

And the *total number of runs* scored in *9 innings*

$$= 47 \frac{\text{runs}}{\cancel{\text{innings}}} \times 9 \cancel{\text{innings}}$$
$$= 423 \text{ runs}$$

Hence the *number of runs scored* in his *tenth innings*

$$= (510 - 423) \text{ runs}$$
$$= 87 \text{ runs}$$

Exercise 3v

1. Erica's marks in eight consecutive Mathematics Examinations were:
94, 83, 75, 52, 71, 68, 75, 49.
(a) Calculate the total marks she scored.
(b) What amount was her average mark?

2. Maria's examination marks in eight subjects were 74, 58, 85, 62, 95, 97, 45 and 69. What amount was her average mark?

3. The heights of a group of boys (in cm) are:
158, 154, 152, 153, 156, 161, 151, 159, 160, 156.
Calculate their mean height.

4. Gemma's marks in five consecutive examinations were 95, 87, 58, 74 and 69.
(a) Calculate the total amount of marks that she scored.
(b) What amount was her mean mark?

5. The heights of 13 men (in cm) are given below:
162, 160, 163, 160, 165, 167, 170, 167, 174, 176, 178, 179, 178.
Determine the mean of the heights.

6. A motorist covered the following distances during one week:
185, 145, 155, 90, 175, 95, 240 (km).
What amount was his daily average?

If his car consumed 217 litres of petrol for the entire week and the cost of petrol is 40 ¢ per litre, what was his average daily cost per kilometre.

7. Beverly's average mark for eight examination papers was 74.5. How many marks did she score altogether?

8. Adam bought five books at \$9.48 each and three books at \$5.32 each. What was the mean amount that Adam paid for a book?

9. In eight completed innings, a batsman's average score was 42.5. After a further innings his average fell to 38.0. How many runs did he score in his ninth innings?

10. My car travels an average 12.3 km on each litre of petrol. What distance will it travel on 105 litres?

11. Ann's mean mark after 6 results was 68. Her mean mark dropped to 59 when she received her seventh result which was for Spanish. What was her Spanish mark?

12. The mean height of the 15 girls in a class is 152 cm and the mean height of the 12 boys is 159 cm. Calculate the mean height of the class.

Using Three-figure Mathematical Tables

Three-figure mathematical tables can also be used to find the *square of a number*. The *square of a number from* 1 *to* 10 can be found directly by reading the value from the *table of squares*. And the *square of a number* outside of this range can also be found using the *table* as shown below.

Example 32

Find the square of each of the following numbers by using three-figure mathematical tables:

(a) 1.4 *(b) 3.95* *(c) 39.5*
(d) 395 *(e) 3 950* *(f) 0.395*
(g) 0.0395

Solution

(a) Now $1.4^2 = 1.40^2 = 1.96$ (*directly* from the *table of squares* from 1 to 10)

(b) Now $3.95^2 = 15.60$ (*directly* from the *table of squares* from 1 to 10)

(c) Now
$$\begin{aligned} 39.5^2 &= (3.95 \times 10)^2 \\ &= 3.95^2 \times 10^2 \\ &= 15.60 \times 100 \\ &= 1\,560 \end{aligned}$$

(d) Now
$$\begin{aligned} 395^2 &= (3.95 \times 100)^2 \\ &= 3.95^2 \times 100^2 \\ &= 15.60 \times 10\,000 \\ &= 156\,000 \end{aligned}$$

(e) Now
$$\begin{aligned} 3\,950^2 &= (3.95 \times 1\,000)^2 \\ &= 3.95^2 \times 1\,000^2 \\ &= 15.60 \times 1\,000\,000 \\ &= 15\,600\,000 \end{aligned}$$

(f) Now
$$\begin{aligned} 0.395^2 &= \left(3.95 \times \frac{1}{10}\right)^2 \\ &= 3.95^2 \times \frac{1}{10^2} \\ &= 15.60 \times \frac{1}{100} \\ &= 0.156 \end{aligned}$$

(g) Now
$$\begin{aligned} 0.039\,5^2 &= \left(3.95 \times \frac{1}{100}\right)^2 \\ &= 3.95^2 \times \frac{1}{100^2} \\ &= 15.60 \times \frac{1}{10\,000} \\ &= 0.001\,560 \end{aligned}$$

From the above *examples*, it can be *seen* that if the number whose *square* is to be found is *not between* 1 *and* 10 *inclusive*, then it has to be written as a number *between* 1 *and* 10 *times* a *power of* 10 or the *reciprocal of a power of* 10, in order to find the *square of the number* using *three-figure mathematical tables.*

Example 33

Calculate the value of $\left(\frac{0.91}{0.13}\right)^2$.

Solution

Now $\left(\frac{0.91}{0.13}\right)^2 = \left(\frac{91}{13}\right)^2 = 7^2 = 49$

$$\begin{array}{r} 13 \\ \times\ 7 \\ \hline 91 \end{array}$$

Alternatively,
$$\begin{aligned} \left(\frac{0.91}{0.13}\right)^2 &= \left(\frac{91 \times \frac{1}{\cancel{100}}}{13 \times \frac{1}{\cancel{100}}}\right)^2 \\ &= \left(\frac{91}{13}\right)^2 \\ &= 7^2 \\ &= 49 \end{aligned}$$

Example 34

Complete the following table of values for the function $y = x^2$*, and hence draw the graph of the function for the domain* $-3 \leqslant x \leqslant 3$.

Table 3.14 Table of values

x	-3	-2	-1	0	1	2	3
$x \times x$	$-3 \times (-3)$		$-1 \times (-1)$	0×0		2×2	
$y = x^2$	9		1	0		4	

Hence determine the value of the function y when:

(a) $x = 1.5$ *(b)* $x = 2.5$
(c) $x = -1.5$ *(d)* $x = -2.5$

Solution

Below can be seen the completed *table of values* of the *function* $y = x^2$, for the *domain* $-3 \leqslant x \leqslant 3$.

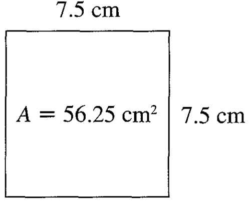


Fig. 3.20 Square

The *area* of the *square* $= (7.5 \text{ cm})^2$

$= 7.5 \text{ cm} \times 7.5 \text{ cm} = 56.25 \text{ cm}^2$

1 *d.p.* + 1 *d.p.* = 2 *d.p.*

$$\boxed{\begin{array}{r} 75 \\ \underline{75} \\ \underline{5\,625} \end{array} \times}$$

Using a Calculator

The following *method* illustrates how a *scientific calculator* was used to find the *square of a number.*

Example 31

Find the square of each of the following numbers by using a calculator:

(a) 25.6 *(b) 147.93* *(c) 0.91*

(d) 36581 *(e) 0.037* *(f) 0.0082*

Solution

Table 3.12

	Input	*Seen on the display of the calculator*
(a)	25.6	25.6
	$\boxed{x^2}$	655.36
(b)	147.93	147.93
	$\boxed{x^2}$	21 883.2849
(c)	0.91	0.91
	$\boxed{x^2}$	0.828 1
(d)	36581	36581
	$\boxed{x^2}$	1 338 169 561
(e)	0.037	0.037
	$\boxed{x^2}$	1.369×10^{-03}
(f)	0.0082	0.008 2
	$\boxed{x^2}$	6.724×10^{-05}

Thus:

(a) $25.6^2 = 655.36$

(b) $147.93^2 = 21\,883.2849$

(c) $0.91^2 = 0.828\,1$

(d) $36\,581^2 = 1\,338\,169\,561$

(e) $0.037^2 = 1.369 \times 10^{-03} = 0.001\,369$

(f) $0.008\,2^2 = 6.724 \times 10^{-05} = 0.000\,067\,24$

From the above *examples*, it can be *seen* that when *powers are displayed* on the screen of the calculator:

(i) If the *power is positive*, then we have to shift the decimal point a number of places to the right, equal to the power displayed. Sometimes we need to add zeros as in the example above in order to keep the place values.

(ii) If the *power is negative*, then we have to shift the decimal point a number of places to the left, equal to the power displayed. Sometimes we need to add zeros as in the example above in order to keep the place values.

In the D.A.L. *scientific calculator* used to find the *square of the numbers*, the first or main key was the $\boxed{x^2}$ *square key*. Thus we had to *input the number* and then *press the* $\boxed{x^2}$ *key* in order to obtain the *square of a number.*

In the case of a *simple* or *non-scientific calculator*, the *square of a number* can be found as shown below.

Table 3.13

	Input	*Seen on the display of the calculator*
(a)	25.6	25.6
	$\boxed{\times}$	
	25.6	25.6
	$\boxed{=}$	655.36
(f)	0.008 2	0.008 2
	$\boxed{\times}$	
	0.008 2	0.008 2
	$\boxed{=}$	0.000 067 24

22. (a) 15.4 cm (b) 23.7 cm (c) 35.9 cm

23. (a) 105.3 cm (b) 143.8 cm (c) 175.8 cm

24. (a) 74 mm (b) 83 mm (c) 92 mm

25. (a) 2.5 m (b) 3.6 m (c) 5.8 m

Calculate the value of each of the following squares:

26. (a) $\left(\frac{0.84}{0.12}\right)^2$ (b) $\left(\frac{0.52}{0.13}\right)^2$

27. (a) $\left(\frac{0.9}{0.3}\right)^2$ (b) $\left(\frac{0.6}{0.2}\right)^2$

28. (a) $\left(\frac{1.8}{2}\right)^2$ (b) $\left(\frac{3.6}{3}\right)^2$

29. (a) $\left(\frac{9.8}{7}\right)^2$ (b) $\left(\frac{1.32}{4}\right)^2$

30. (a) $\left(\frac{2.5}{0.5}\right)^2$ (b) $\left(\frac{3.6}{0.6}\right)^2$

Complete the following table of values for the function $y = x^2$, and hence draw the graph of the function for the giving domain, using a scale of 1 cm ≡ 1 unit on both axes:

31. *Table 3.16 Table of values*

x	0	1	2	3	4	5	6	7
$y = x^2$								

Determine the value of the function y when:

(a) $x = 0.5$ (b) $x = 1.5$

(c) $x = 5.5$ (d) $x = 6.5$

32. *Table 3.17 Table of values*

x	0	−1	−2	−3	−4	−5	−6	−7
$y = x^2$								

Determine the value of the function y when:

(a) $x = -0.5$ (b) $x = -1.5$

(c) $x = -5.5$ (d) $x = -6.5$

33. *Table 3.18 Table of values*

x	−2	−1	0	1	2	3	4	5
$y = x^2$								

Determine the value of the function y when:

(a) $x = -1.5$ (b) $x = 2.5$

(c) $x = 3.5$ (d) $x = 4.5$

34. *Table 3.19 Table of values*

x	−5	−4	−3	−2	−1	0	1	2
$y = x^2$								

Determine the value of the function y when:

(a) $x = -2.5$ (b) $x = -3.5$

(c) $x = -4.5$ (d) $x = 1.5$

35. *Table 3.20 Table of values*

x	−4	−3	−2	−1	0	1	2	3	4
$y = x^2$									

Find the value of the function y when:

(a) $x = -2.5$ (b) $x = -3.5$

(c) $x = 2.5$ (d) $x = 3.5$

Square Root of a Number

The *square root of a number* is defined as that *number* which when *multiplied by itself* gives the *original number*. The *square root of the number* 25 is written as $\sqrt{25} = 25^{\frac{1}{2}}$, where the *sign* $\sqrt{\ }$ and the *power* $\frac{1}{2}$, both mean '*the square root of*'. Thus the *square root of the number* 25, $\sqrt{25} = \pm 5$, since $(+5) \times (+5) = 25$ and $(-5) \times (-5) = 25$. Hence if $25 = (\pm 5)^2$, then $\sqrt{25} = \pm 5$. So both +5 and −5 are *square roots* of the real number 25. That is, a *positive real number* has *two square roots*. For most practical purposes however, we only need to use the *positive square root*, thus the *negative square root* is neglected.

For example:

$\sqrt{25 \text{ cm}^2} = \sqrt{(5 \text{ cm})^2} = 5$ cm, since in reality the *length* of an object *cannot be negative*.

The *square root of a number* can also be thought of as the *length* of the *side of a square*, where the *area* is *equal* to the *number whose square root* is to be found. Thus the *length* of the *side of a square* whose *area* is *equal* to 49 cm² is

$$\sqrt{49 \text{ cm}^2} = \sqrt{7 \text{ cm} \times 7 \text{ cm}} = 7 \text{ cm}.$$

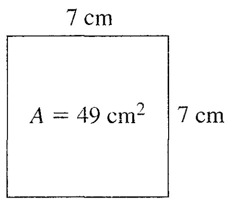


Fig. 3.22 Square

Table 3.15 Table of values

x	-3	-2	-1	0	1	2	3
$x \times x$	$-3 \times (-3)$	$-2 \times (-2)$	$-1 \times (-1)$	0×0	1×1	2×2	3×3
$y = x^2$	9	4	1	0	1	4	9

Using the *table of values*, the *graph* of the *function* was then *drawn* on graph paper.

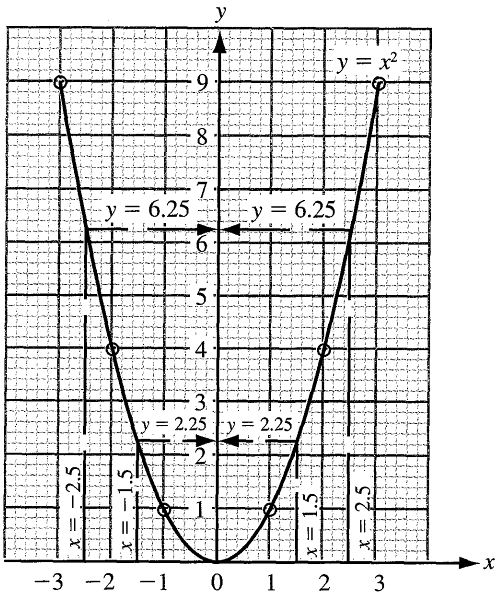


Fig. 3.21 Parabola

From the *graph*:

(a) When $x = 1.5$
then the *value of the function* $y = 2.25$

(b) When $x = 2.5$
then the *value of the function* $y = 6.25$

(c) When $x = -1.5$
then the *value of the function* $y = 2.25$

(d) When $x = -2.5$
then the *value of the function* $y = 6.25$

Exercise 3w

Determine the square of each of the following numbers by multiplication:

1. (a) 1 (b) 2 (c) 3
(d) 8 (e) 9

2. (a) 12 (b) 13 (c) 15
(d) 18 (e) 19

3. (a) 0.1 (b) 0.2 (c) 0.4
(d) 0.7 (e) 0.8

4. (a) 2.3 (b) 2.6 (c) 2.7
(d) 2.8 (e) 2.9

5. (a) 0.03 (b) 0.04 (c) 0.05
(d) 0.07 (e) 0.09

6. (a) 0.001 (b) 0.005 (c) 0.006
(d) 0.008 (e) 0.009

Find the square of each of the following numbers by using a calculator:

7. (a) 15.4 (b) 19.3 (c) 21.5
(d) 27.9 (e) 29.1

8. (a) 121.43 (b) 125.79 (c) 135.68
(d) 145.27 (e) 149.35

9. (a) 0.15 (b) 0.29 (c) 0.34
(d) 0.71 (e) 0.95

10. (a) 15 273 (b) 19 178 (c) 24 319
(d) 35 619 (e) 39 475

11. (a) 0.015 (b) 0.023 (c) 0.047
(d) 0.089 (e) 0.095

12. (a) 0.001 4 (b) 0.001 9 (c) 0.003 8
(d) 0.005 8 (e) 0.009 4

Find the square of each of the following numbers by using three-figure mathematical tables:

13. (a) 1.1 (b) 1.2 (c) 1.5
(d) 1.7 (e) 1.9

14. (a) 2.93 (b) 3.84 (c) 5.76
(d) 8.41 (e) 9.75

15. (a) 14.7 (b) 25.3 (c) 34.1
(d) 45.9 (e) 93.4

16. (a) 143 (b) 247 (c) 531
(d) 754 (e) 949

17. (a) 1 410 (b) 3 170 (c) 5 620
(d) 8 170 (e) 9 340

18. (a) 0.145 (b) 0.217 (c) 0.319
(d) 0.614 (e) 0.713

19. (a) 0.013 8 (b) 0.024 7 (c) 0.041 9
(d) 0.063 4 (e) 0.071 5

Calculate the area of each of the following squares with a side of length:

20. (a) 3 cm (b) 8 cm (c) 10 cm

21. (a) 5.3 cm (b) 6.4 cm (c) 9.6 cm

Example 37

Calculate the positive square root of each of the following quotients:

(a) $\frac{36}{64}$ *(b)* $\frac{81}{25}$

(c) $\frac{144}{169}$

Solution

(a) Now $\sqrt{\frac{36}{64}} = \sqrt{\frac{6^2}{8^2}} = \frac{6}{8} = \frac{3}{4} = 0.75$

(b) Now $\sqrt{\frac{81}{25}} = \sqrt{\frac{9^2}{5^2}} = \frac{9}{5} = 1\frac{4}{5} = 1.8$

(c) Now $\sqrt{\frac{144}{169}} = \sqrt{\frac{12^2}{13^2}} = \frac{12}{13}$

Example 38

Determine the length of a side of the square whose area is 81 cm².

Solution

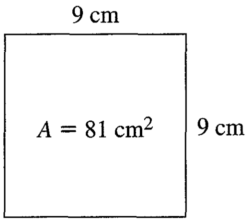


Fig. 3.23 Square

The *length* of a side of the *square*

$$= \sqrt{81 \text{ cm}^2}$$
$$= \sqrt{9 \text{ cm} \times 9 \text{ cm}}$$
$$= 9 \text{ cm}$$

Example 39

Given the function $y = x^2$.

Then the roots of the function, $x = \pm\sqrt{y}$.

Calculate the roots of the function when $y = 0.81$

Solution

The roots of the function, $x = \pm\sqrt{0.81}$

$$= \pm\sqrt{81 \times \frac{1}{100}}$$
$$= \pm 9 \times \frac{1}{10} = \pm 0.9$$

Using a Calculator

The following *method* illustrates how a *scientific calculator* was used to find the *square root of a number*. Most *simple* or *non-scientific calculators* have the *square root function*, hence the method of finding the *square root of a number* would be the same.

Example 40

Find the square root of each of the following numbers by using a calculator:

(a) 2.89 *(b) 90.25* *(c) 122 500*
(d) 0.504 1 *(e) 0.004 225*

Solution

Table 3.21

	Input	*Seen on the display of the calculator*
(a)	$\boxed{\sqrt{\ }}$	2.89
	2.89	1.7
(b)	$\boxed{\sqrt{\ }}$	90.25
	90.25	9.5
(c)	$\boxed{\sqrt{\ }}$	122 500
	122 500	350
(d)	$\boxed{\sqrt{\ }}$	0.504 1
	0.504 1	0.71
(e)	$\boxed{\sqrt{\ }}$	0.004 225
	0.004 225	0.065

Thus:

(a) $\sqrt{2.89} = 1.7$

(b) $\sqrt{90.25} = 9.5$

(c) $\sqrt{122\,500} = 350$

Example 35

Determine the positive square root of each of the following numbers without using tables or calculators:

(a) 25 *(b) 2 500*

(c) 0.25 *(d) 0.0025*

Solution

(a) The *positive square root* of the number 25 $= \sqrt{25} = \sqrt{5 \times 5} = 5$

(b) The *positive square root* of the number 2 500

$$\begin{aligned} &= \sqrt{2\,500} \\ &= \sqrt{25 \times 100} \\ &= \sqrt{5 \times 5 \times 10 \times 10} \\ &= 5 \times 10 \\ &= 50 \end{aligned}$$

(c) The *positive square root* of the number 0.25 (2 *d.p.*)

$$\begin{aligned} &= \sqrt{0.25} \\ &= \sqrt{25 \times \frac{1}{100}} \\ &= \sqrt{5 \times 5 \times \frac{1}{10 \times 10}} \\ &= 5 \times \frac{1}{10} \\ &= 0.5 \text{ (1 d.p.)} \end{aligned}$$

(d) The *positive square root* of the number 0.0025 (4 *d.p.*)

$$\begin{aligned} &= \sqrt{0.0025} \\ &= \sqrt{25 \times \frac{1}{10\,000}} \\ &= \sqrt{5 \times 5 \times \frac{1}{100 \times 100}} \\ &= 5 \times \frac{1}{100} \\ &= 0.05 \text{ (2 d.p.)} \end{aligned}$$

Alternative Method

(a) The *positive square root* of the number 25 $= \sqrt{25} = \sqrt{5^2} = 5$

(b) The *positive square root* of the number 2 500

$$\begin{aligned} &= \sqrt{2\,500} \\ &= \sqrt{5^2 \times 10^2} \\ &= 5 \times 10 \\ &= 50 \end{aligned}$$

(c) The *positive square root* of the number 0.25 (2 *d.p.*)

$$\begin{aligned} &= \sqrt{0.25} \\ &= \sqrt{5^2 \times \frac{1}{10^2}} \\ &= 5 \times \frac{1}{10} \\ &= 0.5 \text{ (1 d.p.)} \end{aligned}$$

(d) The *positive square root* of the number 0.0025 (4 *d.p.*)

$$\begin{aligned} &= \sqrt{0.0025} \\ &= \sqrt{5^2 \times \frac{1}{100^2}} \\ &= 5 \times \frac{1}{100} \\ &= 0.05 \text{ (2 d.p.)} \end{aligned}$$

From the above *examples*, it can be seen that:

(i) The number must be written as a *product of squares* in order to determine its *square root.* And the *square root* of a *squared number* is equal to its *base*. That is $\sqrt{5^2} = 5$.

(ii) The *number of decimal places* in the *square root* is *half* the number of decimal places in the number whose square root is to be found.

Example 36

Calculate the positive square root of each of the following products:

(a) 4×36 *(b)* 9×64

(c) $4 \times 25 \times 36$ *(d)* $9 \times 36 \times 64$

Solution

(a) Now $\sqrt{4 \times 36} = \sqrt{2^2 \times 6^2} = 2 \times 6 = 12$

(b) Now $\sqrt{9 \times 64} = \sqrt{3^2 \times 8^2} = 3 \times 8 = 24$

(c) Now $\sqrt{4 \times 25 \times 36} = \sqrt{2^2 \times 5^2 \times 6^2}$

$= 2 \times 5 \times 6 = 60$

(d) Now $\sqrt{9 \times 36 \times 64} = \sqrt{3^2 \times 6^2 \times 8^2}$

$= 3 \times 6 \times 8 = 144$

where $n \in N$.

For example: $\sqrt{10^6} = (10^6)^{\frac{1}{2}} = 10^{6 \times \frac{1}{2}} = 10^3$.

Example 42

Draw the graph of the function $y = x^2$ for the domain $-3 \leqslant x \leqslant 3$. Hence determine roots of the function by interpolating when:

(a) $y = 3$ *(b) $y = 5.5$*

(c) $y = 8$

Solution

Below can be seen the *table of values* of the *function* $y = x^2$, for the *domain* $-3 \leqslant x \leqslant 3$.

Table 3.22 Table of values

x	-3	-2	-1	0	1	2	3
$y = x^2$	9	4	1	0	1	4	9

Using the *table of values*, the *graph* of the *function* was then *drawn* on graph paper.

From the *graph*:

(a) When $y = 3$
then the *roots of* $y = x^2$ are ± 1.73

(b) When $y = 5.5$
then the *roots of* $y = x^2$ are ± 2.35

(c) When $y = 8$
then the *roots of* $y = x^2$ are ± 2.83

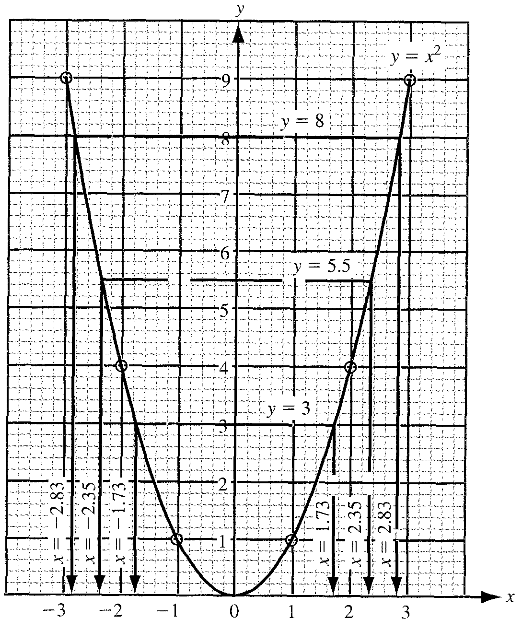


Fig. 3.24 Parabola

Exercise 3x

Determine the positive square root of each of the following numbers without using tables or calculators:

1. (a) 1 (b) 100 (c) 10 000 (d) 0.01 (e) 0.000 1

2. (a) 16 (b) 1 600 (c) 160 000 (d) 0.16 (e) 0.001 6

3. (a) 36 (b) 3 600 (c) 0.36 (d) 0.003 6 (e) 0.000 036

4. (a) 49 (b) 4 900 (c) 0.49 (d) 0.004 9 (e) 0.000 049

5. (a) 144 (b) 14 400 (c) 1 440 000 (d) 0.014 4 (e) 0.000 144

6. (a) 225 (b) 22 500 (c) 0.022 5 (d) 0.000 225 (e) 0.000 002 25

Find the square root of each of the following numbers by using a calculator:

7. (a) 144 (b) 1.69 (c) 1.96 (d) 3.24 (e) 3.61

8. (a) 73.96 (b) 75.69 (c) 79.21 (d) 90.25 (e) 96.04

9. (a) 75 625 (b) 83 521 (c) 84 100 (d) 86 436 (e) 88 209

10. (a) 20 592.25 (b) 23 195.29 (c) 26 049.96 (d) 30 835.36 (e) 39 880.09

11. (a) 0.260 1 (b) 0.396 9 (c) 0.547 6 (d) 0.739 6 (e) 0.940 9

12. (a) 0.002 025 (b) 0.002 209 (c) 0.004 761 (d) 0.006 084 (e) 0.008 649

Find the square root of each of the following numbers by using three-figure mathematical tables:

13. (a) 2.8 (b) 5.7 (c) 6.31 (d) 7.84 (e) 9.36

14. (a) 18.4 (b) 29.8 (c) 48.6 (d) 79.1 (e) 85.7

15. (a) 129 (b) 347 (c) 425 (d) 648 (e) 724

16. (a) 1 470 (b) 2 480 (c) 6 370 (d) 7 560 (e) 8 340

17. (a) 18 400 (b) 25 600 (c) 37 800 (d) 45 700 (e) 84 900

18. (a) 0.247 (b) 0.381 (c) 0.683 (d) 0.742 (e) 0.768

(d) $\sqrt{0.5041} = 0.71$

(e) $\sqrt{0.004225} = 0.065$

In the D.A.L. *scientific calculator* used to find the *square root of the numbers*, the first or main key was the $\boxed{\sqrt{\ }}$ *square root key*. Thus we had to *press* the $\boxed{\sqrt{\ }}$ key and then *input the number* in order to obtain the *square root of a number*.

Using Three-figure Mathematical Tables

Three-figure mathematical tables can also be used to find the *square root of a number*. The *square root of a number from* 1 *to* 100 can be found directly by reading the value from the *tables of square roots*. And the *square root of a number* outside of this range can also be found using the *tables* as shown below.

Example 41

Find the square root of each of the following numbers by using three-figure mathematical tables:

(a) 2.9 *(b) 6.78* *(c) 24.7* *(d) 99.5*
(e) 187 *(f) 1870* *(g) 18700* *(h) 0.359*
(i) 0.0359 *(j) 0.00359*

Solution

(a) Now $\sqrt{2.9} = \sqrt{2.90} = 1.70$ (*directly* from the *table of square roots* from 1 to 10)

(b) Now $\sqrt{6.78} = 2.60$ (*directly* from the *table of square roots* from 1 to 10)

(c) Now $\sqrt{24.7} = 4.97$ (*directly* from the *table of square roots* from 10 to 100)

(d) Now $\sqrt{99.5} = 9.97$ (*directly* from the *table of square roots* from 10 to 100)

(e) Now
$$\begin{aligned}\sqrt{187} &= \sqrt{1.87 \times 100}\\ &= \sqrt{1.87 \times 10^2}\\ &= 1.37 \times 10 = 13.7\end{aligned}$$

(f) Now
$$\begin{aligned}\sqrt{1870} &= \sqrt{18.7 \times 100}\\ &= \sqrt{18.7 \times 10^2}\\ &= 4.32 \times 10\\ &= 43.2\end{aligned}$$

(g) Now
$$\begin{aligned}\sqrt{18700} &= \sqrt{1.87 \times 10000}\\ &= \sqrt{1.87 \times 10^4}\\ &= 1.37 \times 10^2\\ &= 137\end{aligned}$$

(h) Now
$$\begin{aligned}\sqrt{0.359} &= \sqrt{35.9 \times \frac{1}{100}}\\ &= \sqrt{35.9 \times \frac{1}{10^2}}\\ &= 5.99 \times \frac{1}{10}\\ &= 0.599\end{aligned}$$

(i) Now
$$\begin{aligned}\sqrt{0.0359} &= \sqrt{3.59 \times \frac{1}{100}}\\ &= \sqrt{3.59 \times \frac{1}{10^2}}\\ &= 1.89 \times \frac{1}{10}\\ &= 0.189\end{aligned}$$

(j) Now
$$\begin{aligned}\sqrt{0.00359} &= \sqrt{35.9 \times \frac{1}{10000}}\\ &= \sqrt{35.9 \times \frac{1}{10^4}}\\ &= 5.99 \times \frac{1}{10^2}\\ &= 0.0599\end{aligned}$$

From the above *examples*, it can be *seen* that if the *number* whose *square root* is to be found is *not between* 1 *and* 100 *inclusive*, then it has to be written as a number *between* 1 *and* 10 or *between* 10 *and* 100 *times* a *power* of 10^{2n} or the reciprocal of a power of 10^{2n} (i.e. an even power of 10 or the reciprocal of an even power of 10) in order to find the *square root of the number* using *three-figure mathematical tables.*

Note that $\sqrt{10^{2n}} = 10^n$, since $(10^{2n})^{\frac{1}{2}} = 10^{2n \times \frac{1}{2}} = 10^n$,

and that $\dfrac{1}{\sqrt{10^{2n}}} = \dfrac{1}{10^n}$, since $\dfrac{1}{(10^{2n})^{\frac{1}{2}}} = \dfrac{1}{10^{2n \times \frac{1}{2}}} = \dfrac{1}{10^n}$,

Using a Calculator

The following *method* illustrates how a *scientific calculator* was used to find the *reciprocal* of a *non-zero number.*

Example 44

Find the reciprocal of each of the following numbers by using a calculator, stating your answers correct to 3 significant figures:

(a) 75 *(b) 0.129* *(c) 748*

(d) 623.5 *(e) 0.0947*

Solution

Table 3.23

	Input	Seen on the display of the calculator
(a)	75	75
	$\boxed{\frac{1}{x}}$	0.013 333 333
(b)	0.129	0.129
	$\boxed{\frac{1}{x}}$	7.751 937 984
(c)	748	748
	$\boxed{\frac{1}{x}}$	$1.336\,898\,396 \times 10^{-03}$
(d)	623.5	623.5
	$\boxed{\frac{1}{x}}$	$1.603\,849\,238 \times 10^{-03}$
(e)	0.0947	0.0947
	$\boxed{\frac{1}{x}}$	10.559 662 209

Thus:

(a) $\frac{1}{75} = 0.0133$ (*correct* to 3 *s.f.*)

(b) $\frac{1}{0.129} = 7.75$ (*correct* to 3 *s.f.*)

(c) $\frac{1}{748} = 0.00134$ (*correct* to 3 *s.f.*)

(d) $\frac{1}{623.5} = 0.00160$ (*correct* to 3 *s.f.*)

(e) $\frac{1}{0.0947} = 10.6$ (*correct* to 3 *s.f.*)

In the D.A.L. *scientific calculator* used to find the *reciprocal of the numbers* above, the first or main key was the $\boxed{\frac{1}{x}}$ reciprocal key. Thus we had to *input the number* and then *press the* $\boxed{\frac{1}{x}}$ *key* in order to obtain the *reciprocal of a number.*

In the case of a *simple* or *non-scientific calculator,* the *reciprocal of a number* can be found as shown below. However, many basic calculations do have the $\boxed{\frac{1}{x}}$ *reciprocal key.*

Table 3.24

	Input	Seen on the display of the calculator
(a)	1	1
	$\boxed{\div}$	
	75	75
	$\boxed{=}$	0.013 333 3
(e)	1	1
	$\boxed{\div}$	
	0.0947	0.0947
	$\boxed{=}$	10.559 662

Using Three-figure Mathematical Tables

Three-figure mathematical tables can also be used to find the *reciprocal of a number.* The *reciprocal of a number from* 1 *to* 10 can be found directly by reading the value from the *table.* And the *reciprocal of a number* outside of this range can also be found using the *table* as shown below.

Example 45

Find the reciprocal of each of the following numbers by using three-figure mathematical tables:

(a) 1.4 *(b) 8.63* *(c) 649*

(d) 0.347 *(e) 0.0139*

Solution

19. (a) 0.0135 (b) 0.0247 (c) 0.0379
(d) 0.0483 (e) 0.0726

20. (a) 0.00328 (b) 0.00461 (c) 0.00583
(d) 0.00692 (e) 0.00784

Determine the positive square root of each of the following products:

21. (a) 4×81 (b) 9×64 (c) 25×64

22. (a) 49×81 (b) 36×64 (c) 49×64

23. (a) $4 \times 49 \times 64$ (b) $9 \times 25 \times 36$
(c) $16 \times 64 \times 81$

24. (a) $16 \times 25 \times 121$ (b) $4 \times 36 \times 49$
(c) $25 \times 64 \times 81$

25. (a) $25 \times 64 \times 144$ (b) $25 \times 36 \times 169$
(c) $36 \times 49 \times 196$

Calculate the length of a side of each of the following squares whose area is stated below:

26. (a) Area = 25 cm^2 (b) Area = 36 cm^2

27. (a) Area = 49 cm^2 (b) Area = 64 cm^2

28. (a) Area = 121 cm^2 (b) Area = 144 cm^2

29. (a) Area = 289 mm^2
(b) Area = 361 mm^2

30. (a) Area = 655.36 mm^2
(b) Area = 992.25 mm^2

Given the function $y = x^2$ and that the roots of the function, $x = \pm\sqrt{y}$, calculate the roots of the function when:

31. (a) $y = 4$ (b) $y = 25$

32. (a) $y = 36$ (b) $y = 49$

33. (a) $y = 6.25$ (b) $y = 13.69$

34. (a) $y = 20.25$ (b) $y = 34.81$

35. (a) $y = 39.69$ (b) $y = 90.25$

Draw the graph of the function $y = x^2$ for the domain $-9 \leqslant x \leqslant 9$, using a scale of 1 cm to represent 1 unit on the x-axis and 1 cm to represent 5 units on the y-axis. Hence determine the roots of the function $y = x^2$ by interpolating when:

36. (a) $y = 2.5$ (b) $y = 5.0$ (c) $y = 7.5$

37. (a) $y = 10.0$ (b) $y = 12.5$ (c) $y = 15.0$

38. (a) $y = 17.5$ (b) $y = 20.0$ (c) $y = 22.5$

39. (a) $y = 25.0$ (b) $y = 27.5$ (c) $y = 30.0$

40. (a) $y = 32.5$ (b) $y = 35.0$ (c) $y = 37.5$

41. (a) $y = 40.0$ (b) $y = 42.5$ (c) $y = 45.0$

42. (a) $y = 47.5$ (b) $y = 50.0$ (c) $y = 52.5$

43. (a) $y = 55.0$ (b) $y = 57.5$ (c) $y = 60.0$

44. (a) $y = 62.5$ (b) $y = 65.0$ (c) $y = 67.5$

45. (a) $y = 70.0$ (b) $y = 72.5$ (c) $y = 75$

Reciprocal of a Number

The *reciprocal of a non-zero number x* is defined as 1 *divided by the number*. That is, it is the *number* $\frac{1}{x}$, where $x \neq 0$. Thus the *reciprocal of the number* 4 is $\frac{1}{4}$. The *product of any number* and its *reciprocal* is always *equal to* 1. That is, $x \times \frac{1}{x} = 1$.

For example: $4 \times \frac{1}{4} = 1$. Hence the *reciprocal of a number* is also the *multiplicative inverse of the number.*

Example 43

Determine the reciprocal of each of the following numbers without using tables or calculators:

(a) 5 *(b)* 0.25 *(c)* $\frac{1}{25}$ *(d)* $\frac{2}{3}$

Solution

(a) The *reciprocal* of the number $5 = \frac{1}{5} = 0.2$

(b) The *reciprocal* of the number 0.25

$$= \frac{1}{0.25} = \frac{1}{\frac{25}{100}} = 1 \times \frac{100}{25} = 4$$

(c) The *reciprocal* of the number $\frac{1}{25}$

$$= \frac{1}{\frac{1}{25}} = 1 \times \frac{25}{1} = 25$$

(d) The *reciprocal* of the number $\frac{2}{3}$

$$= \frac{1}{\frac{2}{3}} = 1 \times \frac{3}{2} = \frac{3}{2} = 1\frac{1}{2} = 1.5$$

2. (a) Calculate

$$\left(2\frac{1}{2} + 1\frac{2}{3}\right) \div 1\frac{1}{4}$$

(b) Calculate 2.01×0.015 giving your answer

(i) exactly

(ii) correct to 3 significant figures

(iii) in standard form.

(c) There are 840 pupils in a school. The ratio of boys to girls in the school is 5:7.

Calculate

(i) the number of boys

(ii) the ratio of girls to pupils.

Question 1. C.X.C. (Basic). June 1993.

3. (a) The dimensions of a rectangular garden plot, measured to the nearest metre, are given as 9 m long and 6 m wide.

(i) State the range of values for 9 m.

(ii) Calculate the minimum possible area of the plot of land.

(iii) Determine the maximum possible length of fencing that would be needed to fence the plot.

Question 7(a). C.X.C. (Basic). June 1993.

(a) Now $\frac{1}{1.4} = \frac{1}{1.40} = 0.714$ (*directly* from the *table of reciprocals* from 1 to 10)

(b) Now $\frac{1}{8.63} = 0.116$ (*directly* from the *table of reciprocals* from 1 to 10)

(c) Now
$$\begin{aligned}\frac{1}{649} &= \frac{1}{6.49} \times 100\\ &= \frac{1}{6.49} \times \frac{1}{100}\\ &= 0.154 \times \frac{1}{100}\\ &= 0.001\,54\end{aligned}$$

(d) Now
$$\begin{aligned}\frac{1}{0.347} &= \frac{1}{3.47 \times \frac{1}{10}}\\ &= \frac{1}{3.47} \times \frac{10}{1}\\ &= 0.288 \times 10\\ &= 2.88\end{aligned}$$

(e) Now
$$\begin{aligned}\frac{1}{0.0139} &= \frac{1}{1.39 \times \frac{1}{100}}\\ &= \frac{1}{1.39} \times \frac{100}{1}\\ &= 0.719 \times 100\\ &= 71.9\end{aligned}$$

From the above *examples*, it can be *seen* that if the *number is not between* 1 *and* 10 *inclusive*, then it has to be written as a number *between* 1 *and* 10 *times* a *power of* 10 or the reciprocal of a *power* of 10, in order to find the *reciprocal of the number* using *three-figure mathematical tables.*

Exercise 3y

Determine the reciprocal of each of the following numbers without using tables or calculators:

1. (a) 1 (b) 2 (c) 3

2. (a) 0.4 (b) 0.75 (c) 0.625

3. (a) $\frac{1}{12}$ (b) $\frac{1}{38}$ (c) $\frac{1}{57}$

4. (a) $\frac{3}{5}$ (b) $\frac{7}{9}$ (c) $\frac{9}{13}$

Find the reciprocal of each of the following numbers by using a calculator, stating your answers correct to 3 significant figures:

5. (a) 65 (b) 89 (c) 95

6. (a) 0.137 (b) 0.158 (c) 0.196

7. (a) 679 (b) 768 (c) 895

8. (a) 647.8 (b) 784.3 (c) 984.6

9. (a) 0.0347 (b) 0.0768 (c) 0.0985

10. (a) 0.001 39 (b) 0.004 85 (c) 0.008 76

Find the reciprocal of each of the following numbers by using three-figure mathematical tables:

11. (a) 1.5 (b) 6.7 (c) 9.9

12. (a) 5.63 (b) 7.84 (c) 9.48

13. (a) 347 (b) 654 (c) 981

14. (a) 0.145 (b) 0.385 (c) 0.769

15. (a) 0.0147 (b) 0.0478 (c) 0.0635

C.X.C. Past Paper Questions

The following supplementary questions were taken from C.X.C. Past Papers.

Exercise 3z

1. Calculators, slide rules and mathematical tables must NOT be used to answer this question. Show ALL steps clearly.

(a) Calculate the exact value of

$$0.021 \times 3.6$$

Write your answer correct to 2 significant figures.

(b) Calculate

$$\frac{5\frac{2}{3} + 2\frac{3}{7}}{2\frac{5}{6}}$$

Question 1. C.X.C. (Basic). June 1990.

decimetre is $\frac{1}{10}$ *of a metre*, a *centigram* is $\frac{1}{100}$ *of a gram*, and a *millilitre* is $\frac{1}{1\,000}$ *of a litre.*

The following *table* lists the *prefixes* which may be used to indicate *multiples* or *sub-multiples* of the *base units.*

Table 4.1

Prefix	*Symbol*	*Multiplication factor*
kilo (thousand)	*k*	$1\,000 = 10^3$
hecto (hundred)	*h*	$100 = 10^2$
deka (ten)	*da*	$10 = 10^1$
deci (tenth)	*d*	$0.1 = 10^{-1}$
centi (hundredth)	*c*	$0.01 = 10^{-2}$
milli (thousandth)	*m*	$0.001 = 10^{-3}$

Metric System Unit of Length

The *metre* is *the standard unit of length* in the *metric system.* A *metre is slightly longer* than a *yard* used in the traditional imperial system of units. And a *kilometre* is *approximately* equal to *five-eighths* $\left(\frac{5}{8}\right)$ *of a mile.*

The *units* which are *most commonly used* to *measure length* are the *millimetre*, the *centimetre*, the *metre* and the *kilometre.*

Short lengths, such as the *diameter of a coin*, are usually measured in *millimetres.*

Slightly greater lengths, such as the *length of a book*, may be measured in *centimetres*. Even the *height of a person* can be measured in *centimetres.*

Metres are used to measure *greater lengths*, such as the *height of a waterfall.*

The *distance between cities* are measured in *kilometres.*

The *symbols* of the commonly used *units* are:

millimetre = *mm*
centimetre = *cm*
metre = *m*
kilometre = *km*

And the *conversion tables* are:

$$1 \text{ m} = 100 \text{ cm} = 1\,000 \text{ mm}$$

$$1 \text{ km} = 1\,000 \text{ m} = 100\,000 \text{ cm} = 1\,000\,000 \text{ mm}$$

Example 1

(a) Convert to millimetres:
(i) 3.417 m (ii) 485 cm

(b) Change to centimetres:
(i) 7.82 km (ii) 4 871 mm

(c) Express in metres:
(i) 8.45 km (ii) 7 896 cm

(d) Convert into kilometres:
(i) 9 768 m (ii) 8 473 cm

Solution

(a) (i) Now $1 \text{ m} = 1\,000 \text{ mm}$
So $3.417 \text{ m} = 3.417 \times 1\,000 \text{ mm} = 3\,417 \text{ mm}$

(ii) Now $1 \text{ cm} = 10 \text{ mm}$
So $485 \text{ cm} = 485 \times 10 \text{ mm} = 4\,850 \text{ mm}$

(b) (i) Now $1 \text{ km} = 100\,000 \text{ cm}$
So $7.82 \text{ km} = 7.82 \times 100\,000 \text{ cm} = 782\,000 \text{ cm}$

(ii) Now $1 \text{ mm} = \frac{1}{10} \text{ cm}$
So $4\,871 \text{ mm} = 4\,871 \times \frac{1}{10} \text{ cm} = 487.1 \text{ cm}$

(c) (i) Now $1 \text{ km} = 1\,000 \text{ m}$
So $8.45 \text{ km} = 8.45 \times 1\,000 \text{ m} = 8\,450 \text{ m}$

(ii) Now $1 \text{ cm} = \frac{1}{100} \text{ m}$
So $7\,896 \text{ cm} = 7\,896 \times \frac{1}{100} \text{ m} = 78.96 \text{ m}$

(d) (i) Now $1 \text{ m} = \frac{1}{1\,000} \text{ km}$
So $9\,768 \text{ m} = 9\,768 \times \frac{1}{1\,000} \text{ km} = 9.768 \text{ km}$

CHAPTER 4

Measurement

This chapter will teach you how to

- convert units of length, area, volume, capacity and mass in the metric system.
- express a quantity stated in SI units in standard form (or scientific notation).
- calculate areas and perimetres of simple and compound figures.
- determine the volume, density and surface area of simple right solids such as prisms and pyramids.
- use time, calculate average speed and convert between speed units.
- state an answer appropriate to the margin of error.
- use a scale and the measurements on maps.

Metric System

The *metric system* of units is a *measuring* system that uses *decimals* for all of its *calculations*. The *metric system* is the most *precise* and *easy to use method* of *weighing* and *measuring* any *quantity*. The metric units form the basis of the International system of Units.

The *metric system's name* comes from the word *metre*. The *standard units* for *measuring length, mass* and *capacity* in the early metric system were *centimetre*, the *gram* and the *litre*, respectively.

The *base units* of the present *metric system* are the *metre*, the *kilogram* and the *litre*. All *units of length* are based on the *metre*. *Units of mass* are based on the *kilogram*. And *units of capacity* are based on the *litre*.

In order to *measure units* that are *larger* or *smaller* than the *base units*, the words *metre, gram* and *litre* are *combined* with *prefixes*. Each *prefix* represents a *multiple of ten* (which is ten or tens of ten) or *sub-multiple of ten* (which is the reciprocal of ten or the reciprocal of tens of ten).

The kilogram is the only base unit with a prefix (i.e. kilo-) already in its place.

For *units* that are *larger* than the *base units, prefixes* such as *deka* which *means* 10, *hecta* which *means* 100, and *kilo* which *means* 1 000 are used. So a *dekametre* is 10 *metres*, a *hectogram* is 100 *grams*, and a *kilolitre* is 1 000 *litres*.

For *units* that are *smaller* than the *base units, prefixes* such as *deci*, which means *one-tenth* $\left(\frac{1}{10}\right)$, *centi* which means *one-hundredth* $\left(\frac{1}{100}\right)$ and *milli* which means *one-thousandth* $\left(\frac{1}{1\,000}\right)$ are used. So a

Large areas, such as the *area of a park*, are measured in *square metres* or *hectares.*

Very large areas, such as the *size of a country*, are measured in *square kilometres.*

The *symbols* of the commonly used *units* are:

square millimetre = *sq. mm* = *mm²*
square centimetre = *sq. cm* = *cm²*
square metre = *sq. m* = *m²*
square kilometre = *sq. km* = *km²*
hectare = *ha*

And the *conversion tables* are:

$$\begin{aligned} 1\text{ m}^2 &= (100\text{ cm})^2 &&= 10\,000\text{ cm}^2 \\ &= (1\,000\text{ mm})^2 &&= 1\,000\,000\text{ mm}^2 \\ 1\text{ km}^2 &= (1\,000\text{ m})^2 &&= 1\,000\,000\text{ m}^2 \\ 1\text{ ha} &= (100\text{ m})^2 &&= 10\,000\text{ m}^2 \end{aligned}$$

Example 2

(a) Convert to mm²:
(i) 5 m² *(ii) 0.95 m²*

(b) Change to cm²:
(i) 4 m² *(ii) 3.5 m²*

(c) Express in m²:
(i) 9 875 000 mm² *(ii) 642 000 mm²*

(d) Convert into km²:
(i) 4 135 000 m² *(ii) 25 400 000 m²*

(e) Change to ha:
(i) 94 000 m² *(ii) 143 700 m²*

Solution

(a) (i) Now $1\text{ m}^2 = 1\,000\,000\text{ mm}^2$
So $5\text{ m}^2 = 5 \times 1\,000\,000\text{ mm}^2$
$= 5\,000\,000\text{ mm}^2$

(ii) Now $1\text{ m}^2 = 1\,000\,000\text{ mm}^2$
So $0.95\text{ m}^2 = 0.95 \times 1\,000\,000\text{ mm}^2$
$= 950\,000\text{ mm}^2$

(b) (i) Now $1\text{ m}^2 = 10\,000\text{ cm}^2$
So $4\text{ m}^2 = 4 \times 10\,000\text{ cm}^2$
$= 40\,000\text{ cm}^2$

(ii) Now $1\text{ m}^2 = 10\,000\text{ cm}^2$
So $3.5\text{ m}^2 = 3.5 \times 10\,000\text{ cm}^2$
$= 35\,000\text{ cm}^2$

(c) (i) Now $1\text{ mm}^2 = \frac{1}{1\,000\,000}\text{ m}^2$
So $9\,875\,000\text{ mm}^2$
$= 9\,875\,\cancel{000} \times \frac{1}{1\,000\,\cancel{000}}\text{ m}^2$
$= 9.875\text{ m}^2$

(ii) Now $1\text{ mm}^2 = \frac{1}{1\,000\,000}\text{ m}^2$
So $642\,000\text{ mm}^2$
$= 642\,\cancel{000} \times \frac{1}{1\,000\,\cancel{000}}\text{ m}^2$
$= 0.642\text{ m}^2$

(d) (i) Now $1\text{ m}^2 = \frac{1}{1\,000\,000}\text{ km}^2$
So $4\,135\,000\text{ m}^2$
$= 4\,135\,\cancel{000} \times \frac{1}{1\,000\,\cancel{000}}\text{ km}^2$
$= 4.135\text{ km}^2$

(ii) Now $1\text{ m}^2 = \frac{1}{1\,000\,000}\text{ km}^2$
So $25\,400\,000\text{ m}^2$
$= 25\,\cancel{400\,000} \times \frac{1}{1\,\cancel{000\,000}}\text{ km}^2$
$= 25.4\text{ km}^2$

(e) (i) Now $1\text{ m}^2 = \frac{1}{10\,000}\text{ ha}$
So $94\,000\text{ m}^2$
$= 94\,\cancel{000} \times \frac{1}{10\,\cancel{000}}\text{ ha}$
$= 9.4\text{ ha}$

(ii) Now $1\text{ m}^2 = \frac{1}{10\,000}\text{ ha}$
So $143\,700\text{ m}^2$
$= 143\,7\cancel{00} \times \frac{1}{10\,0\cancel{00}}\text{ ha}$
$= 14.37\text{ ha}$

Exercise 4b

1. Convert to mm²:
(a) 7 m² (b) 9 m²

2. Convert to mm²:
(a) 4.5 m² (b) 8.3 m²

3. Convert to mm²:
(a) 0.21 m² (b) 0.65 m²

4. Convert to mm²:
(a) 8.412 m² (b) 3.173 m²

(ii) Now $1 \text{ cm} = \frac{1}{100\,000} \text{ km}$

So $8\,473 \text{ cm} = 8\,473 \times \frac{1}{100\,000} \text{ km}$

$= 0.08473 \text{ km}$

From the *examples* above it can be *seen* that:

(i) When you are *converting from* a *small unit* to a *larger unit*, for example, *centimetre to metre*, you *divide* by a *multiple of* 10 (which is 10 or tens of 10).

(ii) When you are *converting from* a *large unit* to a *smaller unit*, for example, *kilometre to metre*, you *multiply* by a *multiple of* 10 (which is 10 or tens of ten).

Exercise 4a

1. Convert to millimetres:
(a) 4.51 m (b) 37.5 m

2. Convert to millimetres:
(a) 7.85 cm (b) 24.9 cm

3. Convert to millimetres:
(a) 1.82 km (b) 9.153 km

4. Convert to millimetres:
(a) 0.81 m (b) 0.94 cm

5. Convert to millimetres:
(a) 0.63 km (b) 0.75 m

6. Change to centimetres:
(a) 850 mm (b) 94.5 mm

7. Change to centimetres:
(a) 4.5 m (b) 0.93 m

8. Change to centimetres:
(a) 8.1 km (b) 0.08 km

9. Change to centimetres:
(a) 74.3 mm (b) 0.921 m

10. Change to centimetres:
(a) 0.41 km (b) 5.8 m

11. Express in metres:
(a) 4 875 mm (b) 9 421 mm

12. Express in metres:
(a) 394 cm (b) 4 869 cm

13. Express in metres:
(a) 4.51 km (b) 24.3 km

14. Express in metres:
(a) 39 423 mm (b) 49.3 cm

15. Express in metres:
(a) 8.45 km (b) 47.9 mm

16. Convert into kilometres:
(a) 512 475 mm (b) 769 861 mm

17. Convert into kilometres:
(a) 28 372 cm (b) 11 786 cm

18. Convert into kilometres:
(a) 68 475 m (b) 3 147 m

19. Convert into kilometres:
(a) 2 479 000 mm (b) 496 000 cm

20. Convert into kilometres:
(a) 894 000 m (b) 45.3 mm

Metric System Unit for Area

The *square metre* is the *standard unit* for measuring *area* in the *metric system. One square metre* is *equivalent* to the *area of a square* whose *sides* are *one metre* in *length.* That is, *one square metre* =

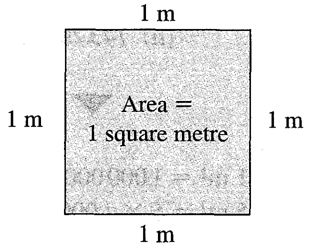


Fig. 4.1 Square

The *units* which are *most commonly used* to *measure area* are the *square millimetre*, the *square centimetre*, the *square metre*, the *hectare* and the *square kilometre*.

Small areas, such as the *area of a button*, are usually measured in *square millimetres.*

Slightly greater areas, such as the *area of a sheet of paper*, may be measured in *square centimetres.*

(ii) Now $1 \text{ m}^3 = 1\,000\,000\,000 \text{ mm}^3$

So $0.8 \text{ m}^3 = 0.8 \times 1\,000\,000\,000 \text{ mm}^3$

$= 800\,000\,000 \text{ mm}^3$

(b) (i) Now $1 \text{ m}^3 = 1\,000\,000 \text{ cm}^3$

So $3 \text{ m}^3 = 3 \times 1\,000\,000 \text{ cm}^3$

$= 3\,000\,000 \text{ cm}^3$

(ii) Now $1 \text{ m}^3 = 1\,000\,000 \text{ cm}^3$

So $0.4 \text{ m}^3 = 0.4 \times 1\,000\,000 \text{ cm}^3$

$= 400\,000 \text{ cm}^3$

(c) (i) Now $1 \text{ cm}^3 = \dfrac{1}{1\,000\,000} \text{ m}^3$

So $8\,475\,000 \text{ cm}^3 = 8\,475\cancel{000} \times \dfrac{1}{1\,000\cancel{000}}$

$= 8.475 \text{ m}^3$

(ii) Now $1 \text{ mm}^3 = \dfrac{1}{1\,000\,000\,000} \text{ m}^3$

So $4\,763\,000\,000 \text{ mm}^3$

$= 4\,763\cancel{000\,000} \times \dfrac{1}{1\,000\cancel{000\,000}} \text{ m}^3$

$= 4.763 \text{ m}^3$

Exercise 4c

1. Convert to mm³:
(a) 7 m³ (b) 9 m³

2. Convert to mm³:
(a) 0.4 m³ (b) 0.5 m³

3. Convert to mm³:
(a) 8.4 m³ (b) 9.7 m³

4. Convert to mm³:
(a) 75.31 m³ (b) 84.19 m³

5. Change to cm³:
(a) 5 m³ (b) 8 m³

6. Change to cm³:
(a) 0.6 m³ (b) 0.9 m³

7. Change to cm³:
(a) 7.1 m³ (b) 4.9 m³

8. Change to cm³:
(a) 27.15 m³ (b) 34.97 m³

9. Express in m³:
(a) 9 495 000 cm³ (b) 84 763 000 cm³

10. Express in m³:
(a) 645 000 cm³ (b) 321 400 cm³

11. Express in m³:
(a) 9 847 000 000 mm³ (b) 4 134 000 000 mm³

12. Express in m³:
(a) 935 000 000 mm³ (b) 347 000 000 mm³

Metric System Unit of Capacity

The *litre* is the *standard unit* used to measure the *capacity of a container* in the *metric system.* A *litre is slightly less* than 2 *pints* in the traditional imperial system of units.

The *units* which are *most commonly used* to *measure* the *capacity of a container* are the *millilitre*, the *centilitre* and the *litre.*

The *contents* of a *small container*, such as a *bottle of eye drops*, are usually measured in *millilitres.*

The *contents* of a *larger container,* such as a *bottle of sweet drink*, may be measured in either *millilitres* or *centilitres.*

The *contents* of still *larger containers*, such as a *water tank*, are measured in *litres.*

The *symbols* of the commonly used units are:

millilitre = *ml*
centilitre = *cl*
litre = *l (or L)*

And the *conversion tables* are:

$1\, l = 100\, cl$
$= 1\,000 \text{ m}l$
$= 1\,000 \text{ cm}^3$
$1 \text{ m}l = 1 \text{ cm}^3$
$1 \text{ m}^3 = 1\,000\, l$

Example 4

(a) Convert to litres:
(i) 7000 ml (ii) 485 000 cm³

(b) Change to cm³ or ml:
(i) 5 l (ii) 37.9 l

(c) Express in m³:
(i) 45 000 l (ii) 349 000 l

Solution

5. Change to cm²:
(a) 2 m² (b) 9 m² (c) 5 m²

6. Change to cm²:
(a) 3.6 m² (b) 8.1 m²

7. Change to cm²:
(a) 0.51 m² (b) 0.83 m²

8. Change to cm²:
(a) 9.314 m² (b) 4.713 m²

9. Express in m²:
(a) 8 475 000 mm² (b) 12 341 000 mm²

10. Express in m²:
(a) 9 345 mm² (b) 17 834 mm²

11. Express in m²:
(a) 483 000 cm² (b) 341 700 cm²

12. Express in m²:
(a) 8 475 cm² (b) 3 149 cm²

13. Convert into km²:
(a) 45 735 000 m² (b) 37 412 000 m²

14. Convert into km²:
(a) 1 425 000 m² (b) 8 375 000 m²
(c) 8 500 000 m²

15. Convert into km²:
(a) 647 000 m² (b) 312 000 m²

16. Convert into km²:
(a) 345 m² (b) 849 m²

17. Express in ha:
(a) 347 000 m² (b) 839 000 m²

18. Express in ha:
(a) 85 000 m² (b) 39 000 m²

19. Express in ha:
(a) 9 475 m² (b) 7 135 m²

20. Express in ha:
(a) 768 m² (b) 847 m²

Metric System Unit for Volume

The *cubic metre* is the *standard unit* for measuring *volume* in the *metric system. One cubic metre* is *equivalent* to the *volume of a cube whose edges are* each 1 *metre* in *length.* That is, *one cubic metre* =

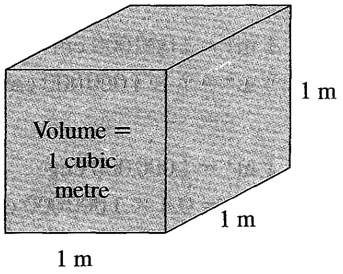


Fig. 4.2 Cube

The *units* which are *most commonly used* to *measure volume* are the *cubic millimetre*, the *cubic centimetre* and the *cubic metre.*

Small volumes, such as the *volume of a coin*, are usually measured in *cubic millimetres.*

Large volumes, such as the *volume of a gas cylinder*, are measured in *cubic centimetres.*

Larger volumes, such as the *volume of a house*, are measured in *cubic metres.*

The *symbols* of the commonly used *units* are:

cubic millimetre = *cu. mm* = *mm³*
cubic centimetre = *cu. cm* = *cm³*
cubic metre = *cu. m* = *m³*

And the *conversion table* is:

$$1 \text{ m}^3 = (100 \text{ cm})^3 = 1\,000\,000 \text{ cm}^3$$
$$= (1\,000 \text{ mm})^3 = 1\,000\,000\,000 \text{ mm}^3$$

Example 3

(a) Convert to mm³:
(i) 5 m³ *(ii) 0.8 m³*

(b) Change to cm³:
(i) 3 m³ *(ii) 0.4 m³*

(c) Express in m³:
(i) 8 475 000 cm³ *(ii) 4 763 000 000 mm³*

Solution

(a) (i) Now $1 \text{ m}^3 = 1\,000\,000\,000 \text{ mm}^3$

So $5 \text{ m}^3 = 5 \times 1\,000\,000\,000 \text{ mm}^3$
$= 5\,000\,000\,000 \text{ mm}^3$

Example 5

(a) Convert to grams:
  (i) 3 kg *(ii) 7.68 kg*

(b) Change to milligrams:
  (i) 7 g *(ii) 8.14 g*

(c) Express in kilograms:
  (i) 4 t *(ii) 6.51 t*

(d) Convert into tonnes:
  (i) 4 768 kg *(ii) 12 140 kg*

Solution

(a) (i) Now 1 kg = 1 000 g
So 3 kg = 3 × 1 000 g = 3 000 g

(ii) Now 1 kg = 1 000 g
So 7.68 kg = 7.68 × 1 000 g = 7 680 g

(b) (i) Now 1 g = 1 000 mg
So 7 g = 7 × 1 000 mg = 7 000 mg

(ii) Now 1 g = 1 000 mg
So 8.14 g = 8.14 × 1 000 mg
= 8 140 mg

(c) (i) Now 1 t = 1 000 kg
So 4 t = 4 × 1 000 kg = 4 000 kg

(ii) Now 1 t = 1 000 kg
So 6.51 t = 6.51 × 1 000 kg = 6 150 kg

(d) (i) Now $1 \text{ kg} = \frac{1}{1\,000} \text{ t}$

So $4\,768 \text{ kg} = 4\,768 = \frac{1}{1\,000} \text{ t}$
$= 4.768 \text{ t}$

(ii) Now $1 \text{ kg} = \frac{1}{1\,000} \text{ t}$

So $12\,140 \text{ kg} = 12\,140 \times \frac{1}{1\,000} \text{ t}$
$= 12.14 \text{ t}$

Exercise 4e

1. Convert to grams:
(a) 5 kg (b) 9 kg

2. Convert to grams:
(a) 0.47 kg (b) 0.891 kg

3. Convert to grams:
(a) 4.39 kg (b) 7.149 kg

4. Convert to grams:
(a) 8.4795 kg (b) 3.1476 kg

5. Change to milligrams:
(a) 4 g (b) 7 g

6. Change to milligrams:
(a) 0.49 g (b) 0.95 g

7. Change to milligrams:
(a) 1.5 g (b) 8.9 g

8. Change to milligrams:
(a) 34.78 g (b) 49.17 g

9. Express in kilograms:
(a) 5 t (b) 7 t

10. Express in kilograms:
(a) 0.8 t (b) 0.9 t

11. Express in kilograms:
(a) 4.31 t (b) 7.64 t

12. Express in kilograms:
(a) 43.29 t (b) 84.17 t

13. Convert into tonnes:
(a) 147 435 kg (b) 849 135 kg

14. Convert into tonnes:
(a) 15 768 kg (b) 24 140 kg

15. Convert into tonnes:
(a) 8 471 kg (b) 3 178 kg

16. Convert into tonnes:
(a) 947 kg (b) 835 kg

Système International d'Unités

The *International System of units*, which is abbreviated *SI* (*Système International d'Unités*) is essentially an *expansion* of the *metric system.* It forms a *coherent system of units* and is based on *seven basic* and *two suppplementary units.* It is used for *measurements* in all branches of *science, technology, industry, commerce* and *everyday life.* The *SI system* is completely *decimal* and completely *coherent*, hence *calculations* based on *measurements* are greatly *simplified.* By the *system* being *coherent*, we mean that all the *derived units* are formed by

(a) (i) Now $1\ \text{m}l = \frac{1}{1\,000}\,l$

So $$\begin{aligned} 7\,000\ \text{m}l &= 7\cancel{000} \times \frac{1}{1\cancel{000}}\,l \\ &= 7\ l \end{aligned}$$

(ii) Now $1\ \text{cm}^3 = \frac{1}{1\,000}\,l$

So $$\begin{aligned} 485\,000\ \text{cm}^3 &= 485\cancel{000} \times \frac{1}{1\cancel{000}}\,l \\ &= 485\ l \end{aligned}$$

(b) (i) Now $1\ l = 1\,000\ \text{cm}^3$

So $$\begin{aligned} 5\ l &= 5 \times 1\,000\ \text{cm}^3 \\ &= 5\,000\ \text{cm}^3 \\ &= 5\,000\ \text{m}l \end{aligned}$$

(ii) Now $1\ l = 1\,000\ \text{m}l$

So $$\begin{aligned} 37.9\ l &= 37.9 \times 1\,000\ \text{m}l \\ &= 37\,900\ \text{m}l \\ &= 37\,900\ \text{cm}^3 \end{aligned}$$

(c) (i) Now $1\ l = \frac{1}{1\,000}\ \text{m}^3$

So $$\begin{aligned} 45\,000\ l &= 45\cancel{000} \times \frac{1}{1\cancel{000}}\ \text{m}^3 \\ &= 45\ \text{m}^3 \end{aligned}$$

(ii) Now $1\ l = \frac{1}{1\,000}\ \text{m}^3$

So $$\begin{aligned} 349\,000\ l &= 349\cancel{000} \times \frac{1}{1\cancel{000}}\ \text{m}^3 \\ &= 349\ \text{m}^3 \end{aligned}$$

Exercise 4d

Convert to litres:

1. (a) 5 000 m*l* (b) 9 000 m*l*

2. (a) 3 000 cm^3 (b) 8 000 cm^3
(c) 6 000 cm^3 (d) 3 500 cm^3

3. (a) 480.95 m*l* (b) 793.84 m*l*

4. (a) 4 765.8 cm^3 (b) 2 175.3 cm^3

5. (a) 39 470 m*l* (b) 45 763 cm^3

Change to cm^3:

6. (a) 4 *l* (b) 8 *l*

7. (a) 5.61 *l* (b) 45.3 *l*

Change to m*l*:

8. (a) 8.75 *l* (b) 24.9 *l*

9. (a) 15.81 *l* (b) 19.48 *l*

Express in m^3:

11. (a) 12 000 *l* (b) 16 000 *l*

12. (b) 685.7 *l* (b) 947.8 *l*
(c) 7 000 *l*

13. (a) 9 475.84 *l* (b) 4 349.15 *l*

14. (a) 15 378 *l* (b) 12 147 *l*

15. (a) 6 470.4 *l* (b) 1 384.5 *l*

Express in c*l*:

16. (a) 425 *l* (b) 943 *l*

17. (a) 48.7 *l* (b) 83.2 *l*

18. (a) 438 m*l* (b) 574 m*l*

19. (a) 51.3 m*l* (b) 34.15 m*l*

20. (a) 45 cm^3 (b) 178.4 cm^3

Metric System Unit of Mass

A *kilogram* is the *standard unit of mass* in the *metric system.* A *kilogram* is *approximately equal* to 2.2 *pounds* in the traditional imperial system of units. The *units* which are *most commonly used* to *measure mass* are the *milligram*, the *gram*, the *kilogram* and the *tonne.*

Very small masses, such as *a capsule* or *tablet*, are usually measured in *milligrams.*

Most packed foods are measured in *grams* or *kilograms.*

Large quantities of products such as *rice* and *wheat*, are sold in *tonnes.*

The *symbols* of the commonly used *units* are:

milligram = *mg*
gram = *g*
kilogram = *kg*
tonne = *t*

And the *conversion tables* are:

1 kg = 1 000 g
1 g = 1 000 mg
1 t = 1 000 kg

(iv) Now
$$\begin{aligned} 3\,475\,000\text{ km} &= 3\,475\,000 \times 10^3\text{ m} \\ &= 3.475 \times 1\,000\,000 \times 10^3\text{ m} \\ &= 3.475 \times 10^6 \times 10^3\text{ m} \\ &= 3.475 \times 10^9\text{ m} \\ &= 3.475\text{ Gm} \end{aligned}$$

(v) Now
$$\begin{aligned} 0.0057\text{ A} &= \frac{5.7}{1000}\text{ A} \\ &= \frac{5.7}{10^3}\text{ A} \\ &= 5.7 \times 10^{-3}\text{ A} \\ &= 5.7\text{ mA} \end{aligned}$$

(vi) Now
$$\begin{aligned} 0.0893\text{ cm} &= 0.0893 \times 10^{-2}\text{ m} \\ &= \frac{893}{10\,000} \times 10^{-2}\text{ m} \\ &= \frac{893}{10^4} \times 10^{-2}\text{ m} \\ &= 893 \times 10^{-4} \times 10^{-2}\text{ m} \\ &= 893 \times 10^{-6}\text{ m} \\ &= 893\ \mu\text{m} \end{aligned}$$

Note that
$$\begin{aligned} 0.0893\text{ cm} &= 0.0893 \times 10^{-2}\text{ m} \\ &= \frac{0.893}{10} \times 10^{-2}\text{ m} \\ &= 0.893 \times 10^{-1} \times 10^{-2}\text{ m} \\ &= 0.893 \times 10^{-3}\text{ m} \\ &= 0.893\text{ mm} \end{aligned}$$

Hence 0.0893 cm = 893 μm = 0.893 mm.

However 0.0893 cm is *preferably written* as 893 μm, rather than 0.893 mm.

(vii) Now
$$\begin{aligned} 0.000\,000\,057\,49\text{ m} &= \frac{57.49}{1\,000\,000\,000} \\ &= \frac{57.49}{10^9}\text{ m} \\ &= 57.49 \times 10^{-9}\text{ m} \\ &= 57.49\text{ nm} \end{aligned}$$

(viii) Now
$$\begin{aligned} &0.000\,000\,000\,61\text{ kg} \\ &= 0.000\,000\,000\,61 \times 10^3\text{ g} \\ &= \frac{610}{1\,000\,000\,000\,000} \times 10^3\text{ g} \\ &= \frac{610}{10^{12}} \times 10^3\text{ g} \\ &= 610 \times 10^{-12} \times 10^3\text{ g} \\ &= 610 \times 10^{-9}\text{ g} \\ &= 610\text{ ng} \end{aligned}$$

Note that
$$\begin{aligned} 0.000\,000\,000\,61\text{ kg} &= 0.000\,000\,000\,61 \times 10^3\text{ g} \\ &= \frac{0.61}{1\,000\,000\,000} \times 10^3\text{ g} \\ &= 0.61 \times 10^{-9} \times 10^3\text{ g} \\ &= 0.61 \times 10^{-6}\text{ g} \\ &= 0.61\ \mu\text{g} \end{aligned}$$

Hence 0.000 000 000 61 kg = 610 ng = 0.61 μg.

However 0.000 000 000 61 kg *is preferably written* as 610 ng, rather than 0.61 μg.

From the above *examples* it can be *seen* that:

(i) When we *express a quantity* as a *multiple* or *submultiple* of its *base unit*, then in general we write it first in the *form* $B \times 10^{3n}$, where $1 \leq B < 1000$ and $n \in Z$.

(ii) The *power* of 10, that is, 3*n then indicates* the *prefix* to be attached to the *base unit.*

(iii) In the case of the unit of mass, which is the kilogram, each prefix is attached to the imperial base unit, which is the gram. So we do not have, for example, h kg, but Mg.

(b) (i) Now
$$\begin{aligned} 85\text{ kg} &= 85 \times 10^3\text{ g} \\ &= 8.5 \times 10 \times 10^3\text{ g} \\ &= 8.5 \times 10^4\text{ g} \end{aligned}$$

(ii) Now
$$\begin{aligned} 413\text{ km} &= 413 \times 10^3\text{ m} \\ &= 4.13 \times 100 \times 10^3\text{ m} \\ &= 4.13 \times 10^2 \times 10^3\text{ m} \\ &= 4.13 \times 10^5\text{ m} \end{aligned}$$

(iii) Now
$$\begin{aligned} 95\text{ mm} &= 95 \times 10^{-3}\text{ m} \\ &= 9.5 \times 10 \times 10^{-3}\text{ m} \\ &= 9.5 \times 10^{-2}\text{ m} \end{aligned}$$

(iv) Now
$$\begin{aligned} 435\text{ mg} &= 435 \times 10^{-3}\text{ g} \\ &= 4.35 \times 100 \times 10^{-3}\text{ g} \\ &= 435 \times 10^2 \times 10^{-3}\text{ g} \\ &= 4.35 \times 10^{-1}\text{ g} \end{aligned}$$

(v) Now
$$4.781\text{ Eg} = 4.781 \times 10^{18}\text{ g}$$

(vi) Now
$$\begin{aligned} 19.53\text{ Gg} &= 19.53 \times 10^9\text{ g} \\ &= 1.953 \times 10 \times 10^9\text{ g} \\ &= 1.953 \times 10^{10}\text{ g} \end{aligned}$$

(vii) Now
$$\begin{aligned} 47.8\ \mu\text{m} &= 47.8 \times 10^{-6}\text{ m} \\ &= 4.78 \times 10 \times 10^{-6}\text{ m} \\ &= 4.78 \times 10^{-5}\text{ m} \end{aligned}$$

(viii) Now
$$\begin{aligned} 374.9\text{ fm} &= 374.9 \times 10^{-15}\text{ m} \\ &= 3.749 \times 100 \times 10^{-15}\text{ m} \\ &= 3.749 \times 10^2 \times 10^{-15}\text{ m} \\ &= 3.749 \times 10^{-13}\text{ m} \end{aligned}$$

simple multiplication or *division* of the *basic units, without* having to introduce any *numerical factor*, even a *power of ten.*

The *seven basic units* are:

(1) The *metre* (*m*) for measuring *length.*

(2) The *kilogram* (*kg*) for measuring *mass.*

(3) The *second* (*s*) for measuring *time.*

(4) The *ampere* (*A*) for measuring *electric current.*

(5) The *kelvin* (*K*) for measuring *temperature.*

(6) The *candela* (*cd*) for measuring *luminous intensity* or *light.*

(7) The *mole* (*mol*) for measuring *amount of a substance.*

The *two supplementary units* are:

(1) The *radian* (*rad*) for measuring *plane angles.*

(2) The *steradian* (*sr*) for measuring *solid angles.*

All *other units* apart from the *seven basic* and *two supplementary units* are called *derived units.*

Some *derived units* are:

(1) The *hertz* (*Hz*) for measuring *frequency.*

(2) The *newton* (*N*) for measuring *force.*

(3) The *joule* (*J*) for measuring *energy, work*, or *quantity of heat.*

(4) The *watt* (*W*) for measuring *power* or *radiant flux.*

(5) The *pascal* (*Pa*) for measuring *pressure* or *stress.*

The following *prefixes* may be used to indicate *multiples* or *sub-multiples* of the *base units.*

Table 4.2

Prefix name	*Symbol*	*Multiplication factor*
yotta	*Y*	1 000 000 000 000 000 000 000 000 = 10^{24}
zetta	*Z*	1 000 000 000 000 000 000 000 = 10^{21}
exa (quintillion)	*E*	1 000 000 000 000 000 000 = 10^{18}
peta (quadrillion)	*P*	1 000 000 000 000 000 = 10^{15}
tera (trillion)	*T*	1 000 000 000 000 = 10^{12}
giga (billion)	*G*	1 000 000 000 = 10^{9}
mega (million)	*M*	1 000 000 = 10^{6}
kilo (thousand)	*k*	1000 = 10^{3}
hecto (hundred)	*h*	100 = 10^{2}
deka (ten)	*da*	10 = 10^{1}
deci (tenth)	*d*	0.1 = 10^{-1}
centi (hundredth)	*c*	0.01 = 10^{-2}
milli (thousandth)	*m*	0.001 = 10^{-3}
micro (millionth)	*μ*	0.000001 = 10^{-6}
nano (billionth)	*n*	0.000 000 001 = 10^{-9}
pico (trillionth)	*p*	0.000 000 000 001 = 10^{-12}
femto (quadrillionth)	*f*	0.000 000 000 000 001 = 10^{-15}
alto (quintillionth)	*a*	0.000 000 000 000 000 001 = 10^{-18}
zepto	*z*	0.000 000 000 000 000 000 001 = 10^{-21}
yocto	*y*	0.000 000 000 000 000 000 000 001 = 10^{-24}

For most practical purposes, the *prefixes* with *positive powers* are used to represent *measurements* in the *macroscopic world*, for example, the flight distance from Georgetown to New York. And the *prefixes* with *negative powers* are used to represent *measurements* in the *microscopic world*, for example, the diameter of a bacterium.

Example 6

(a) Express each of the following quantities as a multiple or sub-multiple of the base unit:

(i) 9500 m *(v) 0.0057 A*

(ii) 4700 kA *(vi) 0.0893 cm*

(iii) 19400 kg *(vii) 0.000 000 057 49 m*

(iv) 3 475 000 km *(viii) 0.000 000 000 61 kg*

(b) Express each of the following quantities in the form $A \times 10^n$*, where* $1 \leqslant A < 10$ *and* $n \in Z$ *(i.e. in standard form or scientific notation):*

(i) 85 kg *(v) 4.781 Eg*

(ii) 413 km *(vi) 19.53 Gg*

(iii) 95 mm *(vii) 47.8 μm*

(iv) 435 mg *(viii) 374.9 fm*

Solution

(a) (i) Now

$$\begin{aligned} 9500\text{ m} &= 9.5 \times 1000\text{ m} \\ &= 9.5 \times 10^3\text{ m} \\ &= 9.5\text{ km} \end{aligned}$$

(ii) Now

$$\begin{aligned} 4700\text{ kA} &= 4700 \times 10^3\text{ A} \\ &= 4.7 \times 1000 \times 10^3\text{ A} \\ &= 4.7 \times 10^3 \times 10^3\text{ A} \\ &= 4.7 \times 10^6\text{ A} \\ &= 4.7\text{ MA} \end{aligned}$$

(iii) Now

$$\begin{aligned} 19400\text{ kg} &= 19400 \times 10^3\text{ g} \\ &= 19.4 \times 1000 \times 10^3\text{ g} \\ &= 19.4 \times 10^3 \times 10^3\text{ g} \\ &= 19.4 \times 10^6\text{ g} \\ &= 19.4\text{ Mg} \end{aligned}$$

Example 7

(a)

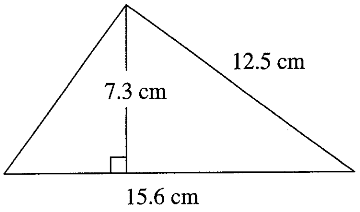


Fig. 4.5 Acute-angled triangle

(b)

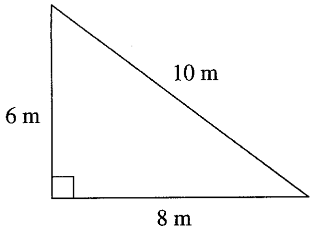


Fig. 4.6 Right-angled triangle

(c)

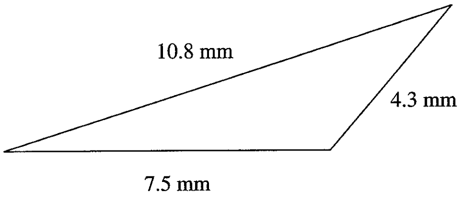


Fig. 4.7 Obtuse-angled triangle

Calculate the area of each triangle shown in the diagrams above.

(d)

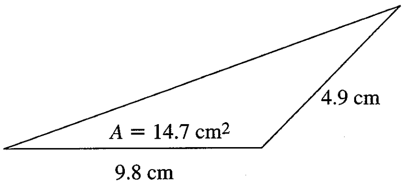


Fig. 4.8 Obtuse-angled triangle

Evaluate the altitude of the triangle of area 14.7 cm², shown in the diagram above.

(e)

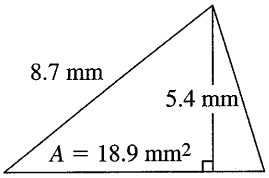


Fig. 4.9 Acute-angled triangle

Determine the base of the triangle of area 18.9 mm², shown in the previous diagram.

(f)

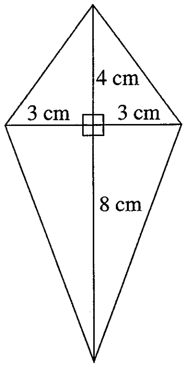


Fig. 4.10 Kite

Calculate the area of the kite shown in the diagram above.

Solution

(a)

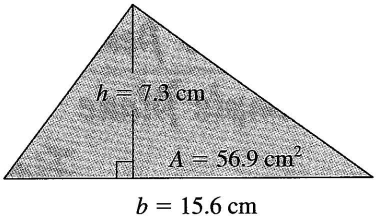


Fig. 4.5 Acute-angled triangle

The *area* of the *triangle*,

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2} \times 15.6 \text{ cm} \times 7.3 \text{ cm} \\ &= 7.8 \text{ cm} \times 7.3 \text{ cm} \\ &= 56.94 \text{ cm}^2 \\ &= 56.9 \text{ cm}^2 \ (\textit{correct} \text{ to } 1 \ \textit{d.p.}) \end{aligned}$$

(b)

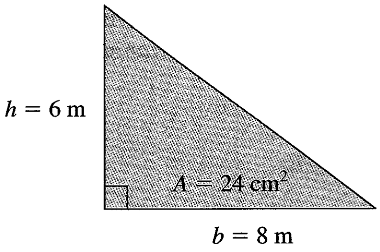


Fig. 4.6 Right-angled triangle

Exercise 4f

Express each of the following quantities as a multiple or sub-multiple of the base unit:

1. 1609 m
2. 1.13×10^{-15} kg
3. 3.156×10^{7} s
4. 0.0459 A
5. 5400 k
6. 0.000004848 rad
7. 3.084×10^{6} m
8. 4.65×10^{-21} kg
9. 3600000000 s
10. 0.0000000048 k

Express each of the following quantities in the form $A \times 10^{n}$, where $1 \leqslant A < 10$ and $n \in Z$:

11. 9.461 Pm
12. 5.08 μg
13. 35 Ms
14. 5930 MK
15. 0.2909 m rad
16. 1055 kJ
17. 7457 kW
18. 3048 kg
19. 0.00254 mm
20. 1475 ps

Areas and Perimeters of Simple Plane Figures

A *plane* is defined as a *flat smooth surface* with *no thickness*. For example: The top of a desk and the surface of a blackboard are two planes.

A *plane figure* is a *shape* that can be drawn with all its points in the *same plane*. For example: *Circles*; and all *polygons* such as triangles, squares and pentagons.

The *area* of a *plane figure* is the *size of the surface enclosed* by its *boundary*.

The *altitude* of a *plane figure* is its *perpendicular distance* (or height). The altitude makes an *angle of* 90° with the *horizontal*. In a *triangle*, the *horizontal* is its *base*.

The Triangle

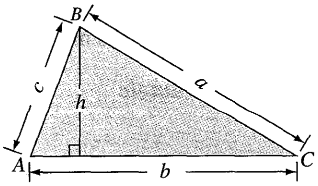


Fig. 4.3 Triangle

$$\text{The } \textit{area of a triangle}, A = \frac{1}{2}bh,$$

where b = The *base* of the *triangle*
and h = The *altitude* of the *triangle*.

The *base of a triangle*, $b = \dfrac{2A}{h}$.

The *altitude of a triangle*, $h = \dfrac{2A}{b}$.

Note that the *base* and the *altitude meet* at 90° (or *one right-angle*).

The *area of a triangle*,

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

the *semi-perimeter of the triangle*,

$$s = \frac{a+b+c}{2}$$

and the *perimeter of the triangle*, $P = a + b + c$.

where *a*, *b and c* are the three *lengths* of the *sides* of the *triangle*.

The *formula* $A = \sqrt{s(s-a)(s-b)(s-c)}$ is called *Heron's formula*.

The Kite

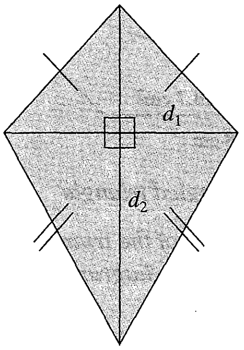


Fig. 4.4 Kite

$$\text{The } \textit{area of a kite}, A = \frac{1}{2}d_1\, d_2,$$

where d_1 = the *length* of *one diagonal* of the *kite*
and d_2 = the *length* of the *second diagonal* of the *kite*.

Considering the *kite* to be formed by an *upper* and a *lower triangle*.

Then the *area* of the *upper triangle*,

$$\begin{aligned} A_1 &= \frac{1}{2}bh \\ &= \frac{1}{2} \times 6 \text{ cm} \times 4 \text{ cm} \\ &= 3 \text{ cm} \times 4 \text{ cm} \\ &= 12 \text{ cm}^2 \end{aligned}$$

And the *area* of the *lower triangle*,

$$\begin{aligned} A_2 &= \frac{1}{2}bh \\ &= \frac{1}{2} \times 6 \text{ cm} \times 8 \text{ cm} \\ &= 3 \text{ cm} \times 8 \text{ cm} \\ &= 24 \text{ cm}^2 \end{aligned}$$

$\therefore$ the *area* of the *kite*, $A = A_1 + A_2$
$= (12 + 24) \text{ cm}^2$
$= 36 \text{ cm}^2$

Exercise 4g

Calculate the area of each of the following triangles:

1.

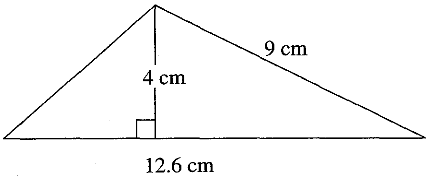


Fig. 4.11 Acute-angled triangle

2.

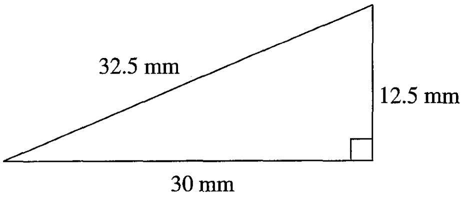


Fig. 4.12 Right-angled triangle

3.

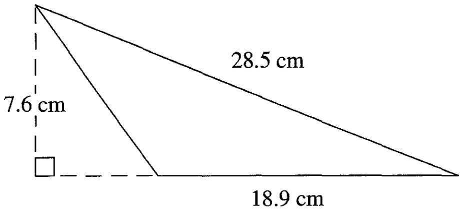


Fig. 4.13 Obtuse-angled triangle

4.

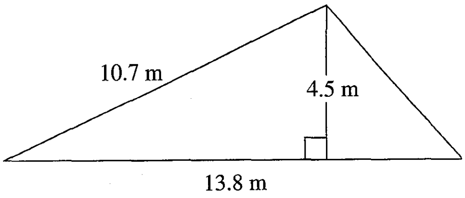


Fig. 4.14 Acute-angled triangle

5.

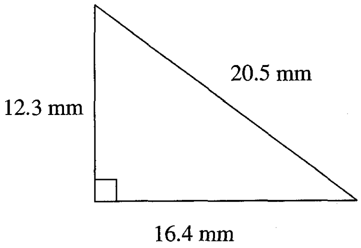


Fig. 4.15 Right-angled triangle

6.

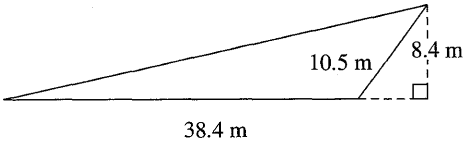


Fig. 4.16 Obtuse-angled triangle

7. Plot the points $A(-3, 0)$, $B(4, 1)$ and $C(2, 5)$ on graph paper. Hence determine the area of triangle ABC.

Calculate the missing measurements for each of the following triangles:

Table 4.3

	Shape	*Area*	*Base*	*Altitude*
8.	Triangle	14 m^2	8 m	
9.		43 cm^2		8.6 cm
10.		45 mm^2	12.5 mm	
11.		34.5 cm^2		13.8 cm

12. The area of a triangle is 20 cm^2. The height of the triangle is 6 cm. Evaluate the length of the base.

13. The area of a triangle is 25.8 mm^2. The length of the base is 8.6 mm. Calculate the altitude of the triangle.

14. The lengths of the three sides of a triangle are 17.1 cm, 22.8 cm and 28.5 cm, respectively. Calculate the area of the triangle correct to one decimal place.

15. A triangle has sides 7.5 mm, 18 mm and 19.5 mm. Calculate the area of the triangle correct to three significant figures.

The *area* of the *triangle*,

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2} \times 8 \text{ cm} \times 6 \text{ cm} \\ &= 4 \text{ cm} \times 6 \text{ cm} \\ &= 24 \text{ cm}^2 \end{aligned}$$

(c)

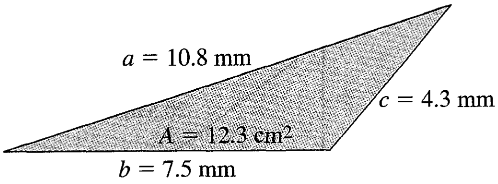


Fig. 4.7 Obtuse-angled triangle

The *semi-perimeter* of the *triangle*,

$$\begin{aligned} s &= \frac{a+b+c}{2} \\ &= \frac{(10.8 + 7.5 + 4.3) \text{ mm}}{2} \\ &= \frac{22.6 \text{ mm}}{2} \\ &= 11.3 \text{ mm} \end{aligned}$$

So the area of the triangle,

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{11.3(11.3 - 10.8)(11.3 - 7.5)(11.3 - 4.3) \text{ mm}^4} \\ &= \sqrt{11.3(0.5)(3.8)(7) \text{ mm}^4} \\ &= \sqrt{150.29 \text{ mm}^4} \\ &= 12.3 \text{ mm}^2 (\textit{correct} \text{ to } 1 \; \textit{d.p.}) \end{aligned}$$

(d)

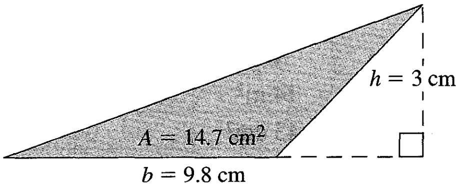


Fig. 4.8 Obtuse-angled triangle

The *altitude* of the *triangle*, $h = \frac{2A}{b}$

$$\begin{aligned} &= \frac{2 \times 14.7 \text{ cm}^{\cancel{2}}}{9.8 \; \cancel{\text{cm}}} \\ &= \frac{14.7}{4.9} \text{ cm} \\ &= 3 \text{ cm} \end{aligned}$$

(e)

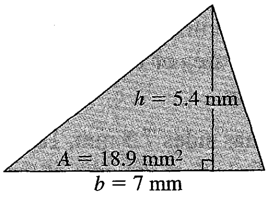


Fig. 4.9 Acute-angled triangle

The *base of the triangle*, $b = \frac{2A}{h}$

$$\begin{aligned} &= \frac{2 \times 18.9 \text{ mm}^{\cancel{2}}}{5.4 \; \cancel{\text{mm}}} \\ &= \frac{18.9}{2.7} \text{ mm} \\ &= 7 \text{ mm} \end{aligned}$$

(f)

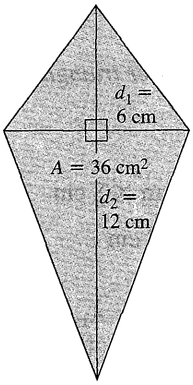


Fig. 4.10 Kite

The *length* of *one diagonal*,

$$d_1 = (3 + 3) \text{ cm} = 6 \text{ cm}$$

And the *length* of the *second diagonal*,

$$d_2 = (4 + 8) \text{ cm} = 12 \text{ cm}$$

∴ the *area* of the *kite*,

$$\begin{aligned} A &= \frac{1}{2}d_1 \, d_2 \\ &= \frac{1}{2} \times 6 \text{ cm} \times 12 \text{ cm} \\ &= 3 \text{ cm} \times 12 \text{ cm} \\ &= 36 \text{ cm}^2 \end{aligned}$$

Alternative Method

(f)

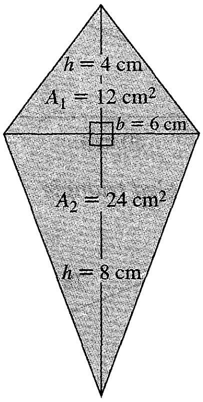


Fig. 4.10 Kite

There are three ways of *solving* the *problem* using this *method.* We can consider the *kite* as being formed by *two triangles*, or being formed by *four triangles.*

The *area of a rectangle*, $A = lb$

and the *perimeter of a rectangle*,

$$P = 2l + 2b = 2(l + b),$$

where l = the *length* of the *rectangle*

and b = the *width* of the *rectangle*.

The *length of a rectangle*, $l = \frac{A}{b}$.

Also, the *length of a rectangle*, $l = \frac{P - 2b}{2}$.

The *width of a rectangle*, $b = \frac{A}{l}$.

Also, the *width of a rectangle*, $b = \frac{P - 2l}{2}$.

Example 8

(a)

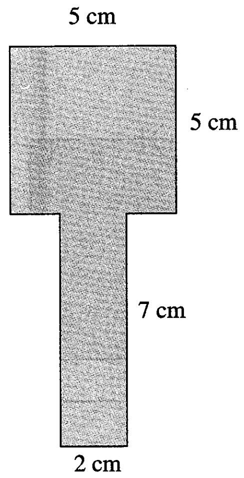


Fig. 4.22 Compound figure

Calculate:

(i) the area of the compound figure

(ii) the perimeter of the compound figure.

(b)

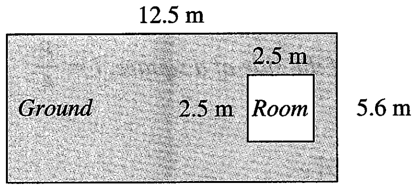


Fig. 4.23 Plane figure

Determine:

(i) the area of the ground, excluding the room

(ii) the perimeter of the ground.

(c) A living room is 4 m by 2.5 m. How many square tiles of side 20 cm will be needed to cover the entire room?

Evaluate:

(i) the perimeter of the living room

(ii) the perimeter of a tile.

Solution

(a)

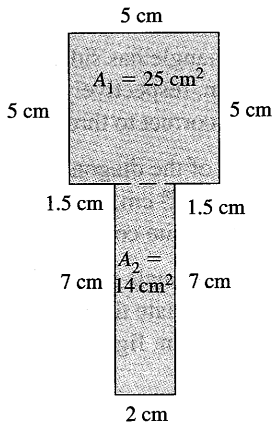


Fig. 4.22 Compound figure

(i) The *area* of the *square*,

$$A_1 = l^2 = (5\text{ cm})^2 = 25\text{ cm}^2$$

The *area* of the *rectangle*,

$$A_2 = lb = 7\text{ cm} \times 2\text{ cm} = 14\text{ cm}^2$$

∴ the *area* of the *compound figure*,

$$A = A_1 + A_2 = (25 + 14)\text{ cm}^2 = 39\text{ cm}^2$$

(ii) The *perimeter* of the *compound figure*,

$$P = (5 + 5 + 1.5 + 7 + 2 + 7 + 1.5 + 5)\text{ cm} = 34\text{ cm}$$

(b)

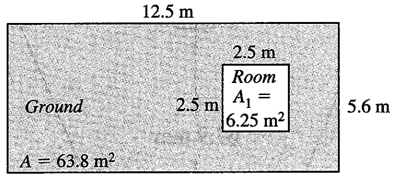


Fig. 4.23 Plane figure

(i) The *area* of the *square room*,

$$A_1 = l^2 = (2.5\text{ m})^2 = 6.25\text{ m}^2$$

The *area* of the *rectangle*,

$$A_2 = lb = 12.5\text{ m} \times 5.6\text{ m} = 70\text{ m}^2$$

16. The lengths of the sides of a triangle are 2 m, 4.8 m and 5.2 m. Determine the area of the triangle.

17. The lengths of the sides of a scalene triangle are 12.9 cm, 17.2 cm and 21.5 cm. Evaluate the area of the triangle correct to one decimal place.

18. A scalene triangle has sides 30.5 mm, 73.2 mm and 79.3 mm, respectively. Evaluate the area of the triangle correct to three significant figures.

19. The lengths of the diagonals of a kite are 12.5 cm and 18.3 cm, respectively. Calculate the area of the kite correct to one decimal place.

20. A kite has diagonals of lengths 25.9 mm and 48.7 mm. Calculate the area of the kite correct to three signficant figures.

21.

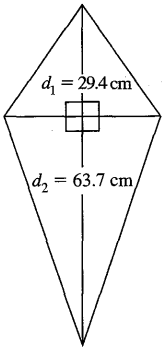


Fig. 4.17 Kite

Calculate the area of the kite shown in the diagram above. State your answer correct to one decimal place.

22.

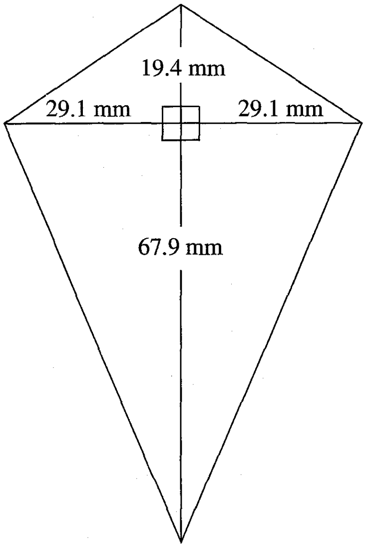


Fig. 4.18 Kite

Determine the area of the kite shown in the diagram above. State your answer correct to three significant figures.

23.

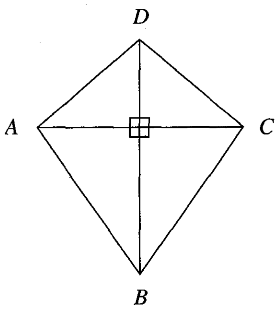


Fig. 4.19 Kite

Evaluate the area of the kite *ABCD* whose diagonals measure 20 cm and 24 cm.

The Square

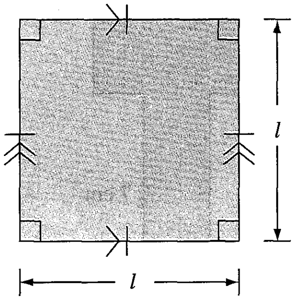


Fig. 4.20 Square

The *area of square*, $A = l^2$

and the *perimeter of a square*, $P = 4l$,

where l = the *length* of a *side* of a *square*.

The *length of the side of a square*, $l = \sqrt{A}$.

Also, the *length of the side of a square*, $l = \frac{P}{4}$.

The Rectangle

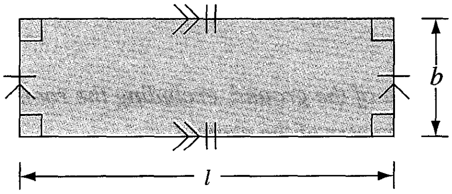


Fig. 4.21 Rectangle

5. A school's gymnasium measuring 30 m by 15 m is to be coverd with square tiles of side 20 cm. How many tiles are required to complete the job?

6. A rectangular carpet measures 5 m by 3 m. Calculate its area. What amount would it cost the owner to clean at 80 ¢ per square metre?

7. Evaluate the area and perimeter of a football field measuring 150 m by 90 m.

8. A rectangular carpet measures 15 m by 9 m. Calculate its area. How much would it cost to clean at 60 ¢ per square metre?

9. Evaluate the area of each of the following rectangles, giving your answer in the unit in brackets:

Table 4.4

	Length	*Breadth*	*Unit*
(a)	20 m	0.5 m	(cm^2)
(b)	0.45 km	0.005 km	(m^2)

10. The area of a rectangle is 86 cm^2 and its length is 12 cm. Determine its width.

11. Calculate the length of a rectangle of area 45 cm^2 and width 5 cm.

12. Evaluate the area of each of the following rectangles, giving your answer in the unit in brackets:

Table 4.5

	Length	*Breadth*	*Unit*
(a)	10 m	0.3 m	(cm^2)
(b)	500 cm	200 cm	(m^2)
(c)	0.5 km	0.2 km	(m^2)
(d)	1.8 cm	1.2 cm	(mm^2)

13. *Table 4.6*

	Shape	*Length*	*Breadth*	*Perimeter*	*Area*
(a)	Rectangle	4 cm		14 cm	
(b)			3 cm		15 cm^2

Determine the missing values in the table above.

14. Fill in the values that are missing in the following table:

Table 4.7

	Shape	*Length*	*Breadth*	*Perimeter*	*Area*
(a)	Rectangle	7 cm		22 cm	
(b)			3 cm	24 cm	
(c)		12 cm			96 cm^2
(d)			7 cm		77 cm^2

15. The area of a rectangle is 48 cm^2. If its length is 12 cm, evaluate its breadth.

16. The perimeter of a rectangle is 32 cm. If its width is 4 cm, evaluate its length.

17. The area of a triangle is 48 cm^2. The height of the triangle is 6 cm. Calculate the length of the base.

18. In the figure shown; determine:
(a) the perimeter
(b) the area.

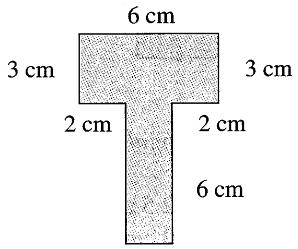


Fig. 4.25 Compound figure

19. Calculate the area of the compound figure:

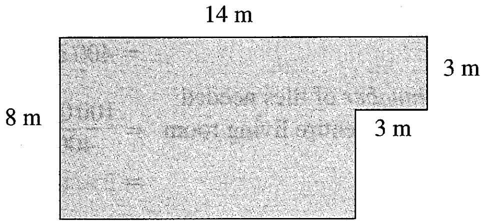
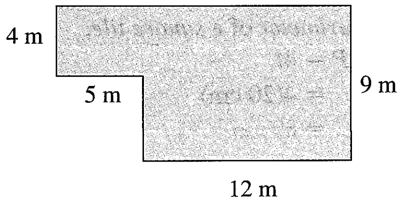


Fig. 4.26 Compound figure

20. Calculate the areas and perimeter of each of the following compound figures:

(a)



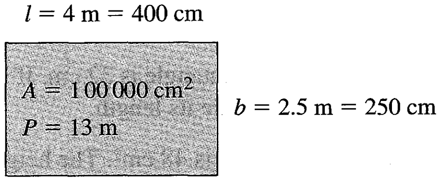
∴ the *area* of the *ground*,

$$\begin{aligned} A &= A_2 - A_1 \\ &= (70 - 6.25)\text{ m}^2 \\ &= 63.75\text{ m}^2 \\ &= 63.8\text{ m}^2 \ (\textit{correct} \text{ to } 1\ \textit{d.p.}). \end{aligned}$$

(ii) The *perimeter* of the *ground*,

$$\begin{aligned} P &= 2(l + b) \\ &= 2(12.5\text{ m} + 5.6\text{ m}) \\ &= 2(18.1\text{ m}) \\ &= 36.2\text{ m} \end{aligned}$$

(c)



Rectangular living room

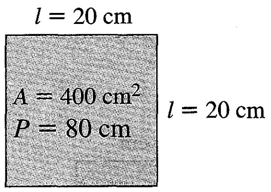


Fig. 4.24 Square tile

The *area* of the *rectangular living room*,

$$\begin{aligned} A &= lb \\ &= 4\text{ m} \times 2.5\text{ m} \\ &= 4 \times 100\text{ cm} \times 2.5 \times 100\text{ cm} \\ &= 400\text{ cm} \times 250\text{ cm} \\ &= 100\,000\text{ cm}^2 \end{aligned}$$

The *area* of a *square tile*, $A = l^2$

$$\begin{aligned} &= (20\text{ cm})^2 \\ &= 400\text{ cm}^2 \end{aligned}$$

∴ the *number* of tiles needed to cover the entire living room $= \dfrac{100\,0\cancel{00}\,\cancel{\text{cm}^2}}{4\cancel{00}\,\cancel{\text{cm}^2}}$

$= 250$ tiles

(i) The *perimeter* of the rectangular *living room*,

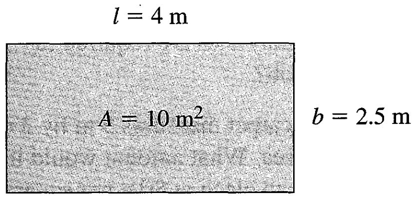
$$\begin{aligned} P &= 2(l + b) \\ &= 2(4\text{ m} + 2.5\text{ m}) \\ &= 2(6.5\text{ m}) \\ &= 13\text{ m} \end{aligned}$$

(ii) The *perimeter* of a *square tile*,

$$\begin{aligned} P &= 4l \\ &= 4(20\text{ cm}) \\ &= 80\text{ cm} \end{aligned}$$

Alternative Method

(c)



Rectangular living room

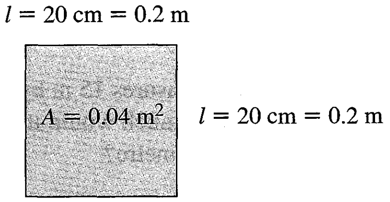


Fig. 4.24 Square tile

The *area* of the rectangular *living room*,

$$\begin{aligned} A &= lb \\ &= 4\text{ m} \times 2.5\text{ m} \\ &= 10\text{ m}^2 \end{aligned}$$

The *area* of a *square tile*, $A = l^2$

$$\begin{aligned} &= (20\text{ cm})^2 \\ &= \left(\frac{20}{100}\text{m}\right)^2 \\ &= (0.2\text{ m})^2 \\ &= 0.04\text{ m}^2 \end{aligned}$$

∴ the *number of tiles* needed to cover the entire living room $= \dfrac{10\,\cancel{\text{m}^2}}{0.04\,\cancel{\text{m}^2}}$

$= \dfrac{1\,000}{4}$ tiles

$= 250$ tiles

Exercise 4h

1. Calculate the area and perimeter of:
(a) a square of side 6 cm
(b) a rectangle measuring 12 cm by 5 cm.

2. How many squares of side 2 cm are required to cover a square of side 8 cm?

3. How many squares of side 5 cm are required to cover a rectangle measurig 35 cm by 20 cm?

4. A school's cafeteria measuring 30 m by 20 m is to be covered with square floor tiles of side 50 cm. How many tiles are needed?

The Rhombus

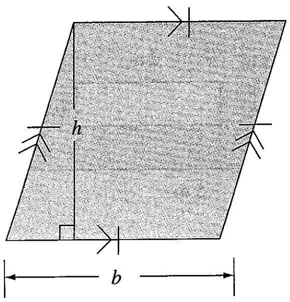


Fig. 4.35 Rhombus

The *area of a rhombus*, $A = bh$

and the *perimeter of a rhombus*, $P = 4b$,

where b = the base of the *rhombus*

and h = the *altitude* (or *perpendicular height*) of the *rhombus*.

The Parallelogram

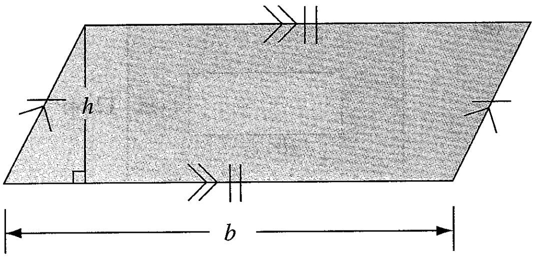


Fig. 4.36 Parallelogram

The *area of a parallelogram*, $A = bh$,

where b = the *base* of the *parallelogram*

and h = the *altitude* of the *parallelogram*.

The Trapezium

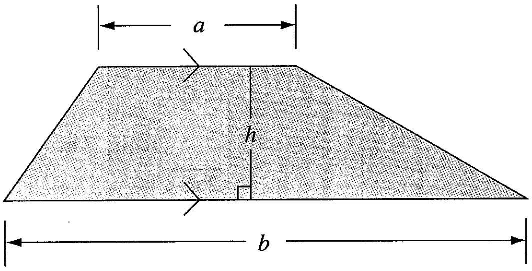


Fig. 4.37 Parallelogram

The *area of a trapezium*, $A = \frac{1}{2}(a + b)h$,

where a = the *length* of *one parallel* side of the *trapezium*,

b = the *length* of the *second parallel* side of the *trapezium*

and h = the *altitude* of the *trapezium*. That is, the *perpendicular distance* between the *two parallel sides* of the *trapezium*.

Example 9

(a)

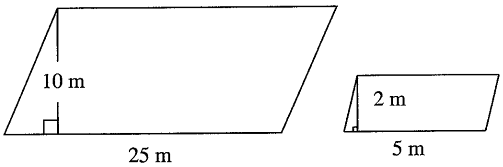


Fig. 4.38 Field Turf

A field is in the shape of a parallelogram with dimensions as shown in Fig. 4.38. The field is to be covered with turfs in the shape of a rhombus with dimensions as given in Fig. 4.38. Calculate the number of turfs needed to cover the field without cutting.

(b)

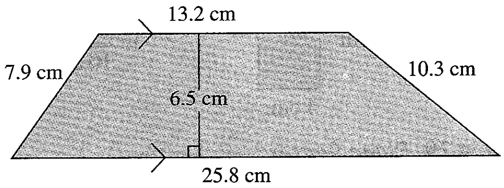


Fig. 4.39 Girder

Calculate the area of the steel girder in the shape of a trapezium with the dimensions given in the diagram above.

Solution

(a)

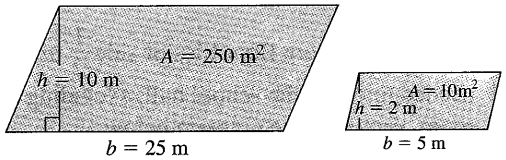


Fig. 4.38 Parallelogram Rhombus

The *area* of the *field* in the shape of a *parallelogram*, $A = bh$

$= 25 \text{ m} \times 10 \text{ m}$

$= 250 \text{ m}^2$

The *area* of a *turf* in the shape of a *rhombus*, $A = bh$

(b)

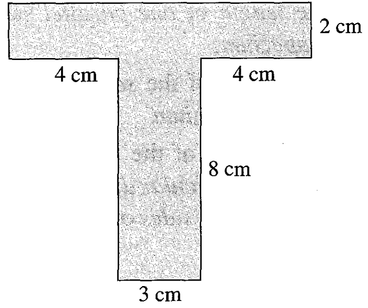


Fig. 4.27 Compound figures

21. Calculate the area of the compound figure:

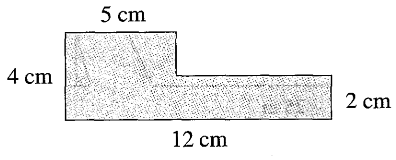


Fig. 4.28 Compound figure

22. Evaluate the area that is shaded in each of the following plane figures:

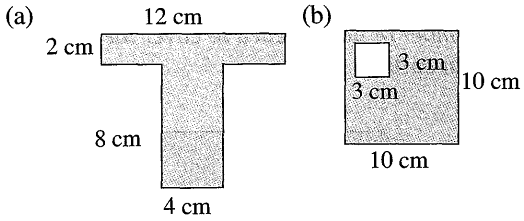


Fig. 4.29 Plane figures

23.

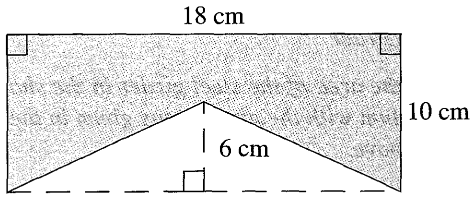


Fig. 4.30 Plane figure

Calculate the area of the shaded figure above.

24. How many square floor tiles of side $\frac{1}{2}$ m are needed to cover the school hall, excluding the room, shown in the diagram below.

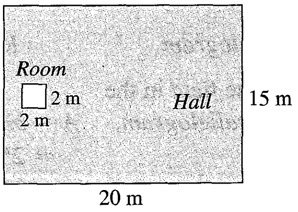


Fig. 4.31 Plane figure

25. In the figure given, calculate the area that is shaded.

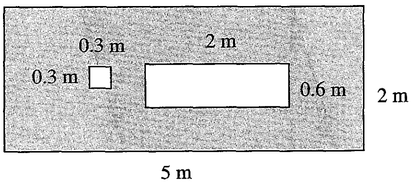


Fig. 4.32 Plane figure

26. Determine the area of the shaded region:

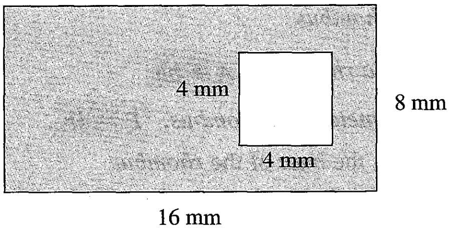


Fig. 4.33 Plane figure

27. Evaluate the area that is shaded in each of the following figures:

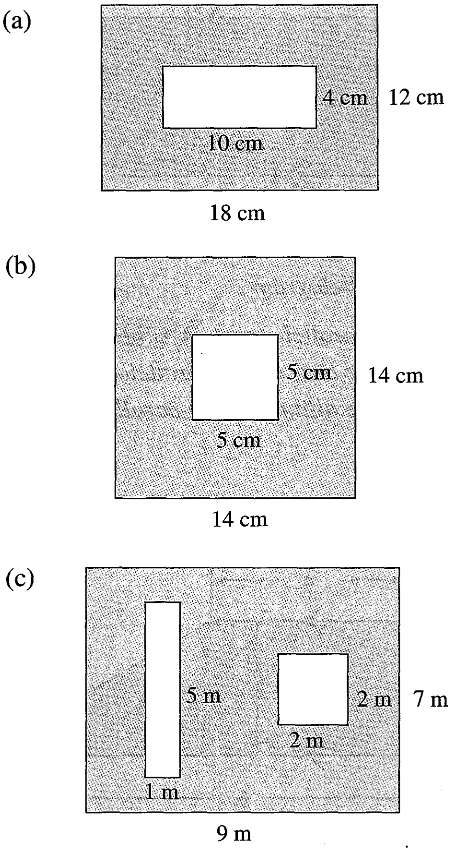
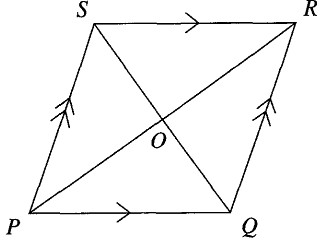


Fig. 4.34 Plane figures

8.



$RO = 6.4$ cm and
$SO = 4.2$ cm.

Fig. 4.47 Rhombus

Calculate the area of each of the following parallelograms:

9.

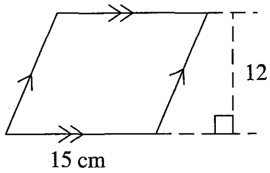


Fig. 4.48 Rhombus

10.

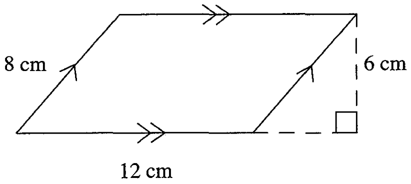


Fig. 4.49 Parallelogram

11.

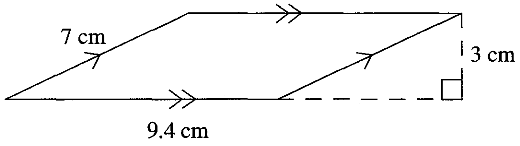


Fig. 4.50 Parallelogram

12.

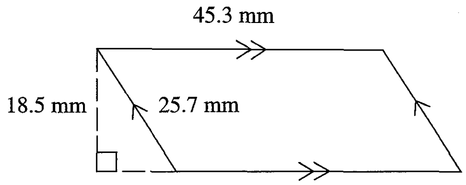


Fig. 4.51 Parallelogram

13. Determine the missing measurements for the given shape:

Table 4.8

	Shape	*Area*	*Base*	*Altitude*
(a)	Parallelogram	26 cm^2	4 cm	
(b)		36 cm^2		7.2 cm

14. Calculate the missing measurements in the table below:

Table 4.9

	Shape	*Area*	*Base*	*Altitude*
(a)	Parallelogram	74.1 mm^2		7.8 mm
(b)		44.2 mm^2	8.5 mm	

15. The area of a parallelogram is 72 cm^2 and its base is 12 cm. Determine its altitude.

16. Determine the base of a parallelogram of area 58.9 mm^2 and altitude 6.2 mm.

17.

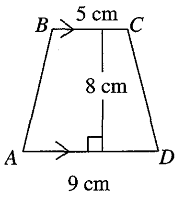


Fig. 4.52 Trapezium

The figure above represents a trapezium of altitude 8 cm. Given that $AD = 9$ cm and $BC = 5$ cm, calculate the area of the trapezium.

Calculate the area of each of the following trapeziums:

18.

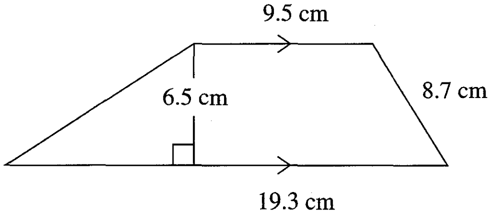


Fig. 4.53 Trapezium

19.

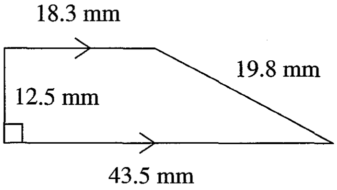


Fig. 4.54 Trapezium

20.

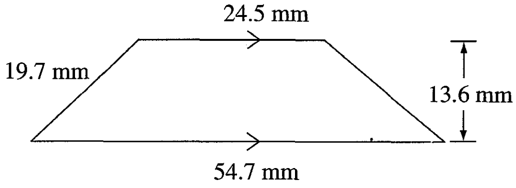


Fig. 4.55 Trapezium

$= 5\text{ m} \times 2\text{ m}$

$= 10\text{ m}^2$

∴ the *number* of *turfs* needed to cover the field $= \frac{250\cancel{\text{m}^2}}{10\cancel{\text{m}^2}}$

$= 25$ turfs

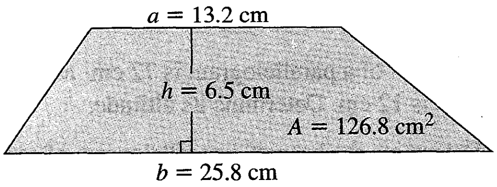


Fig. 4.39 Trapezium

The *area* of the *steel girder* in the shape of a *trapezium*,

$$A = \frac{1}{2}(a + b)h$$
$$= \frac{1}{2}(13.2 + 25.8)\text{ cm} \times 6.5\text{ cm}$$
$$= \frac{1}{2} \times 39\text{ cm} \times 6.5\text{ cm}$$
$$= 19.5\text{ cm} \times 6.5\text{ cm}$$
$$= 126.75\text{ cm}^2$$
$$= 126.8\text{ cm}^2 \text{ (\textit{correct} to 1 \textit{d.p.})}$$

Exercise 4i

Calculate the area and perimeter of each of the following rhombuses:

1.

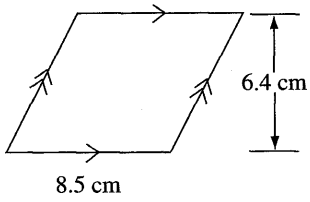


Fig. 4.40 Rhombus

2.

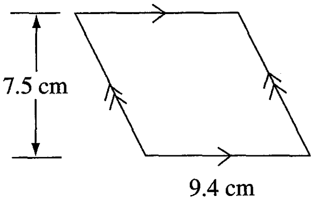


Fig. 4.41 Rhombus

3.

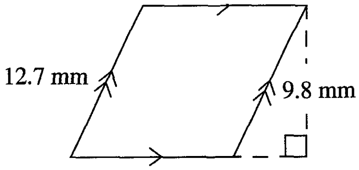


Fig. 4.42 Rhombus

4.

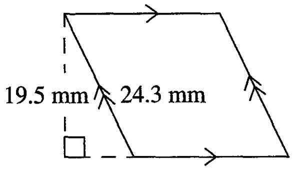


Fig. 4.43 Rhombus

Evaluate the area of each of the following rhombuses:

5.

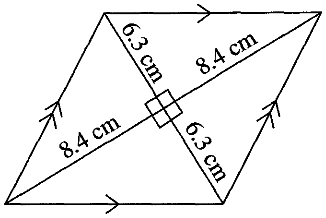


Fig. 4.44 Rhombus

6.

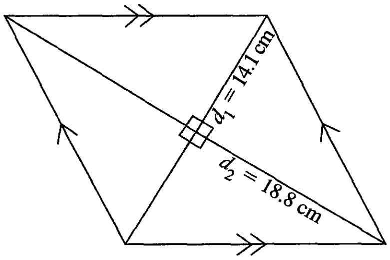
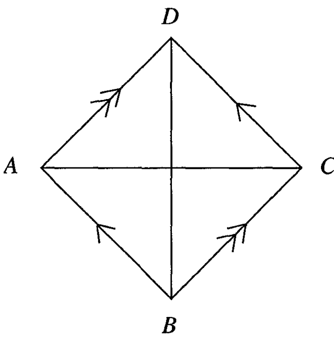


Fig. 4.45 Rhombus

7.



$AC = 10$ cm and
$BD = 13$ cm.

Fig. 4.46 Rhombus

Example 10

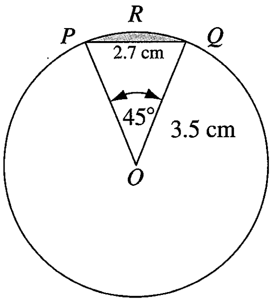


Fig. 4.61 Circle

The diagram above shows a circle centre O and radius 3.5 cm. PQ is a chord of the circle of length 2.7 cm and angle POQ = 45°.

(a) Calculate the area of:
- *(i) the circle*
- *(ii) the minor sector POQR*
- *(iii) the triangle POQ*
- *(iv) the minor segment PQR.*

(b) Evaluate:
- *(i) the circumference of the circle*
- *(ii) the length of the minor arc PQ.*

(Take π as 3.142)

Solution

(a) (i)

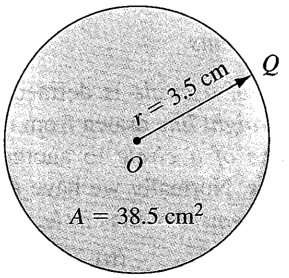


Fig. 4.62 Circle

The *area* of the *circle*,

$$\begin{aligned} A &= \pi r^2 \\ &= 3.142 \times (3.5 \text{ cm})^2 \\ &= 3.142 \times 3.5 \text{ cm} \times 3.5 \text{ cm} \\ &= 38.4895 \text{ cm}^2 \\ &= 38.5 \text{ cm}^2 \text{ (\textit{correct} to 1 \textit{d.p.})} \end{aligned}$$

(ii)

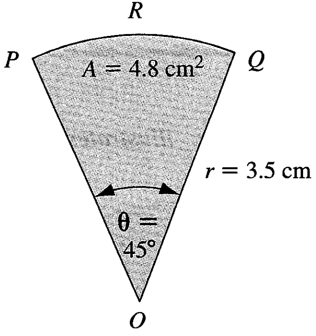


Fig. 4.63 Minor sector of a circle

The *area* of the *minor sector POQR*,

$$\begin{aligned} A &= \pi r^2 \frac{\theta}{360} \\ &= 3.142 \times (3.5 \text{ cm})^2 \times \frac{45}{360} \\ &= 38.4895 \text{ cm}^2 \times \frac{1}{8} \\ &= 4.81 \text{ cm}^2 \text{ (\textit{correct} to 1 \textit{d.p.})} \end{aligned}$$

(iii)

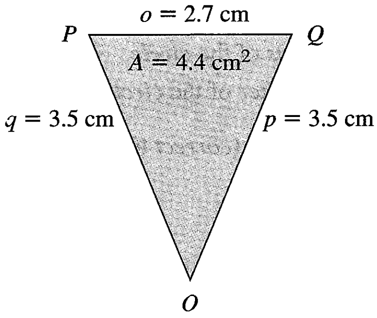


Fig. 4.64 Isosceles triangle

The *semi-perimeter* of the *triangle POQ*,

$$\begin{aligned} s &= \frac{p + o + q}{2} \\ &= \frac{(3.5 + 2.7 + 3.5) \text{ cm}}{2} \\ &= \frac{9.7 \text{ cm}}{2} \\ &= 4.85 \text{ cm} \end{aligned}$$

The *area* of the *triangle POQ*,

$$\begin{aligned} A &= \sqrt{s(s-p)(s-o)(s-q)} \\ &= \sqrt{4.85(4.85 - 3.5)(4.85 - 2.7)(4.85 - 3.5) \text{ cm}^4} \\ &= \sqrt{4.85(1.35)(2.15)(1.35) \text{ cm}^4} \\ &= \sqrt{19.004 \text{ cm}^4} \\ &= 4.359 \text{ cm}^2 \\ &= 4.4 \text{ cm}^2 \text{ (\textit{correct} to 1 \textit{d.p.})} \end{aligned}$$

The diagram below *illustrates* a *circle* with centre O.

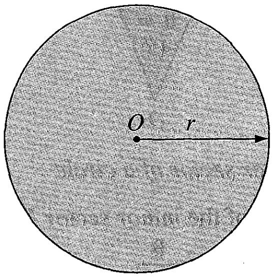


Fig. 4.56 Circle

The *area of a circle*, $A = \pi r^2 = \frac{1}{4}\pi d^2$

and the *circumference of a circle*, $C = 2\pi r = \pi d$,

where r = the *radius* of the *circle*,

d = the *diameter* of the *circle*

and $\pi = \frac{22}{7} = 3.142$ (*correct* to 3 *d.p.*).

The Sector of a Circle

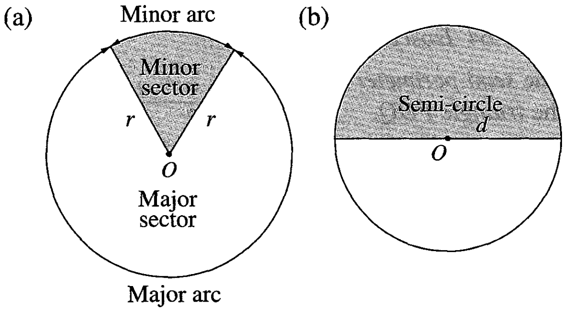


Fig. 4.57 Sectors

The *sector of a circle* is defined by *two radii*. Normally we have a *minor sector* and a *major sector* of a *circle*, except when *both sectors* are *semi-circles*.

We also have a *minor arc* and a *major arc* being defined at the *circumference of the circle*.

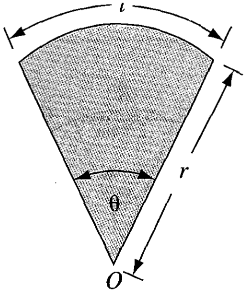


Fig. 4.58 Sector of a circle

The *area of the sector of a circle*, $A = \pi r^2 \frac{\theta}{360}$,

and the *length of the arc*, $l = 2\pi r \frac{\theta}{360}$,

where r = the *radius* of the *circle*

and θ = the *sector angle* in *degrees*.

The Segment of a Circle

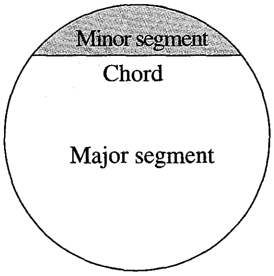


Fig. 4.59 Segments

The *segment of a circle* is defined by a *chord*. A *chord* is a *straight line* drawn from *one point* on the *circumference of a circle* to another *point* on the *circumference*. Normally we have a *minor segment* and a *major segment* of the *circle*.

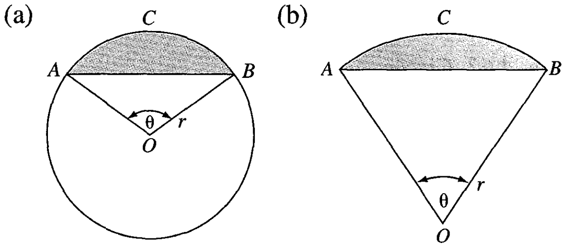


Fig. 4.60 Segments of a circle

The area of the segment ABC = *The area of the sector AOBC* − *The area of the triangle AOB*

Exercise 4j

1.

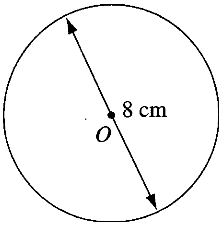


Fig. 4.70 Circle

Using the diagram above:

(a) State the radius of the circle.
(b) Determine the circumference of the circle.
(c) Calculate the area of the circle.
Take π as 3.14

2. Evaluate the radius of a circle with circumference 62.8 cm.
Take π as 3.14

3.

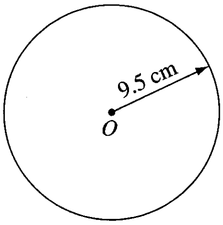


Fig. 4.71 Circle

Calculate the circumference and the area of the circle with radius 9.5 cm. State your answers correct to one decimal place.
Take π as 3.142

4. Calculate the circumference and the area of a silver dollar with diameter 4 cm. State your answers correct to three significant figures.
Use π as 3.14

5. Determine the radius of a disc of area 38.5 cm^2.
Use π as $\frac{22}{7}$

6.

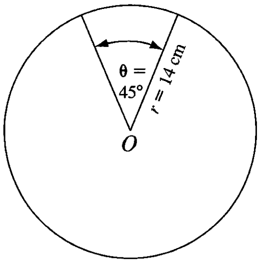


Fig. 4.72 Circle

The previous diagram shows a circle of radius 14 cm with a sector angle of 45°.
Calculate:
(i) the length of the minor arc
(ii) the area of the minor sector.

Take π as $\frac{22}{7}$

7. (a) Calculate the perimeter of the following shape.
(b) Evaluate the area of the following shape.

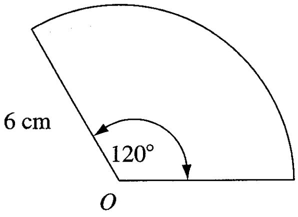


Fig. 4.73 Sector of a circle

Use π as 3.14 and state your answers correct to three significant figures.

8. How far does a wheel of radius 28 cm travel in one revolution?
Take π as 3.14. Give your answer correct to three significant figures.

9. The hour hand on a clock is 12 cm long. What area does it pass over in 5 h?
Take π as 3.14. Give your answer correct to three significant figures.

10. Calculate the perimeter of the sector shown below:

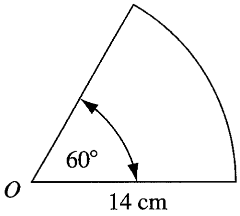


Fig. 4.74 Sector of a circle

11. What distance, in cm, does a wheel of radius 21 cm travel in one complete revolution? How many times will the wheel turn when the bicycle travels a distance of 528 cm.

Use $\pi = \frac{22}{7}$

(iv)

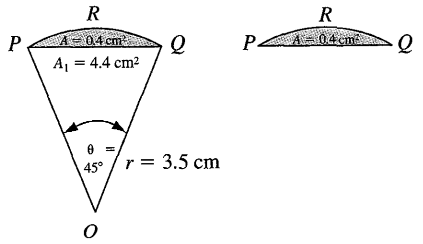


Fig. 4.65 Minor segment of a circle

The *area* of the *minor segment PQR,*

A = *The area of the minor sector POQR − The area of the triangle POQ*

$= (4.8 - 4.4)\ \text{cm}^2$

$= 0.4\ \text{cm}^2$

(b) (i)

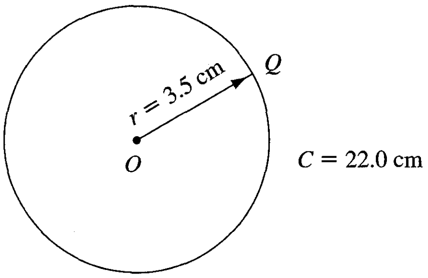


Fig. 4.66 Circle

The *circumference* of the *circle,*

$C = 2\pi r$

$= 2 \times 3.142 \times 3.5\ \text{cm}$

$= 21.994\ \text{cm}$

$= 22.0\ \text{cm}$ (*correct* to 1 *d.p.*)

(ii)

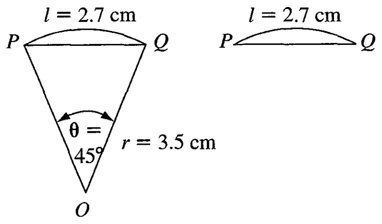


Fig. 4.67 Arc of a circle

The *length* of the *minor arc PQ,*

$$l = 2\pi r \frac{\theta}{360}$$

$$= 2 \times 3.142 \times 3.5\ \text{cm} \times \frac{45}{360}$$

$$= 21.994\ \text{cm} \times \frac{1}{8}$$

$$= 2.749\ \text{cm}$$

$= 2.7\ \text{cm}$ (*correct* to 1 *d.p.*)

Example 11

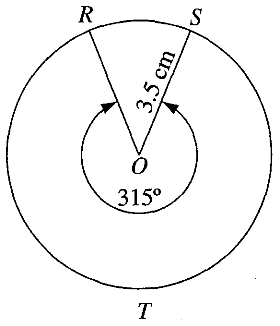


Fig. 4.68 Circle

The diagram above shows a circle centre O and radius 3.5 cm. The major sector angle ROS = 315°.

(a) Calculate the area of the major sector ROST.

(b) Determine the length of the major arc RS. Take π as 3.142

Solution

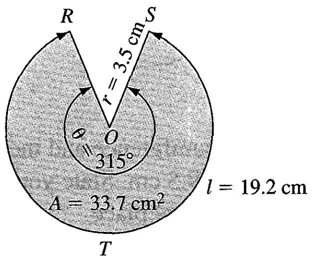


Fig. 4.69 Major sector of a circle

(a) The *area* of the *major sector ROST,*

$$A = \pi r^2 \frac{\theta}{360}$$

$$= 3.142 \times (3.5\ \text{cm})^2 \times \frac{315}{360}$$

$$= 3.142 \times 12.25\ \text{cm}^2 \times \frac{7}{8}$$

$$= 33.678\ \text{cm}^2$$

$= 33.7\ \text{cm}^2$ (*correct* to 1 *d.p.*)

(b) The *length* of the *major arc RS,*

$$l = 2\pi r \frac{\theta}{360}$$

$$= 2 \times 3.142 \times 3.5\ \text{cm} \times \frac{315}{360}$$

$$= 6.284 \times 3.5\ \text{cm} \times \frac{7}{8}$$

$$= 19.24\ \text{cm}$$

$= 19.2\ \text{cm}$ (*correct* to 1 *d.p.*)

(b) Determine the length of:
   - (i) the circumference of the circle centre *O*.
   - (ii) the major arc *PQ*.

Use π as $\frac{22}{7}$ and state your answers correct to one decimal place.

20.

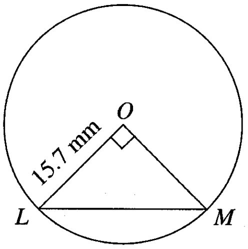


Fig. 4.81 Circle

In the diagram above, the radius of the circle centre *O* is 15.7 mm and the sector angle *LOM* is one right angle.

(a) Calculate the area of:
   - (i) the minor sector *LOM*
   - (ii) the triangle *LOM*
   - (iii) the minor segment bounded by the chord *LM* and the circumference of the circle.

(b) Evaluate the length of:
   - (i) the circumference of the circle centre *O*
   - (ii) the major arc *LM*.

Use π as 3.142 and state your answers correct to one decimal place.

21.

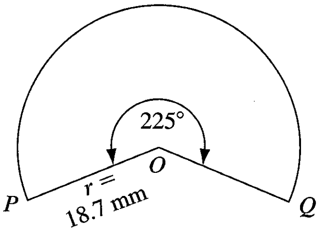


Fig. 4.82 Sector

Using the diagram of a sector given above:

(a) Calculate the area of the major sector *POQ*.

(b) Determine the length of the major arc *PQ*. Use π as 3.142 and state your answers correct to three significant figures.

22.

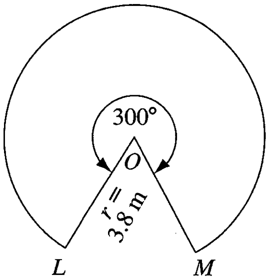


Fig. 4.83 Sector

Using the diagram of the sector of a circle given above:

(a) Calculate the length of the major arc *LM*.

(b) Evaluate the area of the major sector *LOM*.

Use π as 3.142 and state your answers correct to three significant figures.

Area and Perimeter of a Complex Compound Figure

The *method* of *calculating* the *area* and *perimeter* of a *complex compound figure* is illustrated in the example given below.

Example 12

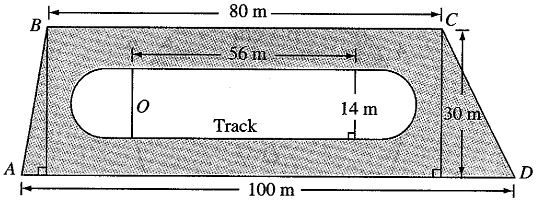


Fig. 4.84 Stadium

The diagram above shows a stadium in the shape of a trapezium with a central track consisting of two semicircular ends of diameter 14 m.

(a) Calculate the area of the shaded portion.

(b) Determine the distance around the track.

Take π as $\frac{22}{7}$

Solution

12. Calculate the radius of the sector shown below:

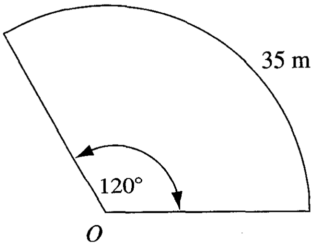


Fig. 4.75 Sector of a circle

13. The minute hand on a clock is 49 mm long. What area does it pass over in 30 min?

Take π as $\frac{22}{7}$

14.

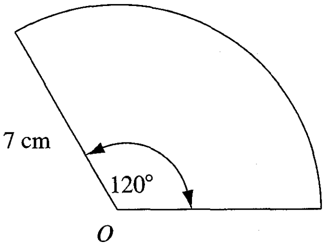


Fig. 4.76 Sector of a circle

(a) Evaluate the perimeter of the sector.

(b) Calculate the area of the sector.

Use π as 3.142

15. The hour hand on a clock is 14 cm long. What area does it pass over in 5 h?

Take π as 3.142

16.

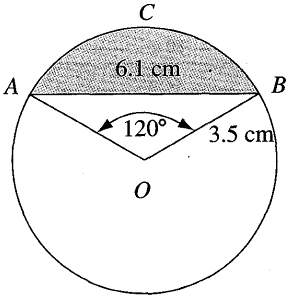


Fig. 4.77 Circle

For the figure above:

(a) Calculate the area of the circle, centre *O*.

(b) Calculate the area of the minor sector *AOBC*.

(c) Calculate the area of the triangle *AOB*.

(d) Determine the area of the segment *ABC*.

Take π as $\frac{22}{7}$

17.

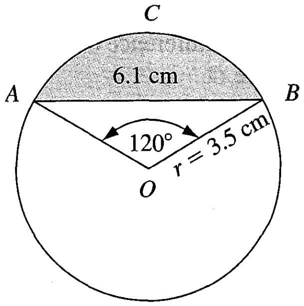


Fig. 4.78 Circle

For the figure above:

(a) Determine the area of the minor sector.

(b) Calculate the length of the minor arc *ACB*.

Take π as $\frac{22}{7}$

18.

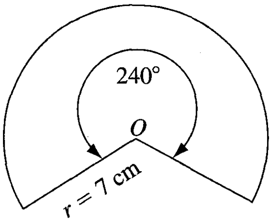


Fig. 4.79 Sector of a circle

Calculate:

(a) the area of the major sector of the circle shown in the diagram above

(b) the length of the major arc.

Use π as $\frac{22}{7}$

19.

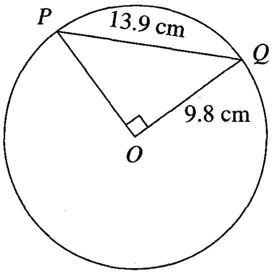


Fig. 4.80 Circle

The diagram above shows a circle centre *O* and radius 9.8 cm. The length of a chord *PQ* is 13.9 cm and the angle *POQ* is one right angle.

(a) Calculate the area of:

  (i) the minor sector *POQ*

  (ii) the triangle *POQ*

  (iii) the minor segment bounded by the chord *PQ* and the circumference of the circle.

The *length* of the *semi-circular* end,

$$\begin{aligned} l &= 2\pi r \frac{\theta}{360} \\ &= \pi d \times \frac{180}{360} \\ &= \pi d \times \frac{1}{2} \\ &= \frac{1}{2}\pi d \\ &= \frac{1}{2} \times \frac{22}{7} \times 14 \text{ m} \\ &= 11 \times 2 \text{ m} \\ &= 22 \text{ m} \end{aligned}$$

Hence the *distance* around the *track*,

$$\begin{aligned} P &= (56 + 22 + 56 + 22) \text{ m} \\ &= 156 \text{ m} \end{aligned}$$

Alternative Method

(b) The *two congruent semi-circular ends* of the track will form a *circle of diameter* 14 metres.

The *circumference* of the *circle*,

$$\begin{aligned} C &= 2\pi r \\ &= \pi d \\ &= \frac{22}{7} \times 14 \text{ m} \\ &= 22 \times 2 \text{ m} \\ &= 44 \text{ m} \end{aligned}$$

Hence the *distance* around the *track*,

$$\begin{aligned} P &= (56 + 56 + 44) \text{ m} \\ &= 156 \text{ m} \end{aligned}$$

Exercise 4k

1.

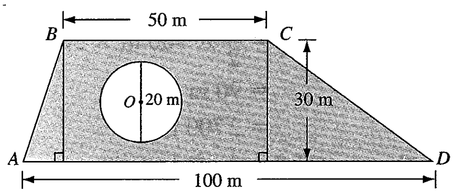


Fig. 4.87 Plane figure

Calculate the area of the shaded portion in the figure shown above correct to three significant figures.

Take π as 3.142

2.

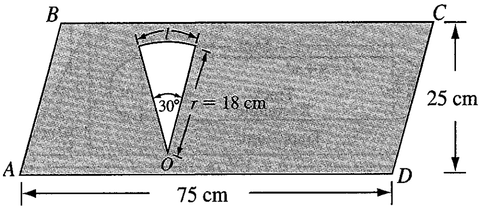


Fig. 4.88 Compound figure

Calculate the area of the shaded region in the diagram above.

Take π as $\frac{22}{7}$

3. The largest possible circle is cut from a sheet of square paper of length 14 cm. What area of paper, in cm², is left?

Use π as $\frac{22}{7}$

4. The shape in the diagram is made up of a semi-circle and a square. Evaluate the length of a side of the square.

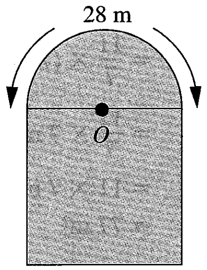


Fig. 4.89 Plane figure

5. A bicycle wheel has a circumference of 400 cm. What is the radius, in cm, of the wheel?

6. Calculate the area of the following ring:

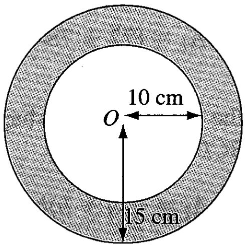


Fig. 4.90 Ring

7. The hour hand of a clock is 20 cm long. What amount of area does it pass over in 3 hours?

(a)

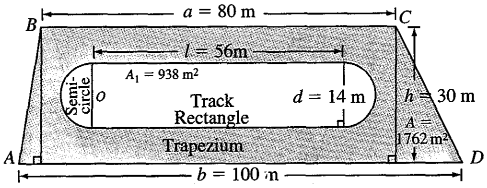


Fig. 4.85 Stadium

The *area* of one *semi-circular* end of the *track*,

$$\begin{aligned} A &= \pi r^2 \frac{\theta}{360} \\ &= \pi r^2 \times \frac{180}{360} \\ &= \pi r^2 \times \frac{1}{2} \\ &= \frac{1}{2}\pi r^2 \\ &= \frac{1}{2}\pi\left(\frac{d}{2}\right)^2 \\ &= \frac{1}{2} \times \frac{22}{7} \times \left(\frac{14\text{ m}}{2}\right)^2 \\ &= \frac{11}{7} \times (7\text{ m})^2 \\ &= \frac{11}{7} \times 7\text{ m} \times 7\text{ m} \\ &= 11 \times 7\text{ m}^2 \\ &= 77\text{ m}^2 \end{aligned}$$

The *area* of the *rectangular* portion of the *track*,

$$\begin{aligned} A &= lb \\ &= ld \\ &= 56\text{ m} \times 14\text{ m} \\ &= 784\text{ m}^2 \end{aligned}$$

∴ the *total area* of the *track*,

$$\begin{aligned} A_1 &= (77 + 784 + 77)\text{ m}^2 \\ &= 938\text{ m}^2 \end{aligned}$$

The *area* of the *stadium* in the shape of a *trapezium ABCD*,

$$\begin{aligned} A_2 &= \frac{1}{2}(a + b)h \\ &= \frac{1}{2}(80 + 100)\text{ m} \times 30\text{ m} \\ &= \frac{1}{2} \times 180\text{ m} \times 30\text{ m} \\ &= 90\text{ m} \times 30\text{ m} \\ &= 2700\text{ m}^2 \end{aligned}$$

Hence the *area* of the *shaded portion*,

$$\begin{aligned} A &= A_2 - A_1 \\ &= (2700 - 938)\text{ m}^2 \\ &= 1762\text{ m}^2 \end{aligned}$$

Alternative Method

(a) The *two congruent semi-circular ends* of the track will form a *circle of diameter* 14 metres.

The *area* of the *circle*,

$$\begin{aligned} A &= \pi r^2 \\ &= \pi\left(\frac{d}{2}\right)^2 \\ &= \frac{22}{7} \times \left(\frac{14\text{ m}}{2}\right)^2 \\ &= \frac{22}{7} \times (7\text{ m})^2 \\ &= \frac{22}{7} \times 7\text{ m} \times 7\text{ m} \\ &= 22 \times 7\text{ m}^2 \\ &= 154\text{ m}^2 \end{aligned}$$

The *area* of the *rectangular* portion of the *track*,

$$\begin{aligned} A &= lb \\ &= ld \\ &= 56\text{ m} \times 14\text{ m} \\ &= 784\text{ m}^2 \end{aligned}$$

∴ the *total area* of the *track*,

$$\begin{aligned} A_1 &= (154 + 784)\text{ m}^2 \\ &= 938\text{ m}^2 \end{aligned}$$

The *area* of the *stadium* in the shape of a *trapezium*,

$$\begin{aligned} A_2 &= \frac{1}{2}(a + b)h \\ &= \frac{1}{2}(80 + 100)\text{ m} \times 30\text{ m} \\ &= \frac{1}{2} \times 180\text{ m} \times 30\text{ m} \\ &= 90\text{ m} \times 30\text{ m} \\ &= 2700\text{ m}^2 \end{aligned}$$

Hence the *area* of the *shaded portion*,

$$\begin{aligned} A &= A_2 - A_1 \\ &= (2700 - 938)\text{ m}^2 \\ &= 1762\text{ m}^2 \end{aligned}$$

(b)

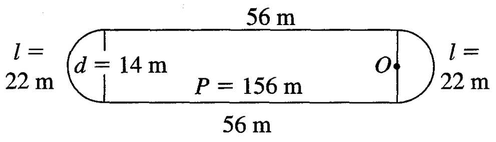


Fig. 4.86 Track

19. Determine the area of the shaded region in the following diagram.

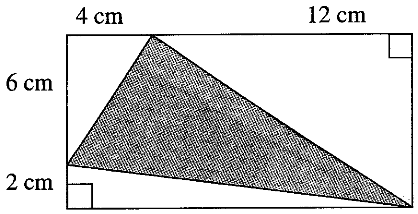


Fig. 4.99 Plane figure

Volume, Density and Surface Area of a Simple Right Solid

A *solid* is a *three-dimensional figure*. For example: a stone and a cube.

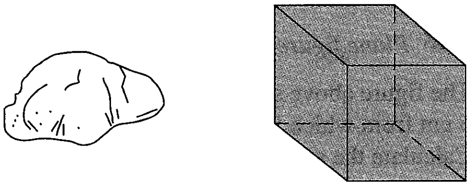


Fig. 4.100 (a) Stone (b) Cube

Simple right solids are basically *prisms* or *pyramids*. A *polyhedron* is a *solid shape* with many *flat faces*. The *flat faces* are all *polygons*. A *polygon* is a flat shape bounded by three or more straight edges. A right *prism* is defined as a *polyhedron* with the *same shape* along its entire *length*, and possessing two identical polygonal *cross-sections* (or *ends*), connecting the sides which are all rectangles. The angle between a polygonal *cross-section* (or *ends*) and a rectangular side is one right angle. Hence a right *prism* is said to be a *uniform solid*. For example: a *cube and* a *cuboid* are right prisms. A *cylinder*, although *not a prism*, is an *uniform solid* with a *circular cross-section.*

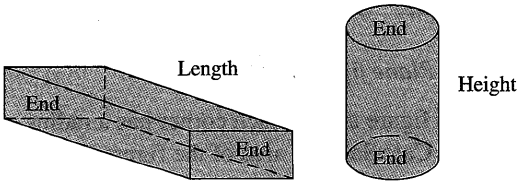


Fig. 4.101 (a) Cuboid (b) Cylinder

A *pyramid* is defined as a *polyhedron* which has a *polygon* as its *base* and all other *faces* meet at one *vertex* called the *apex*. Hence all the *faces* other than the base are *triangles*. A right pyramid is a pyramid for which the line joining the centre of its base to its apex is perpendicular to the base. For example: the *tetrahedron* (a *triangular-based pyramid*) and the *square-based pyramid.*

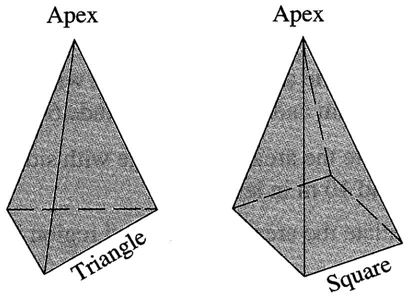


Fig. 4.102 (a) Tetrahedron (b) Square-based pyramid

The Density of a Solid

The *mass* of a *solid* is the *quantity of matter* that it *contains*. The *volume* of a *solid* is the *amount of space* that it *occupies*. The *density* of a *solid* is defined as its *mass per unit volume*. It is a *measure* of the '*lightness*' or '*heaviness*' of a *solid* for a given *volume*. For example: one cubic metre of gold is heavier than 1 cm^3 of oxygen, since the density of gold is 19.32 g/cm^3 and the density of oxygen is 1.14 g/cm^3.

The *density of a solid*, $\rho = \frac{m}{V}$,

where m = the *mass* of the *solid*

and V = the *volume* of the *solid.*

It follows that the *mass of the solid*, $m = \rho V$.

And the *volume of the solid*, $V = \frac{m}{\rho}$.

The Volume and Surface Area of a Right Uniform Solid

The *surfaces* of a *solid* can be *either plane* (*flat*) or *curved* or *both plane* (*flat*) and *curved.*

The *volume of a right uniform solid*, $V = Ah$,

where A = the *area* of the *cross-section* of the *right uniform solid*

and h = the *length, height* or *thickness* of the *right uniform solid.*

8. Determine the perimeter of the following model of a race track:

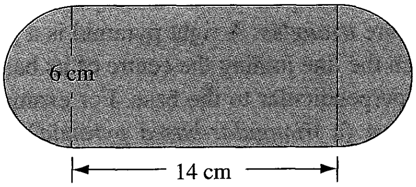


Fig. 4.91 Race track model

9. Calculate the area of a trapezium with parallel sides 12 cm and 16 cm and altitude 6 cm.

10. Calculate the area of a triangle with sides 6 m, 8 m and 10 m in length.

11. Calculate the area of the shaded region in the diagram below. $\left(\text{Take } \pi = \frac{22}{7}\right)$

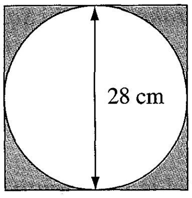


Fig. 4.92 Plane figure

12. Calculate the area of a sector with radius 7 m and sector angle 90°. $\left(\text{Take } \pi = \frac{22}{7}\right)$

13. Evaluate the area of the shaded portion in the diagram below. $\left(\text{Use } \pi \text{ as } \frac{22}{7}\right)$

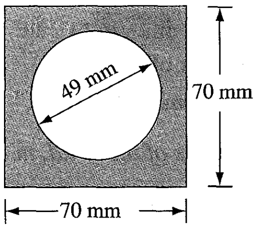


Fig. 4.93 Plane figure

14. Calculate the area of the shaded region in the diagram below. $\left(\text{Use } \pi = \frac{22}{7}\right)$

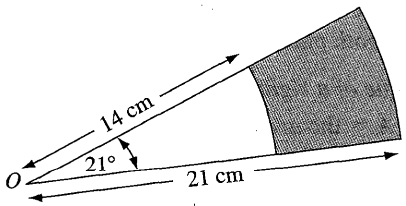


Fig. 4.94 Sector of a circle

15. Determine the area of the shaded portion in the following diagram. $\left(\text{Use } \pi = \frac{22}{7}\right)$

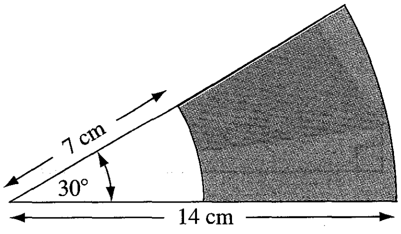


Fig. 4.95 Sector of a circle

16.

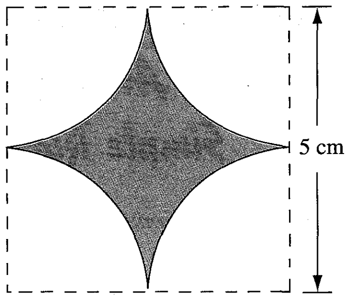


Fig. 4.96 Plane figure

The figure above shows a square of side 5 cm from which four quadrants are cut out. Calculate the area of the shaded region.

17.

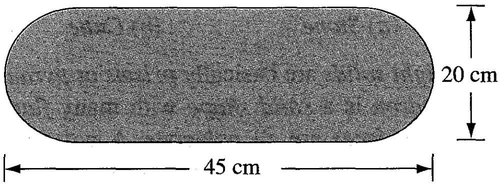


Fig. 4.97 Plane figure

In the figure above, each end consist of a semi-circle. Calculate the total surface area of the figure.

18.

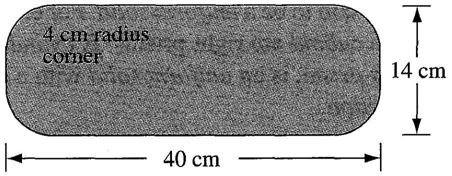


Fig. 4.98 Plane figure

In the figure above, each corner has a radius of 4 cm. Calculate the area of the figure.

The *volume of a triangular prism*, $V = Al$,

where l = the *length* of the *triangular prism*

and A = the *area* of *cross-section* of the *triangular prism*

= the *area* of a *triangle*.

So $A = \frac{1}{2}bh$

or $A = \sqrt{s(s-a)(s-b)(s-c)}$,

where a, b, c, h and s are the *dimensions defined* as in a *triangle*.

Example 14

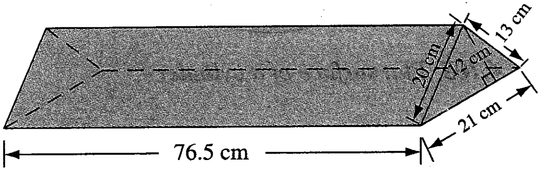


Fig. 4.108 Triangular prism or wedge

The diagram above represents a triangular prism, which is also called a wedge, with measurements as shown.

(a) Calculate:
  (i) the surface area, in cm², of the wedge.
  (ii) the volume, in cm³, of the wedge.

(b) If the wedge is made of iron of density 7.86 g cm⁻³, determine its mass in kilograms correct to three significant figures.

Solution

(a)

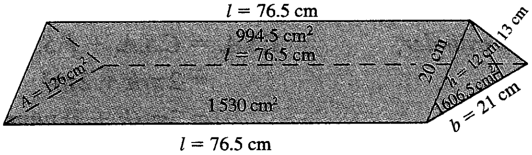


Fig. 4.108 Triangular prism or wedge

(i) The *area* of the *triangular cross-section* of the *wedge*,

$$A = \frac{1}{2}bh$$
$$= \frac{1}{2} \times 21 \text{ cm} \times 12 \text{ cm}$$
$$= 21 \text{ cm} \times 6 \text{ cm}$$
$$= 126 \text{ cm}^2$$

The *area* of the *rectangular base*,

$$A = lb$$
$$= 76.5 \text{ cm} \times 21 \text{ cm}$$
$$= 1606.5 \text{ cm}^2$$

The *area* of the *rectangular back*,

$$A = lb$$
$$= 76.5 \text{ cm} \times 13 \text{ cm}$$
$$= 994.5 \text{ cm}^2$$

The *area* of the *rectangular top*,

$$A = lb$$
$$= 76.5 \text{ cm} \times 20 \text{ cm}$$
$$= 1530 \text{ cm}^2$$

∴ the *surface area* of the *wedge*,

$$T.S.A. = (126 + 1606.5 + 994.5 + 1530 + 126) \text{ cm}^2$$
$$= 4383 \text{ cm}^2$$

(ii) The *volume* of the *wedge*,

$$V = Al$$
$$= 126 \text{ cm}^2 \times 76.5 \text{ cm}$$
$$= 9639 \text{ cm}^3$$

(b) The *mass* of the *iron wedge*,

$$m = \rho V$$
$$= 7.86 \text{ gcm}^{-3} \times 9639 \text{ cm}^3$$
$$= 75762.54 \text{ g}$$
$$= \frac{75762.54}{1000} \text{ kg}$$
$$= 75.76254 \text{ kg}$$
$$= 75.8 \text{ kg } (\textit{correct} \text{ to } 3 \textit{ s.f.})$$

Exercise 4I

1.

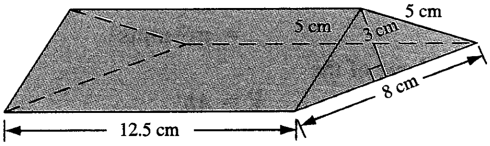


Fig. 4.109 Uniform solid

(a) Calculate the surface area and volume of the uniform solid shown with the given measurements in the diagram above.

(b) The uniform solid is made of zinc of density 7.13 g cm⁻³. Calculate the mass of the solid in grams.

The *total surface area of a right uniform solid*,

$$T.S.A. = C.S.A. + F.S.A. ,$$

where $C.S.A.$ = the *curved surface area* of the *right uniform solid*

and $F.S.A.$ = the *flat surface area* of the *right uniform solid.*

The *curved surface area*, $C.S.A. = Ph$,

where P = *the perimeter* of *cross-section* of the *right uniform solid.*

Example 13

(a) Calculate the volume of a cylinder of radius r units and altitude h units.

(b) Determine the curved surface area of the cylinder.

(c) Determine the flat surface area of the cylinder.

(d) Hence determine the total surface area of the cylinder.

Solution

(a)

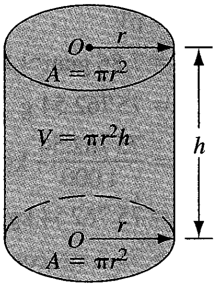


Fig. 4.103 Volume of a cylinder

The *area* of *cross-section* of the *cylinder*, A = The *area* of a *circle centre O, radius r units*

$= \pi r^2$ units²

∴ the *volume* of the *cylinder*, $V = Ah$

$= \pi r^2 h$ units³

(b)

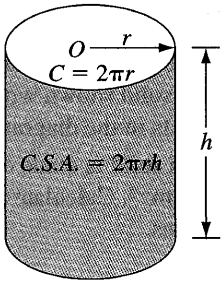


Fig. 4.104 Curved surface of a cylinder

The *perimeter* of *cross-section* of the *cylinder*, P = The *circumference* of *a circle centre O, radius r* units

$= 2\pi r$ units

∴ the *curved surface area of the cylinder*, $C.S.A. = Ph$

$= 2\pi rh$ units²

(c)

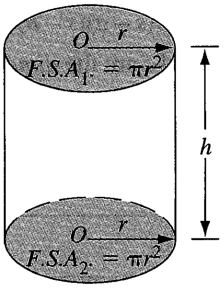


Fig. 4.105 Flat surface of a cylinder

The *area* of *cross-section* of the *cylinder*, $A = \pi r^2$ units²

∴ the *flat surface area* of the *cylinder*, $F.S.A. = 2A$

$= 2\pi r^2$ units²

(d)

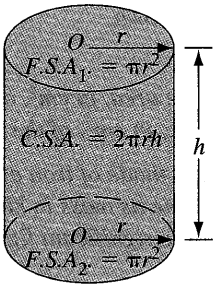


Fig. 4.106 Total surface area of a cylinder

The *total surface area* of the *cylinder*, $T.S.A. = C.S.A. + F.S.A.$

$= 2\pi rh + 2\pi r^2$

$= 2\pi r(h + r)$ units²

The Triangular Prism or Wedge

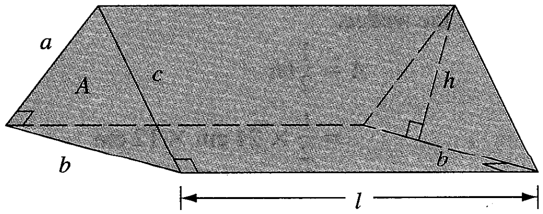


Fig. 4.107 Triangular prism or wedge

(b) the surface area, in cm^2, of the prism

(c) the volume, in cm^3, of the prism

(d) the size of angle *DBC*.

The Cube

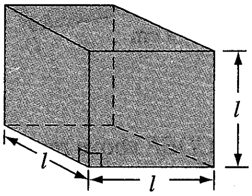


Fig. 4.116 Cube

The *volume of a cube*, $V = l^3$

and the *total surface area of a cube*, $T.S.A. = 6l^2$,

where l = the *length* of an *edge* of the *cube*.

The Cuboid

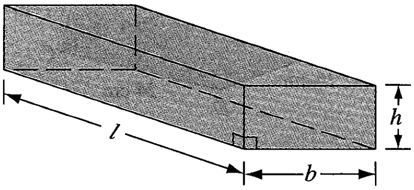


Fig. 4.117 Cuboid

The *volume of a cuboid*, $V = lbh$

and the *total surface area of a cuboid*,

$$\begin{aligned} T.S.A. &= 2bh + 2lb + 2lh \\ &= 2(bh + lb + lh), \end{aligned}$$

where l = the *length* of the *cuboid*,

b = the *breadth* of the *cuboid*

and h = the *height* of the *cuboid*.

Example 15

(a) A Rubic cube has a side of length 6 cm. How many cubes of side 1 cm can be filled in the same space as the Rubic cube?

(b) A cardboard box has dimensions 30 cm by 24 cm by 18 cm. What is the maximum number of Rubic cubes that can be packed for sale in each cardboard box?

Solution

(a)

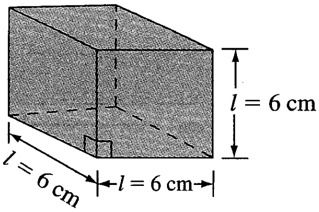


Fig. 4.118 Rubic cube

The *volume* of the *Rubic cube*, $V = l^3$

$$= (6 \text{ cm})^3$$
$$= 216 \text{ cm}^3$$

So 216 *cubes* of side 1 cm can be filled in the *same space* as the *Rubic cube*.

(b)

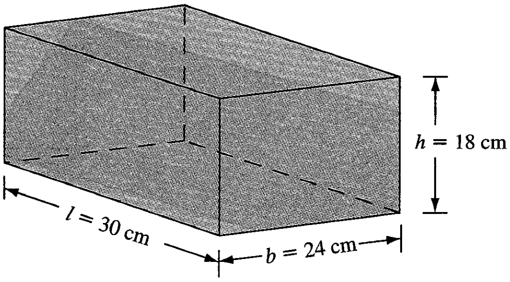


Fig. 4.119 Cardboard box

The *volume* of the *cardboard box* in the shape of a *cuboid*, $V = lbh$

$$= 30 \text{ cm} \times 24 \text{ cm} \times 18 \text{ cm}$$
$$= 12\,960 \text{ cm}^3$$

∴ the *number of Rubic cubes* that can be packed in each *cardboard box*

$$= \frac{12\,960 \cancel{\text{cm}^3}}{216 \cancel{\text{cm}^3}}$$
$$= 60 \text{ Rubic cubes}$$

Alternative Method

(b) The *number of Rubic cubes* that can be packed in each *cardboard box*

$$= \frac{30 \cancel{\text{cm}} \times 24 \cancel{\text{cm}} \times 18 \cancel{\text{cm}}}{6 \cancel{\text{cm}} \times 6 \cancel{\text{cm}} \times 6 \cancel{\text{cm}}}$$
$$= 5 \times 4 \times 3 \text{ Rubic cubes}$$
$$= 60 \text{ Rubic cubes}$$

Exercise 4m

1. (a) How many lead cubes of side 5 mm could be made from a rectangular block of lead measuring 10 cm by 5 cm by 4 cm.

2.

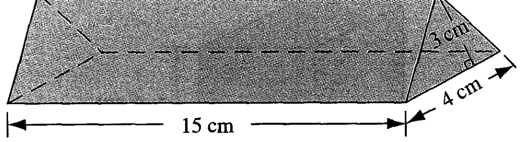


Fig. 4.110 Wedge

(a) Calculate the volume of the wedge shown with the given measurements in the diagram above.

(b) The wedge is made of nickel of density 8.90 g cm^{-3}. Determine the mass of the wedge in grams.

3. A prism of length 12 cm has a right-angled triangular end with edges 3 cm, 4 cm and 5 cm.

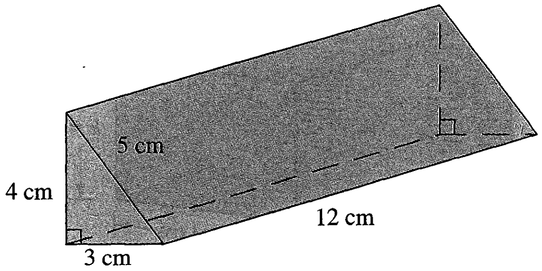


Fig. 4.111 Prism

(a) Determine the total surface area of the prism.

(b) Calculate the volume of the prism.

4.

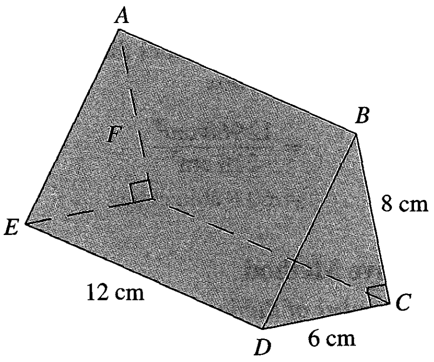


Fig. 4.112 Prism

For the prism above, calculate:

(a) the length, in cm, of *BD*

(b) the surface area, in cm^2, of the prism

(c) the volume, in cm^3, of the prism.

5.

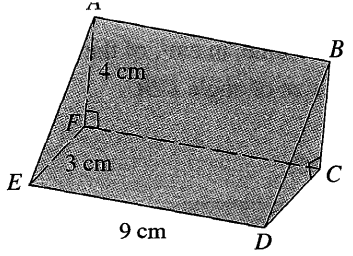


Fig. 4.113 Prism

The figure *ABCDEF* above represents a prism with measurements as shown. *BC* is perpendicular to the plane *FEDC*. Calculate:

(a) the length, in cm, of *BD*

(b) the surface area, in cm^2, of the prism

(c) the volume, in cm^3, of the prism

(d) the size of angle *DBC*.

6.

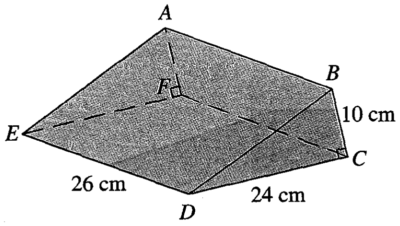


Fig. 4.114 Wedge

The figure *ABCDEF* above represents a wedge with measurements as shown. *BC* is perpendicular to the plane *FEDC*. Calculate:

(a) the length, in cm, of *BD*

(b) the surface area, in cm^2, of the wedge

(c) the volume, in cm^3, of the wedge

(d) the size of angle *BDC*.

7.

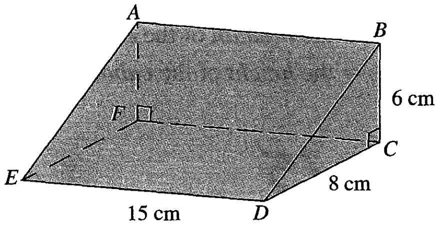


Fig. 4.115 Prism

The figure *ABCDEF* above represents a prism with measurements as shown. *BC* is perpendicular to the plane *FEDC*. Calculate:

(a) the length, in cm, of *BD*

The previous figure *ABCDEFGH* represents a cuboid with $AB = 120$ cm, $EH = 90$ cm and $AE = 45$ cm. M and N are the mid-points of AB and DC, respectively. Calculate the volume of the wedge *AMEDNH*.

The wedge is cut along *EMNH* and removed from the cuboid. Calculate the volume of the solid which remains.

The Cylinder

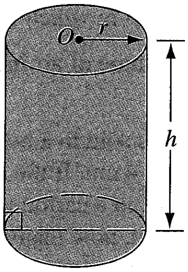


Fig. 4.123 Cylinder

The *volume of a cylinder*, $V = \pi r^2 h$

and the *total surface area of a cylinder*, $T.S.A. = 2\pi r(h + r)$,

where r = the *radius* of the *cross-section* of the *cylinder*

and h = the *altitude* of the *cylinder*.

Example 16

A cylindrical aluminium container of diameter 20 cm and altitude 28 cm has three-fifths of its volume filled with soft drink. How many ice-cubes of length 2 cm can be added to fill the space?

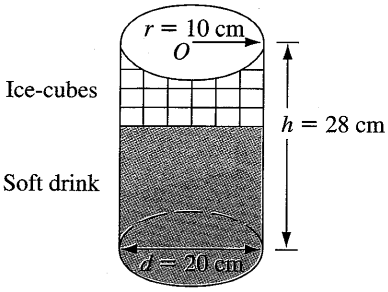


Fig. 4.124 Cylinder

Solution

The *volume* of the *cylindrical container*,

$$V = \pi r^2 h$$
$$= \pi\left(\frac{d}{2}\right)^2 h$$
$$= \frac{22}{7}\left(\frac{20 \text{ cm}}{2}\right)^2 \times 28 \text{ cm}$$
$$= 22 \times (10 \text{ cm})^2 \times 4 \text{ cm}$$
$$= 88 \times 100 \text{ cm}^3$$
$$= 8\,800 \text{ cm}^3$$

The *volume* of the *space* in the container (*two-fifths volume of container*)

$$= \frac{2}{5} \times 88 \times 100 \text{ cm}^3$$
$$= 176 \times 20 \text{ cm}^3$$
$$= 3\,520 \text{ cm}^3$$

The *volume* of *one ice-cube*,

$$V = l^3$$
$$= (2 \text{ cm})^3$$
$$= 8 \text{ cm}^3$$

∴ the *number* of ice-cubes needed to *fill* the *space*

$$= \frac{3\,520 \cancel{\text{cm}^3}}{8 \cancel{\text{cm}^3}}$$
$$= 440 \text{ ice-cubes}$$

Exercise 4n

Take $\pi = 3.142$ where necessary in the following problems and state your answers correct to one decimal place.

1. Calculate the volume of a coin of radius 1.5 cm and thickness 0.2 cm.
2. Calculate the volume of a cylinder of diameter 6 cm and height 10 cm.
3. Calculate the volume and curved surface area of the following uniform solid:

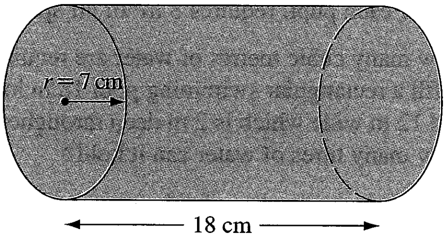


Fig. 4.125 Uniform solid

(b) The density of lead is 11.35 g/cm^3. Determine the mass of the rectangular block of lead.

2. Calculate the volume of:

(a) a concrete block measuring 38 cm by 18 cm by 14 cm.

(b) a car park which is 35 m long and 25 m wide by 12 m high.

(c) air in a room measuring 5 m by 8 m which is 4 m high.

3. A water storage tank is 4 m long, 3 m wide and 2 m deep. How many litres of water will it hold?

4. Determine the volume of each of the following cuboids, giving your answers in the units stated in brackets:

Table 4.10

	Length	*Breadth*	*Height*	*Unit*
(a)	60 mm	30 mm	20 mm	(cm^3)
(b)	500 cm	100 cm	40 cm	(m^3)
(c)	2 cm	50 cm	2 500 mm	(mm^3)

5. Evaluate the volume of air in a hall measuring 5 m by 7 m which is 3 m high.

6. A cologne is sold in a box measuring 8 cm by 5 cm by 4 cm. How many colognes may be packed in a carton measuring 64 cm by 40 cm by 32 cm.

7. A stadium has a section 15 m in length, 12 m in width and 3 m in height. How many people can it hold if each person requires 6 m^3 of air space?

8. A classroom is 12 m in length, 8 m in width and 4 m in height. How many pupils could it hold if each pupil requires 8 m^3 of air space?

9. How many cubic metres of water are required to fill a rectangular swimming pool 18 m long and 12 m wide which is 2 m deep throughout? How many litres of water can it hold?

10. How many lead cubes of side 3 cm could be made from a lead cube of side 27 cm?

11. How many lead cubes of side 5 mm could be made from a rectangular block of lead measuring 20 cm by 10 cm by 8 cm?

12.

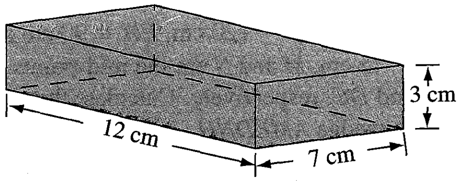


Fig. 4.120 Uniform solid

Calculate the volume and surface area of the uniform solid shown in the diagram above, with the stated measurements.

13. Calculate the volume of air in an apartment measuring 3 m by 5 m which is 4 m in height.

14. Determine the volume of air in a cube of side 9 cm.

15. A famous cereal is sold in a box measuring 10 cm by 5 cm by 4 cm. If the shopkeeper receives the boxes in a carton measuring 60 cm by 30 cm by 24 cm, how many boxes of cereal would be packed in a carton?

16. An apartment is 8 m in length, 5 m in width and 3 m in height. How many people can be invited to a party if each person requires 5 m^3 of air space?

17. A rectangular metal water tank is 5 m in length, 3 m in width and 2 m in depth. Calculate its capacity in

(a) cm^3 (b) m^3 (c) litres

18. How many cubic metres of water are required to fill a rectangular swimming pool measuring 15 m by 12 m which is 2 m deep?

19. Calculate the volume of the following cuboid.

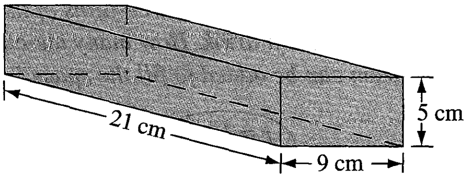


Fig. 4.121 Cuboid

20.

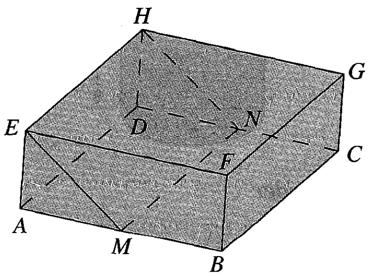


Fig. 4.122 Cuboid

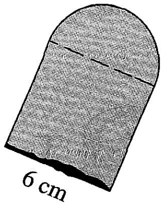


Fig. 4.132 Loaf of bread

11. Evaluate the volume of a cuboid of length 10.5 cm, width 8 cm and height 3.2 cm.

12. Determine the volume of each of the following solids:

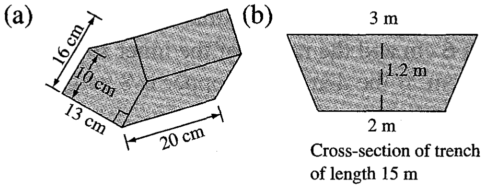


Fig. 4.133 House model Trench

13. The area of the cross-section of the log model shown is 54.3 cm² and its length is 48.1 cm. Evaluate its volume.

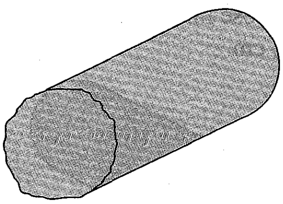


Fig. 4.134 Log model

14.

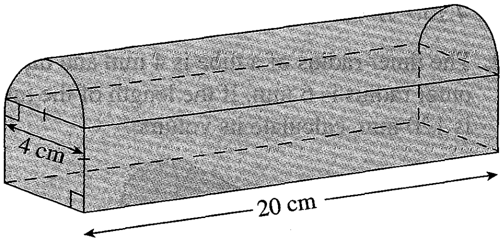


Fig. 4.135 Loaf of bread

A model of a loaf of bread is shown in the diagram above. Calculate the volume of the model.

15. Determine the volume of a cube of height 6 cm.

16. Calculate the volume of a cuboid of length 9 cm, width 5 cm and height 4 cm.

17. Determine the volume of each of the uniform solids whose cross-sections are shown below:

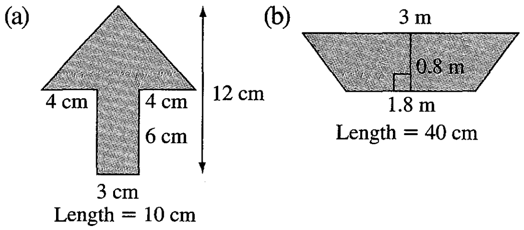


Fig. 4.136 Plane figures

(c) The area of the cross-section of the given solid is 32 cm². Calculate the volume of the solid.

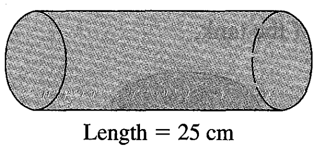


Fig. 4.137 Uniform solid

18.

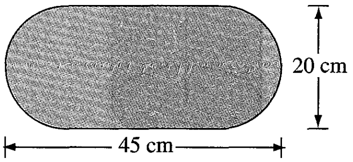


Fig. 4.138 Plane figure

In the figure above, each end consist of a semi-circle. Calculate the total surface area of the figure. If the figure is the base of an aquarium of vertical height 25 cm, calculate the volume of the aquarium.

19.

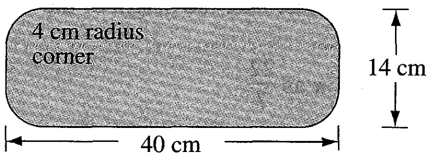


Fig. 4.139 Plane figure

In the figure above, each corner has a radius of 4 cm. Calculate the area of the figure. If the figure is the base of an aquarium of vertical height 30 cm, calculate the volume of the aquarium.

20.

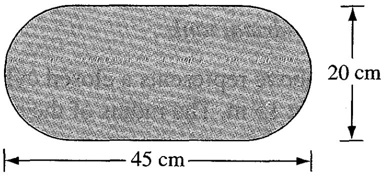


Fig. 4.140 Plane figure

4.

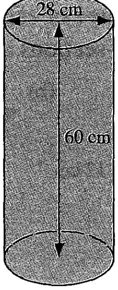


Fig. 4.126 Cylindrical water tank

The figure above, not drawn to scale, represents a water tank in the shape of a cylinder of height 60 cm and diameter 28 cm. Calculate the volume of the tank.

5.

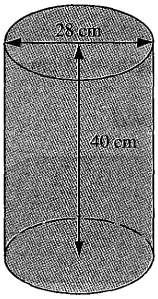


Fig. 4.127 Cylindrical water tank

The figure above, not drawn to scale, represents a water tank in the shape of a cylinder of height 40 cm and diameter 28 cm.

(a) Calculate the volume of the tank.

(b) If water of volume 18.48 *l* is present in the tank, calculate the height of the water in the tank.

Use π as $\frac{22}{7}$

6.

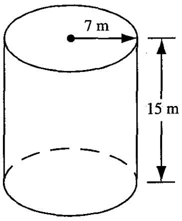


Fig. 4.128 Cylindrical tank

The figure above represents a closed cylindrical tank of height 15 m. The radius of the cross-section is 7 m.

Calculate:

(a) the curved surface area of the tank

(b) the volume of the tank.

Use $\pi = \frac{22}{7}$

7.

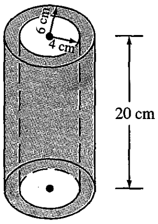


Fig. 4.129 Copper tubing

The diagram above shows a copper tubing of length 20 cm. The radius of the outer rim is 6 cm and the radius of the inner rim is 4 cm. If the density of copper is 8.94 g/cm^3, calculate:

(a) the volume of material used to make the copper tubing

(b) the mass of the copper tubing

(c) the curved surface area of the copper tubing.

8. Determine the volume of a hose of length 530 cm, whose inner radius is 5 cm and outer radius is 6 cm.

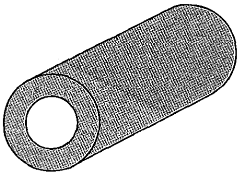


Fig. 4.130 Hose

9. The inner radius of a tube is 4 mm and the outer radius is 6 mm. If the length of the tube is 120 mm, calculate its volume.

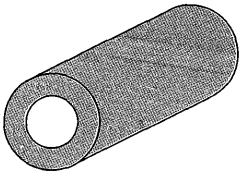


Fig. 4.131 Tube

10. Calculate the volume of a loaf of bread of length 36 cm, whose cross-section consists of a square of side 6 cm surmounted by a semi-circle.

(b) Given that the uniform solids are made of aluminium of density 2.70 g/cm³, calculate the mass of each of the solids above.

The Volume and Surface Area of a Right Pyramid

The *volume of a right pyramid,* $\boxed{V = \frac{1}{3}Ah}$,

where A = the *area* of the *base* of the *pyramid*

and h = the *altitude* (or *perpendicular height*) of the *pyramid.*

The total surface area of the pyramid, T.S.A. = *The area of the base of the pyramid* + *The sum of the areas of the triangles forming the faces of the pyramid*

It should be noted that the *volume of a cone* is found using the *formula* for the *volume of a pyramid.*

Example 17

Calculate the volume of a cone of radius r units and altitude h units.

Solution

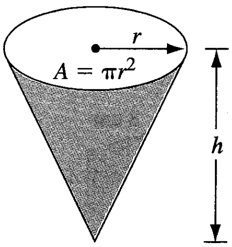


Fig. 4.145 Cone

The *volume of the cone,*

$$V = \frac{1}{3}Ah$$

$$= \frac{1}{3}\pi r^2 h \text{ units}^3.$$

The Tetrahedron

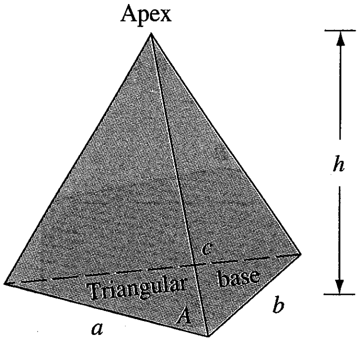


Fig. 4.146 Tetrahedron

The *volume of a tetrahedron,* $V = \frac{1}{3}Ah$,

where h = the *altitude* of the *tetrahedron*

and A = the *area* of the *base* of the *tetrahedron*

= the *area* of a *triangle.*

So $A = \frac{1}{2}bh$

or $A = \sqrt{s(s-a)(s-b)(s-c)}$,

where a, b, c, h and s are the *dimensions defined* as in a *triangle.*

The Square-based Pyramid

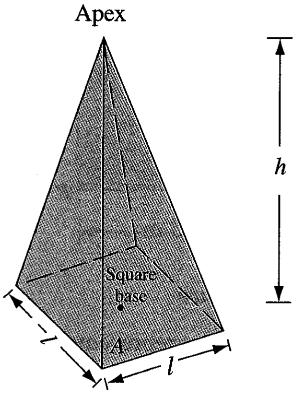


Fig. 4.147 Square-based pyramid

The *volume of a square-based pyramid,* $V = \frac{1}{3}l^2h$,

where l = the *length* of an *edge* of the *square base*

and h = the *altitude* of the *square-based pyramid.*

In the previous figure, each end consists of a semi-circle. Calculate the total surface area of the figure. If the figure is the base of an aquarium of altitude 25 cm, calculate the volume of the aquarium.

21.

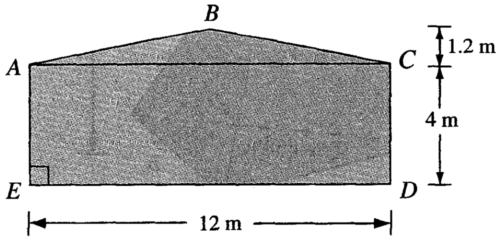


Fig. 4.141 Plane figure

The side of a house is shown in the diagram above. The side of the house is in the shape of a rectangle *ACDE* of width 12 m and height 4 m, and a triangle *ABC* of altitude 1.2 m.

(a) Calculate the total surface area of the side of the house *ABCDE*.

(b) If the house has a length of 25 m, calculate the volume of the house in cubic metres.

22.

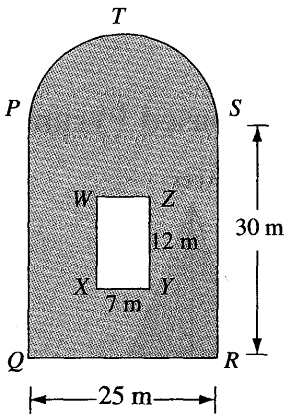


Fig. 4.142 Plane figure

The diagram above represents a plot of land *PQRST* in the shape of a rectangle of sides 30 m and 25 m, with a semicircle at one end.

(a) Calculate in metres, the perimeter of the land.

(b) *WXYZ* is a rectangular flower bed of length 12 m and width 7 m. Calculate in square metres, the area of the shaded region.

(c) The soil in the flower bed is replaced to a depth of 3 m. Calculate in cubic metres, the volume of the soil replaced.

23.

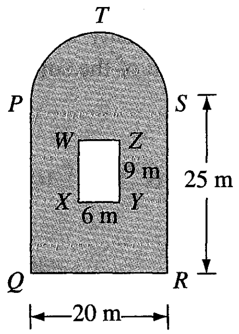


Fig. 4.143 Plane figure

The diagram above represents a plot of land *PQRST* in the shape of a rectangle of sides 25 m and 20 m, with a semicircle at one end.

(a) Calculate in metres, the perimeter of the land.

(b) *WXYZ* is a rectangular flower bed of length 9 m and width 6 m. Calculate in square metres, the area of the shaded region.

(c) The soil in the flower bed is replaced to a depth of 2 m. Calculate in cubic metres, the volume of the soil replaced.

24. (a) Calculate the volume for each of the following uniform solids with their given dimensions:

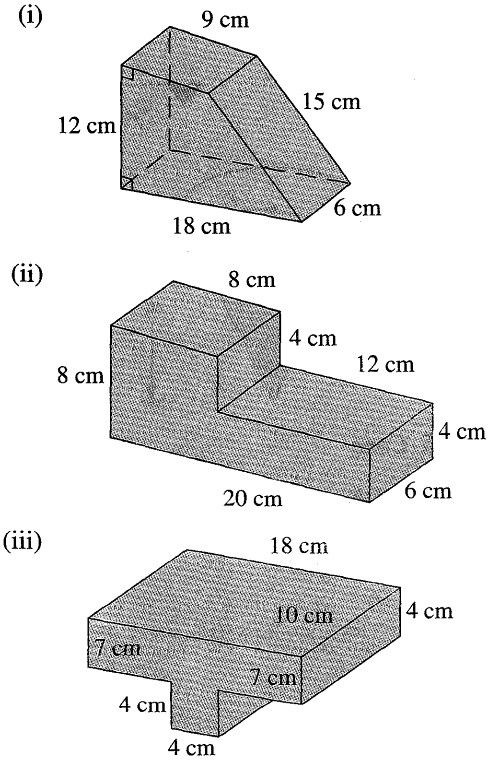


Fig. 4.144 Uniform solids

∴ the *volume* of the *tetrahedron*, $V = \frac{1}{3}Ah$

$$= \frac{1}{3} \times 132.3 \text{ cm}^2 \times 34.5 \text{ cm}$$
$$= 44.1 \text{ cm}^2 \times 34.5 \text{ cm}$$
$$= 1\,521.45 \text{ cm}^3$$
$$= 1\,521.5 \text{ cm}^3 \text{ (\textit{correct} to 1 \textit{d.p.})}$$

(b)

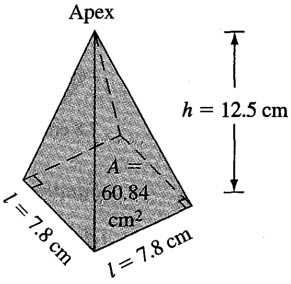


Fig. 4.150 Square-based pyramid

The *area* of the *base* of the *pyramid*, $A = l^2$

$$= (7.8 \text{ cm})^2$$
$$= 60.84 \text{ cm}^2$$

∴ the *volume* of the *square-based pyramid*,

$$V = \frac{1}{3}Ah$$
$$= \frac{1}{3} \times 60.84 \text{ cm} \times 12.5 \text{ cm}$$
$$= 20.28 \text{ cm}^2 \times 12.5 \text{ cm}$$
$$= 253.5 \text{ cm}^3$$

Alternative Method

The *volume* of the *square-based pyramid*,

$$V = \frac{1}{3}l^2h$$
$$= \frac{1}{3} \times (7.8 \text{ cm})^2 \times 12.5 \text{ cm}$$
$$= \frac{1}{3} \times 60.84 \text{ cm}^2 \times 12.5 \text{ cm}$$
$$= 20.28 \text{ cm}^2 \times 12.5 \text{ cm}$$
$$= 253.5 \text{ cm}^3$$

(c)

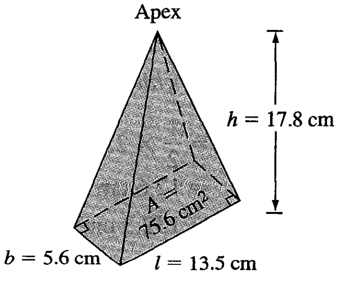


Fig. 4.150 Rectangular-based pyramid

The *area* of the *base* of the *pyramid*,

$$A = lb$$
$$= 13.5 \text{ cm} \times 5.6 \text{ cm}$$
$$= 75.6 \text{ cm}^2$$

∴ the *volume* of the *rectangular-based pyramid*,

$$V = \frac{1}{3}Ah$$
$$= \frac{1}{3} \times 75.6 \text{ cm}^2 \times 17.8 \text{ cm}$$
$$= 25.2 \text{ cm}^2 \times 17.8 \text{ cm}$$
$$= 448.56 \text{ cm}^3$$
$$= 448.6 \text{ cm}^3 \text{ (\textit{correct} to 1 \textit{d.p.})}$$

Alternative Method

The *volume* of the *rectangular-based pyramid*,

$$V = \frac{1}{3}lbh$$
$$= \frac{1}{3} \times 13.5 \text{ cm} \times 5.6 \text{ cm} \times 17.8 \text{ cm}$$
$$= 4.5 \text{ cm} \times 5.6 \text{ cm} \times 17.8 \text{ cm}$$
$$= 448.56 \text{ cm}^3$$
$$= 448.6 \text{ cm}^3 \text{ (\textit{correct} to 1 \textit{d.p.})}$$

(d)

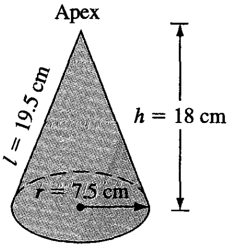


Fig. 4.150 Cone

The *volume* of the *cone*,

$$V = \frac{1}{3}\pi r^2h$$
$$= \frac{1}{3} \times 3.142 \times (7.5 \text{ cm})^2 \times 18 \text{ cm}$$
$$= 3.142 \times 56.25 \text{ cm}^2 \times 6 \text{ cm}$$
$$= 1\,060.425 \text{ cm}^3$$
$$= 1\,060.4 \text{ cm}^3 \text{ (\textit{correct} to 1 \textit{d.p.})}$$

(e) The *curved surface area* of the *cone*,

$$C.S.A. = \pi rl$$
$$= 3.142 \times 7.5 \text{ cm} \times 19.5 \text{ cm}$$
$$= 459.5175 \text{ cm}^2$$
$$= 459.5 \text{ cm}^2 \text{ (\textit{correct} to 1 \textit{d.p.})}$$

The Rectangular-based Pyramid

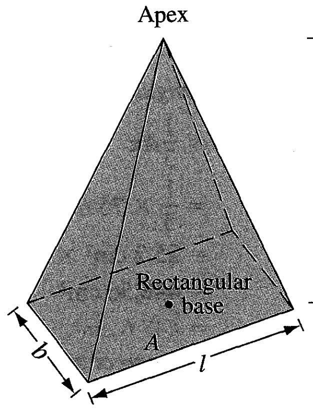


Fig. 4.148 Rectangular-based pyramid

The *volume of a rectangular-based pyramid,*

$$V = \frac{1}{3}lbh,$$

where l = the *length* of the *rectangular base,*
b = the *width* of the *rectangular base*
and h = the *altitude* of the *rectangular-based pyramid.*

The Cone

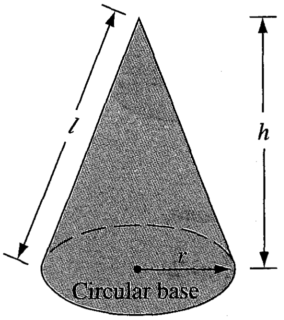


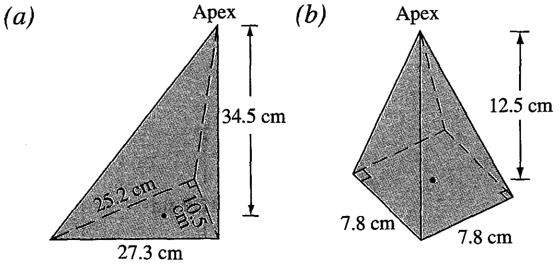
Fig. 4.149 Cone

The *volume of a cone,* $V = \frac{1}{3}\pi r^2 h$
and the *curved surface area of a cone,*
C.S.A. = $\pi r l$,

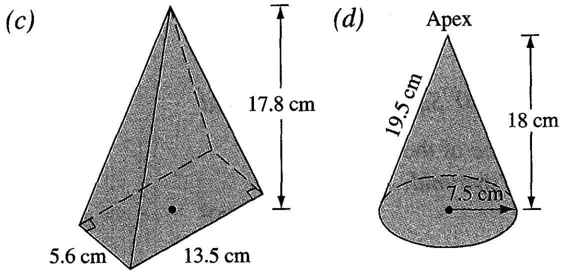
where r = the *radius* of the *circular-base,*
h = the *altitude* of the *cone*
and l = the *slant height* of the *cone.*

Example 18

Calculate the volume of each of the following pyramids and cone:



Triangular-based pyramid *Square-based pyramid*



Rectangular-based pyramid *Cone*

Fig. 4.150 Pyramids and cone

(e) Calculate the curved surface area of the cone.

Solution

(a)

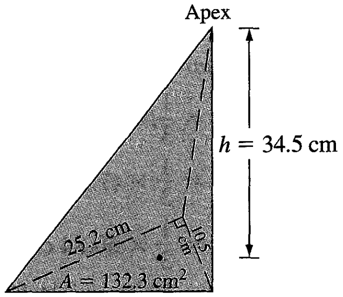


Fig. 4.150 Triangular-based pyramid

The *area* of the *base*
of the *tetrahedron,* $A = \frac{1}{2}bh$

$$= \frac{1}{2} \times 25.2 \text{ cm} \times 10.5 \text{ cm}$$
$$= 12.6 \text{ cm} \times 10.5 \text{ cm}$$
$$= 132.3 \text{ cm}^2$$

11.

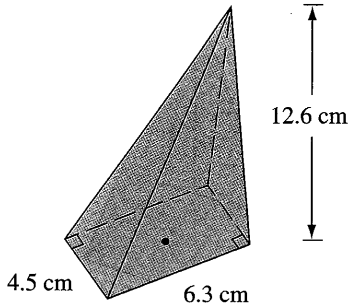


Fig. 4.155 Rectangular-based pyramid

(a) Calculate the volume of the rectagular-based pyramid shown above with the given measurements.

(b) The pyramid is made of platinum of density 21.45 g/cm³. Evaluate the mass of the rectangular-based pyramid.

12.

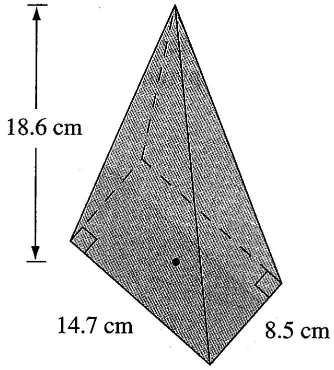


Fig. 4.156 Rectangular-based pyramid

The diagram above shows a rectangular-based pyramid of altitude 18.6 cm and dimensions 8.5 cm by 14.7 cm. The pyramid is made of lead of density 11.35 g/cm³.

(a) Calculate the volume of the pyramid.

(b) Evaluate the mass of the rectangular-based pyramid in kilograms correct to two significant figures.

13.

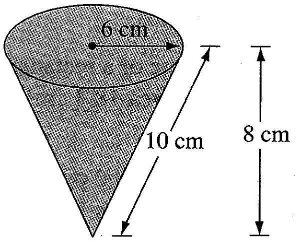


Fig. 4.157 Cone

(a) Calculate the volume of the cone with dimensions shown in the diagram above.

(b) Determine its surface area in cm².
Take π as 3.142

14.

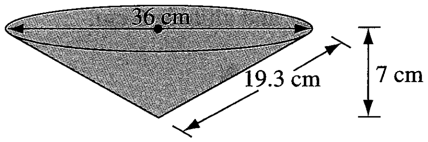


Fig. 4.158 Cone

The diagram above shows a cone of diameter 36 cm, slant height 19.3 cm and altitude 7 cm.

(a) Calculate the maximum volume of water that can be held by the cone.

(b) Evaluate the surface area of the cone.
Take π as 3.142

15.

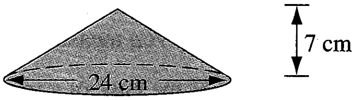


Fig. 4.159 Cone

The diagram above shows a cone of diameter 24 cm and altitude 7 cm. Calculate the maximum volume of sweet drink that can be placed in the cone.
Take π as 3.142

16.

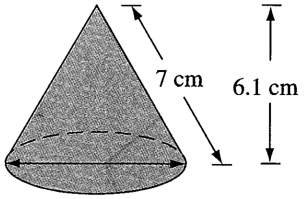


Fig. 4.160 Cone

The figure above consists of a cone of diameter 7 cm, slant height 7 cm and altitude 6.1 cm.

(a) Calculate the curved surface area of the cone.

(b) Evaluate the volume of the cone.
Take π as $\frac{22}{7}$

17.

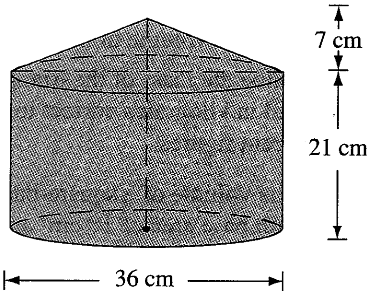


Fig. 4.161 Cone and cylinder

Exercise 4o

1. Calculate the volume of a tetrahedron with base area of 15 cm^2 and altitude 6 cm.

2.

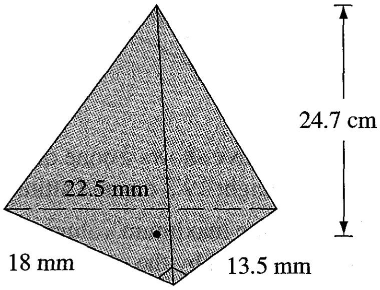


Fig. 4.151 Tetrahedron

Calculate the volume of the tetrahedron shown above with the given measurements.

3. (a) The area of the base of a triangular-based pyramid is 7.5 cm^2 and its altitude is 9.8 cm.
Calculate its volume.

(b) The pyramid is a diamond of density 3.53 g cm^{-3}. Determine the mass of the triangular-based pyramid.

4.

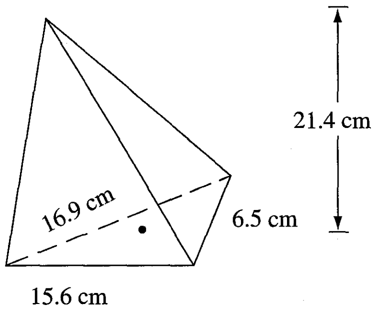


Fig. 4.152 Triangular-based pyramid

The diagram above shows a triangular-based pyramid with base dimensions 6.5 cm, 15.6 cm and 16.9 cm; and with altitude 21.4 cm. The pyramid is made of silver of density 10.5 g cm^{-3}.

(a) Calculate the volume of the tetrahedron.

(b) Determine the mass of the triangular-based pyramid in kilograms correct to three significant figures.

5. Calculate the volume of a square-based pyramid with base area of 18 cm^2 and altitude 7 cm.

6.

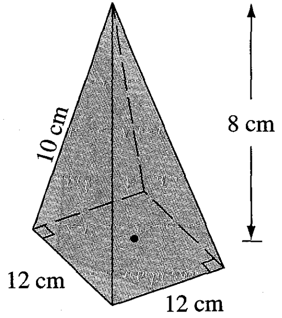


Fig. 4.153 Square-based pyramid

Calculate the volume of the square-based pyramid shown above with the given measurements.

7. (a) The area of the base of a square-based pyramid is 9.0 cm^2 and its altitude is 12.3 cm. Calculate its volume.

(b) The pyramid is made of grey tin of density 5.75 g/cm^3. Determine the mass of the square-based pyramid.

8.

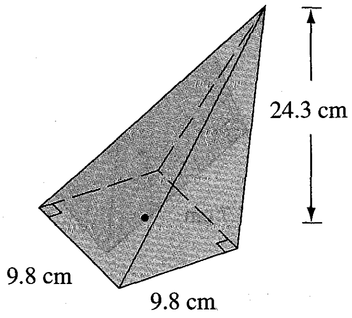


Fig. 4.154 Square-based pyramid

The diagram above shows a square-based pyramid with base length 9.8 cm and altitude 24.3 cm. The pyramid is made of white tin of density 7.31 g/cm^3.

(a) Calculate the volume of the pyramid.

(b) Determine the mass of the square-based pyramid in kilograms correct to two significant figures.

9. Calculate the volume of a rectangular-based pyramid with base area 18.4 cm^2 and altitude 9.3 cm.

10. A rectangular-based metal pyramid of height 5 cm and base dimensions 10 cm by 18 cm; is melted down and rolled into a cylinder of height 7 cm. Calculate the radius of the cylinder in centimetre correct to two significant figures.

Example 20

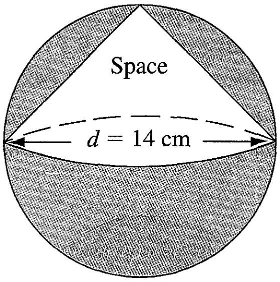


Fig. 4.165 Sphere and cone

A solid sphere of diameter 14 cm contains a conical space in one of its hemisphere as shown in the diagram above.

(a) Calculate the volume of the region shown shaded.

(b) If the density of the shade region in 6 g/cm³, determine its mass in kilograms correct to two significant figures.

(c) Evaluate the surface area of the sphere.

$\left(\text{Use } \pi = \frac{22}{7}\right)$

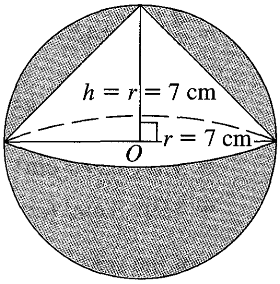


Fig. 4.166 Sphere and cone

Solution

(a) The *volume* of the *sphere*,

$$V_2 = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3} \times \frac{22}{7} \times (7\text{ cm})^3$$
$$= \frac{4}{3} \times \frac{22}{7} \times 7\text{ cm} \times 7\text{ cm} \times 7\text{ cm}$$
$$= 1\,437.\dot{3}\text{ cm}^3$$

The *volume* of the *cone*,

$$V_1 = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3} \times \frac{22}{7} \times (7\text{ cm})^2 \times 7\text{ cm}$$
$$= \frac{1}{3} \times \frac{22}{7} \times 7\text{ cm} \times 7\text{ cm} \times 7\text{ cm}$$
$$= 359.\dot{3}\text{ cm}^3$$

So the *volume* of the *shaded region*,

$$V = V_2 - V_1$$
$$= (1\,437.\dot{3} - 359.\dot{3})\text{ cm}^3$$
$$= 1\,078\text{ cm}^3$$

(b) The *mass* of the *shaded region*,

$$m = \rho V$$
$$= 6\text{ g/cm}^3 \times 1\,078\text{ cm}^3$$
$$= 6\,468\text{ g}$$
$$= \frac{6\,468}{1\,000}\text{ kg}$$
$$= 6.468\text{ kg}$$
$$= 6.5\text{ kg } (\textit{correct} \text{ to } 2\ \textit{s.f.})$$

(c) The *surface area* of the *sphere*,

$$C.S.A. = 4\pi r^2$$
$$= 4 \times \frac{22}{7} \times (7\text{ cm})^2$$
$$= 4 \times \frac{22}{7} \times 7\text{ cm} \times 7\text{ cm}$$
$$= 616\text{ cm}^2$$

Alternative Method

(a) Now the *radius* of the *sphere*,

$$r = \frac{d}{2}$$
$$= \frac{14}{2}\text{ cm}$$
$$= 7\text{ cm}$$

And the *altitude* of the *cone*,

$$h = r = 7\text{ cm } (\textit{radius} \text{ of the } \textit{sphere})$$

So the *volume* of the *cone*,

$$V_1 = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi r^2 r$$
$$= \frac{1}{3}\pi r^3$$

And the *volume* of the *sphere*,

$$V_2 = \frac{4}{3}\pi r^3$$

∴ the *volume* of the *shaded region*, $V = V_2 - V_1$

$$= \frac{4}{3}\pi r^3 - \frac{1}{3}\pi r^3$$

The previous diagram represents a bird cage in the form of a cylinder surmounted by a cone. Calculate the total volume of the bird cage.

Use $\pi = \frac{22}{7}$

18.

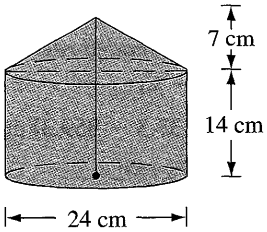


Fig. 4.162 Cone and cylinder

The diagram above represents a bird cage in the form of a cylinder surmounted by a cone. Calculate the total volume of the bird cage.

Take π as $\frac{22}{7}$

Volume and Surface Area of a Sphere

A *sphere* is a *solid* consisting of an *infinite set of points* which are all *equidistant* from a fixed point called the *centre*. The *distance* of a *point* from the *centre* is called its *radius*. For example: a ball used to play cricket and a ball bearings are spheres. For simplicity in calculations, the earth, the sun and planets are considered to be spheres of different fixed radii.

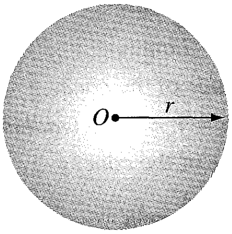


Fig. 4.163 Sphere

The *volume of a sphere*, $V = \frac{4}{3}\pi r^3$

and the *surface area of a sphere*, $C.S.A. = 4\pi r^2$,

where r = *radius* of the *sphere*.

Example 19

(*a*) *Calculate the volume and surface area of a spherical orange of radius 9.5 cm correct to 1 decimal place, using π as 3.142.*

(*b*) *If three-quarters of the orange is juice, what volume juice can you get from 8 such oranges?*

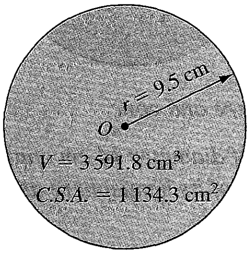


Fig. 4.164 Spherical orange

Solution

(a) The *volume* of the *spherical orange*,

$$
\begin{aligned}
V &= \frac{4}{3}\pi r^3 \\
&= \frac{4}{3} \times 3.142 \times (9.5\text{ cm})^3 \\
&= \frac{4}{3} \times 3.142 \times 857.375\text{ cm}^3 \\
&= 3\,591.829\,7\text{ cm}^3 \\
&= 3\,591.8\text{ cm}^3\ (\textit{correct}\text{ to }1\ \textit{d.p.})
\end{aligned}
$$

The *surface area* of the *spherical orange*,

$$
\begin{aligned}
C.S.A. &= 4\pi r^2 \\
&= 4 \times 3.142 \times (9.5\text{ cm})^2 \\
&= 4 \times 3.142 \times 90.25\text{ cm}^2 \\
&= 1\,134.262\text{ cm}^2 \\
&= 1\,134.3\text{ cm}^2\ (\textit{correct}\text{ to }1\ \textit{d.p.})
\end{aligned}
$$

(b) The *volume* of *juice* that can be obtained from 8 *such oranges*,

$$
\begin{aligned}
&= 8 \times \frac{3}{4} \times 3\,591.8\text{ cm}^3 \\
&= 6 \times 3\,591.8\text{ cm}^3 \\
&= 21\,550.8\text{ cm}^3
\end{aligned}
$$

9.

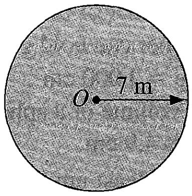


Fig. 4.169 Sphere

In this question use $\pi = \frac{22}{7}$.

The diagram above represents a sphere of radius 7 m.

Calculate:

(a) the distance around the sphere

(b) the volume of the sphere.

10.

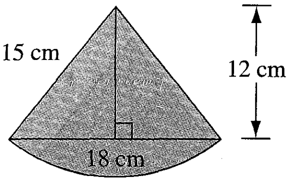


Fig. 4.170 Cone and hemisphere

In this question use π as 3.142

The diagram above is made up of a hemisphere of diameter 18 cm surmounted by a cone. The slant height of the cone is 15 cm and its altitude is 12 cm.

(a) Calculate the total surface area of the solid.

(b) Determine the volume of the solid.

11. Assume the earth to be a sphere of radius 6×10^6 m and mass 6×10^{24} kg.

(a) Calculate the volume of the earth.

(b) Calculate the density of the earth. Express your answers in standard form.

$\left(\text{Take } \pi = \frac{22}{7}\right)$

Time

Time is the *measurement* of a *period* or *instant* in which something *happens* in the *past*, *present*, or *future*.

Traditionally we accepted the *length* of *one day* to be equal to 24 *hours*, divided into *two* 12-*hour periods*. The *first* 12-*hour period* was between *midnight* and *noon* (*midday*). Times between *midnight* and *noon* were denoted by the *symbol a.m.* (or *ante meridiem*) meaning *before noon*. The *second* 12-*hour period* was between *noon* and *midnight*. Times between *noon* and *midnight* were denoted by the *symbol p.m.* (or *post meridiem*) meaning *after noon*. So the *faces of clocks* and *watches* were *numbered* from 1 to 12 as seen in the diagram shown below.

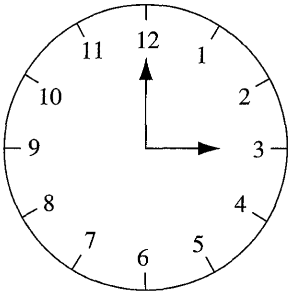


Fig. 4.171 12-hour clock

In *recent times* we have started to use a 24-*hour clock*. So times between *midnight* and *noon* are equivalent to 00 *hours* to 12 *hours*. And times between *noon* and *midnight* are equivalent to 12 *hours* to 24 *hours*. The *face* of a *clock* using the 24-*hour period* can be seen illustrated in the diagram below.

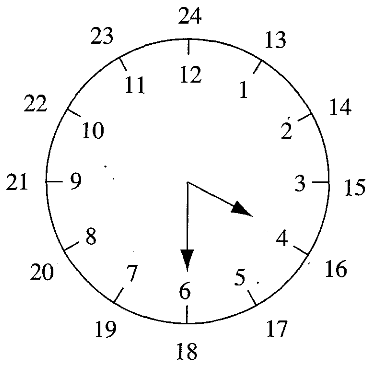


Fig. 4.172 24-hour clock

Thus 3 a.m. is equivalent to 03:00 *h*, 4.30 *p.m.* is equivalent to 16:30 *h* and 11.45 *p.m.* is equivalent to 23:45 *h*.

As stated above, *one day* is equal to 24 *hours*. Further, 1 *hour* was divided into 60 *minutes*; and 1 *minute* was divided into 60 *seconds*.

The *standard abbreviations* for the *units of time* are:

second = *s*
minute = *min*
hour = *h*
day = *d*

$$= \pi r^3$$
$$= \frac{22}{7} \times (7\text{ cm})^3$$
$$= \frac{22}{7} \times 7 \times 7 \times 7\text{ cm}^3$$
$$= 22 \times 49\text{ cm}^3$$
$$= 1\,078\text{ cm}^3$$

Exercise 4p

1. (a) Calculate the volume of a sphere of diameter 17 cm.
   (b) Evaluate the surface area of a sphere of radius 8.5 cm.
   (c) Determine the distance around the equator of the earth, assuming that it is a sphere of radius 6 370 km.
   (Use $\pi = 3.142$)

2. Calculate the volume and surface area of the earth assuming that it is a sphere of radius 6 370 km. Give your answers in standard form correct to 3 significant figures.
(Use $\pi = 3.142$)

3. An orange is 14 cm in diameter. If $\frac{3}{7}$ of it is juice, what volume of juice can you get from 5 such oranges?
$\left(\text{Take } \pi \text{ as } \frac{22}{7}\right)$

4. (a) An apple is 7 cm in diameter. If $\frac{2}{7}$ of it is juice, what volume of juice can you get from 9 such apples.
   (b) If $\frac{3}{11}$ of the juice is poured into a cylindrical container of height 21 cm and diameter 3 cm, how many ice-cubes, each of side 2 cm, can be added before the juice begins to overflow?
   $\left(\text{Take } \pi \text{ as } \frac{22}{7}\right)$

5. (a) Calculate each of the following to three significant figures:
      (i) The volume of a cylindrical glass container of height 30 cm and diameter 28 cm.
      (ii) The number of litres the glass container can hold given that 1 litre = 1 000 cm^3.
      (iii) The volume of a spherical globe of radius 4.9 cm.
   (b) Calculate the least number of these globes that can be put into the glass container so as to cause the water to overflow, if the glass contains 7 l of water.
   $\left(\text{Take } \pi \text{ as } \frac{22}{7}\right)$

6. A spherical object has an outer diameter of 30 cm and an inner diameter of 26 cm. The material of which it is made has a density of 7 g/cm^3. Calculate to the nearest kilogram, the mass of the object. (Take π as 3.142)

7.

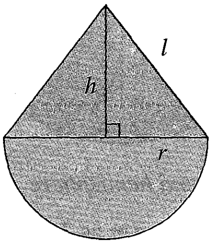


Fig. 4.167 Hemisphere and cone

The diagram above is made up of a hemisphere of radius 6 cm surmounted by a cone. The slant height of the cone is 10 cm and its altitude is 8 cm.
(a) Calculate the total surface area of the solid.
(b) Determine the volume of the solid.
(Take $\pi = 3.142$)

8.

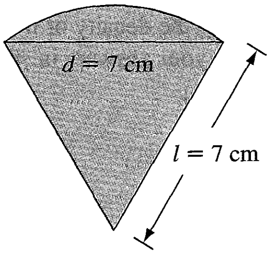


Fig. 4.168 Cone and hemisphere

The figure above consists of a cone surmounted by a hemisphere. The diameter of the hemisphere is 7 cm and the slant height of the cone is also 7 cm. Calculate the total surface area of the solid.
$\left(\text{Use } \pi = \frac{22}{7}\right)$

Exercise 4q

The table shown below is an extract from a timetable of bus schedules.

Table 4.12

	Time of departure	*Time of arrival*
1.	12.05 a.m.	1.23 a.m.
2.	5.45 a.m.	9.52 a.m.
3.	8.25 a.m.	10.45 a.m.
4.	12.45 a.m.	2.25 a.m.
5.	8.35 a.m.	10.20 a.m.
6.	10.19 a.m.	11.12 a.m.
7.	12.30 p.m.	3.45 p.m.
8.	3.48 p.m.	9.54 p.m.
9.	8.27 p.m.	11.45 p.m.
10.	12.15 p.m.	4.05 p.m.
11.	5.43 p.m.	9.25 p.m.
12.	8.07 p.m.	11.01 p.m.
13.	6.35 a.m.	1.25 p.m.
14.	11.45 a.m.	2.20 p.m.
15.	11.40 a.m.	1.20 p.m.
16.	9.25 p.m.	6.24 a.m. (next day)
17.	10.35 p.m.	3.25 a.m. (next day)
18.	11.45 p.m.	1.35 a.m. (next day)
19.	9.25 p.m.	1.28 p.m. (next day)
20.	10.35 p.m.	2.15 p.m. (next day)
21.	11.55 p.m.	2.20 p.m. (next day)

Calculate the length of time for the journey, in hours and minutes, between each pair of times.

The table shown below is an extract from a train schedule.

Table 4.13

	Departure time		*Arrival time*	
	Scheduled	*Actual*	*Scheduled*	*Actual*
22.	04:15 h		07:35 h	07:38 h
23.	09:25 h		10:45 h	10:49 h
24.	10:30 h	10:33 h	11:45 h	11:47 h
25.	04:45 h	04:48 h	09:35 h	
26.	09:35 h	09:38 h	11:15 h	11:17 h
27.	10:45 h		12:30 h	
28.	12:15 h		15:30 h	
29.	14:25 h	14:48 h	20:30 h	20:41 h
30.	19:20 h	19:27 h	20:45 h	20:53 h
31.	12:15 h		16:05 h	
32.	17:40 h	17:43 h	21:25 h	
33.	20:05 h		22:05 h	22:08 h
34.	05:35 h		13:30 h	
35.	11:45 h		15:15 h	15:20 h
36.	08:50 h		14:20 h	

Calculate the actual length of time for each journey, in hours and minutes, between each pair of times.

37. Aileen left Georgetown at 06:45 h and arrived at Turkeyen Campus at 08:59 h. Determine the time it took Aileen to reach Turkeyen.

38. Janet left California at 14:21 h and arrived at St. Augustine Campus at 15:08 h. Determine the time it took Janet to reach St. Augustine.

39. Robert left Bridgetown at 17:15 h and arrived in Bathsheba at 18:27 h. Evaluate the time it took Robert to reach Bathsheba.

40. Lawrence left Kingston at 12:12 h and arrived in Port Maria at 15:05 h. Calculate the time it took Lawrence to reach Port Maria.

Average Speed

The *speed of a body* is defined as its *rate of change* of *distance* with *time*. That is, *speed* is the *quotient* of *distance divided by time*. When a body is *travelling* for any *reasonable distance*, then its *speed* would *vary* from *time to time*. So we speak about the *average speed* of the body for a *particular time interval*. Thus:

$$\text{The average speed} = \frac{\text{The distance travelled}}{\text{The time taken}}$$

That is, the *average speed*, $s = \frac{d}{t}$.

And the *distance travelled*, $d = st$.

So the *time taken*, $t = \frac{d}{s}$,

And the *conversion table* is

60 *s* = 1 *min*
60 *min* = 1 *h*
24 *h* = 1 *d*

In many mathematical problems dealing with time it is necessary to determine a *time difference*. It has been found to be most convenient to find a *time difference* using the 24-*hour clock*. Hence this method is shown in the examples below.

The time difference (or the length of time), t = The time of actual arrival − The time of actual departure

Example 21

Calculate the length of time for the journey, in hours and minutes, between the following pairs of times extracted from a timetable of an airline schedule.

Table 4.11

	Time of departure	*Time of arrival*
(a)	*3.39 a.m.*	*6.43 a.m.*
(b)	*7.38 a.m.*	*11.18 a.m.*
(c)	*12.15 p.m.*	*2.30 p.m.*
(d)	*1.15 p.m.*	*3.12 p.m.*
(e)	*10.45 a.m.*	*7.32 p.m.*
(f)	*9.12 p.m.*	*6.45 a.m. (next day)*
(g)	*8.45 p.m.*	*7.30 p.m. (next day)*

Solution

(a) The *time* of *arrival* = 06:43 h
The *time* of *departure* = − 03:39 h
∴ the *length of time* for the *journey*, t = 3 h 4 min

(b) The *time* of *arrival* = $\overset{10}{\cancel{11}}:\overset{78}{18}$ h
The *time* of *departure* = − 07:38 h
∴ the *length of time* for the *journey*, t = 3 h 40 min

(c) The *time* of *arrival* = 14:30 h
The *time* of *departure* = − 12:15 h
∴ the *length of time* for the *journey*, t = 2 h 15 min

(d) The *time* of *arrival* = $\overset{14}{\cancel{15}}:\overset{72}{12}$ h
The *time of departure* = − 13:15 h
∴ the *length of time* for the *journey*, t = 1 h 57 min

(e) The *time* of *arrival* = $\overset{18}{\cancel{19}}:\overset{92}{32}$ h
The *time* of *departure* = − 10:45 h
∴ the *length of time* for the *journey*, t = 8 h 47 min

(f) The *time equivalent* for *midnight* = $\overset{23}{\cancel{24}}:\overset{60}{00}$ h
The *time* of *departure* = − 21:12 h
∴ the *length of time* to *midnight* = 2 h 48 min
So the *length of time* for the *journey*, = 2 h 48 min
\+ 6 h 45 min
t = 9 h 33 min

(g) The *time equivalent* for *midnight* = $\overset{23}{\cancel{24}}:\overset{60}{00}$ h
The *time* of *departure* = − 20:45 h
∴ the *length of time* to *midnight* = 3 h 15 min
So the *length of time* for the *journey*, = 3 h 15 min
\+ 19 h 30 min
t = 22 h 45 min

From the *examples* shown above, it can be *seen* that:

(i) The *length of time* for a *journey* is given in *hours and minutes* for example, 22 h 45 min, and *not* 22:45 h, which is a *common mistake* made by *students.*

(ii) The *length of time* for a *journey* is *not* a *decimal quantity*. That is 22 h 45 min is *not equivalent* to 22.45 h, which is another *common mistake* made by *students*. This *problem* probably stems from the *a.m. – p.m. method* of writing *time*.

Zenoland at 07:05 h. It travelled at an average speed of 60 km per hour. Calculate the distance from Zenoland to Xanadu.

(b) The car then left Zenoland at 08:30 h and arrived in Dell View at 10:35 h. If Dell View is 125 km from Zenoland, calculate the average speed of the car on the journey from Zenoland to Dell View.

7. (a) A car left Penal at 02:35 h and arrived in Place X at 03:10 h. It travelled at an average speed of 45 km per hour. Calculate the distance from Penal to Place X.

(b) The car then left Place X at 15:25 h and arrived in Couva at 16:05 h. If Place X is 25 km from Couva, calculate the average speed of the car on the journey from Place X to Couva.

8. A cyclist left Turkeyen Campus at 16:25 h and arrived in Place X at 17:20 h. If Place X is 11 km from Turkeyen Campus, calculate the average speed of the cyclist on the journey from Turkeyen Campus to Place X.

9. Determine the average speed, in km/h, for a journey of 180 km in 4 h.

10. What distance, in km, will a car travel in 3 h at 60 km/h?

11. What time, in hours, will it take to travel 192 km at an average speed of 32 km/h?

12. A motorist travels for one hour at an average speed of 72 km/h and then for 2 h at an average speed of 90 km/h. Calculate his average speed for the whole journey.

13. I normally drive the 35 km to school at an average speed of 60 km/h. Today I am 10 minutes late leaving home, calculate my average speed if I am to arrive on time.

14. A man leaves home at 07:10 h to travel 250 km to Town X. He travels 150 km at an average speed of 75 kmh^{-1}.

(a) Calculate the time he takes to travel the 150 km.

He then takes 1 h and 40 min to travel the final 100 km.

(b) Calculate his average speed for the whole journey.

(c) Determine the time at which he arrives at Town X.

an aeroplane's flight from the U.S.A. to Guyana with intermediate stops in Trinidad and Tobago.

Table 4.13

		Local time	*Time spent travelling*	*Distance between airports*	*Average speed in km/h*
Depart Arrive	U.S.A. Tobago	07:25 12:35	*a*	*b*	510
Depart Arrive	Tobago Trinidad	12:55 13:10	15 min	75 km	300
Depart Arrive	Trinidad Guyana	16:45 17:35	*c*	560 km	*d*

NOTES: Local time is the time in the given country.

(a) Calculate the values of *a*, *b*, *c* and *d* in the table above.

(b) Excluding the times the aeroplane spent on the ground, calculate the average speed for the journey from the U.S.A. to Guyana.

16. (a) A cyclist left Georgetown at 07:55 h and arrived at Turkeyen Campus at 08:40 h. He travelled at an average speed of 17 km per hour. Calculate the distance from Georgetown to Turkeyen Campus.

(b) The cyclist left Turkeyen Campus at 16:35 h and arrived in Place X at 19:20 h. If Place X is 34.1 km from Turkeyen Campus, calculate the average speed of the cyclist on the journey from Turkeyen Campus to Place X.

17. A Tristar left Airport A at 08:30 h and arrived at Airport B at 13:15 h the same day. The distance from A to B is 3562.5 km. Calculate the average speed at which the Tristar travelled.

18. (a) The Earth-Sun distance is 1.5×10^{11} m and the speed of light in vacuum is 3×10^{8} ms^{-1}. Calculate the time it takes for a ray of light to reach the earth from the sun.

(b) A light ray reflected from the moon reaches the earth in 1.33 s. Calculate the Earth-Moon distance correct to one significant figure.

19. In Havana, each night at 21:00 h a real cannon shot is fired. If you are 19.8 km away and hear that familiar noise, at what time would you set your watch? Assume that the speed of sound in air is 330 ms^{-1}.

where s = the *average speed,*
d = the *distance travelled*
and t = the *time taken.*

Example 22

(a) A bullet takes 3 s to travel a distance of 1 200 m. Calculate the average speed of the bullet.

(b) A Tristar jet travels for 6 h at an average speed of 650 km/h. What was the distance covered in that peroid?

(c) What amount of time will a car take to travel 84 km at an average speed of 48 km per hour?

(d) (i) A maxi taxi left Couva at 08:21 h and arrived in Port-of-Spain at 09:45 h. Calculate the time taken to travel from Couva to Port-of-Spain.

(ii) The maxi taxi left Port-of-Spain at 11:54 h and arrived in St. James at 12:17 h. Determine the time taken to travel from Port-of-Spain to St. James.

Solution

(a) The *distance travelled* by the bullet,

$$d = 1\,200 \text{ m}.$$

The *time taken,* $t = 3$ s.

$\therefore$ the *average speed* of the bullet, $s = \frac{d}{t}$

$$= \frac{1\,200 \text{ m}}{3 \text{ s}}$$

$$= 400 \text{ m/s}$$

(b) The *average speed* of the Tristar, $s = 650$ km/h

The *time taken,* $t = 6$ h

$\therefore$ *the distance covered,* $d = st$

$$= 650 \text{ km/}\cancel{\text{h}} \times 6\,\cancel{\text{h}}$$

$$= 3\,900 \text{ km}$$

(c) The *distance travelled,* $d = 84$ km

The *average speed,* $s = 48$ km/h

$\therefore$ the *time taken,* $t = \frac{d}{s}$

$$= \frac{84\,\cancel{\text{km}}}{48\,\cancel{\text{km}}/\text{h}}$$

$$= 1\frac{3}{4} \text{ h}$$

(d) (i) The time
Port-of-Spain = 09:45 h
The *time* the maxi taxi *departed* from Couva = 08:21 h

$\therefore$ the *time taken* to *travel* from *Couva to Port-of-Spain*

$$\begin{array}{r} 09{:}45 \text{ h} \\ -\ 08{:}21 \text{ h} \\ \hline = 1 \text{ h } 24 \text{ min} \end{array}$$

(ii) The *time* the maxi taxi *arrived* in St. James = 12:17 h
The *time* the maxi taxi *departed* from Port-of-Spain = 11:54 h

$\therefore$ the *time taken* to *travel* from *Port-of-Spain* to *St. James*

$$\begin{array}{r} \overset{11}{\cancel{12}}{:}\overset{77}{17} \text{ h} \\ -\ 11{:}54 \text{ h} \\ \hline = 23 \text{ min} \end{array}$$

Note that 1 h = 60 min.

Exercise 4r

1. Calculate the average speed, in km/h, for a journey of distance 270 km in time $4\frac{1}{2}$ h.

2. What distance will a car travel in $3\frac{1}{4}$ h at an average speed of 120 km/h?

3. What length of time will it take to travel 96 km at an average speed of 24 km/h?

4. A cyclist travels the 30 km from Town A to Town B at an average speed of 25 km/h and immediately continues the journey to Town C, Which is a further 60 km away at an average speed of 20 km/h.

Determine the average speed for the whole journey.

5. An airport timetable reads as follows:

Piarco	depart	8:30 a.m.
Timehri	arrive	9:20 a.m.
	depart	10:35 a.m.
Grantley Adams	arrive	12:05 p.m.

(a) What length of time does the journey from Piarco (Trinidad) to Timehri (Guyana) take?

(b) What amount of time does the journey from Timehri (Guyana) to Grantley Adams (Barbados) take?

Convert each of the following speeds to kilometres per hour.

16. 5 m/s
17. 15.5 m/s
18. 24.5 m/s
19. 32.5 m/s
20. 40 m/s

Estimated Margin of Error for a Given Measurement

Suppose the *length* of a *steel rod* was measured to be 5.34 m *accurate* to the *nearest* $\frac{1}{100}$ m. Then the *error* interval involved in the *measurement* of the length is ±0.005 m.

Since the *absolute error* involved in the measurement of a length is defined as *half* the *smallest unit of length*, and $\frac{0.01 \text{ m}}{2} = 0.005$ m.

Thus the *greatest possible length* of the steel rod, $l_{\max} = (5.34 + 0.005)$ m
$= 5.345$ m

And the *least possible length* of the steel rod, $l_{\min} = (5.34 - 0.005)$ m
$= 5.335$ m

This means that the *actual length* of the steel rod can be *more than or equal to* 5.335 m, but *less than or equal to* 5.345 m. That is, the *greatest possible error* involved in the *measurement* is +0.005 m and the *least possible error* involved in the *measurement* is −0.005 m. Hence the *estimated margin of error* for the *measurement* of each length is ±0.005 m.

Calculations involving Numbers Derived from a Set of Measurements

Example 24

(a) The lengths of three rods were measured as 145.6 cm, 134.5 cm and 125.7 cm accurate to the nearest $\frac{1}{10}$ cm. State the total length of the three rods appropriate to the margin of error.

(b) The length and breadth of a rectangle were measured as 7.29 and 3.17 m accurate to the nearest $\frac{1}{100}$ m. State the area of the rectangle appropriate to the margin of error.

Solution

(a) The *total length* of the three rods using the *measurements given*,

$$l = (145.6 + 134.5 + 125.7) \text{ cm}$$
$$= 405.8 \text{ cm}$$

The *error interval* involved in the *measurement of each length*

$$= \pm 0.05 \text{ cm}$$

So the *greatest possible total length* of the three rods.

$$l_{\max} = (145.65 + 134.55 + 125.75) \text{ cm}$$
$$= 405.95 \text{ cm}$$

And the *least possible total length* of the three rods.

$$l_{\min} = (145.55 + 134.45 + 125.65) \text{ cm}$$
$$= 405.65 \text{ cm}$$

Comparing these *three lengths*, we see that the *digits* 8, 9 *and* 6 in the *fourth significant figure* is *worthless*. We can therefore only *state* our *answer* as 406 cm *correct* to 3 *significant figures*.

(b) The *area* of the *rectangle* using the *measurements given*,

$$A = lb$$
$$= 7.29 \text{ m} \times 3.17 \text{ m}$$
$$= 23.1093 \text{ m}^2$$

The *error interval* involved in the *measurement of each length*

$$= \pm 0.005 \text{ m}$$

So the *greatest possible area* of the *rectangle*,

$$A_{\max} = l_{\max} \times b_{\max}$$
$$= 7.295 \text{ m} \times 3.175 \text{ m}$$
$$= 23.161\,625 \text{ m}^2$$

And the *least possible area* of the *rectangle*,

$$A_{\min} = l_{\min} \times b_{\min}$$
$$= 7.285 \text{ m} \times 3.165 \text{ m}$$
$$= 23.057\,025 \text{ m}^2$$

Comparing these *three areas*, we see that the *digits* in the *third significant figure* is *worthless*. We can

20. A car left Couva at 06:30 h and arrived in Port-of-Spain at 07:15 h. If Port-of-Spain is 24 km from Couva, calculate the average speed of the car on the journey from Couva to Port-of-Spain.

21. A car left Port-of-Spain at 06:35 h and arrived at Place X at 07:20 h. It travelled with an average speed of 52 km/h. Calculate the distance from Port-of-Spain to Place X.

22. A car left Couva at 09:30 h and arrived in Port-of-Spain at 10:15 h. If Port-of-Spain is 24 km from Couva, calculate the average speed of the car on the journey to Port-of-Spain.

23. A car left Couva at 17:58 h and arrived in Port-of-Spain at 18:28 h. If Port-of-Spain is 24 km from Couva, calculate the average speed of the car on the journey from Couva to Port-of-Spain.

Converting Units of Speed

Sometimes it may be necessary to *convert* from *kilometres per hour* (*km/h or kmh*$^{-1}$) into *miles per hour* (*m.p.h.*) and vice versa. Or from *kilometres per hour* (*km/h or kmh*$^{-1}$) into *metres per second* (*m/s or ms*$^{-1}$).

We will therefore need to use the *conversion tables* shown below:

$$1 \text{ km} \approx \frac{5}{8} \text{ mile} \Rightarrow 1 \text{ mile} \approx \frac{8}{5} \text{ km}$$

$$1 \text{ km} = 1\,000 \text{ m} \Rightarrow 1 \text{ m} = \frac{1}{1\,000} \text{ km}$$

$$1 \text{ h} = 60 \times 60 \text{ s} = 3\,600 \text{ s}$$

Example 23

(a) Convert the speed 160 km/h into miles per hour.

(b) Convert the speed 75 m.p.h. into kilometres per hour.

(c) Convert the speed 180 km/h into metres per second.

(d) Convert the speed 30 m/s into kilometres per hour.

Solution

(a) The *speed* 160 km/h $\approx 160 \times \frac{5}{8}$ m.p.h.

$= 20 \times 5$ m.p.h.

$= 100$ m.p.h.

(b) The *speed* 75 m.p.h. $\approx 75 \times \frac{8}{5}$ km/h

$= 15 \times 8$ km/h

$= 120$ km/h

(c) The *speed* 180 km/h $= \dfrac{180 \times 1\,0\cancel{0}\cancel{0}}{3\,6\cancel{0}\cancel{0}}$ m/s

$= \dfrac{1\,800}{36}$ m/s

$= 50$ m/s

(d) The *speed* 30 m/s $= \dfrac{3\cancel{0} \times 3\,6\cancel{0}\cancel{0}}{1\,\cancel{0}\cancel{0}\cancel{0}}$ km/h

$= 108$ km/h

Alternative Method

(c) The *speed* 180 km/h $= 180 \times \frac{5}{18}$ m/s

$= 10 \times 5$ m/s

$= 50$ m/s

(d) The speed 30 m/s $= 30 \times \frac{18}{5}$ km/h

$= 6 \times 18$ km/h

$= 108$ km/h

Exercise 4s

Convert each of the following speeds into miles per hour:

1. 16.8 km/h
2. 46.4 km/h
3. 60 km/h
4. 125.6 km/h
5. 148 km/h

Express each of the following speeds in kilometres per hour:

6. 24.5 m.p.h.
7. 39.5 m.p.h.
8. 80 m.p.h.
9. 125 m.p.h.
10. 160 m.p.h.

Change each of the following speeds into metres per second:

11. 18.9 km/h
12. 72 km/h
13. 91.8 km/h
14. 120.6 km/h
15. 163.8 km/h

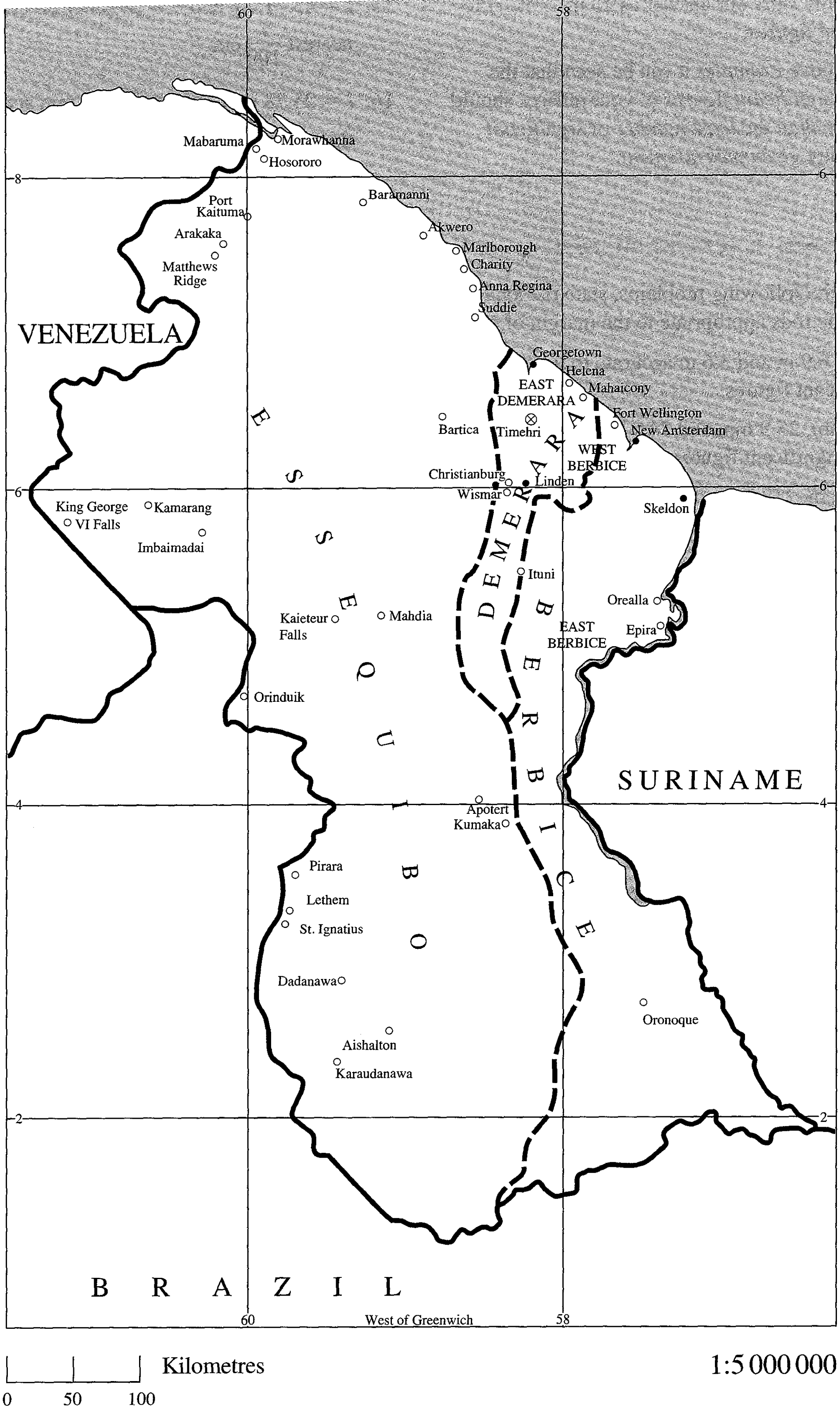


Fig. 4.173 Map of Guyana

therefore only *state* our answer as 23 m^2 *correct* to 2 *significant figures.*

From the above *examples* it can be *seen* that the number of *significant figures* in your *answer* should not be *more than* the *least number* of *significant figures* in any *given measurement.*

Exercise 4t

In each of the following problems, state the total length of the rods appropriate to the margin of error:

1. 5.8 m, 6.9 m and 3.6 m accurate to two significant figures.
2. 18.7 mm, 25.9 mm and 14.5 mm accurate to three significant figures.
3. 124.5 cm, 118.7 cm and 109.6 cm accurate to four significant figures.
4. 3.5 m, 9.4 m and 4.7 m accurate to the nearest $\frac{1}{10}$ m.
5. 175.6 mm, 84.7 mm and 75.9 mm accurate to the nearest $\frac{1}{10}$ mm.
6. 225.6 cm, 149.8 cm and 95.4 cm accurate to the nearest $\frac{1}{10}$ cm.
7. 3.54 m, 2.35 m and 4.79 m accurate to three significant figures.
8. 25.43 mm, 18.97 mm and 16.89 mm accurate to the nearest $\frac{1}{100}$ mm.
9. 85.4 cm, 74.3 cm and 53.2 cm accurate to the nearest $\frac{1}{10}$ cm.
10. 7.58 m, 4.75 m and 6.21 m accurate to the nearest $\frac{1}{100}$ m.

In the following problems, state the area of the rectangle appropriate to the margin of error:

11. $l = 6.74$ m and $b = 3.59$ m, accurate to three significant figures.
12. $l = 85.4$ cm and $b = 25.7$ cm accurate to three significant figures.
13. $l = 125.7$ mm and $b = 62.3$ mm acccurate to the nearest $\frac{1}{10}$ mm.
14. $l = 8.54$ m and $b = 2.32$ m accurate to three significant figures.
15. $l = 43.71$ cm and $b = 19.48$ cm accurate to the nearest $\frac{1}{100}$ cm.
16. $l = 25.78$ mm and $b = 14.63$ mm accurate to the nearest $\frac{1}{100}$ mm.
17. $l = 8.51$ m and $b = 3.45$ m accurate to the nearest $\frac{1}{100}$ m.
18. $l = 95.4$ cm and $b = 64.3$ cm accurate to the nearest $\frac{1}{10}$ cm.
19. $l = 60.3$ mm and $b = 24.1$ mm accurate to three significant figures.
20. $l = 9.41$ m and $b = 6.81$ m accurate to three significant figures.

Measurement on Maps and Scale Drawings

It is obviously an impossible task to draw the *actual* (or full size) *map* of a country or the *exact drawing* of a house or car on a sheet of paper. In order to draw these shapes on paper, we therefore have to use a *scale* as in the drawing of graphs on graph paper. We are then able to draw a '*replica*' or *model* or *scale drawing* of the shape on paper. Our *model* or *scale drawing* will then be *similar* to or '*resemble*' the actual shape.

Fig. 4.173 shows a *sketch map* of Guyana. The *scale* of the sketch map of Guyana is 1:5 000 000. The *scale* is given as the *ratio* of a *length on the map* to the *actual distance on the ground.* The *scale* given also means that 1 cm measured on the *map* is *equal* to 5 000 000 cm or 50 km measured on the *ground.*

Example 25

Using the sketch map of Guyana, determine the actual distance between:

(a) Georgetown and Skeldon

(b) Georgetown and Port Kaituma

(c) Georgetown and Christianburg.

Solution

Exercise 4u

Using the sketch map of Guyana, determine to the nearest kilometre the actual distance between:

1. (a) Georgetown and Matthews Ridge
   (b) Georgetown and Morawhanna
   (c) Georgetown and Charity.

2. (a) Georgetown and Bartica
   (b) Georgetown and Wismar
   (c) Georgetown and Kaieteur Falls.

3. (a) Georgetown and Mahdia
   (b) Georgetown and Orealla
   (c) Georgetown and King George VI Falls.

4. (a) Georgetown and New Amsterdam
   (b) Georgetown and Suddie
   (c) Georgetown and Oronoque.

5. (a) Georgetown and Anna Regina
   (b) Georgetown and Ituni
   (c) Georgetown and Timehri Airport.

6. (a) Georgetown and Lethem
   (b) Georgetown and St. Ignatius
   (c) Georgetown and Orinduik.

The scale of a sketch map of Trinidad is 1:500 000. Determine the actual distance between:

7. Port-of-Spain and Couva, if the distance on the map is 5.1 cm.

8. Port-of-spain and Tacarigua, if the distance on the map is 2.8 cm.

9. Port-of-Spain and St. Joseph (on the East coast), if the distance on the map is 13.4 cm.

10. Port-of-Spain and San Fernando, if the distance on the map is 8.3 cm.

11. Port-of-Spain and Guayaguayare, if the distance on the map is 15.3 cm.

12. Port-of-Spain and Rio Claro, if the distance on the map is 10.6 cm.

13. Port-of-Spain and Tortuga, if the distance on the map is 6.6 cm.

14. Port-of-Spain and Pointe-a-Pierre, if the distance on the map is 7.2 cm.

15. Port-of-Spain and Blanchisseuse, if the distance on the map is 5.4 cm.

16. Port-of-Spain and Salibea, if the distance on the map is 10.3 cm.

17. Port-of-Spain and Penal, if the distance on the map is 10.8 cm.

18. Port-of-Spain and Sangre Grande, if the distance on the map is 8.5 cm.

The scale of a sketch map of Jamaica is 1:1 000 000. Determine the distance on the map between:

19. Kingston and Montego Bay, if the actual distance is 125 km.

20. Kingston and Mandeville, if the actual distance is 71 km.

21. Kingston and Spanish Town, if the actual distance is 11 km.

22. Kingston and Savanna-la-Mar, if the actual distance is 139 km.

23. Kingston and St. Ann's Bay, if the actual distance is 59 km.

The scale of a sketch map of Barbados is 1:200 000. Determine the distance on the map between:

24. Bridgetown and St. Lawrence, if the actual distance is 1.8 km.

25. Bridgetown and Speightstown, if the actual distance is 14.4 km.

26. Bridetown and Bathsheba, if the actual distance is 13.2 km.

27. Bridgetown and Harrison Point, if the actual distance is 21 km.

28. Bridgetown and Martin's Bay, if the actual distance is 14.6 km.

29. The scale on a road map is 1:15 000.
   (a) What is the distance, in metres, between two villages represented by 8.5 cm?
   (b) What is the actual area of a stadium represented on the map by a rectangle 9 cm long and 6 cm wide?
   (c) State the area of the stadium in hectares.

30. The area of a cricket pitch is 90 m^2. On a map the area of the pitch is 0.9 cm^2.
   (a) What is the area of the cricket pitch in cm^2?
   (b) Calculate the ratio of the areas given.
   (c) Determine the scale of the map.

State your answers correct to the nearest kilometre.

(a) The *distance on the map* between *Georgetown* and *Skeldon* = 3.1 cm

So the *actual distance* between *Georgetown* and *Skeldon* = 3.1 cm × 5 000 000
= 15 500 000 cm
= 155 000 m
= 155 km

(b) The *distance on the map* between *Georgetown* and *Port kaituma* = 4.9 cm

So the *actual distance* between *Georgetown* and *Port Kaituma* = 4.9 × 50 km
= 245 km

(c) The *distance on the map* between *Georgetown* and *Christianburg* = 1.9 cm

So the *actual distance* between *Georgetown* and *Christianburg* = 1.9 × 50 km
= 95 km

The *distance* between each pair of places calculated above, is the *direct distance*, or as they say, '*the distance as the crow flies*'.

Example 26

The scale on a road map is 1:25 000.

(a) What is the distance, in metres, between two villages represented by 3.5 cm?

(b) What is the actual area of a playing field represented on the map by a rectangle 0.5 cm long and 0.3 cm wide?

(c) State the area of the playing field in hectares.

Solution

(a) The *actual distance* between the two villages = 3.5 cm × 25 000
= 87 500 cm
= 875 m

(b) The *actual length* of the field, *l* = 0.5 cm × 25 000
= 12 500 cm
= 125 m

The *actual width* of the field, *b* = 0.3 cm × 25 000
= 7 500 cm
= 75 m

So the *actual area* of the *rectangular* playing field,

$A = lb$
= 125 m × 75 m
= 9 375 m^2

Alternative Method

(b) The *ratio* of a *length on the map* to the *actual distance on the ground*, $1:n$ = 1:25 000

So the *ratio* of an *area on the map* to the *actual area on the ground*, $1:n^2$ = $1:25\,000^2$
= 1:625 000 000

And the *area* of the *rectangular field* on the *map*, $A = lb$
= 0.5 cm × 0.3 cm
= 0.15 cm^2

So the *actual area* of the playing field = 0.15 cm^2 × 625 000 000
= 93 750 000 cm^2

$$= \frac{93\,750\,0\cancel{00}}{1\cancel{00} \times 1\cancel{00}}\ \text{m}^2$$

= 9 375 m^2

(c) The *area* of the playing field in *hectares*, = 9 375 m^2

$$= \frac{9\,375}{10\,000}$$

= 0.937 5 ha

For a *model*:

(i) The *ratio* of a *length on the model* to the *actual length* = $1:n$

(ii) The *ratio* of an *area on the model* to the *actual area* = $1:n^2$

(iii) The *ratio* of a *volume on the model* to the *actual volume* = $1:n^3$

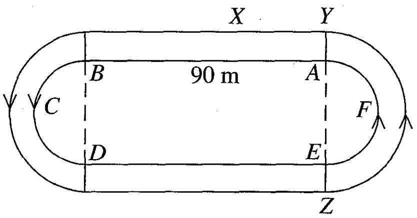
5. **(Do not use tables for this question).**

A map is drawn to scale of 1:20 000.

(i) Calculate the length on the map which represents a distance of 153 m on the ground.

(ii) A rectangular field is shown on the map. Its dimensions on the map can be measured only to the nearest 0.1 mm. You read the measurements on the map as 8.6 mm and 5.2 mm. Calculate the largest possible area of the field (on the ground).

Question 5. C.X.C. (Basic). June 1981.



6. (i) The inner boundary *ABCDEF* of an athletic track consists of two straight parts each 90 m long and two semi-circular ends as shown in the diagram. If the perimeter of the inner boundary is 400 m, calculate the diameter *AE* of the inner semi-circle.

(ii) The track is 3.5 m wide. An athlete starts the 400 m run at *X*, and remains in the outer lane. He finishes at *Y*.

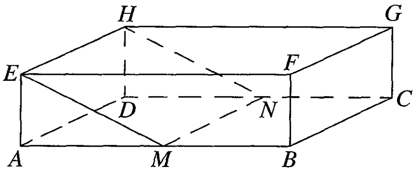
Calculate (a) the outer diameter *YZ*

(b) the distance *XY*.

$\left(\text{Take } \pi \text{ to be } \frac{22}{7}\right)$

Question 3. C.X.C. (Basic). June 1982.

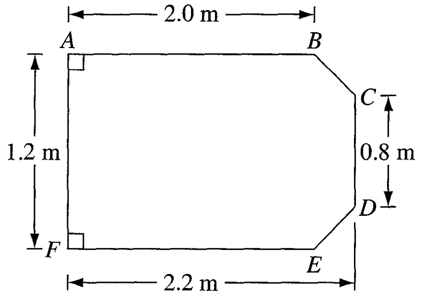
7. (i) A bus timetabled to depart at 15:40 for a journey which is scheduled to take $3\frac{1}{2}$ hours. It left at 16:45 and arrived at its destination 45 minutes later than scheduled. How long did this journey take?



(ii) (a) The figure *ABCDEFGH* above represents a cuboid with $AB = 80$ cm, $EH = 60$ cm, and $AE = 30$ cm. *M* and *N* are the midpoints of *AB* and *DC*, respectively. Calculate the volume of the wedge *AMEDNH*.

(b) The wedge is cut along *EMNH* and removed from the cuboid. Calculate the volume of the solid which remains.

Question 4. C.X.C. (Basic). June 1983.



8. (i) In the figure above (**which is not drawn to scale**) *AF* is parallel to *CD* and $AF = BE$. Calculate the area of the enclosed region *ABCDEFA*.

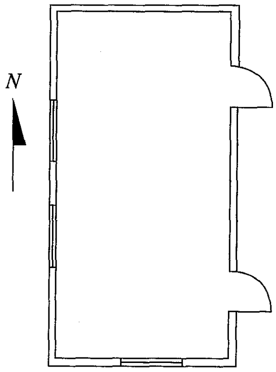
(ii) A rectangular wooden beam of length 5 metres has a cross-section 20 cm by 15 cm. The wood has a density of 600 kg per cubic metre.

(a) Calculate the volume of the beam in cubic metres.

(b) Express the answer for (a) in standard form.

(c) Calculate the mass of the beam in kilograms.

Question 4. C.X.C. (Basic). June 1984.



Plan of Assembly Hall
Scale 1:1 000

9. The figure above shows a plan of a school's Assembly Hall.

(a) How many windows are there in the Hall?

(b) By making suitable measurements calculate, in square metres, the area of the Hall.

Question 10(ii). C.X.C. (Basic). June 1984.

31. A scale of 1:40 was used to make a scale drawing of a rectangular room.

(a) The dimensions of the scale drawing of the room are 9 cm by 6 cm. What is the actual area of the room in cm^2?

(b) The area of the scale drawing of the room: The actual area of the room = ?

32. The surface area of the model of a cylindrical tank is 15 cm^2. The model is built using a scale of 1:30. What is the area in m^2 of the actual tank?

33. The scale of a model space city is 5 cm:250 m.

(a) What distance does 1 cm represent?

(b) What area does 1 cm^2 represent?

(c) What volume does 1 cm^3 represent?

34. The scale of a model skyscraper is 5 cm:150 m.

(a) What distance is represented by 1 cm?

(b) What area is represented by 1 cm^2?

(c) What volume is represented by 1 cm^3?

35.

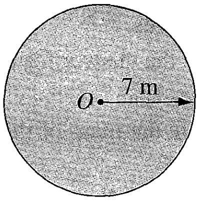


Fig. 4.174 Sphere

The previous figure represents a sphere of radius 7 m. A small model of the sphere is made. The radius of the model is 14 cm. Calculate:

(a) the scale used to make the model

(b) the area of the curved surface of the model, given that the area of the curved surface of the sphere is 616 m^2

(c) the volume of the model, given that the volume of the sphere is $1\,437\frac{1}{3}$ m^3.

$\left(\text{Use } \pi = \frac{22}{7}\right)$

C.X.C. Past Paper Questions

The following supplementary questions were taken from C.X.C. Past Papers.

Exercise 4v

1. A machine travels at 10 m/s. Express, in standard form, the number of metres it travels in 1 hour.

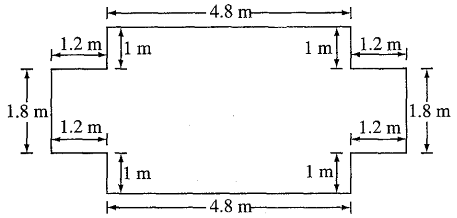
Question 2(i). C.X.C. (Basic). June 1979.

2. The radius of the base of a cylindrical tank is 25 cm. If the water level rises 10 cm/s, calculate the change in volume of the water in the tank after 35 s.

Find how many cones of height 30 cm and base radius 14 cm can be completely filled from this water, and what volume of water is left over.

$\left(\text{Take } \pi \text{ to be } \frac{22}{7}\right)$

Question 8. C.X.C. (Basic). June 1979.



3. The above diagram (which is not drawn to scale) shows the shape of the floor of a room. At each of the 12 corners, there is a right angle.

(i) What is the area, in square metres, of the floor?

(ii) How many square tiles of side 20 cm are needed to cover the floor?

Question 2. C.X.C. (Basic). June 1980.

4. A drop of oil of volume 0.5 cm^3 is placed on water in a cylindrical trough of area of cross-section 200 cm^2. The oil spreads evenly to cover the whole surface of the water. What is the thickness of the film of oil formed by the drop?

(i) If the same drop is placed instead in another trough, it forms a film of thickness 0.0005 cm. What is the area of the cross-section of this trough?

(ii) If the drop is placed in troughs of differing areas of cross-section how does the thickness of the film formed by the drop vary as the area is changed?

(iii) Write a formula showing the relation between the area of cross-section, A cm^2, and the thickness, t cm, of films formed by the drop.

Question 5. C.X.C. (Basic). June 1980.

In the previous diagram, **not drawn to scale**, represents a flower bed in the shape of a sector *BAC* of a circle. *A* is the centre, $AB = 10$ m and angle $BAC = 72°$. (Take π as 3.14).

(a) Calculate:

(i) the length in metres, of the arc *BC*

(ii) the area in square metres, of the sector *BAC*.

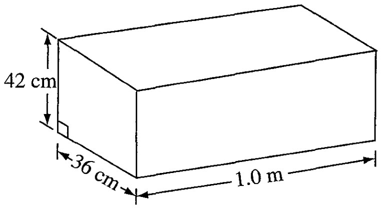
(b) The flower bed is to be fenced by five strands of wire all around it. The wire is sold in rolls of single strand 20 m long. Calculate the number of rolls needed to fence the flower bed.

(c) The surface of the flower bed is to be covered with top soil 15 cm deep. Calculate, cm^3, the volume of soil required.

Question 8. C.X.C. (Basic). June 1990.

16. (a) The shortest distance between two towns, *A* and *B*, on a map is 2.5 cm. The map is drawn to a scale of 1:2 500 000. Calculate, in kilometres, the actual shortest distance between the two towns *A* and *B*.

(b)



The diagram above, **not drawn to scale,** shows a water trough with a cross-section 36 cm wide and 42 cm high. The length of the trough is 1.0 m.

Calculate:

(i) the volume, in cm^3, of the trough

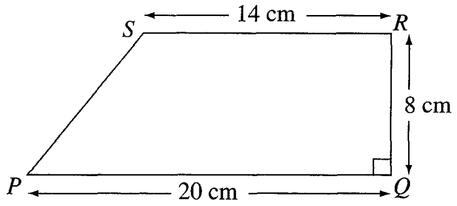
(ii) the number of litres of water required to fill the trough

(iii) the depth of water in the trough when it contains 72 litres of water.

(Note: 1 litre = 1 000 cm^3)

Question 10. C.X.C. (Basic). June 1991.

17. (a)



In the previous trapezium, not drawn to scale,

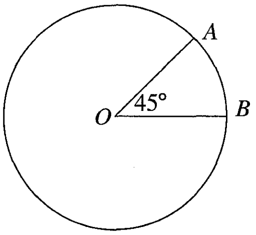
$PQ = 20$ cm, $QR = 8$ cm, $RS = 14$ cm and angle $PQR = 90°$.

(i) Calculate the area of the trapezium.

(ii) Calculate the length of *PS*.

Question 3(a). C.X.C. (Basic). June 1992.

18. **Note:** Use $\pi = \frac{22}{7}$ to answer this part of the question.



The figure above, **not drawn to scale**, represents a circle of diameter 9.8 cm, centre *O*. Angle $AOB = 45°$.

Calculate to one decimal place

(i) the circumference of the circle

(ii) the area of the circle

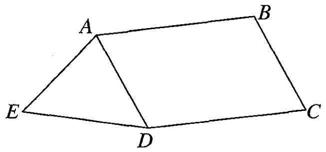
(iii) the area of the MINOR sector *AOB*.

Question 7(b). C.X.C. (Basic). June 1993.

19. (a) A cylindrical object of height 21 cm has an outer diameter of 28 cm and a inner diameter of 24 cm. The material of which it is made has a density of 6 g/cm^3. Calculate, to the nearest kilogram, the mass of the object.

Question 6(a). C.X.C. (General). June 1985.

20. (a)



The diagram above (**not drawn to scale**) Shows a right triangular prism with $AB = 15$ cm, $AD = AE = 10$ cm and $ED = 12$ cm.

Calculate the volume of the prism.

Question 2(a). C.X.C. (General). June 1986.

10. (a) Calculate each of the following to two significant figures:

(i) The volume of a cylindrical tin of height 20 cm and diameter 28 cm.

(ii) The number of litres the tin can hold given that 1 litre = 1 000 cm^3.

(iii) The volume of a spherical ball of radius 4.2 cm.

(b) Calculate the least number of balls that can be put in the tin so as to cause the water to overflow, if the tin contains 9 litres of water.

Question 7. C.X.C. (Basic). June 1985.

11. A milk container is $\frac{5}{8}$ full. Milk is poured in at a rate of 5 litres per minute. After 14 minutes the container is $\frac{4}{5}$ full.

Calculate the number of litres of milk which the container can hold.

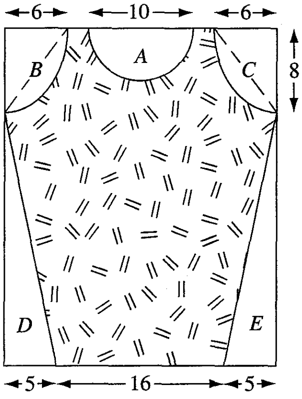
Question 3. C.X.C. (Basic). June 1986.

12. The front of a doll's blouse is cut from a rectangular piece of material 32 cm by 26 cm as shown in the diagram below (**not drawn to scale**).

The region *A* is a semi-circle and each of the regions *B* and *C* is made up of a triangle and semi-circle as shown in the diagram.

All measurements are in centimetres.

Take π to be 3.14.



Calculate to the nearest whole number of square centimetres, the area of:

(a) region *D*

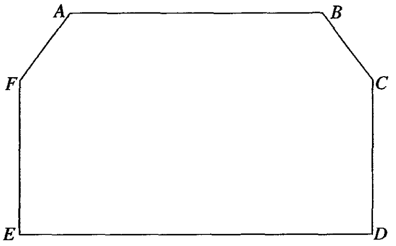
(b) region *A*

(c) region *B*

(d) the front of the blouse.

Question 5. C.X.C. (Basic). June 1986.

13.



The figure *ABCDEF* is an accurate scale drawing of a pane of glass where *AB* and *ED* represent the top and bottom edges, respectively.

(a) Measure accurately and state, in centimetres,

(i) the length of *AB*

(ii) the distance between *AB* and *ED*.

(b) Given that *CD* = 3.0 cm and *ED* = 7.0 cm, calculate in cm^2 the area of the figure *ABCDEF*.

(c) Given that the top edge *AB* of the actual pane of glass measures 2.5 m, calculate the scale used in the drawing.

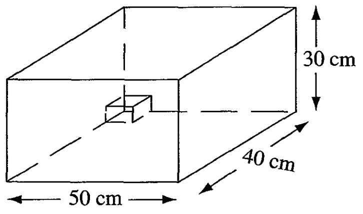
(d) Using your answer to part (c), calculate for the actual pane of glass:

(i) The length of the bottom edge in metres

(ii) The actual area of the glass in square metres.

Question 4. C.X.C. (Basic). June 1988.

14.



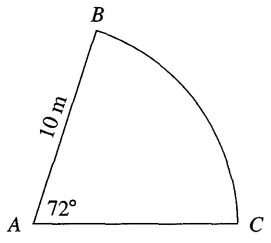
The figure above, not drawn to scale, represents a fish-tank in the shape of a cuboid of height 30 cm.

(i) Calculate, in cm^3, the volume of the tank.

(ii) If there are 40 litres of water in the tank, calculate the height of water in the tank.

Question 5(a). C.X.C. (Basic). June 1989.

15.



Thus:

The net monthly salary = *The gross monthly salary* − *The monthly deductions*

The net annual salary = *The net monthly salary* × 12

Example 1

(a) The gross annual salary of a teacher is $44 772. What is his gross monthly salary?

(b) A civil servant is employed at a gross monthly salary of $1 875. How much is her gross annual salary?

(c) The gross monthly salary of a manager is $6 715. Calculate his net annual salary after deductions of $1 976 were made monthly.

(d) A quantity surveyor earns $75 600 annually. Deductions of $2 845 are made each month. Calculate his net monthly salary.

Solution

(a) The teacher's *gross annual salary* = $44 772

∴ the teacher's *gross monthly salary* = *The gross annual salary* ÷ 12

$$= \frac{\$44\,772}{12}$$

= $3 731

Hence the teacher's *gross monthly salary* is $3 731.

(b) The civil servant's *gross monthly salary* = $1 875

∴ the civil servant's *gross annual salary* = *The gross monthly salary* × 12

= $1 875 × 12

= $22 500

Hence the civil servant's *gross annual salary* is $22 500.

(c) The manager's *gross monthly salary* = $6 715

And the *monthly deductions* = $1 976

∴ the manager's *net monthly salary* = *The gross monthly salary* − *The monthly deductions*

= $(6 715 − 1 976)

= $4 739

So the manager's *net annual salary* = *The net monthly salary* × 12

= $4 739 × 12

= $56 868

Hence the manager's *net annual salary* is $56 868.

(d) The surveyor's *gross annual salary* = $75 600

∴ the surveyor's *gross monthly salary* $= \frac{\textit{The gross annual salary}}{12}$

$$= \frac{\$75\,000}{12}$$

= $6 300

So the surveyor's *net monthly salary* = *The gross monthly salary* − *The monthly deductions*

= $(6 300 − 2 845)

= $3 455

Hence the surveyor's *net monthly salary* is $3 455.

Exercise 5a

1. A teacher is paid an annual salary of $32 160. What is his gross monthly salary?
2. A clerk is paid an annual salary of $22 320. What is her gross monthly salary?
3. An engineer earns an annual salary of $58 236. Calculate his gross monthly salary.
4. An accountant earns an annual salary of $78 252. Find his gross monthly salary.
5. A member of parliament is paid an annual salary of $150 720. Determine her gross monthly salary.
6. The gross monthly salary of a marine biologist is $6 543. Calculate her annual salary.
7. The gross monthly salary of an environmentalist is $5 149. Calculate her annual salary.
8. A meterologist is paid a monthly salary of $4 841. Find the amount that he is paid annually.
9. An archaeologist earns $9 147 monthly. Determine the amount that he is paid annually.
10. An architect is paid $4 179 monthly. Find the amount that he earns annually.
11. The gross monthly salary of a manager is $5 875. Calculate her net annual salary after deductions of $976 were made monthly.

CHAPTER 5

Consumer Arithmetic

This chapter will teach you how to

- calculate gross monthly salary, net monthly salary, annual salary, basic wage, overtime wage and a commission.
- solve problems dealing with income tax, sales tax (or VAT), land and building taxes, water rates, gas rates and posting a letter or parcel.
- to calculate percentage profit, percentage loss, percentage change, discount, hire purchase price and a mortgage.
- to calculate and understand an electricity bill, a telephone bill and convert currencies.
- to calculate simple interest, compound interest and depreciation.

Salary

Normally, government employees, for example, teachers and civil servants are paid a *'flat pay'* or *fixed amount of money each month* for services rendered during that *period.* These *monthly paid employees* are said to *receive* a *salary*. Thus the *gross annual salary* of a salaried employee can be obtained by simply *multiplying* their *gross monthly salary by* 12. It follows then that their *gross monthly salary* can be obtained by *dividing* their *gross annual salary by* 12. These facts can be seen *illustrated* by the *function machine* and the *reverse function-machine* shown below.

(a)

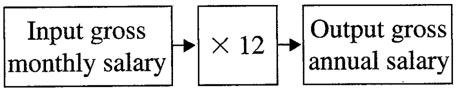


Fig. 5.1 Function-machine

(b)

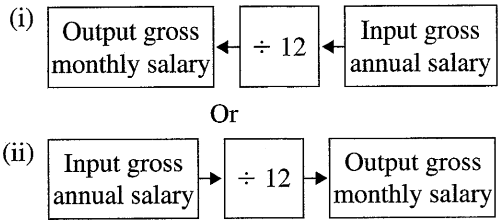


Fig. 5.2 Reverse function-machine

Thus:

The gross annual salary = *The gross monthly salary* × 12

The gross monthly salary = *The gross annual salary* ÷ 12

Further, the *'take home pay'* or *net monthly salary* of each monthly paid employee will vary according to the *income tax claims* that the employee can make. These individual *claims* would obviously cause the *deductions* from each employee to vary and hence also the employee's *net monthly salary*.

5. Calculate the basic wage for the following cane boiler.

Table 5.2

Name	*Number of hours worked*	*Basic rate of pay*
Mr. Bachan	80	$9.35

Calculate the basic week for each of the following workers.

Table 5.3

	Basic wage	*Basic rate*
6.	$328.70	$8.65
7.	$274.05	$7.83

Calculate the basic fortnight for each of the following workers.

Table 5.4

	Basic wage	*Basic rate*
8.	$527.20	$6.59
9.	$566.20	$7.45

10. An operator works a basic fortnight at a basic rate of $8.95 and earns $671.25. Determine the basic fortnight for the operator.

Calculate the basic rate for each of the following factory workers.

Table 5.5

	Basic wage	*Basic week*
11.	$254	40 hours
12.	$306.25	35 hours

Calculate the basic rate for each of the following factory operators.

Table 5.6

	Basic wage	*Basic fortnight*
13.	$788	80 hours
14.	$633.75	75 hours

15. A man works a basic week of 32 hours and earns $172.48. Find his basic rate of pay.

16. A man's wage for a 35-hour week is $263.90. Calculate his hourly rate of payment.

Overtime Wage—Gross Wage

Workers who are *normally paid hourly* are sometimes called upon to *work extra hours* on a *daily basis*, if for example, the shift worker replacing him is absent. The *extra hours worked* is called the *overtime* and it is an *addition* to the *basic week worked.* Because there is a *demand* for the worker at this time, since the work in a factory or industry must be *maintained continuously*, *overtime* is paid for at a *higher rate* than the *basic rate.*

The *overtime rates* are as follows:

(i) *The overtime rate at time-and-a-quarter* = 1.25 × *The basic rate* = 125% *of the basic rate*

(ii) *The overtime rate at time-and-a-half* = 1.5 × *The basic rate* = 150% *of the basic rate*

(iii) *The overtime rate at double-time* = 2 × *The basic rate* = 200% *of the basic rate*

(iv) *The overtime rate at triple-time* = 3 × *The basic rate* = 300% *of the basic rate*

The overtime wage = *The overtime rate* × *The overtime worked*

For *every week* or *fortnight* then, the payroll clerk in *calculating* the *gross wage* (*total wage*) will need to know how many *basic hours* and how many *overtime hours* were *accumulated* by *each worker* during that *period.*

Thus:

The gross wage = *The basic wage* + *The overtime wage*

The overtime wage = *The gross wage* − *The basic wage*

The basic wage = *The gross wage* − *The overtime wage*

$$\textit{The overtime worked} = \frac{\textit{The overtime wage}}{\textit{The overtime rate}}$$

Example 3

(a) Anita working as a sales clerk is paid a basic rate of $3.75. During the rush Christmas season she works 6 hours overtime on Friday at time-and-a-quarter, 8 hours overtime on

12. The gross monthly salary of a permanent secretary is $6583. Find his net annual salary after deductions of $1475 were made monthly.

13. The gross annual salary earned by a teacher is $45 600. Determine his net monthly salary if deductions of $872 are made each month.

14. The gross annual salary earned by a civil servant is $54 240. Find his net monthly salary if deductions of $1475 are made per month.

15. A quantity surveyor earns $70 740 annually. Deductions of $2016 are made each month. Calculate his net monthly salary.

Basic Wage

Normally, people who work in factories and industries, for example, operators, boilers, cane cutters, drivers and brick layers are paid a *given amount each hour* for work done during that *period.* The *amount of money normally paid* for *each hour of work* is called the *basic rate.*

Quite a number of these workers *normally work* a 5-*day week* at 8 *hours per day*. This *normal* 40-*hour week* (or otherwise stated) is called the *basic week.* And the *amount of money earned* during a *basic week* is called the *basic wage.*

Some workers *normally work* a 10-*day fortnight* (2 *weeks*) at 8 *hours per day*. This *normal* 80-*hour fortnight* (or otherwise stated) is called the *basic fortnight.* And the *amount of money earned* during a *basic fortnight* is called the *basic wage.*

Thus:

$$\textit{The basic wage} = \begin{array}{l}\textit{The basic}\\ \textit{rate}\end{array} \times \begin{array}{l}\textit{The basic week}\\ \textit{or fortnight}\end{array}$$

$$\begin{array}{l}\textit{The basic week}\\ \textit{or fortnight}\end{array} = \frac{\textit{The basic wage}}{\textit{The basic rate}}$$

$$\textit{The basic rate} = \frac{\textit{The basic wage}}{\textit{The basic week or fortnight}}$$

Example 2

(a) A refinery operator works a basic week of 35 hours and his basic rate is $14.75. What is his basic wage for that week?

(b) A seamstress works for a basic wage of $236.25 and her basic rate is $6.75. Calculate her basic week.

(c) A driver is paid $703 for a basic fortnight of 76 hours. Calculate his basic rate.

Solution

(a) The operator's *basic wage*

$$= \textit{The basic rate} \times \textit{The basic week}$$
$$= \$14.75 \times 35$$
$$= \$516.25$$

Hence the operator's *basic wage* is $516.25.

(b) The seamstress' *basic week*

$$= \frac{\textit{The basic wage}}{\textit{The basic rate}}$$
$$= \frac{\cancel{\$}236.25}{\cancel{\$}6.75}$$
$$= 35 \text{ hours}$$

Hence the seamstress' *basic week* is 35 hours.

(c) The driver's *basic rate*

$$= \frac{\textit{The basic wage}}{\textit{The basic fortnight}}$$
$$= \frac{\$703}{76}$$
$$= \$9.25$$

Hence the driver's *basic rate* is $9.25.

Exercise 5b

1. Calculate the basic wage for the following factory worker.

Table 5.1

Name	*Number of hours worked*	*Basic rate of pay*
Sharon	38	$5.40

2. Robin starts work each day at 7.30 a.m. and finishes at 4.30 p.m. He has a 45-minute lunch break. How many hours does he work in a normal 5-day week? Find his basic wage if his rate of pay is $7.25 per hour.

3. A girl works a basic week of 40 hours and her basic rate is $6.25 per hour. Calculate her basic wage for the week.

4. Mr. Rayburn starts work each day at 8:00 a.m. and finishes at 4:00 p.m. He has a 30-minute lunch break. How many hours does he work in a normal 10-day fortnight. Calculate his basic wage if his basic rate of pay is $8.75.

working day was 8 hours and anytime worked in excess of this was paid for at time-and-a-half, with Saturday work being paid at double-time and Sunday working being paid at triple-time. She was paid $5.60 per hour normally. Calculate:

(a) her wage for the normal working week
(b) her overtime wage for working from Monday to Friday
(c) her overtime wage for Saturday
(d) her overtime wage for Sunday
(e) her gross wage for the week.

4. In a factory all employees work a basic week of 38 hours. Any overtime worked during weekdays is paid for at time-and-a-half. Overtime worked on Saturday is paid for at double-time, whilst on Sunday it is paid for at triple-time. If the basic rate is $8.96 per hour, determine the gross wage of a man who worked 15 hours overtime from Monday to Friday, 3 hours overtime on Saturday and 2 hours overtime on Sunday.

5. In an urea manufacturing plant all employees work a basic week of 40 hours. Any overtime worked during weekdays is paid for at time-and-a-quarter. Overtime worked on Saturday is paid for at time-and-a-half, whilst on Sunday it is paid for at double-time. If the basic rate is $14.60 per hour and a man worked 15 hours overtime during weekdays, 6 hours overtime on Saturday and 5 hours overtime on Sunday, calculate:

(a) his basic wage
(b) his overtime wage
(c) his gross wage for the week.

6. At a chemical factory the basic week is 40 hours. During a particular week Mr. James earned a gross wage of $518.00, but $148.00 was for overtime.

(a) Calculate Mr. James' basic rate of payment.
(b) If overtime was paid for at double-time, calculate how many hours Mr. James worked overtime.

7. A man is paid $9.60 per hour for a 40-hour week and he is paid at a time-and-a-half for overtime. Find how many hours of overtime he worked when his wage for a certain week was $528.00.

8. At the Brechin Castle sugar factory, the normal working week is 40 hours. During a certain week Mr. Ramnath earned a total of $516.20, but $160.20 was for overtime.

Calculate how much per hour he is normally paid.

9. A woman is paid $8.50 per hour for a 35-hour week and she is paid double-time for overtime. Determine how many hours she worked when her wage for a certain week was $450.50.

10. At a factory the basic week is 40 hours. During a particular week Mr. Ali earned a gross wage of $491.15, but $156.75 was for overtime.

(a) Calculate Mr. Ali's basic rate of payment.
(b) If overtime was paid for at time-and-a-quarter, find how many hours Mr. Ali worked overtime.

Commission—Gross Wage

Salespersons such as those dealing in cloth, insurance, medicine and cars are paid a *commission* which is *calculated* as a *percentage* of the *total value of the product sold* by them. Sometimes the *commission* is *paid* only on the *value of products sold above* a *given amount*. Of course the *commission* offered will *vary* according to the *employer* and to the *cost of the product* sold. This *commission* is *paid* in *addition* to the *basic wage*.

Thus:

The commission = *x% of the total value of the product sold*
The gross wage = *The basic wage + The commission*
The commission = *The gross wage − The basic wage*

Example 5

A car salesman is paid a basic wage of $600. In addition, he is paid a commission of 1.5 per cent of the value of the cars sold. During a certain week he sold cars valued at $97 600 and $68 700.

Saturday at time-and-a-half, and 5 hours overtime on Sunday at double-time. Calculate Anita's overtime wage for that particular week.

(b) Anita works a basic week of 40 hours. Evaluate her basic wage.

(c) Hence determine Anita's gross wage for that particular week during the rush Christmas season.

Solution

(a) Anita's *overtime wage* for *Friday* = 1.25 × *The basic rate* × *The overtime worked*
= 1.25 × $3.75 × 6
= $28.125
= $28.13 (*correct* to the *nearest cent*)

Anita's *overtime wage* for *Saturday* = 1.5 × *The basic rate* × *The overtime worked*
= 1.5 × $3.75 × 8
= $45.00

Anita's *overtime wage* for *Sunday* = 2 × *The basic rate* × *The overtime worked*
= 2 × $3.75 × 5
= $37.50

∴ Anita's *overtime wage* for that *particular week* = $(28.13 + 45.00 + 37.50)
= $110.63

Hence Anita's *overtime wage* is $110.63.

(b) Anita's *basic wage* = *The basic rate* × *The basic week*
= $3.75 × 40
= $150.00

Hence Anita's *basic wage* is $150.00.

(c) Anita's *gross wage* for that *particular week* = *The basic wage* + *The overtime wage*
= $(150.00 + 110.63)
= $260.63

Hence Anita's *gross wage* is $260.63.

Example 4

At a factory the basic week is 40 hours. During a particular week Mr. Riley earned a gross wage of $513.88. However $159.48 was for overtime.

(a) Calculate Mr. Riley's basic rate of payment.

(b) If overtime was paid for at time-and-a-half, determine how many hours Mr. Riley worked overtime.

Solution

(a) Mr. Riley's *basic wage* = *The gross wage* − *The overtime wage*
= $(513.88 − 159.48)
= $354.40

∴ Mr. Riley's *basic rate of payment* = $\frac{\textit{The basic wage}}{\textit{The basic week}}$
= $\frac{\$354.40}{40}$
= $8.86

Hence Mr. Riley's *basic rate* is $8.86.

(b) Mr. Riley's *overtime rate* at *time-and-a-half* = 1.5 × *The basic rate*
= 1.5 × $8.86
= $13.29

∴ the overtime worked = $\frac{\textit{The overtime wage}}{\textit{The overtime rate}}$
= $\frac{\cancel{\$}159.48}{\cancel{\$}13.29}$
= 12 hours

Hence Mr. Riley worked 12 hours *overtime*.

Exercise 5c

1. A secretary works a 35-hour week for which she is paid $262.50. She works 6 hours overtime on Saturday which is paid for at time-and-a-half, and 4 hours overtime on Sunday which is paid for at double-time. Calculate her gross wage for the week.

2. In an engineering firm all employees work a basic week of 40 hours. Any overtime worked from Monday to Friday is paid for at time-and-a-quarter. Overtime worked on Saturday is paid for at time-and-a-half, whilst on Sunday it is paid for at double-time. If the basic rate is $14.80 per hour, find the gross wage of a man who worked 12 hours overtime from Monday to Friday, 2 hours overtime on Saturday and 5 hours overtime on Sunday.

3. During a certain week Maureen worked $9\frac{1}{2}$ hours Monday to Friday each day, together with 6 hours on Saturday and $4\frac{1}{2}$ hours on Sunday. The normal

Income Tax

In most Caribbean countries *income tax* is *levied* by the *Board of Inland Revenue* and is a large source of *revenue* for the *government*. If an individual *earns* an *income* which is *less than or equal to a minimum amount*, then he *does not* have to pay *income tax*. If however, an individual *earns* an *income* which is *greater than the minimum amount*, then he *has* to pay *income tax*. The *total amount of money* a person earns *before tax* is *levied* is called the *gross income*. Each individual has a number of *allowances*. An *allowance* is that part of the *gross income* that is *non-taxable* (i.e. *tax-free income*).

Some *normal allowances* are as follows:

(i)	*Personal allowance* —	an allowance for the income earner.
(ii)	*Spouse allowance* —	an allowance for the husband or wife who is not working.
(iii)	*Child allowance* —	an allowance for the children that are at school or university.
(iv)	*Dependent relative allowance* —	an allowance for relatives dependent on the taxpayer.
(v)	*National insurance allowance* —	an allowance for the national insurance payments made.
(vi)	*Insurance premium allowance* —	an allowance for the premiums paid on whole life or deferred annuity policies.
(vii)	*Credit union shares allowance* —	an allowance for the purchase of credit union shares.
(viii)	*Government bonds allowance* —	an allowance for the purchase of government bonds.
(ix)	*Mortgage interest allowance* —	an allowance for the interest paid to a bank for the purchase of a house.

After the income earner has *deducted* all his *legal allowances*, then the *amount remaining* is called the *taxable income*. This *taxable income* is then *taxed* at *varying rates*. The *amount of money remaining* after *paying tax* is called the *net income*.

Thus:

The taxable income = The gross income − The total tax-free income (or allowances)

The net income = The gross income − The tax paid

Example 6

A teacher's gross income is $42 500 per annum. He is married and his wife is not employed. They have two children at school and one child at university. He pays $200 per month towards credit union shares and $900 per month towards mortgage interest. The tax-free allowances and tax rates are as follows:

Table 5.7

Tax-free allowances	*Tax rates*
Personal allowance = $1 500	*5 ¢ on the first $12 000*
Spouse allowance = $1 000	*15 ¢ on the next $8 000*
Child (at school) allowance = $200	*35 ¢ on the next $20 000*
Child (at university) allowance = $500	*40 ¢ on the remaining chargeable income*
Credit union shares allowance = 25% of total payment	
Mortgage interest allowance = total amount paid	
National insurance allowance = $30 per month	

Calculate:

(a) his total tax-free income

(b) his taxable income

(c) the tax he pays per annum

Calculate for that week:

(a) the commission he received

(b) his gross wage

Solution

(a) The *total value* of the cars sold = $(97 600 + 68 700)

= $166 300

∴ the *commission* the car salesman received = *1.5% of the total value of the cars sold*

= 1.5% of $166 300

$= \frac{1.5}{1\cancel{00}} \times \$166\,3\cancel{00}$

= $2 494.50

Hence the *commission* received was $2 494.50.

(b) The car salesman *gross wage* = *The basic wage + The commission*

= $(600 + 2 494.50)

= $3 094.50

Hence the *gross wage* was $3 094.50.

Exercise 5d

1. A man receives a monthly salary of $3 500 together with a commission of 5% on all sales over $5 000 per month. Calculate his gross salary in a month in which his sales amounted to $40 000.

2. A sales assistant is paid a basic wage of $125 per week. In addition, she is paid a commission of 2.5% on the value of the goods she sells. How much commission will she be paid on sales amounting to $1 476 and what is her gross wage for that week?

3. A sales girl is paid a basic wage of $140 per week. In addition, she is paid a commission of 5% on the value of goods she sells. How much commission will she be paid on sales amounting to $1 525 and what is her gross wage for the week?

4. A saleswoman is paid a basic wage of $225 per week. In addition, she is paid a commission of 3% on the value of good she sells above $5 500. How much commission will she be paid on sales amounting to $12 500 and what are her earnings for that week?

5. A salesman is paid a salary of $2 500 per month and a commission of 10% on all sales above $7 000.

(a) Calculate the salesman's gross salary if his sales for a particular month is $16 500.

(b) If his sales for a particular month is $5 325, what is his gross salary.

6. A saleswoman sold $950 worth of goods during a certain week. If she is paid a commission of 3% on total sales, calculate:

(a) her commission

(b) her gross wage if she is paid $150.00 normally per week.

7. A sales assistant is paid a basic wage of $175 per week. In addition, she is paid a commission of 1.25% on the value of the goods she sells. Her sales for a particular week amounted to $3 459.
Determine:

(a) the commission that she will be paid

(b) her gross wage for that week.

8. An agent selling pharmaceuticals is paid a basic wage of $580 per week. In addition, he is paid a commission of 5% on his sales. In a particular week he made sales totalling $9 875.
Determine:

(a) his commission

(b) his gross wage for that week.

9. The gross wage for a salesman during a particular week is $646. If his basic wage is $475 and he is paid a commission of 2.5% of the total value of goods sold, calculate:

(a) the commission that he was paid

(b) the total value of the goods sold that week.

10. The gross wage for an agricultural sales assistant during a particular week was $697.55. If his basic wage is $490 and he is paid a commission of 3.5% of the total value of the agricultural products sold, calculate:

(a) the commission that he was paid

(b) the total value of the agricultural products sold.

2.

Single person's allowance	$1 800
Married man's allowance	$2 500
Child under 11 years old	$700
Child 11–16 years old	$900
Child over 16 years old, if in full time education	$1 100
Dependent relative	$400
National insurance	$150

A married man with one child aged 15 years and a second child aged 18 years who is attending college full time, earns $48 120 per annum. He has a dependent relative whom he helps to support. If he also gets an allowance of $150 for national insurance, calculate the amount he pays in income tax per annum if it is levied at 25%.

3.

Single person's allowance	$1 800
Married man's allowance	$2 500
Child under 11 years old	$700
Child 11–16 years old	$900
Child over 16 years old, if in full time education	$1 200
Dependent relative	$400
National insurance	$250

A married man with one child aged 19 years who is attending college, earns $36 720 per annum. He has a dependent relative whom he helps to support. If he also gets an allowance of $250 for national insurance, calculate the amount he pays in income tax per annum when it is levied at 20%.

4. A clerk's annual gross salary is $24 600. She contributes 3% of her salary to a pension scheme and her company contributes 5%. Her contribution to the pension scheme is a non-taxable allowance. Other non-taxable allowances and the income tax rates on taxable income are given below.

Table 5.8

Non-taxable allowances	*Income tax rates on annual taxable income*
$50 per month for national insurance	15% on first $5 000
$75 per month for medical insurance	25% on remainder
$3 600 per annum for personal allowance	

Calculate, to the nearest cent, for the clerk:

(a) the monthly amount the company contributes to her pension scheme

(b) the total amount of her annual salary that is not taxed

(c) her annual taxable income

(d) the tax she pays monthly.

5. A nurse's annual gross salary is $29 460. He contributes 2% of his salary to a medical scheme. His contribution to the medical scheme is a non-taxable allowance.

Other non-taxable allowances and the income tax rates on taxable income are given in the table below.

Table 5.9

Non-taxable allowances	*Income tax rates on annual taxable income*
$37 per month for national insurance	20% on first $4 000
$3 000 per annum for personal allowance	25% on remainder
$1 800 per annum for his wife	
$1 400 per annum for his children	

Calculate for the nurse:

(a) the total amount of his annual salary that is not taxed

(b) his annual taxable salary

(c) the tax he pays annually.

6. A mechanic's annual gross salary is $25 200. He contributes 3% of his salary to a medical scheme and his company contributes 5%. His contribution to the medical scheme is a non-taxable allowance. Other non-taxable allowances and the income tax rates on taxable income are given in the table below.

Table 5.10

Non-taxable allowances	*Income tax rates on annual taxable income*
$125 per month for national insurance	20% on first $4 000
$3 000 per annum for personal allowance	25% on remainder

(d) the tax he pays per month

(e) his net income.

Solution

(a) His *personal allowance* = \$1 500

His *wife's allowance* = \$1 000

The *allowance* for *two children* at *school* = \$200 × 2 = \$400

The *allowance* for *one child* at *university* = \$500

His *credit union shares allowance* = 25% of \$200 × 12

$= \frac{25}{1\cancel{00}} \times \$24\cancel{00}$

= \$600

His *mortgage interest allowance* = \$900 × 12

= \$10 800

His *national insurance allowance* = \$30 × 12

= \$360

∴ his *total tax-free income* = \$(1 500 + 1 000 + 400 + 500 + 600 + 10 800 + 360)

= \$15 160

Hence the teacher's *total tax-free income* is \$15 160.

(b) The teacher's *gross income* = \$42 500

∴ his *taxable income* = *The gross income − The total tax-free income*

= \$(42 500 − 15 160)

= \$27 340

Hence the teacher's *taxable income* is \$27 340.

(c) The *tax* paid on the *first* \$12 000 = 5% of \$12 000

$= \frac{5}{1\cancel{00}} \times \$120\cancel{00}$

= \$600

The *tax* paid on the *next* \$8 000 = 15% of \$8 000

$= \frac{15}{1\cancel{00}} \times \$8\cancel{000}$

= \$1 200

The *tax* paid on the *remaining* \$7 340 = 35% of \$7 340

$= \frac{35}{10\cancel{0}} \times \$734\cancel{0}$

= \$2 569

∴ the *tax* he pays *per annum* = \$(600 + 1 200 + 2 569)

= \$4 369

Hence the teacher's *annual tax* is \$4 369.

(d) The *tax* he pays *per month* $= \dfrac{\textit{The tax paid per annum}}{12}$

$= \dfrac{\$4\,369}{12}$

= \$364.08

Hence the teacher's *monthly tax* is \$364.08.

(e) His *net income* = *The gross income − The tax paid*

= \$(42 500 − 4 369)

= \$38 131

Hence the teacher's *net income* is \$38 131.

Exercise 5e

1.

Single person's allowance	\$1 500
Married man's allowance	\$2 500
Child under 11 years old	\$400
Child over 16 pursuing full time education	\$700
Dependent relative	\$250
National insurance	\$225

Use the table above to answer the following question.

A married man with one child aged 17 attending university full time and a second aged 9 earns \$25 600 per annum. He supports a dependent relative and also claims his national insurance allowance.
Calculate:

(a) his total allowance

(b) his total taxable income

(c) the amount he pays in tax per annum when it is levied at 40%

(d) the amount he pays in tax per month.

Percentage Profit and Percentage Loss

Business people normally buy and sell articles at different prices. The amount of money that they pay for an article is called the *cost price* (abbreviated *C.P.*) or *buying price* or *original price*. All business people are in the business to make a *profit*. Hence a *percentage of the cost price* is normally *added* to the *cost price* of the article before being *sold*. The price that the consumer pay for the article is called the *selling price* (abbreviated *S.P.*).

Thus:

$$\begin{aligned}\textit{The profit} &= \textit{The selling price} - \textit{The cost price}\\ &= S.P. - C.P.\\ \textit{The profit }\% &= \frac{\textit{The profit}}{\textit{The cost price}} \times 100\%\\ &= \frac{S.P. - C.P.}{C.P.} \times 100\%\\ \textit{The selling price} &= \textit{The cost price} + \textit{The profit}\end{aligned}$$

Sometimes a business person has to sell an article for an amount that is *less than what was paid* for it, because the article was damaged or out of style, for example. In such a case the business person is said to incur a *loss*.

Thus:

$$\begin{aligned}\textit{The loss} &= \textit{The cost price} - \textit{The selling price}\\ &= C.P. - S.P.\\ \textit{The loss }\% &= \frac{\textit{The loss}}{\textit{The cost price}} \times 100\%\\ &= \frac{C.P. - S.P.}{C.P.} \times 100\%\\ \textit{The selling price} &= \textit{The cost price} - \textit{The loss}\end{aligned}$$

From the above *formulae* it can be *seen* that:

(i) The *profit per cent* is calculated as a *percentage* of the *cost price* normally.

(ii) The *loss per cent* is calculated as a *percentage* of the *cost price* normally.

Example 7

A shopkeeper buys 25 cricket balls at a total cost of $150.

(a) He sells them for $8 each. What was his percentage profit?

(b) He sells them for $5 each. What was his percentage loss?

Solution

(a) The *cost price* of the 25 balls $= \$150$

And the *selling price* of the 25 balls $= \$8 \times 25 = \$200$

∴ his *profit* on 25 balls

$$\begin{aligned}&= S.P. - C.P.\\ &= \$(200 - 150)\\ &= \$50\end{aligned}$$

So his *profit* %

$$\begin{aligned}&= \frac{S.P. - C.P.}{C.P.} \times 100\%\\ &= \frac{\overset{1}{\cancel{\$50}}}{\underset{3}{\cancel{\$150}}} \times 100\%\\ &= 33\frac{1}{3}\%\end{aligned}$$

Hence the shopkeeper's *percentage profit* was $33\frac{1}{3}\%$.

(b) The *cost price* of the 25 balls $= \$150$

And the *selling price* of the 25 balls $= \$5 \times 25 = \$125$

∴ his *loss* on 25 balls

$$\begin{aligned}&= C.P. - S.P.\\ &= \$(150 - 125)\\ &= \$25\end{aligned}$$

So his *loss* %

$$\begin{aligned}&= \frac{C.P. - S.P.}{C.P.} \times 100\%\\ &= \frac{\overset{1}{\cancel{\$25}}}{\underset{6}{\cancel{\$150}}} \times 100\%\\ &= 16\frac{2}{3}\%\end{aligned}$$

Hence the shopkeeper's *percentage loss* was $16\frac{2}{3}\%$.

Calculate, to the nearest cent, for the mechanic:

(a) the monthly amount the company contributes to his medical scheme

(b) the total amount of his annual salary that is not taxed

(c) his annual taxable income

(d) the tax he pays monthly.

7. The following are examples of tax allowances for a particular year:

Personal allowance	$2 500
Spouse allowance	$1 800
Child allowance	$700 each
Life insurance premiums:	an allowance of 40% of the annual rate of payment

Calculate the chargeable (taxable) income on an annual salary of $36 000 for a man, his wife and two children, with a monthly insurance premium of $200 and no other claims.

8. A man earns $3 000 and his wife earns $1 000 per month. They have two children. National insurance of 5% of all earnings must be paid before taxes are deducted. Allowances and tax rates are as follows:

Table 5.11

Tax-free allowances	*Rates of taxable income*
$2 000 per annum for each adult	10% on first $2 000
$500 per annum per child	20% on next $2 000
Earned income relief – 10% of husband's salary	30% on next $4 000
Non-taxable income – 50% of wife's salary	40% on the remainder

Calculate:

(a) the amount they paid for national insurance

(b) the total tax-free personal allowance for his family

(c) their total non-taxable allowance

(d) the amount they paid in tax for that year.

9. A man's annual gross salary is $38 400. He contributes 3% of his salary to a medical scheme and his company contributes 5%. His contribution to the medical scheme is a non-taxable allowance. Other non-taxable allowances and the income tax rates on taxable income are given in the table below.

Table 5.12

Non-taxable allowances	*Income tax rates on annual taxable income*
$95 per month for national insurance	20% on first $4 000
$3 500 per annum for personal allowance	25% on remainder

Calculate, correct to the nearest cent, for the man:

(a) the monthly amount the company contributes to his medical scheme

(b) the total amount of his annual salary that is not taxed

(c) his annual taxable income

(d) the tax he pays monthly.

10. Use the following table of tax-free allowances and tax rates to solve the following questions:

Table 5.13

Tax-free allowances	*Tax rates*	
$1 500 Personal allowance	First $2 000	5%
$1 000 Spouse	Next $3 000	15%
$400 Child (each)	Next $5 000	25%
$150 National insurance (per month)	Over $10 000	35%

(a) Mr. Raman is married with five children. He earns $45 600 annually.
Determine:

  (i) his tax-free allowance
  (ii) his taxable income
  (iii) the tax he pays
  (iv) his net income.

(b) If Mr. Raman was unmarried, with no children, and earned the same annual salary, determine:

  (i) his tax free allowance
  (ii) his taxable income
  (iii) the tax he pays
  (iv) his net income.

= 92 × $12.09
= $1 112.28

Hence the *selling price* of the stove was $1 112.28.

Exercise 5f

1. Calculate the % loss:
   Cost price of an article = $28
   Selling price of the article = $21

2. Albert bought his bicycle for $275. He sold it for $350.
   (a) What was the amount of his profit?
   (b) What was the amount of his percentage profit?

3. A dealer buys 50 apples for $40 and sells them for $1.20 each. Calculate his percentage profit.

4. Mrs. Jones bought 25 mangoes for $7.50. She sold 12 mangoes for 60 cents each and the remainder for 55 cents each. Calculate her percentage profit or loss. State whether she made a profit or loss.

5. Mr. Roberts bought a gas cooker for $945. He sold it to a customer for $803.25 due to damage.
   Calculate:
   (a) the loss (b) the percentage loss.

6. A shopkeeper buys a stove from a manufacturer for $860.
   Calculate:
   (a) the selling price if he makes a profit of 15%
   (b) the selling price if he incurs a loss of 15%.

7. A businessman bought a personal computer for $10 768.
   (a) Calculate the selling price of the personal computer if he made a profit of 12%
   (b) The computer's casing was damaged in transporting it to the customer. Determine the selling price of the personal computer if he incurred a loss of 2% on the cost price.

8. An entrepreneur buys a computer game from a manufacturer for $975. Calculate the selling price if he makes a profit of:
   (a) 25% (b) 12.5%

9. An entrepreneur buys a compact disc from a manufacturer for $1 245. Calculate the selling price if he makes a profit of:
   (a) 30% (b) 15%

10. A businesswoman bought a refrigerator from a manufacturer for $1 378. Calculate:
    (a) the selling price if she makes a profit of 17.5%
    (b) the selling price if she incurs a loss of 3.5%.

Percentage Change

If a quantity is *increased* or *decreased* by *x*% of itself, then the *new percentage* is (100 ± *x*)%.

Problems dealing with a *percentage change* can be *solved* using the *unitary method.* However, the *method illustrated* in the examples following uses the *concept* of a *multiplication factor* to *solve* the *problems.*

An *increase in salary* and the *percentage profit* are examples of a *percentage increase*, while a *decrease in salary* and the *percentage loss* are examples of a *percentage decrease*.

Example 9

(a) A businessman sold a refrigerator for $2 745 making a profit of 15% on the cost price. Calculate the cost price of the refrigerator to the businessman.

(b) A businesswoman sold a refrigerator for $2 149 incurring a loss of 12% on the cost price. Determine the cost price of the refrigerator to the businesswoman.

State your answers correct to the nearest cent.

Solution

(a) The *selling price* of the refrigerator (i.e. 115% *of the cost price*) = $2 745

∴ the *multiplication factor* = $\frac{100}{115}$

Alternative Method

(a) The *cost price* of 1 ball $= \dfrac{\$150}{\$25} = \$6$

And the *selling price* of 1 ball $= \$8$

$\therefore$ his *profit* on 1 ball $= S.P. - C.P.$

$= \$(8 - 6)$

$= \$2$

Hence his *profit %* $= \dfrac{S.P. - C.P.}{C.P.} \times 100\%$

$= \dfrac{\overset{1}{\cancel{\$2}}}{\underset{3}{\cancel{\$6}}} \times 100\%$

$= 33\frac{1}{3}\%$

Hence the shopkeeper's *percentage profit* was $33\frac{1}{3}\%$.

(b) The *cost price* of 1 ball $= \dfrac{\$150}{\$25} = \$6$

And the *selling price* of 1 ball $= \$5$

$\therefore$ his *loss* on 1 ball $= C.P. - S.P.$

$= \$(6 - 5)$

$= \$1$

Hence his *loss %* $= \dfrac{C.P. - S.P.}{C.P.} \times 100\%$

$= \dfrac{\$1}{\$6} \times 100\%$

$= 16\frac{2}{3}\%$

Hence the shopkeeper's *percentage loss* was $16\frac{2}{3}\%$.

Example 8

A businesswoman bought a stove for \$1 209.

(a) Calculate the selling price of the stove if she made a profit of 11%.

(b) The stove was damaged in transporting it to the customer. Determine the selling price of the stove if she incurred a loss of 8% on the cost price.

Solution

(a) The *cost price* of the stove $= \$1\,209$

$\therefore$ the *profit* made on the stove $= 11\%$ of $\$1\,209$

$= \dfrac{11}{100} \times \$1\,209$

$= 11 \times \$12.09$

$= \$132.99$

So the *selling price* of the stove $=$ *The cost price + The profit*

$= \$(1\,209 + 132.99)$

$= \$1\,341.99$

Hence the *selling price* of the stove was \$1 341.99.

(b) The *cost price* of the stove $= \$1\,209$

The *loss* incurred on the stove $= 8\%$ of $\$1\,209$

$= \dfrac{8}{100} \times \$1\,209$

$= 8 \times \$12.09$

$= \$96.72$

So the *selling price* of the stove $=$ *The cost price − The loss*

$= \$(1\,209 - 96.72)$

$= \$1\,112.28$

Hence the *selling price* of the stove was \$1 112.28.

Alternative Method

(a) The *cost price* of the stove $= \$1\,209$

$\therefore$ the *selling price* of the stove (i.e. 111% *of the cost price*) $= 111\%$ of $\$1\,209$

$= \dfrac{111}{100} \times \$1\,209$

$= 111 \times \$12.09$

$= \$1\,341.99$

Hence the *selling price* of the stove was \$1 341.99.

(b) The *cost price* of the stove $= \$1\,209$

$\therefore$ the *selling price* of the stove (i.e. 92% *of the cost price*) $= 92\%$ of $\$1\,209$

$= \dfrac{92}{100} \times \$1\,209$

11. When petrol was $2.40 per litre, I used 1 200 litres per annum. The price increased by 150%, so I reduced my yearly consumption by 25%.
Determine:

(a) the new price per litre of petrol

(b) the amount of my reduced annual consumption

(c) the amount by which my petrol bill is more (or less) for the year.

12. A boy's mass increased by 12% between his tenth and eleventh birthdays. If his mass was 52 kg on his tenth birthday, what amount was his mass on his eleventh birthday?

13. There are 30 teachers in a school. It is anticipated that the number of teachers next year will increase by 10%. How many teachers should there be next year?

14. Mrs. Frank earns $528 per week from which income tax is deducted at 40%. Evaluate her net pay.

15. The number of children attending a school is 8% fewer this year than last year. If 550 attended last year, how many are attending this year?

16. Mr. Carter was 125 kg when he decided to go on a diet. He lost 12% of his mass in the first month and a further 8% of his original mass in the second month. What amount was his mass after the two months of dieting?

17. When petrol was $1.48 per litre, I used 900 litres per annum. The price increased by 10%, so. I reduced my yearly consumption by 10%. Calculate:

(a) the new price for a litre of petrol

(b) the amount of my reduced annual petrol consumption

(c) the amount by which my petrol bill is more (or less) for the year.

18. Determine the % error in the following:

(a) Measured mass = 980 g
Actual mass = 1 000 g

(b) Estimated cost = $29.40
Actual cost = $25.00

19. Miss Marie earns $350 per week from which income tax is deducted at 30%. Calculate the amount that she actually takes home.

20. The price of gasolene was increased from $0.90 to $1.30 per litre.
What amount is the percentage increase in the price of the gas?

21. A seamstress charges $225 to sew a dress but gives a discount of 9% for cash. What amount is the cash price?

22. After a 10% increase a teacher's salary was $1 430. Calculate her salary if a 20% increase had been given instead.

23. Determine the annual income tax due on a taxable income of $36 000, if the tax rate is 35%.

24. Last year I paid $520.00 as income tax. If it is increased by 12% this year, calculate the amount of tax I now pay.

25. Miss Vitra earns $450 per week from which income tax is deducted at 30%. Determine the amount she actually gets.

26. After a 10% increase, a teacher's monthly salary was $2 400. Calculate the teacher's salary if a 15% increase had been given instead.

27. After an increase of 10% a clerk's salary was $1 320. Calculate her salary if an increase of 15% was given instead.

28. Last year I paid $520.00 as income tax. If it is increased by 15% this year, determine how much tax I now pay.

29. After a 4% increase an accountant's salary was $7 280. Calculate his salary if a 10% increase had been given instead.

30. After a 20% increase, a physicist's monthly salary was $7 200. Calculate the physicist's salary if a 25% increase had been given instead.

31. Mr. Capildeo's salary after a 5% increase was $6 255.

(a) Calculate his salary if a 3% increase had been given instead.

(b) Calculate his original salary.

Discount

Sometimes a store may have a sale because the stocks need to be *updated* and *modernized* and therefore the old stocks must be *sold* urgently. Thus a *discount* is

So the *cost price* of the refrigerator to the businessman (i.e. 100%) $= \$2745 \times \frac{100}{115}$

$= \$23.8696 \times 100$

$= \$2386.96$

(*correct* to the *nearest cent*)

Hence the *cost price* of the refrigerator was $2386.96.

(b) The *selling price* of the refrigerator (i.e. 88% *of the cost price*) = $2149

∴ the *multiplication factor* $= \frac{100}{88}$

So the *cost price* of the refrigerator to the businesswoman (i.e. 100%) $= \$2149 \times \frac{100}{88}$

$= \$24.4205 \times 100$

$= \$2442.05$

(*correct* to the *nearest cent*)

Hence the *cost price* of the refrigerator was $2442.05.

Example 10

A teacher's salary was $3300 after she had received an increase of 10%. Calculate the teacher's salary if she had received an increase of 20% instead.

Solution

The teacher's salary after the 10% *increase* (i.e. 110% *of her original salary*) = $3300

So the teacher's salary if a 20% *increase* had been given instead (i.e. 120% *of her original salary*) $= \$3300 \times \frac{12\cancel{0}}{11\cancel{0}}$

$= \$300 \times 12$

$= \$3600$

Hence the teacher's *new salary* would have been $3600.

Exercise 5g

1. Find the cost price:
Selling price of an article = $96
Profit % = 40%

2. A salesman buys a stove from a manufacturer. The salesman sells the stove for $1825.00 at a profit of 25%. How many dollars did the salesman pay the manufacturer for the stove?

3. A shopkeeper buys a television from a manufacturer. The shopkeeper sells the television for $2700.00 at a profit of 20%. What amount of money did the shopkeeper pay the manufacturer for the stove?

4. A merchant sold a pen for $6.90, thereby making a profit of 15% on the cost to her. Calculate:

(a) the cost price of the pen to the merchant to the nearest cent

(b) the selling price the merchant should request in order to make a 25% profit instead.

5. A salesman bought a computer from a manufacturer. The salesman then sold the computer for $15600 making a profit of 25%. What amount did the salesman pay the manufacturer for the computer?

6. A businesswoman sold a gas cooker for $1209.60 making a profit of 12% on the cost price. Calculate the cost price of the gas cooker.

7. An entrepreneur sold a damaged bed sheet for $130.50 thereby making a loss of 13% on the cost price. Determine the cost price of the bedsheet.

8. There are 150 shops at a mall, 56% of which sell toys. How many shops do not sell toys?

9. A girl's mass increased by 12% between her tenth and fourteenth birthdays. If her mass was 45 kg on her tenth birthday, what was her mass on her fourteenth birthday?

10. Miss Reyes earns $3500 per month from which income tax is deducted at 30%. Calculate her net pay.

(b) If the shopkeeper gives 10% discount for cash, what amount does a customer pay for the stove?

(c) If the shopkeeper sells the stove at a loss of 10% of the cost price, determine its selling price.

8. A boutique is offering a 15% discount for cash. Calculate the cash price for a dress with a marked price of $125.

9. A store is offering 18% discount for cash. Calculate the cash price for a pants with a marked price of $170.

10. A shopkeeper buys a television from a manufacturer. The shopkeeper sells the television for $2 700 at a profit of 20%.

(a) How much money did the shopkeeper pay the manufacturer for the television?

(b) If the shopkeeper gives 10% discount for cash, how much money does a customer pay for the television?

11. A salesman buys a stove from a manufacturer. The salesman sells the stove for $1 825.00 at a profit of 25%.

(a) What amount did the salesman pay the manufacturer for the stove?

(b) If the salesman gives 5% discount for cash, what amount does a customer pay for the stove?

Sales Tax—VAT

In a number of Caribbean countries, for example, Trinidad and Tobago, and Barbados, a *sales tax* is added to the price of an article. Both prices are supposed to be indicated, on each vatable item, before its purchase. In Trinidad and Tobago the magnitude of this *sales tax* (or *value added tax*; abbreviated *VAT*) is 15%.

In general, VAT is a tax paid on goods and services, to a supplier, for the government.

Example 13

In a Caribbean country the value added tax payable on items purchased is 15%. The price of an automatic washer without VAT is $3 473. Calculate the price of the washer to a customer inclusive of VAT.

Solution

The *price* of the automatic washer without VAT (i.e. 100%) $= \$3473$

$\therefore$ the *price* of the washer *inclusive of VAT* (i.e. 115% *of the pre-VAT price*) $= \$3473 \times \frac{115}{100}$

$= \$34.73 \times 115$

$= \$3993.95$

Hence the price of the washer to a customer *inclusive of VAT* was $3 993.95.

Example 14

The price of a lawnmower inclusive of 15% VAT is $1 050. Calculate the price of the lawnmower exclusive of VAT, correct to the nearest cent.

Solution

The *price* of the lawnmower *inclusive of VAT* (i.e. 115% *of the pre-VAT price*) $= \$1050$

$\therefore$ the *price* of the lawnmower *exclusive of VAT* (i.e. 100% *of the pre-VAT price*) $= \$1050 \times \frac{100}{115}$

$= \$9.1304 \times 100$

$= \$913.04$ (*correct* to the *nearest cent*)

Hence the *price* of the lawnmower *exclusive of VAT* was $913.04.

Exercise 5i

1. A DVD game player is priced at $2 800 plus value added tax (VAT) at 15%. How many dollars does the DVD game player actually cost the customer?

2. A man buys a television set, at a price exclusive of sales tax, for $2 124. If sales tax of 12% is charged, how much money did the man pay?

offered on certain articles and the consumer is able to purchase the articles at a *reduced price* called the *sale price.* A *cash discount* is a *discount* given if the amount owing is paid in *cash.*

Normally the *discount* is calculated as a *percentage* of the *selling price* (or *marked price*).

Thus:

The discount = *x% of the selling price*
The discounted price = *The selling price − The discount*
= *(100 − x)% of the selling price*

Example 11

A television set has a selling price of $1950. A 10% discount is offered for cash. What amount is its cash price to the customer?

Solution

The *selling price* of the T.V. = $1950

∴ the *discount* on the T.V. = *10% of the selling price*

$= \frac{1\cancel{0}}{10\cancel{0}} \times \$195\cancel{0}$

= $195

So the *cash price* to the customer = *The selling price − The discount*

= $(1950 − 195)

= $1755

Hence the *cash price* of the television set was $1755.

Alternative Method

The *selling price* of the T.V. = $1950

∴ the *discount* on the T.V. = 10% of $1950

∴ the *cash price* to the customer = (100 − x)% *of the selling price*

= (100 − 10)% of $1950

= 90% of $1950

$= \frac{9\cancel{0}}{10\cancel{0}} \times \$195\cancel{0}$

= $1755

Hence the *cash price* of the television set was $1755.

Example 12

In a sale a cassette recorder was sold for $2071 after a discount of 5% was given. Calculate the marked price of the cassette recorder.

Solution

The *discounted* (or sale) *price* of the cassette recorder (i.e. 95% *of the marked price*) = $2071

∴ the *marked price* of the cassette recorder (i.e. 100%) $= \$2071 \times \frac{100}{95}$

= $21.80 × 100

= $2180

Hence the *marked price* of the cassette recorder was $2180.

Exercise 5h

1. A boutique is offering a 15% discount for cash. Calculate the cash price for a dress with a marked price of $125.
2. A tailor charges $560 for a suit of clothes and gives a discount of 12% for cash. Calculate the discounted price for a suit of clothes.
3. Mr. Khan bought a refrigerator for $2560. Calculate the selling price of the refrigerator if he adds a profit of 20%.

   A 10% discount is offered for cash. Calculate its cash price.
4. An artist charges $980 for a portrait but gives 5% discount for cash. Calculate the cash price of your portrait?
5. A seamstress charges $225 to sew a dress but gives a discount of 9% for cash. What amount is the cash price?
6. A tailor charges $375 to sew a suit but gives a discount of 8% for cash. What amount is the cash price?
7. A shopkeeper buys a stove from a manufacturer. The shopkeeper sells the stove for $2500 at a profit of 20%.
   - (a) What amount did the shopkeeper pay the manufacturer for the stove?

$= \frac{88}{100} \times \$6980$

$= 88 \times \$69.80$

$= \$6142.40$

Hence the *cash price* of the television was $\$6142.40$.

(b) The *deposit* $= \$628.20$

And the *amount payable* = *The monthly instalment* × *The number of months*

$= \$344.06 \times 24$

$= \$8257.44$

∴ the *hire purchase price* for the television = *The deposit* + *The amount payable*

$= \$(628.20 + 8257.44)$

$= \$8885.64$

Hence the *hire purchase price* of the television was $\$8885.64$.

(c) The *difference* between the *hire purchase price* and the *marked price* for the television = *The hire purchase price* − *The marked price*

$= \$(8885.64 - 6980)$

$= \$1905.64$

Hence the *difference* is $\$1905.64$.

Note that the *difference paid* is the *interest charged* on the *outstanding balance*.

(d) The *interest charged* $= \$1905.64$

And the *outstanding balance* = *The amount payable* − *The interest charged*

$= \$(8257.44 - 1905.64)$

$= \$6351.18$

Or the *outstanding balance* = *The marked price* − *The down payment*

$= \$(6980 - 628.20)$

$= \$6351.80$

∴ the *percent interest charged* on the *outstanding balance* $= \dfrac{\textit{The interest charged}}{\textit{The outstanding balance}} \times 100\%$

$= \frac{\cancel{\$}1905.64}{\cancel{\$}6351.18} \times 100\%$

$= 30\%$

Hence the *percentage interest charged* on the *outstanding balance* was 30%.

Example 16

A housewife purchased a video recorder with a cash price of \$2980 under hire purchase terms. She paid an initial down payment of 20% of the cash price and interest which is equivalent to 15% of the outstanding balance is charged.

The balance is paid in 18 equal monthly instalments.
Calculate for the video recorder:

(a) the hire purchase price

(b) the amount of each monthly instalment

(c) the difference between the hire purchase price and the cash price.

Solution

(a) The *cash price* of the video recorder $= \$2980$

∴ the *initial down payment* = *x% of the cash price*

$= 20\%$ of $\$2980$

$= \frac{2\cancel{0}}{10\cancel{0}} \times \$298\cancel{0}$

$= \$596$

So the *outstanding balance* = *The cash price* − *The deposit*

$= \$(2980 - 596)$

$= \$2384$

And the *interest charged* on the *outstanding balance* = *x% of the outstanding balance*

$= 15\%$ of $\$2384$

$= \frac{15}{100} \times \$2384$

$= 15 \times \$23.84$

$= \$357.60$

∴ the *amount payable* = *The outstanding balance* + *The interest charged*

$= \$(2384 + 357.60)$

$= \$2741.60$

Hence the *hire purchase price* for the video recorder = *The down payment* + *The amount payable*

$= \$(596 + 2741.60)$

$= \$3337.60$

Hence the *hire purchase price* of the video recorder was $\$3337.60$.

3. An airline ticket to New York is priced at \$1 232.10 inclusive of 11% sales tax. What amount would the airline ticket cost exclusive of sales tax?

4. A refrigerator is priced at \$3 800 plus value added tax (VAT) at 15%. What amount does the refrigerator actually cost the customer?

5. A computer is priced at \$5 800 plus value added tax (VAT) at 15%. What amount does the computer actually cost the customer?

6. The customs duty on imported vehicles is 30% of the price paid by the importer.
   (a) Calculate the customs duty on a car for which the price paid by the importer is \$8 500.
   (b) Calculate the price paid by the importer for a bus for which the amount paid, including customs duty, is \$15 600.

7. An airline ticket to Miami is priced at \$994.75 inclusive of 15% sales tax. What amount would the airline ticket cost exclusive of sales tax?

8. A stove is priced at \$1 495 inclusive of 15% VAT. What amount is the price exclusive of VAT?

9. A woman buys a stove for \$1 035 exclusive of VAT. If VAT of 15% is charged, what amount did she actually pay for the stove.

10. A man buys a new Mazda 323 car for \$36 900 exclusive of sales tax. If sales tax of 12% is charged, how much money did the man actually pay for the car?

11. The customs duty on imported vehicles is 25% of the imported price.
    (a) Calculate the customs duty on a car for which the imported price is \$16 800.
    (b) Calculate the imported price of a truck for which the amount paid, inclusive of customs duty, is \$69 750.

12. The customs duty on imported vehicles is 35% of the imported price.
    (a) Calculate the customs duty on a van for which the imported price is \$12 500.
    (b) Calculate the imported price of a truck for which the amount paid, inclusive of customs duty, is \$61 560.

Hire Purchase

Often we are unable to purchase necessary goods, such as furniture and appliances, by paying cash immediately. Hence many people are attracted to certain business places that are licenced to trade in *hire purchase* (abbreviated *H.P.*) goods. Under the *hire purchase terms* a customer is allowed to purchase goods by making a *small deposit* (or *down payment*). *Interest* is then added to the *outstanding balance* and the customer is allowed to pay this *sum* (i.e. the *amount payable*) in a given number of *equal monthly instalments*. Of course, the larger the *initial deposit*, the smaller would be the outstanding balance and *interest payable*, and vice versa.

In some modern *hire purchase* agreements, *no deposit* (or *down payment*) is necessary. Of course, the *interest payable* is much higher.

Thus:

The hire purchase price = *The down payment + The amount payable*

The amount payable = *The outstanding balance + The interest charged*

Example 15

The marked price of a television set is \$6 980. If the consumer pays cash, then the price is 12% below the marked price. If the set is bought on hire purchase, then the buyer pays a down payment of \$628.20 and 24 monthly instalments of \$344.06 each.
Determine for the television:

(a) the cash price

(b) the hire purchase price

(c) the difference between the hire purchase price and the marked price

(d) the percent interest charged on the outstanding balance.

Solution

(a) The *cash price* for the television = $(100 - x)\%$ *of the marked price*

= (100 − 12)% of \$6 980

= 88% of \$6 980

3. (a) A refrigerator can be bought on hire purchase by making a deposit of $500 and 18 monthly instalments of $56.50 each. Calculate the hire purchase cost of the refrigerator.

(b) The actual marked price of the refrigerator is $1 260. This includes a sales tax of 20%. Calculate the selling price of the refrigerator if no sales tax is included.

4. A video game set can be bought on hire purchase by making a deposit of $190 and 12 monthly instalments of $171 each. Calculate the hire purchase cost of the video game set. The actual marked price of the video game set is $1 900. This includes a sales tax of 15%. Calculate the selling price of the video game set if no sales tax is included.

5. The marked price of a freezer is $3 000.00. There is a discount of 15% for cash payment. To obtain the freezer on hire purchase, a deposit of $595.00 and 18 monthly instalments of $159.50 each are required.
Calculate:

(a) the cash price

(b) the total amount paid if bought on hire purchase

(c) the difference between the cash price and the hire purchase price as a percentage of the marked price.

6. The marked price of a car is $49 500. A person can pays deposit of 30% and interest at 12% per annum is charged on the outstanding balance. The total amount payable is to be paid in $2\frac{1}{2}$ years.
Calculate:

(a) the amount of each monthly instalment

(b) the hire purchase price of the car.

7. The retail price of a television set is $2 500. If the buyer pay cash, the price is 10% below the retail price. If the set is bought on hire purchase, the buyer pays a downpayment of $500 and 18 monthly instalments of $150.

(a) Determine the difference between the hire purchase price and the cash price.

(b) Calculate the difference as a percentage of the retail price.

8. (a) A stove can be bought on hire purchase by making a deposit of $650 and 12 monthly instalments of $195 each. Calculate the hire purchase price of the stove.

(b) The actual marked price of the stove is $2 400. This includes a sales tax of 12.5%. Calculate the selling price of the stove if no sales tax is included.

(c) Calculate the difference between the hire purchase price and the marked price as a percentage of the outstanding balance.

9. A freezer can be bought on hire purchase by making a deposit of 15% of the cash price which is $2 975. Interest which is equivalent to 20% of the outstanding balance is charged. The amount payable is paid in 12 monthly instalments.

Calculate for the freezer.

(a) the deposit

(b) the hire purchase price

(c) the difference between the hire purchase price and the cash price

(d) the difference as a percentage of the cash price.

10. A computer can be bought on hire purchase by making a deposit of the 15% on the marked price which is $2 975. Interest which is equivalent to15% of the outstanding balance is charged. The amount payable is paid in 12 equal monthly instalments.

(a) Calculate for the freezer:

  (i) the deposit
  (ii) the amount of each instalment
  (iii) the hire purchase price.

(b) If a discount of 12% is given for cash, calculate:

  (i) the cash price
  (ii) the difference between the hire purchase price and the cash price
  (iii) the difference as a percentage of the marked price.

11. A housewife purchased a video recorder with a cash price of $2 700 under hire purchase terms. She paid an initial deposit of 20% of the cash price and interest at 18% per annum on the outstanding balance is charged. The amount payable is paid in 12 equal monthly instalments.

(b) *The amount of each monthly instalment* $= \dfrac{\textit{The amount payable}}{\textit{The number of monthly instalments}}$

$= \dfrac{\$2\,741.60}{18}$

$= \$152.31$ (*correct* to the *nearest cent*)

Hence *each monthly instalment* was $152.31.

(c) The *difference* between the *hire purchase price* and the *cash price* for the video recorder = *The hire purchase price − The cash price*

$= \$(3\,337.60 - 2\,980)$

$= \$357.60$

Hence the *difference* is $357.60.

Note that the *difference paid* is the *interest charged* on the *outstanding balance.*

Example 17

A chest freezer can be purchased cash for $3 475 or on hire purchase for a deposit 25% and 24 equal monthly instalments of $130.31.

Determine for the chest freezer:

(a) the hire purchase price

(b) the interest charged

(c) the percentage interest charged.

Solution

(a) The *cash price* for the freezer $= \$3\,475$

∴ the *deposit* = *x% of the cash price*

$= 25\%$ of $\$3\,475$

$= \dfrac{25}{100} \times \$3\,475$

$= 25 \times \$34.75$

$= \$868.75$

And the amount payable = *The monthly instalment × The number of months*

$= \$130.31 \times 24$

$= \$3\,127.44$

So the *hire purchase price* for the freezer = *The deposit + The amount payable*

$= \$(868.75 + 3\,127.44)$

$= \$3\,996.19$

Hence the *hire purchase price* of the freezer was $3 996.19.

(b) The *interest charged* = *The hire purchase price − The cash price*

$= \$(3\,996.19 - 3\,475)$

$= \$521.19$

Hence the *interest charged* was $521.19.

(c) The *outstanding balance* = *The cash price − The deposit*

$= \$(3\,475 - 868.75)$

$= \$2\,606.25$

Hence the *outstanding balance* was $2 606.25.

And the *percentage interest charged* $= \dfrac{\textit{The interest charged}}{\textit{The outstanding balance}} \times 100\%$

$= \dfrac{\cancel{\$}521.19}{\cancel{\$}2\,606.25} \times 100\%$

$= \dfrac{52\,119}{2\,606.25}$

$= 20\%$

Hence the *percentage interest charged* was 20%.

Exercise 5j

1. The retail price of a television set is $4 500. If the buyer pays cash, the price is 10% below the retail price. If the set is bought on hire purchase, the buyer pays a downpayment of $675 and 24 monthly instalments of $212.50.
   (a) Determine the difference between the hire purchase price and the cash price.
   (b) Calculate this difference as a percentage of the retail price.

2. (a) A computer can be bought on hire purchase by making a deposit of $1 360 and 40 monthly instalments of $442 each. Calculate the hire purchase price of the computer.
   (b) The actual marked price of the computer is $15 600. This includes a sales tax of 12.5%. Calculate the selling price of the computer if no sales tax is included.

(e) The *total amount paid* for the house = *The deposit* + *The total amount paid*

= $(15 000 + 540 000)

= $555 000

Hence the *total amount paid* for the house was $555 000.

Exercise 5k

1. A country house is on sale for $90 000 cash and a bank offers an 85% mortgage. Calculate the deposit necessary.

2. A luxury apartment is priced at $225 000. If a bank offers a 90% mortgage, calculate the deposit required.

3. A house on sale costs $175 000. What amount is borrowed if the deposit is:

(a) 10% of the sale price

(b) 15% of the sale price.

4. A townhouse is on sale for $150 000 cash. Calculate the amount borrowed if the deposit is:

(a) 12.5% of the cash price

(b) 17.5% of the cash price.

5. A condominium is on sale for $275 000. It is possible to buy the condominium by making a 10% deposit and taking a bank mortgage. Calculate:

(a) the deposit

(b) the amount borrowed

(c) the total amount paid to the bank, if monthly payments of $3 403 are made over a 25-year period.

6. A country house can be bought for $95 000. It is possible to purchase the country house by making an 8% deposit and taking a mortgage. Determine:

(a) the deposit

(b) the amount borrowed

(c) the total amount of money paid to the bank, if monthly payments of $1 395 are made over a 15-year period.

7. A town house costing $185 000 can be bought by making a 10% deposit and taking a bank mortgage for the remaining amount. Calculate:

(a) the deposit

(b) the amount borrowed

(c) the total amount paid to the bank after 15 years if the monthly payment was $2 728

(d) the interest paid to the bank.

8. A flat house costing $125 000 can be bought by making a deposit of 15% and taking a bank mortgage for the remaining amount. Determine:

(a) the deposit

(b) the amount borrowed

(c) the total amount paid to the bank after 10 years if the monthly payment was $2 125

(d) the interest paid to the bank.

9. A luxury apartment is priced at $235 000. An 85% mortgage can be obtained over a 20-year period. Calculate:

(a) the deposit payable

(b) the loan amount needed

(c) the total amount of money paid to the bank if each monthly payment was $2 829

(d) the total amount paid for the house.

10. A bungalow is priced at $95 000. A 90% mortgage can be obtained over a 12-year period. Determine:

(a) the deposit

(b) the amount of the loan required

(c) the total amount paid to the bank if each monthly payment was $1 520

(d) the actual amount paid for the house.

Rates—Land and Building Taxes

Rates are annual taxes paid by owners of land and buildings in a town or city and are levied by the *local government.* Each building or piece of land within the government's boundary is given a *rateable value* depending on its location, size and general condition. This *rateable value* is always much less than the real value of the land or building. Each land or building owner is then charged *rates* which are a *percentage* of the *assessed valuation* (or *rateable value*).

Calculate for the video recorder:

(a) the hire purchase price
(b) the amount of each monthly instalment
(c) the difference between the hire purchase price and the cash price
(d) the difference as a percentage of the cash price.

12. A chest freezer can be purchased with cash for \$2 845 or on hire purchase for a deposit of 25% and 18 equal monthly instalments of \$142.25.

Calculate for the chest freezer:

(a) the hire purchase price
(b) the interest charged
(c) the percentage interest charged on the outstanding balance.

13. (a) A television can be bought on hire purchase by making a deposit of \$600 and 24 monthly instalments of \$110.50 each. Calculate the hire purchase price of the television.

(b) The actual marked price of the television is \$2 435.00. This includes a sales tax of 15%. Calculate the sale price of the television if no sales tax is included.

Mortgage

It is extremely difficulty for someone to buy a car or to build a house in these hard times with their own immediate cash, because of the high cost involved. It is therefore normal for the head of a family to buy a car or to build a house by taking a *mortgage loan* from a commercial bank or mortgage finance company. Under the *mortgage agreement*, the car or the house is legally in the hands of the bank, until the loan is completely repaid with interest.

These day it is normal to obtain a 90% *mortgage* from a bank to buy a house. By a 90% *mortgage*, we mean that the purchaser of a house must first make a deposit of 10% of the cost of the house, and the bank on approval will issue a loan to cover the 90% *balance*. *Mortgage loans* are normally taken for a long period of time, say 10 to 25 years, and therefore the amount payable is far more than the amount borrowed from the bank. Under the *mortgage loan agreement*, equal monthly instalments will have to be made to the bank for 10 to 25 years.

Example 18

A house costing \$150 000 can be bought by making a 10% deposit and taking a bank mortgage. Calculate:

(a) the deposit
(b) the amount borrowed
(c) the total amount paid to the bank, if monthly payments of \$2 250 are made over a 20-year period
(d) the amount of interest paid to the bank
(e) the total amount paid for the house.

Solution

(a) The *deposit* = *x% of the cost of the house*

$= 10\%$ of \$150 000

$= \frac{1\cancel{0}}{1\cancel{0}\cancel{0}} \times \$1500\cancel{0}\cancel{0}$

$= \$15\,000$

Hence the *deposit* was \$15 000.

(b) The *amount borrowed* = *The cost of the house − The deposit*

$= \$(150\,000 - 15\,000)$

$= \$135\,000$

Hence the *amount borrowed* was \$135 000.

Alternatively, the 90% *mortgage* = 90% *of the cost of the house*

$= \frac{9\cancel{0}}{1\cancel{0}\cancel{0}} \times \$15000\cancel{0}$

$= \$135\,000$

(c) The *total amount paid* to the bank = *The monthly instalment × The number of months*

$= \$2\,250 \times 12 \times 20$

$= \$540\,000$

Hence the *total amount paid* to the bank was \$540 000.

(d) The *amount of interest paid* to the bank = *The total amount paid − The amount borrowed*

$= \$(540\,000 - 135\,000)$

$= \$405\,000$

Hence the *amount of interest paid* to the bank was \$405 000.

In a certain district the assessed valuation of property is:

Land : \$12 748 000

Buildings: \$78 947 000

The district council has plans to spend \$18 339 000 next year.

(a) What amount is the total rateable value of the property?

(b) Calculate the rate the council should charge to cover exactly its planned spending.

10. In a town the rateable value for all the property is \$94 768 000. What value should the rate be if the total expenses for the town in a particular year are \$23 692 000?

11. A householder pays \$175 in rates in a particular year when the rate was levied at \$0.35 in the \$1. What amount was the rateable value of the house?

12. The rates charged by a local council are 47 ¢ in the \$1. Calculate the rateable value of a house if the rates payable per annum is \$171.55.

13. The rates charged in a district are 29%. Determine the total rateable value of the property in that district if the rates payable per annum is \$17 568 000.

14. A shopping mall owner paid \$2 475 in rates when the rate was levied at \$0.45 in \$1. Determine the rateable value of the shopping mall.

15. A grocery owner pays \$1 840 in rates during a particular year when rates were levied at 32 ¢ in the \$1. Determine the assessed value of the grocery.

Water Rates

In most countries water is provided by the state as a *public service*. In turn, the users of this service, that is, people who use water daily in their houses, factories, industries and for agricultural purposes, for example, are asked to pay a *token charge*, because it is not necessarily the cost of providing this *essential service*. This *water tax* is usually charged at *varying rates* depending on the *volume of water consumed*, and it is normally payable annually, half-yearly or quarterly.

Example 22

Mr. Da Silva used 105 m^3 of water for the first half of 2002. In 2002, water rates for domestic users for half a year were as follows:

\$2.50 per cubic metre for the first 25 m^3

\$2.00 per cubic metre for the next 50 m^3

\$1.50 per cubic metre for amounts in excess of 75 m^3

5% discount on bills paid before July 7.

Calculate the amount Mr. Da Silva paid for the half year, assuming that the bill was paid before July 7.

Solution

The *cost* for the *first* 25 m^3 of water used = *The cost per unit* × *The number of units used*

= \$2.50 × 25

= \$62.50

The *cost* for the *next* 50 m^3 of water used = *The cost per unit* × *The number of units used*

= \$2.00 × 50

= \$100.00

The *cost* for the *remaining* 30 m^3 of water used = *The cost per unit* × *The number of units used*

= \$1.50 × 30

= \$45.00

∴ the *amount* Mr. Da Silva was *billed* for the water used = \$(62.50 + 100.00 + 45.00)

= \$207.50

So the *amount* Mr. Da Silva *paid* for the half year *after the discount* = *(100 − x)% of the amount billed*

= (100 − 5)% of \$207.50

= 95% of \$207.50

= 0.95 × \$207.50

= \$197.125

= \$197.13 (*correct* to the *nearest cent*)

Hence Mr. Da Silva *paid* \$197.13 for the half year.

Thus:

$$\textit{The rates payable per annum} = \textit{The rate charged} \times \textit{The rateable value}$$

$$\textit{The rate charged} = \frac{\textit{The rates payable per annum}}{\textit{The rateable value}}$$

$$\textit{The rateable value} = \frac{\textit{The rates payable per annum}}{\textit{The rate charged}}$$

Example 19

The rateable value of a house in Bel Air is $1 625. Given that the rates charged by the local council (or government) for that area are 25 ¢ in the $1, determine the amount of money the owner pays in rates per annum.

Solution

The *rate charged* = 25 ¢ in the $1 = 25% = 0.25

∴ the *rate payable per annum* for the house

$$= \textit{The rate charged} \times \textit{The rateable value}$$

$$= 0.25 \times \$1\,625$$

$$= \$406.25$$

Hence the owner pays $406.25 in *rates* per annum for the house.

Example 20

The total rateable value of all the property in a town is $9 768 000. What is the value of the minimum rate that will allow the local council to realize $3 418 800 per annum?

Solution

$$\textit{The rate charged} = \frac{\textit{The rates payable per annum}}{\textit{The rateable charged}}$$

$$= \frac{\cancel{\$}3\,418\,8\cancel{00}}{\cancel{\$}9\,768\,0\cancel{00}}$$

$$= 0.35$$

Hence the *minimum rate* levied is $0.35 in the $1 or 35 ¢ in the $1.

Example 21

The rates charged by a local council are 43 ¢ in the $1. Calculate the rateable value of a house if the rate payable per annum is $375.00.

Solution

$$\textit{The rateable value}\text{ of the house} = \frac{\textit{The rates payable per annum}}{\textit{The rate charged}}$$

$$= \frac{\$375}{0.43}$$

= $872.09 (*correct* to the *nearest cent*)

Hence the *rateable value* of the house is $872.09.

Exercise 51

1. The rateable value of a house is $4 500. Calculate the rates payable by the householder, for a particular year, when the rates are $0.21 in the $1.
2. The rate for all properties in Georgetown is 25%. What amount is paid in rates for a property, if its rateable value is $4 500?
3. The rateable value of a house is $3 500. Determine the rates payable by the houseowner when the rates are $0.23 in the $1.
4. The assessed valuation of a business place is $9 840. Determine the amount paid in rates when the rates were 22 ¢ in the $1.
5. The rateable value of a cinema is $6 425. Calculate the amount paid in rates when the rates were 27 ¢ in the $1.
6. The rateable value for all the property in a city is $96 864 000. What must be the value of the rates if the total expenses for the city for a particular year are $32 104 000?
7. What value of rate should be charged to raise $5 150 000 from a total rateable value of $11 845 000?
8. The total rateable value of the property in a city is $80 000 000.
   - (a) What amount of money would be obtained from a rate of 5%?
   - (b) What value of rate is needed to collect $16 000 000?

remaining 5 m^3 of gas used = *The cost per unit* × *The number of units used*

= \$1.05 × 5

= \$5.25

The *total cost* for the 85 m^3 of gas used = \$(25.50 + 47.50 + 5.25)

= \$78.25

The *government tax* (*VAT*) = *x% of the total cost*

= 15% of \$78.25

= 0.15 × \$78.25

= \$11.74 (*correct* to the *nearest cent*)

∴ the *amount* Mrs. O'Neil was *billed* for the gas used = \$(78.25 + 11.74)

= \$89.99

So the *amount* Mrs. O'Neil *paid* for the half year *after the discount* = (100 − *x*)*% of the amount billed*

= (100 − 3)% of \$89.99

= 97% of \$89.99

= 0.97 × \$89.99

= \$87.29 (*correct* to the *nearest cent*)

Hence Mrs. O'Neil *paid* \$87.29 for the half year.

Exercise 5n

1. Mr. Albert used 75 m^3 of domestic gas for the first half of 2000. In 2000, gas rates for domestic users for a half year were as follows:

   \$0.75 per cubic metre for the first 60 m^3

   \$0.90 per cubic metre for amounts in excess of 60 m^3.

   Government tax (VAT) = 15%.

   10% discount on bills paid before July 14.

   Calculate the amount Mr. Albert paid for the half year, assuming that the bill was paid before July 14.

2. A household used 70 m^3 of gas for the first half of 2001. In 2001, gas rates for domestic users for a half year were as follows:

   \$1.10 per cubic metre for the first 40 m^3

   \$1.20 per cubic metre for amounts in excess of 40 m^3.

   Government tax (VAT) = 15%.

   10% discount on bills paid within 2 weeks of billing.

   Determine the amount the household paid for the half year, assuming that the bill was paid within the 2-week period.

3. A hotel used 1 574 m^3 of gas for the first quarter of 2002. In 2002, gas rates for business places for a quarter year were as follows:

   \$1.25 per cubic metre for the first 500 m^3

   \$2.25 per cubic metre for the next 500 m^3

   \$3.25 per cubic metre for the next 500 m^3

   \$4.25 per cubic metre for amounts in excess of 1 500 m^3.

   Government tax (VAT) = 15%.

   7% discount on bills paid within 2 weeks of billing.

   Determine the amount the hotel owner paid for the quarter year, assuming that the bill was paid within the 2-week period.

4. A school used 125 m^3 of gas in 2001. In 2001, gas rates for government building for a year were as follows:

   \$0.75 per cubic metre for the first 40 m^3

   \$0.85 per cubic metre for the next 60 m^3

   \$0.95 per cubic metre for amount in excess of 100 m^3.

   Evaluate the amount the government paid for that year in gas rates for the school.

5. A steel plant used 63 400 m^3 of gas for the first quarter of 2002. In 2002, gas rates for commercial users for a quarter year were as follows:

   \$0.65 per cubic metre for the first 20 000 m^3

   \$0.55 per cubic metre for the next 40 000 m^3

   \$0.45 per cubic metre for the remaining units.

   Government tax (VAT) = 15%.

   10% discount on bills paid within 3 weeks of billing.

   Calculate the amount the steel plant owner paid for the quarter year, assuming that the bill was paid within the 3-week period.

Posting a Parcel

We all are charged a *fee* for posting a letter or parcel. The *cost of posting* a letter or parcel will vary

Exercise 5m

1. Mrs. Franka used 85 m^3 of water for the first quarter of 2000. In 2000, water rates for domestic users for a quarter year were as follows:

   \$1.25 per cubic metre for the first 20 m^3

   \$1.00 per cubic metre for the next 60 m^3

   \$0.75 per cubic metre for amounts in excess of 80 m^3.

   6% discount on bills paid before April 7. Calculate the amount Mrs. Franka paid for the quarter year, assuming that the bill was paid before April 7.

2. Mr. Khan used 125 m^3 of water for 2001. In 2001, water rates for domestic users for a year were as follows:

   \$1.50 per cubic metre for the first 50 m^3

   \$1.25 per cubic metre for the next 50 m^3

   \$1.00 per cubic metre for amounts in excess of 100 m^3

   5% discount on bills paid before January 9. Determine the amount Mr. Khan paid for the year, assuming that the bill was paid before January 9.

3. A hotel used 1 825 m^3 of water for the first half of 2000. In 2000, water rates for commercial users for a half year were as follows:

   \$2.25 per cubic metre for the first 500 m^3

   \$2.75 per cubic metre for the next 500 m^3

   \$3.25 per cubic metre for amounts in excess of 1 000 m^3.

   9% discount on bills paid before July 14. Calculate the amount the hotel owner paid for the half year, assuming that the bill was paid before July 14.

4. A ministry of education used 1 342 m^3 of water for the year 2001. In 2001, water rates for government buildings for a year were as follows:

   \$1.75 per cubic metre for the first 600 m^3

   \$1.50 per cubic metre for the next 600 m^3

   \$1.25 per cubic metre for amounts in excess of 1 200 m^3.

   Evaluate the amount the government paid for that year in water rates for the ministry.

5. A sweet drink factory used 3 285 m^3 of water for the first half of 2002. In 2002, water rates for commercial users for a half year were as follows:

   \$2.50 per cubic metre for the first 1 000 m^3

   \$2.25 per cubic metre for the next 1 000 m^3

   \$2.00 per cubic metre for amounts in excess of 2 000 m^3.

   12% discount on bills paid before July 15. Determine the amount the factory owner paid for the half year, assuming that the bill was paid before July 15.

Gas Rates

In some countries gas for domestic and/or commercial purposes is pipe borne from the source of manufacture to the receivers or users. The gas used is usually charged at *varying rates* depending on the *volume of gas consumed* and it is normally payable monthly, quarterly or half-yearly.

Example 23

Mrs. O' Neil used 85 m^3 of domestic gas for the first half of 2003. In 2003, gas rates for domestic users for a half year were as follows:

\$0.85 per cubic metre for the first 30 m^3

\$0.95 per cubic metre for the next 50 m^3

\$1.05 per cubic metre for amounts in excess of 80 m^3.

Government tax (VAT) = 5%.

3% discount on bills paid before July 14. Calculate the amount Mrs. O' Neil paid for the half year assuming that the bill was paid before July 14.

Solution

The *cost* for the *first* 30 m^3 of gas used = *The cost per unit* × *The number of units used*

= \$0.85 × 30

= \$25.50

The *cost* for the *next* 50 m^3 of gas used = *The cost per unit* × *The number of units used*

= \$0.95 × 50

= \$47.50

4. The rates for posting parcels are as follows:

Parcels not exceeding 600 g	$0.30
Each additional 600 g or part thereof	$0.25
Registration fee for registering a parcel	$4.25

Determine the cost of posting:

(a) an unregistered parcel of mass 1 500 g

(b) a registered parcel of mass 3 kg.

5. The rates for posting parcels are as follows:

Parcels not exceeding 500 g	$0.90
Each additional 500 g or part thereof up to a maximum of 3 000 g	$0.85
Registration fee for registering a parcel	$3.95

Calculate the cost of posting:

(a) an unregistered parcel of mass 1 kg

(b) a registered parcel of mass 2.7 kg.

Electricity Bills

All householders using electrical appliances must pay an *electricity bill* for the energy used. Electricity is charged for according to the number of *units of energy* used in a given period, for example, a month or a quarter, and it is measured in *kilowatt-hours* (abbreviated *kWh*).

There are a number of *variables* appearing in an *electricity bill.* For example, there may be both a *fuel charge* and an *energy charge* at different varying rates. Then the units may be charged for according to whether the time of day is a *peak time* (i.e. when many units of power are being used) or *off-peak time* (i.e. when less units of power are being used). There may also be a *standing charge* for the *rental* of the *meter.* Of course, *value added tax* (abbreviated *VAT*) can also be added. Also in some countries a *discount* is given for payment within a given time. In any given problem, the *variables* being used will be clearly stated.

The *abbreviations* of the commonly used *units* are:

watt : *W*
kilowatt : *kW*
watt-hour : *Wh*
kilowatt-hour: *kWh*

And the *conversion tables* are:

$$1\,kW = 1000\,W$$
$$1\,kWh = 1000\,Wh$$
$$1\,kWh = 1\,unit$$

Example 25

How many units of electricity would:

(a) a 5 kW heater use in 7 hours

(b) a 60 W bulb use in 84 hours

(c) a 175 W refrigerator use in 24 hours?

Solution

(a)
$$\text{The } \textit{number of units of electricity} \text{ used by the 5 kW heater} = \textit{The power rating in kW} \times \textit{The number of hours used}$$
$$= 5\text{ kW} \times 7\text{ h}$$
$$= 35\text{ kWh}$$
$$= 35\text{ units}$$

Hence 35 *units of electricity* were used.

(b)
$$\text{The } \textit{number of units of electricity} \text{ used by the 60 W bulb} = \textit{The power rating in kW} \times \textit{The number of hours used}$$
$$= \frac{6\cancel{0}}{1\,00\cancel{0}}\text{ kW} \times 84\text{ h}$$
$$= \frac{504}{100}\text{ kWh}$$
$$= 5.04\text{ units}$$

Hence 5.04 *units of electricity* were used.

(c)
$$\text{The } \textit{number of units of electricity} \text{ used by the 175 W refrigerator} = \textit{The power rating in kW} \times \textit{The number of hours used}$$
$$= \frac{175}{1\,000}\text{ kW} \times 24\text{ h}$$
$$= \frac{4\,2\cancel{00}}{1\,0\cancel{00}}\text{ kWh}$$
$$= 4.2\text{ units}$$

Hence 4.2 *units of electricity* were used.

according to the distance or country to which it is going. We pay this *fee* by *purchasing* stamps of the *equivalent value* in and affixing them to the letter or parcel. This method of posting is called *unregistered posting*. Sometimes there is a need to *register* a letter or parcel in order to ensure safe passage to the addressee. Therefore we have to pay an additional charge for *registration*. In such a case, the general post office is held liable if the letter or parcel is not received by the addressed party.

Example 24

The rates for posting letters or parcels to Guyana in 2002 were as follows:

Letters or parcels not exceeding 10 g	*$0.50*
Each additional 10 g or part thereof up to a maximum of 2 500 g	*$0.20*
Registration fee for registering a parcel	*$2.50*

Calculate the cost of posting:

(a) an unregistered letter of mass 45 g

(b) a registered parcel of mass 1.5 kg.

Solution

(a) The *cost for posting* the *first* 10 g = *The fixed charge*
= $0.50

The *cost for posting* the *next* 30 g = *The charge per unit* × *The number of units*
= $0.20 × 3
= $0.60

The *cost for posting the remaining* 5 g = *The charge per unit*
= $0.20

∴ the *cost for posting* the *unregistered* letter *of mass* 45 g = $(0.50 + 0.60 + 0.20)
= $1.30

Hence the *cost of posting* the *unregistered letter* is $1.30.

(b) The *mass* of the parcel = 1.5 kg
= 1.5 × 1 000 g
= 1 500 g

The *cost for posting* the first 10 g = *The fixed charge*
= $0.50

The *cost for posting* the *remaining* 1 490 g = *The cost per unit* × *The number of units*
= $0.20 × 149
= $29.80

The registration fee = $2.50

∴ the *cost for posting* the *registered* parcel *of mass* 1.5 kg = $(0.50 + 29.80 + 2.50)
= $32.80

Hence the *cost of posting* the *registered* parcel is $32.80.

Exercise 5o

1. The rates for posting letters to the US Virgin Islands in 2002 were as follows:

Letters not exceeding 10 g	$0.50
Each additional 10 g or part thereof	$0.20
Registration fee for registering a letter	$2.25

Calculate the cost of posting:

(a) an unregistered letter of mass 27 g

(b) a registered letter of mass 35 g.

2. The rates for posting letters to Zimbabwe in 2002 were as follows:

Letters not exceeding 10 g	$1.00
Each additional 10 g or part thereof	$0.60
Registration fee for registering a letter	$2.75

Determine the cost of posting:

(a) an unregistered letter of mass 39 g

(b) a registered letter of mass 25 g.

3. The rates for posting parcels are as follows:

Parcels not exceeding 600 g	$0.30
Each additional 600 g or part thereof up to a maximum of 3 500 g	$0.25
Registration fee for registering a parcel	$2.75

Evaluate the cost of posting:

(a) an unregistered parcel of mass 1 350 g

(b) a registered parcel of mass 2.5 kg.

(c) The *fuel charge* = *The fuel charge per unit* × *The number of fuel units used*
= 45 ¢ × 2 228
= 100 260 ¢
= $1 002.60

Hence the *fuel charge* is $1 002.60.

(d) The *amount* the businessman was *billed* for the electricity used = $(522.44 + 1 002.60)
= $1 525.04

Hence the businessman was *billed* $1 525.04 for the energy used.

(e) The *actual amount* Mr. Belmar *paid* if a 10% *discount* was given = *(100 − x)% of the amount billed*
= (100 − 10)% of $1 525.04
= 90% of $1 525.04
= 0.9 × $1 525.04
= $1 372.536
= $1 372.54 (*correct* to the *nearest cent*)

Hence Mr. Belmar *paid* $1 372.54.

Exercise 5p

1. How many units of electricity would a 3 kW water heater use in 12 hours?

2. How many units of electricity would a 75 W bulb use in 100 hours?

3. How many units of electricity would a 150 W refrigerator use in 24 hours?

4. How many units of electricity would a 100 W bulb use in 600 hours?

5. How many units of electricity would a 125 W hair blower use in 2.5 hours?

6. Calculate the number of kilowatt-hour a 7 kW industrial pump used in 15 hours.

7. Calculate the number of kilowatt-hour a 60 W bulb used in 125 hours.

8. Calculate the number of kilowatt-hour a 175 W refrigerator used in 48 hours.

9. Calculate the number of kilowatt-hour a 120 W drill used in 9.5 hours.

10. Calculate the number of kilowatt-hour a 225 W freezer used in 168 hours.

11. Calculate the quarterly electricity bill for the following household:

Table 5.16

Name	*Number of units used*		*Cost per peak unit*
	Peak	*Off-peak*	
Mrs. Keate	576	1 420	12 ¢

Assume that there is a standing charge of $31.50, and that off-peak units are sold at half price.

12. Determine the quarterly electricity bill for the following household:

Table 5.17

Name	*Number of units used*	*Standing charge*	*Cost per unit*
Mr. Patton	1 428	$36.40	8.49 ¢

13. Evaluate the quarterly electricity bill for Mrs. Robinson

Table 5.18

Name	*Number of units used*	*Standing charge*	*Cost per unit*
Mrs. Robinson	1 695	$35.20	11.65 ¢

14. Calculate the quarterly electricity bill for the following household:

Table 5.19

Name	*Number of units used*	*Standing charge*	*Cost per unit*
Mr. Farouk	1 748	$34.50	15 ¢

15. Determine the quarterly electricity bill for the following household:

Table 5.20

Name	*Number of units used*	*Standing charge*	*Cost per unit*
Mr. Quan Soon	1 983	$31.75	16.5 ¢

Example 26

Calculate the quarterly electricity bill for the following household. Assume that there is a standing charge of $35.90 and that off-peak units are sold at half price.

Table 5.14

Electricity Bill			
Name	*Number of units used*		*Cost per peak unit*
	Peak	*Off-peak*	
Mr. Deygoo	245	793	15 ¢

Solution

The *cost* for the *peak units* used = *The cost per peak unit* × *The number of peak units used*

= 15 ¢ × 245

= 3 675 ¢

= $36.75

The *cost* for the *off-peak units* used = *The cost per off-peak unit* × *The number of off-peak units used*

= $\frac{15}{2}$ ¢ × 793

= 7.5 ¢ × 793

= 5 947.5 ¢

= $59.475

= $59.48 (*correct* to the *nearest cent*)

The *standing charge* = $35.90

∴ the *quarterly electricity bill* for Mr. Deygoo = $(36.75 + 59.48 + 35.90)

= $132.13

Hence the *quarterly electricity bill* for Mr. Deygoo is $132.13.

Example 27

Charges for electricity in a Caricom country are made up of a fixed fuel charge of 45 cents per kWh and an energy charge which is computed under three schemes as follows:

Scheme A. Homes	*18 cents per kWh*
Scheme B. Schools	*22 cents per kWh*
Scheme C. Business places	*24 cents per kWh for the first 1 000 units* *23 cents per kWh for the remaining units.*

Table 5.15

Electricity Bill					
Meter reading (kWh)		*kWh used*	*Scheme*	*Energy charge ($)*	*Fuel charge ($)*
Present	*Previous*				
6403	*4175*		*C*		

Calculate:

(a) the number of kilowatt-hour used

(b) the energy charge in dollars

(c) the fuel charge in dollars

(d) the amount the businessman was billed for the electricity used

(e) the actual amount Mr. Belmar paid if a 10% discount was given for cash.

Solution

(a) The *number of* kilowatt-hour used = *The present meter reading* − *The previous meter reading*

= (6403 − 4175) kWh

= 2228 kWh

Hence 2228 kWh were used.

(b) The *energy charge* for the *first* 1 000 units = *The energy charge per unit* × *The number of energy units used*

= 24 ¢ × 1 000

= 24 000 ¢

= $240

The *energy charge* for the *remaining* 1 228 units = *The energy charge per unit* × *The number of energy units used*

= 23 ¢ × 1 228

= 28 244 ¢

= $282.44

∴ the *total energy charge* = $(240 + 282.44)

= $522.44

Hence the *energy charge* is $522.44.

per unit and an energy charge computed under three schemes as follows:

Scheme A.	Homes	15 cents per kWh
Scheme B.	Schools	20 cents per kWh
Scheme C.	Business places	30 cents per kWh.

The meter reading of a certain school reads as follows:

Table 5.25

Meter reading (kWh)		*Units used*	*Scheme*	*Energy charge*	*Fuel charge*
Present	*Previous*				
47 936	34 372		B		

Calculate:

(a) the number of units used

(b) the energy charge

(c) the fuel charge

(d) the total amount the school had to pay for the electricity used.

22. Charges for electricity in a Caricom country are made up of a fixed fuel charge of 45 cents per kWh and an energy charge computed under three schemes as follows:

Scheme A.	Homes	18 cents per kWh
Scheme B.	Schools	22 cents per kWh
Scheme C.	Business places	24 cents per kWh.

The meter reading of a certain household reads as follows:

Table 5.26

Meter reading (kWh)		*kWh used*	*Scheme*	*Energy charge ($)*	*Fuel charge ($)*
Present	*Previous*				
7 201	7 076		A		

Calculate:

(a) the number of kilowatt-hour

(b) the energy charge in dollars

(c) the fuel charge in dollars

(d) the amount the householder had to pay for the electricity used

(e) the actual amount the householder paid if a discount of 10% was given for cash.

23. In May 2004, Mr. Gomes' electricity bill was calculated on the following information:

Table 5.27

Present meter reading (kWh)	*Last meter reading (kWh)*	*Rate*	*Fuel charge*
9 045	7 008	\$0.17 for the first 500 units. \$0.13 for the next 1 000 units. \$0.10 for the remaining units.	0.36 ¢ per unit

Monthly rental for meter = \$30.00

Discount of 10% for cash payment within 2 weeks of billing.

Calculate Mr. Gomes' actual electricity bill for May assuming the bill was paid in cash, within the 2-week period.

24. The electricity bill for a certain business place is calculated as follows:

Table 5.28

Meter reading (kWh)		*kWh used*	*Unit rating*	*Energy charge ($)*	*Fuel charge ($)*
Present	*Previous*				
46 571	27 571		A		

Energy charge is computed under three ratings as follows:

A. Industrial and commercial operations at 8 cents per unit.

B. Government and educational institutions at 10 cents per unit.

C. Private homes at 15 cents per unit.

In addition there is a fixed rate for fuel charge at 25 cents per unit.

Calculate:

(a) the number of kilowatt-hour used

(b) the energy charge

(c) the fuel charge

(d) the amount the business had to pay for the electricity used.

16. Calculate the total payment due to the Electricity Commission using the tables given below when the meter readings are as follows:

Previous (*kWh*)	*Present* (*kWh*)
12037	13245

Table for calculating the bill:

Table 5.21

Units used (*kWh*)	*Rate* (¢)
First 50	13
Next 50	10
Next 250	6
Next 1000	3

17.

Fixed charge	$33 per month
First 50 kWh	12 ¢ per kWh
Next 250 kWh	9 ¢ per kWh
All over 300 kWh	6 ¢ per kWh
Discount	A discount of 10% is given on all bills paid within 14 days of billing.

A householder used 754 kWh in 3 months. What amount did he pay if the bill was paid within 14 days of billing?

18. In June 2001, Mr. Raman's electricity bill was calculated on the following information:

Table 5.22

Present meter reading (*kWh*)	*Last meter reading* (*kWh*)	*Rate*	*Fuel charge*
82796	80759	$0.17 for the first 500 units. $0.13 for the next 1000 units. $0.10 for the remaining units.	$0.056 per unit

Monthly rental for meter = $30.00

Discount of 10% for cash payment within 2 weeks of billing.

Calculate Mr. Raman's actual electricity bill for June assuming the bill was paid in cash within the 2-week period.

19. *Table 5.23*

Electricity Bill 03:01:31
Meter Reading

Previous	*Present*
1542	1735

Energy Charge:
First 100 units cost 40 cents per unit.
Next 50 units cost 35 cents per unit.
Remaining units cost 30 cents per unit.

Fuel surcharge = 15 cents per unit
Meter rental = $25.00

(a) Calculate the number of units used.
(b) Evaluate the fuel charge for the units used.
(c) Determine the energy charge for the units used.
(d) Hence evaluate the electricity bill payable.

20. The charges for electricity in a certain country consist of a fixed fuel charge of 35 cents per kWh and an energy charge computed under three schemes as follows:

Scheme A.	Homes	12 cents per kWh
Scheme B.	Schools	18 cents per kWh
Scheme C.	Business places	25 cents per kWh.

The meter reading of a certain school reads as follows:

Table 5.24

Meter reading (*kWh*)		*Units used*	*Scheme*
Present	*Previous*		
27635	18425		B

Calculate:

(a) the number of kilowatt-hour used
(b) the fuel charge (in $)
(c) the energy charge (in $)
(d) the amount the school had to pay for the electricity used (in $).

21. Charges for electricity in a Caricom country are made up of a fixed fuel charge of 45 cents

Example 28

In May Mr. Charles' telephone bill was calculated from the following information:

Table 5.32

Long distance calls to	*Duration of calls in minutes*	*Fixed charge for 3 min or less*	*Charge per additional minute*
Japan	5	$25.50	$8.75
Puerto Rico	2	$8.40	$2.95
Utah, U.S.A.	3	$20.10	$6.85

Monthly rental for telephone = *$39.00*

Rebate received on rental for 2 weeks when the telephone was not working = *$14.50*

Calculate Mr. Charles' actual telephone bill for May.

Solution

The *cost of the call* to Japan = *The fixed charge* + *The charge per additional minute* × *The number of additional minutes*

= $(25.50 + 8.75 × 2)

= $(25.50 + 17.50)

= $43.00

The *cost of the call* to Puerto Rico = *The fixed charge*

= $8.40

The *cost of the call* to Utah = *The fixed charge*

= $20.10

The *cost* of the *monthly rental* = $39.00

∴ the *amount* Mr. Charles was *billed* for May = $(43.00 + 8.40 + 20.10 + 39.00)

= $110.50

The *rebate* received due to the telephone not working = $14.50

∴ Mr. Charles' *actual telephone bill* for May = *The total cost* − *The rebate*

= $(110.50 − 14.50)

= $96.00

Hence Mr. Charles' *actual telephone bill* was $96.00.

Example 29

Table 5.33

Telephone Bill	
Monthly rental	*= $35.00*
No. of metered units used	*= 2 123*
No. of operator-controlled units used	*= 245*
Government tax (VAT)	*= 15%*
Cost per metered unit	*= 23 cents*
Cost per operator-controlled unit	*= 35 cents*

(a) Calculate the cost for the metered units used.

(b) Evaluate the cost for the operator-controlled unit used.

(c) Determine the government tax.

(d) Hence evaluate the telephone bill payable.

Solution

(a) The *cost* for the *metered units* used = *The cost per metered unit* × *The number of metered units used*

= 23 ¢ × 2 123

= 48 829 ¢

= $488.29

Hence the *cost* of the *metered units* is $488.29.

(b) The *cost* for the *operator-controlled units* used = *The cost per operator-controlled units* × *The number of operator-controlled units used*

= 35 ¢ × 245

= 8 575 ¢

= $85.75

Hence the *cost* of the *operator-controlled units* is $85.75.

(c) The *total cost* for *rental* and *units used* = $(35.00 + 488.29 + 85.75)

= $609.04

The *government tax (VAT)* = *x% of the total cost*

= 15% of $609.04

= $91.356

= $91.36 (*correct* to the *nearest cent)*

Hence the *government tax* is $91.36.

25. In April 2003, Mr. Rogers' electricity bill was calculated on the following information:

Table 5.29

Present meter reading (kWh)	*Last meter reading (kWh)*	*Rate*	*Fuel charge*
45 784	43 747	\$0.17 for the first 500 units. \$0.13 for the next 1 000 units. \$0.10 for the remaining units.	0.56 ¢ per unit

Monthly rental for meter = \$30.00

Discount of 10% for cash payment within 2 weeks of billing.

Calculate Mr. Rogers' actual electricity bill for April assuming the bill was paid in cash within the 2-week period.

26. In July 2002, Mr. Joseph's electricity bill was calculated on the following information:

Table 5.30

Present meter reading (kWh)	*Last meter reading (kWh)*	*Rate Charge*	*Fuel*
8 971	7 901	\$0.19 for the first 500 units. \$0.15 for the next 1 000 units. \$0.10 for the remaining units.	\$0.36 per unit

Monthly rental for meter = \$30.00

Discount of 10% for cash payment within 2 weeks of billing.

Calculate Mr. Joseph's actual electricity bill for July assuming the bill was paid in cash within the 2-week period.

27. Charges for electricity in a Caricom country are made up of a fixed fuel charge of 35 cents per kWh and an energy charge computed under three schemes as follows:

Scheme A.	Homes	15 cents per kWh
Scheme B.	Schools	20 cents per kWh
Scheme C.	Business places	25 cents per kWh

Table 5.31

Meter reading (kWh)		*kWh used*	*Scheme*	*Energy charge (\$)*	*Fuel charge (\$)*
Present	*Previous*				
72 471	47 523		B		

Calculate:

(a) the number of kilowatt-hour used

(b) the energy charge in dollars

(c) the fuel charge in dollars

(d) the amount the school had to pay for the electricity used

(e) the actual amount the school paid if a discount of 10% was given for cash.

Telephone Bills

In these modern times many homes have at least one telephone. Hence these householders will have to pay a *telephone bill* each month for the services provided. There are a number of *variables* appearing on a *telephone bill*. For example, there is a *varying charge* depending on the *distance and duration of each call*. Then the *cost of a call* will depend on whether it is *local* or *foreign*. The *charge for a call* will depend on whether it is *peak time* or *off-peak time*, and whether the number called was reached by *direct dialling* or *operator assisted*. Then usually there is a *standing charge* for the *rental* of the *telephone*. Also *value added tax* (*VAT*) can be included.

Monthly rental for telephone = $22.50

Rebate received on rental for 3 weeks when the telephone was not working = $18.00

Calculate Ms. Khan's actual telephone bill for December.

11. In January Mr. Amin's telephone bill was calculated on the following information:

Table 5.42

Long distance calls to	*Duration of call in minutes*	*Fixed charge for 3 min or less*	*Charge per additional minute*
Ontario	25	$17.65	$5.90
New York	37	$15.40	$5.35
Paris	19	$19.20	$6.50

Monthly rental for telephone = $25.50

Rebate received on rental for 3 weeks when the telephone was not working = $12.00

Calculate Mr. Amin's actual telephone bill for January.

12. In September Miss Anna's telephone bill was calculated on the following information:

Table 5.43

Long distance calls to	*Duration of call in minutes*	*Fixed charge for 3 min or less*	*Charge per additional minute*
Boston	12	$15.60	$5.10
Toronto	15	$17.40	$5.70
Miami	9	$12.50	$4.10

Monthly rental for telephone = $35.00

Government tax (VAT) = 15%

Calculate Miss Anna's telephone bill for September.

13. Mary Lou's telephone bill for June 2003 is shown below. Telephone subscribers are charged a monthly service fee of $29.50 which covers up to a maximum of 25 local calls per month. A charge of 20 cents per call is made for each local call in excess of 25 calls. A tax of 75% is payable on all overseas calls.

Table 5.44

Telephone Bill		
Name: Mary Lou	*Account No.* RT0079	
Previous reading May 31, 2003	*Present reading June 30, 2003*	*Number of local calls*
5 093 calls	5 207 calls	

Charges	
	$
Arrears	
Service fee	29.50
Local calls	
Overseas calls	158.00
Tax on overseas calls	
Total	361.55

(a) Calculate:

- (i) the number of local calls made in June
- (ii) the amount due for local calls in June
- (iii) the tax on overseas calls in June
- (iv) the arrears brought forward from May.

(b) Mary Lou was charged $15.60 for local calls July, 2003. Calculate the total number of calls she made in July, 2003.

Foreign Exchange

Most *Caribbean countries* use *currencies* based on *dollars* ($) and *cents* (¢),

where $1.00 = 100 ¢

In the *United Kingdom* their *currency* is based on *pounds* (£) and *pence* (*p*),

where £1.00 = 100 p

When *converting* from one *currency* to another, we *convert* from the *currency* on the *left-hand-side* to the *currency* on the *right-hand-side* of the *equation*. A bureau de change or foreign exchange counter in a bank or cambio are places where currencies

(d) The *telephone bill payable* = *The total cost for the rental and units used* + *The government tax*

= $(609.04 + 91.36)

= $700.40

Hence the *telephone bill payable* is $700.40.

Exercise 5q

1. Calculate the quarterly telephone bill for the following household.

Table 5.34

Name	*Number of units used*	*Standing charge*	*Cost per unit*
Mr. Ryan	1745	$35.00	15 ¢

2. Determine the quarterly telephone bill for the following household.

Table 5.35

Name	*Number of units used*	*Standing charge*	*Cost per unit*
Mrs. Roberts	1473	$33.50	13 ¢

Calculate the monthly telephone bill for each of the following persons:

Table 5.36

	Name	*First 3 min or part thereof*	*Each additional minute*	*Number of minutes*	*Service charge*
3.	Mr. Korada	$3.95	$1.45	12	$24.20
4.	Miss Ines	$9.60	$4.10	$2\frac{1}{2}$	$30.20

5. Determine the quarterly telephone bill for the following household:

Table 5.37

Name	*Number of units used*	*Standing charge*	*Cost per unit*
Mr. Gibson	985	$25.80	12.5 ¢

Calculate the monthly telephone bill for the following persons:

Table 5.38

	Name	*First 3 min or part thereof*	*Each additional minute*	*Number of minutes*	*Service charge*
6.	Miss Taylor	$4.85	$1.70	124	$28.50
7.	Mr. Jackson	$5.40	$1.95	153	$35.70

8. Calculate the monthly telephone bill for the following household:

Table 5.39

Name	*First 3 min or part thereof*	*Each additional minute*	*Number of minutes*	*Service charge*
Mr. Yuri	$18.90	$6.70	35	$15.50

Table 5.40

Telephone Bill	03:01:31
Monthly rental	= $36.00
No. of metered units used	= 4245
No. of operator-controlled units used	= 543
Government tax (VAT)	= 15%
Cost per metered unit	= 14 cents
Cost per operator-controlled unit	= 18 cents

(a) Calculate the cost for the metered units used.

(b) Evaluate the cost for the operator-controlled units used.

(c) Determine the government tax.

(d) Hence evaluate the telephone bill payable.

10. In December Ms. Khan's telephone bill was calculated on the following information:

Table 5.41

Long distance calls to	*Duration of call in minutes*	*Fixed charge for 3 min or less*	*Charge per additional minute*
Xanadu	12	$42.75	$15.00
Paris	15	$17.25	$6.00
Moscow	10	$28.50	$10.00

(b) (i) Given that

$$\begin{aligned}
\text{TT } \$11.50 &= £1.00\\
\text{Then}\quad \text{TT } \$1.00 &= \frac{£1.00}{11.50}\\
\text{So}\quad \text{TT } \$3\,162.50 &= \frac{£1.00}{11.50} \times 3\,162.50\\
&= £275\\
\text{And the } \textit{tax paid} &= 10\% \text{ of } £275\\
&= \frac{1\cancel{0}}{10\cancel{0}} \times £275\\
&= £27.50
\end{aligned}$$

$$\begin{aligned}
\therefore \text{ the } \textit{amount} \text{ of } \textit{sterling} \text{ he } \textit{received} &= £(275 - 27.50)\\
&= £247.50
\end{aligned}$$

Hence the *tax paid* was £27.50 and the *amount of money received* was £247.50.

(ii) The *amount* of money *remaining*

$$\begin{aligned}
&= £(247.50 - 225)\\
&= £22.50\\
\text{So the } \textit{tax paid} &= 10\% \text{ of } £22.50\\
&= \frac{1\cancel{0}}{10\cancel{0}} \times £22.50\\
&= £2.25
\end{aligned}$$

$$\begin{aligned}
\therefore \text{ the } \textit{amount} \text{ of money to be } \textit{converted} &= £(22.50 - 2.25)\\
&= £20.25\\
\text{Given that}\quad £1.00 &= \text{TT } \$11.50\\
\text{Then}\quad £20.25 &= \text{TT } \$11.50 \times 20.25\\
&= \text{TT } \$232.88 \quad (\textit{correct} \text{ to the } \textit{nearest cent})
\end{aligned}$$

∴ the amount of *TT dollars* he *received* was TT $232.88

Hence the *tax paid* was £2.25 and the *amount of money received* was TT $232.88.

Alternative Method

(ii) The *amount* of money *remaining*

$$\begin{aligned}
&= £(247.50 - 225)\\
&= £22.50\\
\text{Given that } £1.00 &= \text{TT } \$11.50\\
\text{Then}\quad £22.50 &= \text{TT } \$11.50 \times 22.50\\
&= \text{TT } \$258.75\\
\text{So the } \textit{tax paid} &= 10\% \text{ of TT } \$258.75\\
&= \frac{1\cancel{0}}{10\cancel{0}} \times \text{TT } \$258.75\\
&= \text{TT } \$25.88 \quad (\textit{correct} \text{ to the } \textit{nearest cent})
\end{aligned}$$

$$\begin{aligned}
\therefore \text{ the } \textit{amount} \text{ of } \textit{TT dollars} \text{ he } \textit{received} &= \text{TT } \$(258.75 - 25.88)\\
&= \text{TT } \$232.87
\end{aligned}$$

Hence the *tax paid* was TT $25.88 and the *amount of money received* was TT $232.87.

Exercise 5r

1. An American tourist changed US $925 into Trinidad and Tobago currency at the exchange rate US $1.00 = TT $6.30

(a) Calculate the amount of TT dollars he received.

(b) The tourist spent TT $2 677.50 and changed the remaining TT dollars into American currency at the same exchange rate. Calculate the amount of US dollars he received.

2. An American tourist changed US $700 into Trinidad and Tobago dollars at a rate of

US $1.00 = TT $6.30

She spent, in Trinidad, TT $1 858. She then travelled to Barbados where she changed her Trinidad and Tobago dollars to Barbados dollars, the exchange rate being TT $3.19 = BDS $1.00.

(a) How many Trinidad and Tobago dollars did she receive?

(b) How many Barbados dollars did she get?

3. The table that follows gives the exchange rates for some Caribbean currencies against the United States dollar.

Table 5.45

Territory	*Amount for US $1.00*
Eastern Caribbean	EC $2.70
Guyana	G $179.00
Jamaica	J $61.00
Trinidad and Tobago	TT $6.30

Calculate:

(a) the amount in Jamaican dollars one would get for US $200 of different countries can be bought and sold (i.e. exchanged). The rate of exchange between two currencies tells us the amount of currency of a country that is equivalent to an amount of currency of another country.

Example 30

The rates of exchange at a bank are as follows:

US \$1.00 = TT \$6.30
CAN \$1.00 = TT \$4.90
£1.00 = TT \$11.50
TAX = 10 cents on the dollar
or
10 pence on the pound.

(a) (i) In May 2004, an American tourist changed US \$1000 into Trinidad and Tobago currency. Calculate the tax paid for this transaction and the amount of TT dollars he received.

(ii) The tourist spent TT \$4200 while vacationing and converted the remaining TT dollars into Canadian currency. Calculate the tax paid for this transaction and the amount of Canadian dollars he received.

(b) (i) A Trinidadian visiting England changes TT \$3162.50 into British sterling. Calculate the tax paid for this transaction and the amount of sterling he received.

(ii) He spent £225 while visiting and converted the remaining sterling into TT currency. Calculate the tax paid for this transaction and the amount of TT dollars he received.

Solution

(a) (i)

$$\begin{aligned}
\text{Given that US \$1.00} &= \text{TT \$6.30}\\
\text{Then US \$1000} &= \text{TT \$6.30} \times 1000\\
&= \text{TT \$6300}\\
\text{So the } \textit{tax paid} &= 10\% \text{ of TT \$6300}\\
&= \frac{1\cancel{0}}{10\cancel{0}} \times \text{TT \$630}\cancel{0}\\
&= \text{TT \$630}\\
\text{And the } \textit{amount} \text{ of } \textit{TT dollars} \text{ he } \textit{received} &= \text{TT \$}(6300 - 630)\\
&= \text{TT \$5670}
\end{aligned}$$

Hence the *tax paid* was TT \$630 and the *amount of money received* was TT \$5670.

(ii)

$$\begin{aligned}
\text{The } \textit{amount} \text{ of money } \textit{remaining} &= \text{TT \$}(5670 - 4200)\\
&= \text{TT \$1470}\\
\text{So the } \textit{tax paid} &= 10\% \text{ of TT \$1470}\\
&= \frac{1\cancel{0}}{10\cancel{0}} \times \text{TT \$147}\cancel{0}\\
&= \text{TT \$147}\\
\therefore \text{ the } \textit{amount} \text{ of money to be } \textit{converted} &= \text{TT \$}(1470 - 147)\\
&= \text{TT \$1323}\\
\text{Given that TT \$4.90} &= \text{CAN \$1.00}\\
\text{Then TT \$1.00} &= \text{CAN } \frac{\$1.00}{4.90}\\
\text{So TT \$1323} &= \text{CAN } \frac{\$1.00}{4.90} \times 1323\\
&= \text{CAN \$270}
\end{aligned}$$

Hence the *tax paid* was TT \$147 and the *amount of money received* was CAN \$270.

Alternative Method

(ii)

$$\begin{aligned}
\text{The } \textit{amount} \text{ of money } \textit{remaining} &= \text{TT \$}(5670 - 4200)\\
&= \text{TT \$1470}\\
\text{Given that TT \$4.90} &= \text{CAN \$1.00}\\
\text{Then TT \$1.00} &= \text{CAN } \frac{\$1.00}{4.90}\\
\text{So TT \$1470} &= \text{CAN } \frac{\$1.00}{4.90} \times 1470\\
&= \text{CAN \$300}\\
\text{And the } \textit{tax paid} &= 10\% \text{ of CAN \$300}\\
&= \frac{1\cancel{0}}{10\cancel{0}} \times \text{CAN \$30}\cancel{0}\\
&= \text{CAN \$30}\\
\therefore \text{ the amount of } \textit{CAN dollars} \text{ he } \textit{received} &= \text{CAN \$}(300 - 30)\\
&= \text{CAN \$270}
\end{aligned}$$

Hence the *tax paid* was CAN \$30 and the *amount of money received* was CAN \$270.

13. A bank gives nineteen dollars and seventy-five cents in Jamaican currency (J $19.75) for one Belize dollar (BEL $1.00). A tourist took J $1 481.25 to the bank to exchange for Belize dollars. Bank changes amounted to BEL $5.50. Calculate the amount in Belize dollars, the tourist received for this transaction.

14. The rates of exchange at the bank are as follows:

EC $1.00 = BDS $0.75

US $1.00 = BDS $1.98

(a) A traveller changed EC $2 000 to Barbados currency. Calculate the amount received.

(b) Of the amount she received she spent BDS $510 and exchanged the remainder for US currency. Calculate the amount in US currency she received for this exchange. (Assume that the buying rate and selling rate for BDS $1.00 are the same.)

15. The rates of exchange at a bank are as follows:

US $1.00 = TT $6.15

CAN $1.00 = TT $4.70

(a) A Canadian changed CAN $1 300 into Trinidad and Tobago currency. Calculate the amount of TT dollars she received.

(b) The tourist spent TT $3 957.50 and converted the remaining TT dollars into American currency. Calculate the amount of American dollars she received.

16. The rate of exchange at a bank is as follows:

£1.00 = TT $11.10

(a) A Trinidadian visiting England changes TT $3 885 into British sterling. Calculate the amount of sterling he received.

(b) He spent £249 while visiting and converted the remaining sterling into TT currency. Calculate the amount of TT dollars he received.

17. The rates of exchange at the bank are as follows:

EC $1.00 = BDS $0.75

US $1.00 = BDS $1.98

(a) A traveller changes EC $3 000 to Barbados currency. Calculate the amount received.

(b) Of the amount she received she spent BDS $1 250 and exchanged the remainder for US currency. Calculate the amount in US currency she received for this transaction. (Assume that the buying rate and selling rate for BDS $1.00 are the same.)

18. Exchange rate US $1.00 = TT $6.10. How many TT$ can be exchanged for US $1 200.00?

19. The list of exchange rates states that
US $1.00 = TT $6.30 and
US $1.00 = 110 yen.

(a) How many TT dollars can 550 yen be exchange for?

(b) How many yens are worth TT $315?

20. In August 2004, an American tourist changed US $1 200 of his American Travellers' cheques for Trinidad and Tobago currency. Two-fifths of this amount was in $20 cheques and the remainder in $10 cheques. He was shown the information below:

Table 5.47

US $1.00 = TT $6.12

1% tax is charged on the total foreign exchange transaction.

TT $0.45 stamp duty is charged for each cheque.

Calculate in Trinidad and Tobago currency:

(a) the tax the tourist had to pay

(b) the amount the tourist received for US $1 200 after paying tax and stamp duty.

NOTE: US means United States and
TT means Trinidad and Tobago.

21. Given that US $1.00 (one United States dollars) is equivalent to TT $6.24 (six dollars and twenty-four cents in Trinidad and Tobago currency). Calculate the amount in US currency that is equivalent to TT $936.

Simple Interest

During the adult life of a person it is a normal process to *borrow money* form the bank or *deposit money* into the *bank*. When we *borrow money* from

(b) the amount in United States dollars one would get for TT $315

(c) the amount in Eastern Caribbean dollars one would get for G $8 950.

4. The rates of exchange at a bank are as follows:

US $1.00 = TT $6.30
CAN $1.00 = TT $4.90

(a) A tourist changed US $900 to Trinidad and Tobago currency. Calculate the amount she receives.

(b) She spends TT $3 710 and changed the remaining Trinidad and Tobago currency to Canadian currency. Calculate the amount of money she received.

NOTE: TT means Trinidad and Tobago
US means Unites States
CAN means Canada

5. Given that TT $1.00 = EC $0.43
and TT $6.30 = US $1.00
Convert (a) TT $125.00 to EC $
(b) EC $850.94 to US $.

6. The rates of exchange at a bank are as follows:

EC $1.00 = BDS $0.75
US $1.00 = BDS $1.98

(a) A traveller changed EC $1 800 to Barbados currency. Calculate the amount received.

(b) Of the amount she received she spent BDS $756 and exchanged the remainder for US currency. Calculate the amount in US currency she received for this exchange. (Assume that the buying rate and selling rate for BDS $1.00 are the same.)

7. Foreign exchange

US $1.00 = TT $6.15
EC $1.00 = TT $2.30

Using the exchange rate above, change:

(a) US $125 to TT $

(b) TT $4 600 to EC $.

8. In July 2004, a Canadian tourist changed CAN $1 500 of her Canadian travellers' cheques for Trinidad and Tobago currency. One-third of this amount was in $50 cheques and the remainder was in $100 cheques.

Table 5.46

CAN $1.00 = TT $4.70

13 cents on the dollar is charged for tax on the total foreign exchange transaction.

TT $0.30 stamp duty is charged per cheque.

Calculate, in Trinidad and Tobago currency,

(a) the tax the Canadian tourist had to pay

(b) the stamp duty the Canadian tourist had to pay

(c) the amount of money the Canadian tourist received from the bank after the transaction.

NOTE: CAN means Canada
TT means Trinidad and Tobago.

9. A bank gives two dollars and seventy-five cents in Eastern Caribbean currency (EC $2.75) for one United States dollar (US $1.00).
Given that 1% tax is charged on all foreign exchange transactions, calculate the amount, in E.C. currency, which a tourist receives in exchange for US $1 200.00.

10. (a) A tourist changed US $1 200 to Trinidad and Tobago currency. Calculate the amount she received at the exchange rate
US $1.00 = TT $6.30.

(b) Calculate the amount she would receive at the exchange rate US $1.00 = TT $5.76.

11. If J $0.34 = GUY $1.00
and J $47.00 = CAN $1.00,
express CAN $125 in J $.

12. The rates of exchange at a bank are as follows:

US $1.00 = TT $6.15
CAN $1.00 = TT $4.70

(a) A tourist changed US $1 200 to Trinidad and Tobago currency. Calculate the amount she received.

(b) If the tourist changed the remaining TT $587.50 to Canadian currency, calculate the amount she received.

the bank, then the *amount borrowed* is called the *principal*. The bank in turn will charge us a sum of

Also the

(c) The simple interest on \$5 400 invested at 8.75% per annum is \$1 890.
Determine the period of investment.

Solution

(a) The *simple interest*, $I = \$1\,680$

The *rate per cent per annum*, $R = 5\%$

The *time in years*, $T = 6$ months

$$= \frac{6}{12} \text{ years}$$

$$= 0.5 \text{ year}$$

∴ the *principal*, $P = \frac{100I}{RT}$

$$= \frac{100 \times \$1\,680}{5 \times 0.5}$$

$$= \$67\,200$$

Hence the *amount of money invested* was \$67 200.

(b) The *principal*, $P = \$8\,500$

The *simple interest*, $I = \$3\,867.50$

The *time in years*, $T = 6\frac{1}{2}$ years

$$= 6.5 \text{ years}$$

∴ the *rate per cent per annum*, $R = \frac{100I}{PT}$

$$= \frac{1\cancel{00} \times \$3\,867.50}{\$8\,5\cancel{00} \times 6.5}$$

$$= 7\% \text{ per annum}$$

Hence the *rate per cent annum* is 7%.

(c) The *principal*, $P = \$5\,400$

The *simple interest*, $I = \$1\,890$

The *rate per cent per annum*, $R = 8.75\%$ per annum

The *time in years*, $T = \frac{100I}{PR}$

$$= \frac{1\cancel{00} \times \cancel{\$}1\,890}{\cancel{\$}5\,4\cancel{00} \times 8.75}$$

$$= 4 \text{ years}$$

Hence the *period of investment* is 4 years.

Simple Interest Table

The *simple interest table* is a *ready reckoner* that is used by business people to calculate quickly the *simple interest* due on a sum of money *invested* or *loaned.*

The *table* below is an *extract* from a *ready reckoner* showing the *appreciation* (i.e. the *rise in value*) of \$1 for *periods* from 1 *year to* 15 *years* and *rates of simple interest* from 8% to 15%.

Table 5.48

Years	8%	9%	10%	11%	12%	13%	14%	15%
1	1.08	1.09	1.10	1.11	1.12	1.13	1.14	1.15
2	1.16	1.18	1.20	1.22	1.24	1.26	1.28	1.30
3	1.24	1.27	1.30	1.33	1.36	1.39	1.42	1.45
4	1.32	1.36	1.40	**1.44**	**1.48**	1.52	1.56	1.60
5	1.40	1.45	1.50	1.55	1.60	1.65	1.70	1.75
6	1.48	1.54	1.60	1.66	1.72	1.78	1.84	1.90
7	1.56	1.63	1.70	1.77	1.84	1.91	1.98	2.05
8	1.64	1.72	1.80	1.88	1.96	2.04	2.12	2.20
9	1.72	1.81	1.90	1.99	2.08	**2.17**	2.26	2.35
10	1.80	1.90	2.00	2.10	2.20	2.30	2.40	2.50
11	1.88	1.99	2.10	2.21	2.32	2.43	2.54	2.65
12	1.96	2.08	2.20	2.32	2.44	2.56	2.68	2.80
13	2.04	2.17	2.30	2.43	2.56	2.69	2.82	2.95
14	2.12	2.26	2.40	2.54	2.68	2.82	2.96	3.10
15	**2.20**	2.35	2.50	2.65	2.80	2.95	3.10	3.25

Example 33

Using the simple interest table above, calculate the simple interest earned when:

(a) \$9 845 is invested at 8% per annum simple interest for 15 years

(b) \$25 831 is invested at 13% per annum simple interest for 9 years

(c) \$3 471 is invested at 11.5% per annum simple interest for 4 years.

Solution

From the *simple interest table*:

(a) The *appreciation of* \$1 after 15 years at 8% *per annum simple interest* = \$2.20

Calculate:

(a) the sum of money Mr. Isaacs had to pay the bank as simple interest

(b) the total amount of money repaid to the bank

(c) the amount of money paid monthly.

10. Mrs. Cappola borrowed \$5 760 from a bank at 8.25% per annum simple interest for 9 months. Evaluate:

(a) the simple interest Mrs. Cappola paid the bank for the money borrowed

(b) the total amount of money that was repaid to the bank

(c) the amount of each monthly instalment.

11. Determine the principal that will earn \$294 in 5 years at 6% per annum simple interest.

12. Determine the principal that will earn \$200 as simple interest after 8 years at 5% per annum.

13. Mr. Frank's bank pays interest at 8% per annum on money he has on deposit. What amount is in his account if the simple interest for 7 months is \$42?

14. Calculate the amount of money invested at 11% per annum, when \$330 simple interest was collected after 2 years.

15. Calculate the amount of money invested at 8.5% per annum, when \$2 488.80 simple interest was collected after 3 years.

16. Calculate the amount of money invested at 9.25% per annum, when \$5 781.25 simple interest was collected after 5 years.

17. Mrs. Wood borrowed a sum of money from a bank at 12.5% per annum for 6 years and paid \$6 375 simple interest. Calculate the sum of money Mrs. Wood borrowed from the bank.

18. Mr. Jonah took a loan from a bank at 13.5% per annum for 6 months and paid \$360.45 simple interest. Calculate the amount of money Mr. Jonah borrowed from the bank.

19. Mrs. Kanhai borrowed a sum of money from the bank for 9 months at 13.5% per annum and paid \$866.70 simple interest. Determine the sum of money Mrs. Kanhai borrowed from the bank.

20. Mr. Kallicharran took a loan from the bank at 11.25% per annum for 9 months and paid \$270 simple interest. Calculate the amount of money Mr. Kallicharran borrowed from the bank.

21. Calculate the rate per cent per annum if \$250 simple interest is paid when \$800 is invested for 5 years.

22. The simple interest on \$15 000 for 9 years is \$6 750. Calculate the rate per cent per annum.

23. The simple interest on \$12 000 for $2\frac{1}{2}$ years is \$3 750. Calculate the rate per cent per annum.

24. The simple interest on \$15 000 for $3\frac{3}{4}$ years is \$4 950. Calculate the rate per cent per annum.

25. Calculate the rate per cent per annum if \$5 760 simple interest is paid when \$12 800 is invested for 6 years.

26. Calculate the rate per cent per annum if \$3 240 simple interest is paid when \$12 000 is invested for 2.5 years.

27. Mr. Singh invested \$9 840 in a bank for $5\frac{1}{2}$ years and received \$5 141.40 simple interest. Calculate the rate per cent per annum that his investment achieved.

28. Mrs. Phillips deposited \$8 400 in a bank for $4\frac{3}{4}$ years and collected \$3 391.50 simple interest. Determine the rate per cent per annum that her investment achieved.

29. Mr. Jerome borrowed \$12 600 from a bank for 5 years and paid \$7 875 simple interest. Calculate the rate per cent per annum that he had to pay for the loan.

30. Mrs. Rainer borrowed \$15 800 from a bank for 3 years and had to pay \$5 806.50 simple interest. Determine the rate per cent per annum that she had to pay in order to obtain the loan.

31. Calculate the number of years in which \$560 will earn \$112 simple interest at 4% per year.

32. Determine the number of years in which \$500 invested at 9% per annum will earn \$450 as simple interest.

∴ the *appreciation of* \$9 845 after 15 years at 8% *per annum simple interest*,

$$\begin{aligned} A &= \$2.20 \times 9\,845 \\ &= \$21\,659 \end{aligned}$$

So the *simple interest*,

$$\begin{aligned} I &= A - P \\ &= \$(21\,659 - 9\,845) \\ &= \$11\,814 \end{aligned}$$

Hence the *simple interest earned* was \$11 814.

(b) The *appreciation of* \$1 after 9 years at 13% *per annum simple interest* $= \$2.17$

∴ the *appreciation of* \$25 831 after 9 years at 13% *per annum simple interest*,

$$\begin{aligned} A &= \$2.17 \times 25\,831 \\ &= \$56\,053.27 \end{aligned}$$

So the *simple interest*,

$$\begin{aligned} I &= A - P \\ &= \$(56\,053.27 - 25\,831) \\ &= \$30\,222.27 \end{aligned}$$

Hence the *simple interest earned* was \$30 222.27.

(c) The *appreciation of* \$1 after 4 years at 11.5% *per annum simple interest*

$$\begin{aligned} &= \frac{\$(1.44 + 1.48)}{2} \\ &= \frac{\$2.92}{2} \\ &= \$1.46 \end{aligned}$$

∴ the *appreciation of* \$3 471 after 4 years at 11.5% *per annum simple interest*,

$$\begin{aligned} A &= \$1.46 \times 3\,471 \\ &= \$5\,067.66 \end{aligned}$$

So the *simple interest*,

$$\begin{aligned} I &= A - P \\ &= \$(5\,067.66 - 3\,471) \\ &= \$1\,596.66 \end{aligned}$$

Hence the *simple interest earned* was \$1 596.66.

Exercise 5s

1. \$9 600 is invested at 8% per annum simple interest for 5 years.
   (a) What is the amount of simple interest payable?
   (b) Calculate the amount accruing for the investment.

2. A man invested \$600 at 8% per annum s interest for 5 years. Calculate:
   (a) the simple interest payable
   (b) the total amount of money the man collected at the end of the 5-year period.

3. Determine the amount of money accruing after \$450 was invested for 4 years at 6% per annum simple interest.

4. (a) Calculate the simple interest collected by a bank if \$1 500 is loaned for 3 years at 8% per annum.
   (b) What amount of money did the borrower actually pay the bank?
   (c) What amount of money was paid monthly?

5. A man invested \$10 500 for 7 years at $5\frac{1}{2}\%$ per annum simple interest.
   (a) Determine the simple interest paid.
   (b) Calculate the amount of money he collected after the period of 7 years.

6. (a) Calculated the simple interest collected by a bank if \$7 650 is loaned for 4 years at 6.5% per annum.
   (b) Hence determine the total amount collected by the bank at the end of the 4-year period.
   (c) What was the amount of money paid monthly?

7. Mr. Ford invested \$12 450 in a bank at 7.25% per annum simple interest for 6 years. Calculate:
   (a) the interest he was paid
   (b) the total of amount of money he would have received at the end of the period of investment.

8. Mrs. Ricky borrowed \$5 340 from a bank at 9.5% per annum simple interest for 5 years. Determine:
   (a) the sum of money paid in interest to the bank
   (b) the total amount of money repaid to the bank
   (c) the value of each monthly instalment.

9. Mr. Isaacs borrowed \$3 680 from a bank for 6 months at 8.75% per annum.

hire purchase price, calculate the amount payable to the bank each month.

46. A man wishes to invest \$3 500. He can buy savings bonds which pay simple interest at the rate of 12% per annum or he can start a savings account which pays simple interest at the rate of 10% per annum. Calculate the difference in the amounts of the two investments at the end of 6 years.

The Simple Interest Table following shows the appreciation of \$1.00 for periods from 20 years to 25 years at rates per cent per annum from 10% to 14%.

Table 5.49

Year	10%	11%	12%	13%	14%
20	3.000	3.200	3.400	3.600	3.800
21	3.100	3.310	3.520	3.730	3.940
22	3.200	3.420	3.640	3.860	3.080
23	3.300	3.530	3.760	3.990	3.220
24	3.400	3.640	3.880	3.120	3.360
25	3.500	3.750	4.000	3.250	3.500

Using the table, calculate:

47. The amount of money a teacher would have to pay a bank if he borrows \$125 000 for 25 years at 12% per annum simple interest.

48. The simple interest a clerk would have to pay a bank if she borrows \$65 000 for 21 years at 11% per annum.

49. The amount of money a civil servant would have to pay the bank if he borrows \$155 000 for 22 years at 11% per annum simple interest.

50. The simple interest a mechanic would have to pay if he borrows \$45 000 for 24 years at 12% per annum.

51. The amount of money an accountant would have to pay the bank if he borrows \$325 000 for 25 years at $12\frac{1}{2}$% per annum simple interest.

52. The simple interest an engineer would have to pay a bank if she borrows \$250 000 for 21 years at 13.5% per annum.

53. The amount of money a manger would have to repay a bank if he borrowed \$175 000 for 20 years at $10\frac{1}{2}$% per annum simple interest.

54. The amount of money a biologist would have to repay a bank if she borrowed \$187 000 for 23 years at 11.5% per annum simple interest.

55. The simple interest a chemist would have to repay a bank if he borrowed \$195 000 for 20 years at 13% per annum.

Compound Interest

When we invest money, the amount invested is called the *principal*. Each year the *principal* achieves an *interest*. If the *interest payable* is *reinvested* at the end of each year in the same *fixed deposit* (or *time deposit*), then the *principal* at the *beginning* of each new year is *greater* than the *principal* of the *previous year*. Their *difference* will be the previous year's *interest*. Thus the *interest payable* at the end of each new year is *greater* than the *interest paid* at the end of the *previous year*. Money *invested* in this way is said to attract *compound interest*, since *interest* also attracts *interest*.

In the *argument* above, it is *assumed* that the *rate per cent per annum* is at *least* the value of *preceding years*. Normally, at this level, the *rate per cent per annum* is *constant* for *each year*.

Thus the *formula* $I = \frac{PRT}{100}$ can be used repeatedly to *calculate* the *interest payable* at the *end of each year*. Or the *interest payable* can be *calculated* as a *percentage* of the principal.

Also the *compound interest formula* is:

$$A = P\left(1 + \frac{R}{100}\right)^n,$$

where A = the *amount of money accruing after n years*,
P = the *principal*,
R = the *rate per cent per annum*
and n = the *number of years* for which the *money was invested*.

Also the *compound interest*, $C.I. = A - P$.

The last *two formulae* can be used to *calculate* the *compound interest* payable for an investment, especially when the *period of investment*, is *greater than* 3 *years*.

33. The simple interest on $12 000 invested at 8% per annum is $6 720. Calculate the number of years for which the sum was invested.

34. The simple interest earned on $9 800 invested at 8.25% per annum is $3 638.25. Determine the number of years for which the principal was invested.

35. $4 353.75 simple interest was collected when $12 900 was invested at 6.75% per annum. Calculate the number of years for which the principal was invested.

36. $855 simple interest was paid when $4 500 was invested at 4.75% per annum. Determine the period of investment.

37. Mr. Joseph borrowed $18 900 at 13.5% per annum and repaid $12 757.50 simple interest. Calculate the period of the loan.

38. (a) The marked price of a car is $49 500. A person can pay a deposit of 30% and simple interest at 12% per annum is charged on the outstanding balance. The total amount payable is to be paid in $2\frac{1}{2}$ years.

Calculate:

(i) the amount of each instalment
(ii) the hire purchase price of the car.

(b) The total amount of $49 500 can be borrowed from the bank at 11% per annum for a period of 3 years.

Calculate:

(i) the total amount to be repaid to the bank
(ii) the amount of each monthly instalment.

39. The cash price of a refrigerator is $2 845.00

(a) It can be bought on hire purchase if a deposit of $710 is first paid. Simple interest at 10% per annum for 2 years is then added to the outstanding balance. Calculate the total amount paid for the refrigerator.

(b) The refrigerator can be also be bought by borrowing the cash price from a bank at 10% simple interest and the principal and interest must be repaid at the end of 2 years. Calculate the amount to be paid for the refrigerator by this arrangement.

(c) Which arrangement is better and by now much?

40. A woman wishes to invest $2 500. She can purchase savings bonds which pay simple interest at the rate of 7% per annum or she can start a savings account which pays simple interest at the rate of 8.5% per annum. Calculate the difference between the amounts of the two investments at the end of 6 years.

41. $6 000 was put in a fixed deposit account on 1st January, 1984 for 6 months. The rate of simple interest was 7.5% per annum. On 1st July, 1984 the total amount received was reinvested for a further 6 months at 7% per annum. Calculate the final amount received at the end of the year.

42. (a) $6 000 was put in a fixed deposit account on 1st January, 1984 for 1 year. Calculate the total amount received at the end of the period, if the rate of simple interest was 7.5% per annum.

(b) $6 000 was also put in a fixed deposit account at a different bank on 1st January, 1984 for 6 months. The rate of simple interest was 7.5% per annum. On 1st July, 1984 the total amount was reinvested for a further 6 months at 7% per annum. Calculate the final amount received at the end of the year.

(c) State whether (a) or (b) was the better investment, giving a reason for your choice.

43. A woman invests $3 000 in government bonds for 5 years at 7% simple interest.

(a) Calculate the total interest she receives for the 5 years.

(b) Calculate the sum that must be invested in bonds to obtain a total interest of $3 500 in 5 years.

44. $9 000 was put in a fixed deposit account on 1st January, 1985 for 6 months. The rate of simple interest was 8.5% per annum. On 1st July, 1985 the total amount received was reinvested for a further 6 months at 8.0% per annum simple interest. Calculate the final amount received at the end of the year.

45. A television can be bought on hire purchase by paying a down payment of $850 and 18 monthly instalments of $195 each. Calculate the hire purchase price of the television.

The television can be bought for cash by taking a bank loan for 12 months at 11% per annum simple interest. If the cash price is 90% of the

The *amount accruing*, $A = P\left(1 + \frac{R}{100}\right)^n$

$$= \$10000\left(1 + \frac{8}{100}\right)^3$$
$$= \$10000\,(1 + 0.08)^3$$
$$= 10000\,(1.08)^3$$
$$= \$12597.12$$

Table 5.50

Input	*Seen on the display of the calculator*
1.08	1.08
$\boxed{x^y}$	
$\boxed{3}$	3.
$\boxed{=}$	1.259712
$\boxed{\times}$	
$\boxed{10000}$	10000
$\boxed{=}$	12597.12

And the *compound interest earned*, $C.I. = A - P$

$$= \$(12597.12 - 10000)$$
$$= \$2597.12$$

Using Three-Figure Mathematical Tables

The following *method* illustrates how *three-figure mathematical tables* were used to *solve* the *problem.*

The *amount accruing*, $A = P\left(1 + \frac{R}{100}\right)^n$

$$= \$10000\left(1 + \frac{8}{100}\right)^3$$
$$= \$10000\,(1 + 0.08)^3$$
$$= \$10000\,(1.08)^3$$
$$= \$12600 \text{ (correct to 3 s.f.)}$$

Table 5.51

Number	*Operation*	*Log*
10000 1.08^3	3(0.033)	4.000 + 0.099
1.26×10^4 = 12600		4.099

Note that $10000 = 10^4$

And $\log 10000 = \log 10^4$
$= 4 \log 10$
$= 4.000$

Also $\log 1.08 = 0.033$

And $\log 1.08^3 = 3 \log 1.08$
$= 3(0.033)$
$= 0.099$

Also antilog $0.099 = 1.26$

And antilog $4.099 = 1.26 \times 10^4$
$= 12600$

So the *compound interest earned*, $C.I. = A - P$

$$= \$(12600 - 10000)$$
$$= \$2600 \text{ (correct to 3 s.f.)}$$

From the above *example* it can be *seen* that when *three-figure mathematical tables* are used, then the *answer* can only be *computed correct* to 3 *significant figures*.

Compound Interest Table

The *compound interest table* is a *ready reckoner* that is used by business people to calculate quickly the *compound interest* due on a sum of money *invested* or *loaned.*

The *table* below is an *extract* from a *ready reckoner* showing the *appreciation* (i.e. the *rise in value*) of \$1 for *periods* from 1 *year* to 10 *years* and rates of compound interest from 8% to 15%.

Table 5.52

Years	8%	9%	10%	11%	12%	13%	14%	15%
1	1.08	1.09	1.10	1.11	1.12	1.13	1.14	1.15
2	1.17	1.19	1.21	1.23	1.25	1.28	1.30	1.32
3	1.26	1.30	1.33	1.37	1.40	1.44	1.48	1.52
4	1.36	1.41	1.46	1.52	1.57	**1.63**	**1.69**	1.75
5	1.47	1.54	1.61	1.69	**1.76**	1.84	1.93	2.01
6	1.59	1.68	1.77	1.87	1.97	2.08	2.19	2.31
7	1.71	1.83	1.95	2.08	2.21	2.35	2.50	2.66
8	1.85	1.99	2.14	2.30	2.48	2.66	2.85	3.06
9	2.00	2.17	2.36	2.56	2.77	3.00	3.25	3.52
10	2.16	**2.37**	2.59	2.84	3.11	3.39	3.71	4.05

Example 34

Calculate the compound interest earned and the amount accruing when \$10 000 is invested at 8% per annum for 3 years.

Solution

The *principal* at the *beginning* of the *first year*,

$$P_1 = \$10000$$

So the *interest* for the *first year*,

$$\begin{aligned} I_1 &= \frac{PRT}{100} \\ &= \frac{\$10\,0\cancel{00} \times 8 \times 1}{1\cancel{00}} \\ &= \$800 \end{aligned}$$

∴ the *principal* at the *beginning* of the *second year*,

$$\begin{aligned} P_2 &= P_1 + I_1 \\ &= \$(10\,000 + 800) \\ &= \$10\,800 \end{aligned}$$

So the *interest* for the *second year*,

$$\begin{aligned} I_2 &= \frac{P_2RT}{100} \\ &= \frac{\$10\,8\cancel{00} \times 8 \times 1}{1\cancel{00}} \\ &= \$864 \end{aligned}$$

∴ the *principal* at the *beginning* of the *third year*,

$$\begin{aligned} P_3 &= P_2 + I_2 \\ &= \$(10\,800 + 864) \\ &= \$11\,664 \end{aligned}$$

So the *interest* for the *third year*,

$$\begin{aligned} I_3 &= \frac{P_3RT}{100} \\ &= \frac{\$11\,664 \times 8 \times 1}{100} \\ &= \$116.64 \times 8 \\ &= \$933.12 \end{aligned}$$

Thus the *compound* interest, *C.I.*

$$\begin{aligned} &= I_1 + I_2 + I_3 \\ &= \$(800 + 864 + 933.12) \\ &= \$2\,597.12 \end{aligned}$$

Hence the *compound interest earned* was \$2 597.12.

The *amount accruing*,

$$\begin{aligned} A &= P_3 + I_3 \\ &= \$(11\,664 + 933.12) \\ &= \$12\,597.12 \end{aligned}$$

Alternatively,

$$\begin{aligned} A &= P_1 + C.I. \\ &= \$(10\,000 + 2\,597.12) \\ &= \$12\,597.12 \end{aligned}$$

Hence the *amount accruing* is \$12 597.12.

Alternative Method

The *principal* at the *beginning* of the *first year*,

$$P_1 = \$10\,000$$

So the *interest* for the *first year*,

$$\begin{aligned} I_1 &= 8\% \text{ of } \$10\,000 \\ &= 0.08 \times \$10\,000 \\ &= \$800.00 \end{aligned}$$

∴ the *principal* at the *beginning* of the *second year*,

$$\begin{aligned} P_2 &= P_1 + I_1 \\ &= \$(10\,000 + 800) \\ &= \$10\,800 \end{aligned}$$

So the *interest* for the *second year*,

$$\begin{aligned} I_2 &= 8\% \text{ of } \$10\,800 \\ &= 0.08 \times \$10\,800 \\ &= \$864.00 \end{aligned}$$

∴ the *principal* at the *beginning* of the *third year*,

$$\begin{aligned} P_3 &= P_2 + I_2 \\ &= \$(10\,800 + 864) \\ &= \$11\,664 \end{aligned}$$

So the *interest* for the *third year*,

$$\begin{aligned} I_3 &= 8\% \text{ of } \$11\,664 \\ &= 0.08 \times \$11\,664 \\ &= \$933.12 \end{aligned}$$

Thus the *compound interest earned*,

$$\begin{aligned} C.I. &= I_1 + I_2 + I_3 \\ &= \$(800.00 + 864.00 \\ &\quad + 933.12) \\ &= \$2\,597.12 \end{aligned}$$

And the *amount accruing*,

$$\begin{aligned} A &= P_1 + C.I. \\ &= \$(10\,000 + 2\,597.12) \\ &= \$12\,597.12 \end{aligned}$$

Using a Calculator

The following *method* illustrates how a *scientific calculator* was used to *solve* the *problem.*

The *first key* is the $\boxed{x^y}$ *power key*, therefore we had to *press* $\boxed{x^y}$ in order to find the *value* of a quantity *raised to a given power.*

9. A man placed $11 500 in a fixed deposit for 5 years at 8% per annum.

(a) Calculate the total amount received at the end of the period under compound interest.

(b) Determine the total amount received at the end of the period under simple interest.

(c) State the difference in the interest received.

10. A woman wishes to invest $2 500. She can purchase savings bond which pays simple interest at the rate of 7% per annum or she can start a savings account which pays compound interest at the same rate. Calculate the difference between the amounts of the two investments at the end of 6 years.

11. Mr. Roland invests $9 500 in a bank for 3 years and receives simple interest at 7.5% per annum. If he invests his money in bonds for the same period he will receive compound interest at 4.5% per annum. Calculate the interest Mr. Roland received in both cases. State which investment is the better one and the difference in interest received.

12. A man wishes to invest $2 500. He can buy savings bonds which pay simple interest at the rate of 7% per annum or he can start a savings account which pays compound interest at the same rate. Calculate, to the nearest cent, the difference in the amounts of the two investments at the end of 3 years.

13. A man wishes to invest $3 500. He can buy savings bonds which pay simple interest at the rate of 12% per annum or he can start a savings account which pays compound interest at the same rate. Calculate the difference in the amounts of the two investments at the end of 6 years.

14. A woman wishes to invest $5 700. She can buy savings bonds which pay simple interest at the rate of 8.5% per annum or he can start a savings account which pays compound interest at the same rate.

Calculate the difference in the amounts of the two investments at the end of 5 years.

The Compound Interest Table below shows the appreciation of $1.00 for periods from 5 years to 10 years at rates per cent per annum compound interest from 10% to 14%.

Table 5.53

Year	10%	11%	12%	13%	14%
5	1.611	1.685	1.762	1.842	1.925
6	1.772	1.870	1.974	2.082	2.195
7	1.949	2.076	2.211	2.353	2.502
8	2.144	2.305	2.476	2.658	2.853
9	2.358	2.558	2.773	3.004	3.252
10	2.594	2.839	3.106	3.395	3.707

Using the table, calculate:

15. The amount of money an accountant receives from a bank if he invests $120 000 for 9 years at 10% per annum compound interest.

16. The compound interest a teacher would have to pay a bank if he borrows $69 000 for 8 years at 12% per annum compound interest.

17. The amount of money a businessman receives from a bank if he invests $145 000 for 8 years at 11% per annum compound interest.

18. The compound interest a merchant would have to pay if he borrows $54 000 for 9 years at 12% per annum compound interest.

19. The amount of money a salesman receives from a bank if he invests $60 000 for 5 years at 13% per annum compound interest.

20. The compound interest a civil servant would have to pay a bank if she borrows $75 000 for 6 years at 14% per annum compound interest.

21. The amount of money an engineer receives from a bank if he invests $50 000 for 7 years at 11.5% per annum compound interest.

22. The compound interest a teacher would have to pay a bank if she borrows $150 000 for 10 years at 12.5% per annum compound interest.

23. The amount of money a manager collects from a bank after 7 years if she invests $75 000 at 10.5% per annum compound interest.

24. The amount of money an entrepreneur receives from a bank after 5 years if he invests $29 700 at 13.5% per annum compound interest.

25. The compound interest an investor earns when he deposits $125 000 in a bank for 9 years at 11.5% per annum compound interest.

Example 35

Using the compound interest table above, calculate the compound interest earned when:

(a) $3 512 is invested at 9% per annum compound interest for 10 years

(b) $25 600 is invested at 12% per annum compound interest for 5 years

(c) $3 540 is invested at 13.5% per annum compound interest for 4 years.

Solution

From the *compound interest table*:

(a) The *appreciation of* $1 after 10 years at 9% *per annum compound interest* = $2.37

∴ the *appreciation of* $3 512 after 10 years at 9% *per annum compound interest*, $A = \$2.37 \times 3\,512$

$= \$8\,323.44$

So the *compound interest*, $C.I. = A - P$

$= \$(8\,323.44 - 3\,512)$

$= \$4\,811.44$

Hence the *compound interest earned* was $4 811.44.

(b) The *appreciation of* $1 after 5 years at 12% *per annum compound interest* = $1.76

∴ the *appreciation of* $25 600 after 5 years at 12% *per annum compound interest*, $A = \$1.76 \times 25\,600$

$= \$45\,056$

So the *compound interest*, $C.I. = A - P$

$= \$(45\,056 - 25\,600)$

$= \$19\,456$

Hence the *compound interest earned* was $19 456.

(c) The *appreciation of* $1 after 4 years at 13.5% *per annum compound interest*

$$= \frac{\$(1.63 + 1.69)}{2}$$

$$= \frac{\$3.32}{2}$$

$= \$1.66$

∴ the *appreciation of* $3 540 after 4 years at 13.5% *per annum compound interest*, $A = \$1.66 \times 3\,540$

$= \$5\,876.40$

So the *compound interest*, $C.I. = A - P$

$= \$(5\,876.40 - 3\,540)$

$= \$2\,336.40$

Hence the *compound interest earned* was $2 336.40.

Exercise 5t

1. Calculate the compound interest on investing $600 for 2 years at [illegible] per annum.

2. Calculate the compound interest on investing $600 for 3 years at [illegible] per annum.

3. Calculate the compound interest on investing $8 500 for 3 years at [illegible] per annum.

4. Calculate the compound interest earned when $12 000 is invested for 5 years at 8% per annum.

5. Calculate the compound interest earned when $14 000 is invested for 5 years at 7% per annum.

6. Which is the better investment?

(a) $1 200 at 9% simple interest for 2 years.

(b) $1 200 at 8% compound interest for 2 years.

7. $8 000 was put in a fixed deposit account on 1st January, 2002 for 6 months. The rate of interest was 7.5% per annum. On 1st July, 2002 the total amount received was reinvested for a further 6 months at 7% per annum. Calculate the final amount received at the end of the year.

8. A man wishes to invest $3 500. He can buy savings bonds which pay simple interest at the rate of 12% per annum or he can start a savings account which pays compound interest at the same rate.

Calculate, the difference in the amounts of the two investments at the end of 3 years.

the car *depreciates* for the 3-*year period*,

$$\begin{aligned} D &= P_1 - P_4 \\ &= \$(27\,000 - 23\,149.12) \\ &= \$3\,850.88 \end{aligned}$$

Hence the car *depreciates* by \$3 850.88.

Alternatively,
the depreciation,

$$\begin{aligned} D &= D_1 + D_2 + D_3 \\ &= \$(1\,350 + 1\,282.50 + 1\,218.38) \\ &= \$3\,850.88 \end{aligned}$$

Using a Calculator

The following *method* illustrates how a *scientific calculator* was used to *solve* the *problem.*

(a) The *book value* of the car *after* 3 *years*,

$$\begin{aligned} A &= P\left(1 - \frac{R}{100}\right)^n \\ &= \$27\,000\left(1 - \frac{5}{100}\right)^3 \\ &= \$27\,000(1 - 0.05)^3 \\ &= \$27\,000(0.95)^3 \\ &= \$23\,149.125 \\ &= \$23\,149.13 \text{ (correct to the nearest cent)} \end{aligned}$$

Table 5.54

Input	*Seen on the display of the calculator*
0.95	0.95
$\boxed{x^y}$	
$\boxed{3}$	3.
$\boxed{=}$	0.857 375
$\boxed{\times}$	
$\boxed{27\,000}$	27 000
$\boxed{=}$	23 149.125

(b) The *amount* by which the car *depreciates* for the 3-year *period*,

$$\begin{aligned} D &= P - A \\ &= \$(27\,000 - 23\,149.13) \\ &= \$3\,850.87 \end{aligned}$$

Tables

The following *method* illustrates how *three-figure mathematical tables* were used to *solve* the *problem.*

(a) The *book value* of the car *after* 3 *years*,

$$\begin{aligned} A &= P\left(1 - \frac{R}{100}\right)^n \\ &= \$27\,000\left(1 - \frac{5}{100}\right)^3 \\ &= \$27\,000(1 - 0.05)^3 \\ &= \$27\,000(0.95)^3 \\ &= \$23\,200 \text{ (correct to 3 s.f.)} \end{aligned}$$

Table 5.55

Number	*Operation*	*Log*
27 000 0.95^3	$3(\bar{1}.978)$	4.431 + $\bar{1}.934$
2.32×10^4 = 23 200		4.365

Note that 27 000 $= 2.7 \times 10^4$

$$\begin{aligned} &\text{And} && \log 27\,000 &&= \log(2.7 \times 10^4) \\ &&&&&= \log 2.7 + \log 10^4 \\ &&&&&= 4 + \log 2.7 \\ &&&&&= 4.431 \\ &\text{Also} && \log 0.95 &&= \bar{1}.978 \\ &\text{And} && \log 0.95^3 &&= 3 \log 0.95 \\ &&&&&= 3(\bar{1}.978) \\ &&&&&= \bar{1}.934 \\ &\text{Also} && \text{antilog } 0.365 &&= 2.32 \\ &\text{And} && \text{antilog } 4.365 &&= 2.32 \times 10^4 \\ &&&&&= 23\,200 \end{aligned}$$

(b) The *amount* by which the car *depreciates* for the 3-*year period*,

$$\begin{aligned} D &= P - A \\ &= \$(27\,000 - 23\,200) \\ &= \$3\,800 \text{ (correct to 3 s.f.)} \end{aligned}$$

From the above *example*, it can be *seen* that when *three-figure mathematical tables* are used to *compute* the *book value* of an *asset*, then the *answer* can only be *stated correct* to 3 *significant figures.*

Depreciation

Most working people own *assets*, for example, cars and personal computers, which *decrease in value*, that is, *depreciate* due to wear and deterioration in general with the passing of time. In the case of a vehicle, for example, it is necessary to know the *value* of the vehicle in order to insure the *vehicle* each year. There are two ways of insuring a vehicle; either third-party or fully comprehensive. *Insurance companies* use a *mathematical formula* to calculate the value of a vehicle called the *book value*, because it is not necessarily the actual value of the asset. The method that they use is called the *reducing balance method.* In the *reducing balance method*, the *depreciation* of an asset for a particular year is *calculated* as a *percentage* of the *book value* of the asset at the beginning of each year. And the *book value* at the beginning of the next year is obtained by *subtracting* this *depreciation* from the previous year's *book value* of the asset.

It is also worth noting that some finance companies *lend money* and calculate the *interest payable* each year using the *reducing balance method.* Hence the *interest payable* each year *decreases.*

The *depreciation formula* is: $A = P\left(1 - \frac{R}{100}\right)^n$,

where A = the *book value after n years*,
P = the *initial cost of the asset*,
R = the *rate of depreciation per annum*
and n = the *number of years* for which the *asset was depreciated.*

Also the *amount* by which the *asset depreciates*,
$D = P - A.$

It can be seen that the *depreciation formula* is very similar to the *compound interest formula*. In the case of *depreciation* the *percentage* is *subtracted*, and in the case of *compound interest* which is an *appreciation*, the *percentage* is *added.*

Example 36

A Mazda 323 car was bought in January 1986 for \$27 000. An insurance company decides to calculate its depreciation per annum as 5%.

(a) its book value 3 years later

(b) the amount by which it depreciates for the same period.

Solution

(a) The *initial cost* of the car, $P_1 = \$27\,000$

So the *depreciation* for the *first year*, $D_1 = 5\%$ of \$27 000

$= \frac{5}{1\cancel{00}} \times \$27\,0\cancel{00}$

$= \$1\,350$

∴ the *book value* of the car at the *beginning* of the *second year*, $P_2 = P_1 - D_1$

$= \$(27\,000 - 1\,350)$

$= \$25\,650$

So the *depreciation* for the *second year*, $D_2 = 5\%$ of \$25 650

$= \frac{5}{10\cancel{0}} \times \$25\,65\cancel{0}$

$= 5 \times \$256.50$

$= \$1\,282.50$

∴ the *book value* of the car at the *beginning* of the *third year*, $P_3 = P_2 - D_2$

$= \$(25\,650 - 1\,282.50)$

$= \$24\,367.50$

So the *depreciation* for the *third year*, $D_3 = 5\%$ of \$24 367.50

$= 0.05 \times \$24\,367.50$

$= \$1\,218.375$

$= \$1\,218.38$ (*correct* to the *nearest cent*)

∴ the *book value* of the car at the *beginning* of the *four year*, $P_4 = P_3 - D_3$

$= \$(24\,367.50 - 1\,218.38)$

$= \$23\,149.12$

Hence the *book value* of the car *after* 3 *years* is \$23 149.12.

5. Calculate the book value of a new truck costing $48 000 after 6 years if it depreciates each year by 10%.

6. An insurance company estimates that the value of a car depreciates by 15% each year provided that it is not involved in an accident.

Mr. Gypsy's car was valued by the company at $42 000 on 1st January, 2001. Calculate the book value of his car on 1st January, 2004, assuming that the car drove accident free for that period.

7. A pick-up truck depreciates in value at a rate of 10% per annum. What amount will be the value of the truck in 2 years time, if it is now worth $60 000?

8. (a) A rare stamp appreciates in value by 10% each year. If it is bought for $50, what amount will it be worth in 3 years time?
   (b) Evaluate the book value after 3 years if it depreciates by 10% each year.

9. A man bought a new caravan for $75 600. The insurance company decides to depreciate the caravan each year by 9%. Calculate the book value of the car after 6 years.

10. A woman bought a new trailer for $36 700. The insurance company decides to depreciate the trailer each year by 12.5%. Calculate the book value of the trailer at the end of 10 years.

11. A man bought a car for $60 320. After using it for 2 years he decided to trade in the car. The company estimated a depreciation of 15% for the first year of its use and a further 15% on its reduced value for the second year.
    (a) Calculate the value of the car after 2 years.
    (b) Express the value of the car after 2 years as a percentage of the original value.
    (c) Express the depreciation after the 2-year period as a percentage of the original value.

12. A woman bought a car for $78 450. After using it for 3 years she decided to trade in the car. The company estimated a depreciation of 12% for the first year of its use and a further 12% on its reduced values for the second and third years.
    (a) Calculate the value of the car after the 3-year period.
    (b) Express the value of the car after 3 years as a percentage of its original value.
    (c) Express the depreciation after the 3 years as a percentage of its original value.

The Depreciation Table below shows the book value of an asset costing $1 for periods from 1 year to 10 years at rates of depreciation from 5% to 12%.

Table 5.57

Years	5%	6%	7%	8%	9%	10%	11%	12%
1	0.950	0.940	0.930	0.920	0.910	0.900	0.890	0.880
2	0.903	0.884	0.865	0.846	0.828	0.810	0.792	0.774
3	0.857	0.831	0.804	0.779	0.754	0.729	0.705	0.681
4	0.815	0.781	0.748	0.716	0.686	0.656	0.627	0.600
5	0.774	0.734	0.696	0.659	0.624	0.590	0.558	0.528
6	0.735	0.690	0.647	0.606	0.568	0.531	0.497	0.464
7	0.698	0.648	0.602	0.558	0.517	0.478	0.442	0.409
8	0.663	0.610	0.560	0.513	0.470	0.430	0.394	0.360
9	0.630	0.573	0.520	0.472	0.428	0.387	0.350	0.316
10	0.599	0.539	0.484	0.434	0.389	0.349	0.312	0.279

Using the table, calculate the book value and depreciation of:

13. A computer costing $14 760 after 3 years depreciating at 5% per annum.

14. A lathe costing $9 470 after 5 years depreciating at 7% per annum.

15. A car costing $98 500 after 10 years depreciating at 12% per annum.

16. A motor bike costing $8 600 after 8 years depreciating at 9.5% per annum.

17. A maxi taxi costing $159 000 after 7 years depreciating at 11.5% per annum.

18. A plane costing M $2.50 after 6 years depreciating at 8.5% per annum.

C.X.C. Past Paper Questions

The following supplementary problems were taken from C.X.C. Past Papers.

Depreciation Table

The *depreciation table* is a *ready reckoner* that is used by business people to calculate quickly the *depreciation* of an asset or *interest payable* on a *loan* using the *reducing balance method.*

The *table* below is an *extract* from a *ready reckoner* showing the *book value* of an asset *costing* \$1 for *periods* from 1 *year* to 10 *years* at *rates of depreciation* from 5% to 12%.

Table 5.56

Years	5%	6%	7%	8%	9%	10%	11%	12%
1	0.950	0.940	0.930	0.920	0.910	0.900	0.890	0.880
2	0.903	0.884	0.865	0.846	0.828	0.810	0.792	0.774
3	**0.857**	0.831	0.804	0.779	0.754	0.729	0.705	0.681
4	0.815	0.781	0.748	0.716	0.686	0.656	0.627	0.600
5	0.774	0.734	0.696	0.659	0.624	0.590	0.558	0.528
6	0.735	0.690	0.647	0.606	0.568	0.531	0.497	0.464
7	0.698	0.648	0.602	0.558	0.517	0.478	0.442	0.409
8	0.663	**0.610**	**0.560**	0.513	0.470	0.430	0.394	0.360
9	0.630	0.573	0.520	0.472	0.428	0.387	0.350	0.316
10	0.599	0.539	0.484	0.434	0.389	0.349	0.312	0.279

Example 37

Using the depreciation table above, calculate the book value and depreciation of:

(a) a computer costing \$12 578 after 3 years depreciating at 5% per annum.

(b) a car costing \$69 745 after 8 years depreciating at 6.5% per annum.

Solution

(a) The *book value* of an asset *costing* \$1 after 3 years *depreciating* at 5% *per annum* $= \$0.857$

∴ the *book value* of the computer *costing* \$12578 after 3 years *depreciating* at 5% *per annum*,

$$A = \$0.857 \times 12578$$
$$= \$10779.35$$

Hence the *book value* of the computer is \$10779.35.

The *amount by* which the computer *depreciates* for the 3-year *period*,

$$D = P - A$$
$$= \$(12578 - 10779.35)$$
$$= \$1798.65$$

Hence the *depreciation* of the computer is \$1798.65.

(b) The *book value* of an asset *costing* \$1 after 8 years *depreciating* at 6.5% *per annum*

$$= \frac{\$(0.610 + 0.560)}{2}$$
$$= \frac{\$1.17}{2}$$
$$= \$0.585$$

∴ the *book value* of the car *costing* \$69745 after 8 years *depreciating* at 6.5% *per annum*

$$= \$0.585 \times 69745$$
$$= \$40800.825$$
$$= \$40800.83$$

(*correct* to the *nearest cent*)

Hence the *book value* of the car is \$40800.83.

The *amount* by which the car *depreciates* for the 8-year *period*,

$$D = P - A$$
$$= \$(69745 - 40800.83)$$
$$= \$28944.17$$

Hence the *depreciation* of the car is \$28944.17.

Exercise 5u

1. Calculate the book value of a new maxi taxi costing \$44 000 after 5 years if it depreciates each year by 13%.
2. Calculate the book value of a new mini bus costing \$43 000 after 5 years if it depreciates each year by 12%.
3. A car is bought for \$38 600. The insurance company decided to calculate the depreciation each year as 12.5% of the book value at the beginning of the year. Determine the value of the car at the end of 5 years.
4. A lathe was bought for \$9 800. The insurance company decides to calculate the depreciation each year as 5.5% of the book value at the beginning of the year. Calculate the book value of the lathe at the end of 10 years.

6. *Rate of interest*

	No. of yrs.	5%	6%	7%	8%	9%	10%
Value of $1 at certain rates of compound interest	1	1.050	1.060	1.070	1.080	1.090	1.100
	2	1.103	1.124	1.145	1.166	1.188	1.210
	3	1.158	1.191	1.225	1.260	1.295	1.331
	4	1.216	1.262	1.311	1.360	1.412	1.464

(ii) A businessman borrowed $7 500 for 3 years and repaid the loan at 8% compound interest. Using the table above calculate the interest he had to pay on the loan.

Question 7(ii). C.X.C. (Basic). June 1982.

7. The table below gives the exchange rates for some Caribbean currencies against the United States dollar.

Territory	*Amt. for* US $1.00
Eastern Caribbean	EC $2.70
Guyana	G $3.00
Jamaica	J $2.25
Trinidad and Tobago	TT $2.40

Calculate:

(i) the amount in Jamaican dollars one would get for US $200

(ii) the amount in United States dollars one would get for TT $360

(iii) the amount in Eastern Caribbean dollar one would get for G $50.

Question 5. C.X.C. (Basic). June 1984.

8. A man's gross income for 1993 was $18 000. His wife was unemployed. Their two children, aged 13 and 17 were both at school. He paid 6 cents out of every dollar of his gross income for national insurance. **National insurance payments are non-taxable**. Other tax free allowances and tax rates are given in the tables below:

Tax Free personal allowances	*Tax rates on taxable income*
Employee $1 200	10% on first $4 000
Unemployed spouse $800	20% on next $4 000
Child under 11 years at school $200	40% on the remainder
Child between 11 and 16 years at school $250	
Child over 16 years at school $300	

Calculate:

(i) the amount he paid for national insurance

(ii) the total tax free personal allowances for his family

(iii) his total non-taxable allowance

(iv) the amount he paid in tax for 1983.

Question 7. C.X.C. (Basic). June 1984.

9. (a) (i) A refrigerator can be bought on hire purchase by making a deposit of $480 and 15 monthly instalments of $80 each. Calculate the hire purchase cost of the refrigerator.

(ii) The actual marked price of the refrigerator is $1 400. This includes a sales tax of 12%. Calculate the sale price of the refrigerator if no sales tax is included.

(b) The rates of exchange at a bank are as follows:

EC $1.00 = BDS $0.75
and US $1.00 = BDS $1.98

(i) A traveller changes EC $1 600 to Barbados currency. Calculate the amount received.

(ii) Of the amount she received she spent BDS $210 and exchanged the remainder for US currency. Calculate the amount in US currency she received for this exchange.
(Assume that the buying rate and selling rate for BDS $1.00 are the same.)

Question 8. C.X.C. (Basic). June 1985.

10. (a) The customs duty on imported vehicles is 30% of the imported price.

(i) Calculate the customs duty on a car for which the imported price is $8 500.

(ii) Calculate the imported price of a bus for which the amount paid, including customs duty, is $15 600.

Exercise 5v

1. A man's wage for a 35-hour week is $262.50. Calculate, without using tables, his hourly rate of payment.

Question 2(ii). C.X.C. (Basic). June 1979.

2. (i) If J $0.67 = Guy $1.00 and J $1.44 = Can $1.00, find:
   (a) Can $55.00 in J $.
   (b) J $50.00 in Guy $.

(ii) A shopkeeper buys a stove from a manufacturer. The shopkeeper sells the stove for $150.00 at a profit of 20%.
   (a) How much did the shopkeeper pay the manufacturer for the above?
   (b) If the shopkeeper gives 10% discount for cash, how much does a customer pay for the stove?

Question 3. C.X.C. (Basic). June 1980.

3. A man earns $1 200 per month and his wife earns $800 per month. They have four children. National insurance of 5% of all earnings must be paid before taxes are deducted. Allowances and taxes are calculated on their combined salaries. Tax free allowances and tax rates are as follows:

Tax free allowances	*Rates on taxable income*
$1 000 per annum for each adult	10% on first $2 000
$500 per annum per child	20% on next $2 000
Earned income relief 10% of husband's salary	30% on next $4 000
Non-taxable income 50% of wife's salary	40% on the remainder

Calculate:

(i) the total taxable annual income
(ii) the total tax paid for the whole year.

Question 2. C.X.C. (Basic). June 1981.

4. (i) A man wishes to invest $1 500. He can buy savings bonds which pay simple interest at the rate of 8% per annum or he can start a savings account which pays compound interest at the same rate. Calculate, to the nearest cent, the difference between the amounts of the two investments at the end of 3 years.

(ii) The table is an extract from a ready reckoner showing the cost of a number of articles at 43 ¢ per article:

11	4.73	53	22.97	95	40.85	220	94.60
12	5.16	54	23.22	96	41.28	224	96.32
13	5.59	55	23.65	97	41.71	250	107.50
14	6.02	56	24.08	98	42.14	280	120.40
15	6.45	57	24.51	99	42.57	300	129.00
16	6.88	58	24.94	100	43.00	350	150.50
17	7.31	59	25.37	101	43.43	365	156.95
18	7.74	60	25.80	102	43.86	400	172.00
19	8.17	61	26.23	103	44.29	450	193.50
20	8.60	62	26.66	104	44.72	480	206.40

Use the table to state the cost of:

(a) 13 ball point pens at 43 ¢ per pen
(b) 73 ball point pens at 43 ¢ per pen
(c) 373 ball point pens at 43 ¢ per pen
(d) 5 173 ball point pens at 43 ¢ per pen

To gain full marks intermediate values used must be shown.

Question 8. C.X.C. (Basic). June 1981.

5. The cash price of a living room suite is $2 800.

(i) It can be bought on hire purchase if a deposit of $400 is first paid. Simple interest at 10% per annum for 2 years is then added to the difference between the deposit and the cash price. The amount must then be paid off in equal monthly instalments over the two-year period. Calculate:
   (a) the monthly instalment
   (b) the total amount paid for the suite.

(ii) It can also be bought by borrowing the cash price from a bank at 10% simple interest and the principal and interest are repaid at the end of 2 years. Calculate the amount to be paid for the suite by this arrangement.

(iii) Which arrangement is less costly and by how much?

Question 4. C.X.C. (Basic). June 1982.

15. The telephone b ill for Mary James for April, 1992 is given below. Telephone subscribers are charged a monthly service fee of $27.50 which covers up to a maximum of 30 local calls per month. A charge of 20 cents per call is made for each local call in excess of 30 calls. A tax of 75% is payable on all overseas calls.

Telephone Bill

Name: Mary Jones Account No. J0052

Previous Reading *March* 31, 1992	*Present Reading* *April* 30, 1992	*Number of* *local calls*
4325	4402	

Charges		
	$	¢
Arrears		
Service fee	25	50
Local calls		
Overseas calls	80	00
Tax on overseas calls		
Total	209	40

(a) Calculate:

- (i) the number of local calls made in April
- (ii) the amount due for local calls in April
- (iii) the tax on overseas calls in April
- (iv) the arrears from March.

(b) Mary was charged $13.00 for local calls in May, 1992. Calculate the total number of local calls she made in May, 1992.

Question 6. C.X.C. (Basic). June 1992.

16. The table below, shows the flying times an aeroplane takes in travelling between countries on its route from Jamaica to Trinidad and the distance between these countries.

From	*To*	*Time (in minutes)*	*Distance (in statute miles)*
Jamaica	Puerto Rico	90	710
Puerto Rico	Antigua	35	290
Antigua	Barbados	45	320
Barbados	Trinidad	30	210

(a) Calculate:

- (i) the total flying time, in hours and minutes, of the journey from Jamaica to Trinidad.
- (ii) the average speed, in miles per hour, for the journey from Jamaica to Puerto Rico.

(b) An aeroplane leaves Jamaica at 11:45 h local time, It stops for 45 minutes at each of the countries along the route. Given that the time in Jamaica is one hour behind the time in Trinidad, calculate the local time at which the plane arrives in Trinidad.

Question 7. C.X.C. (Basic). June 1992.

17. (a) Mary borrowed $3 000 from her Credit Union. The Credit Union charges interest at the rate of 20% per annum on the loan balance at the end of each year. Mary paid $200 per month on her account. Calculate:

- (i) the amount she repaid during the first year
- (ii) the interest charged at the end of the first year
- (iii) the amount owed at the beginning of the second year
- (iv) the least number of payments required to pay off the loan and interest during the second year.

(b) Calculate the rate of interest per annum if $405 is the simple interest gained on $2 700 in one year.

Question 9. C.X.C. (Basic). June 1992.

18. (i) A shopkeeper buys a stove from a manufacturer. The shopkeeper sells the stove for $150.00 at a profit of 20%.

- (a) How much did the shopkeeper pay the manufacture for the stove?
- (b) If the shopkeeper gives 10% discount for cash, how much does a customer pay for the stove?

Question 2(i). C.X.C. (General). June 1980.

CHAPTER 6

Algebra 1

This chapter will teach you how to

- represent a number using a symbol, construct an algebraic expression, and substitute for a numeral in an algebraic expression.
- add, subtract, multiply and divide algebraic terms and algebraic fractions.
- define and use the distributive law and binary operators.
- define factorization and to factorize using the distributive law and H.C.F.
- define an equation, an inequation, and simultaneous equations; solve linear equations, linear inequations, and simultaneous linear equations; form and solve word problems.
- use the laws of indices.
- solve an equation where the unknown quantity is in the index.
- use standard form (or scientific notation) to solve problems dealing with the basic arithmetic operations, power's and roots.
- use the logarithm theory with logarithm and antilogarithm to solve problems.
- solve an equation using logarithm.

Introduction

Algebra is the *generalization* and representation in *symbolic form* of meaningful results and patterns in arithmetic and other areas of mathematics.

For example:
If the cost of one book is \$65.00,
then the cost of 10 books is \$650.00 (in *arithmetic*)
and the cost of x books is \$65.00x (in *algebra*).

The statement that the cost of 10 books is \$650.00 is said to be a *particular statement.* And the statement that the cost of x books is \$65.00x is said to be a *general statement.* We can substitute any natural number for x and hence find the cost for that number of books at \$65.00 each by multiplying.

Using a Symbol to Represent a Number

It is a normal *process* in algebra to *translate* from *word statements* to *algebraic statements* using *symbols.* We can choose any *symbol* we like, usually from the *English* or *Greek alphabets*, to represent a *quantity*, unless otherwise stated. For example: Let *two numbers* be x and y, such that $x > y$. Then the *four basic operations* can be seen represented below.

The *sum* of the two numbers $= x + y$.
The *difference* of the two numbers $= x - y$.
The *product* of the two numbers $= x \times y = xy$.

The *quotient* of the two numbers $= x \div y = \frac{x}{y}$, if the *quotient* is *greater than* 1.

Or the quotient of the two numbers $= y \div x = \frac{y}{x}$, if the *quotient* is *less than* 1.

Example 1

Translate each of the following word phrases into algebraic expressions using the symbols given.

(a) *Five times a number x.*

(b) *Seven times a number x, plus a second number y.*

(c) *Six times a number x, minus a second number y.*

(d) *Half times the product of x and y.*

(e) *Three times the product of two numbers x and y, divided by a third number z.*

Solution

(a) *Five times* a *number x* $= 5 \times x$

$= 5x$

(b) *Seven times* a *number x, plus* a *second number y* $= 7 \times x + y$

$= 7x + y$

(c) *Six times* a *number x, minus* a *second number y* $= 6 \times x - y$

$= 6x - y$

(d) *Half times* the *product of x and y* $= \frac{1}{2} \times x \times y$

$= \frac{1}{2}xy$

$= \frac{xy}{2}$

(e) *Three times* the *product* of *two numbers x and y, divided* by a *third number z* $= 3 \times x \times y \div z$

$= 3xy \div z$

$= \frac{3xy}{z}$

Exercise 6a

Translate each of the following word phrases into algebraic expressions using the symbols given.

1. Nine times a number x.

2. Twelve times a number x, plus a second number y.

3. Eleven times a number x, minus a second number y.

4. Three-quarters the product of two numbers x and y.

5. Five times the product of two numbers x and y, divided by a third number z.

6. Seven times the product of two numbers x and y, less five, divided by thrice a third number z.

7. Half the product of two numbers x and y, minus five times a third number z.

8. Three times a number x, minus four times a second number y, divided by seven times a third number z.

9. The square of the sum of two numbers x and y.

10. The cube of the sum of two numbers x and y.

11. Three times a number a, added to four times a second number b, divided by double a third number c.

12. Half the product of two numbers a and b, added to thrice a third number c.

13. The square of thrice a number a, take away double a second number b.

14. The cube of double a number a, take away thrice a second number b.

15. Nine times the product of two numbers a and b, less five times a third number c, divided by a fourth number d.

Translate each of the following algebraic expressions into word phrases:

16. $7x$

17. $9x + y$

18. $5x - y$

19. $\frac{xy}{2}$

20. $(x - y)^2$

21. $(x - y)^3$

22. $\frac{2a + 3b}{4c}$

23. $\frac{ab}{2} - 3c$

24. $(4a - 3b)^2$

25. $(3a - 4b)^3$

Translate each of the following word phrases into algebraic expressions using the symbols given:

26. Five times a number x, minus four times a second number y, divided by twice a third number z.

27. Half the sum of two numbers x and y, divided by twice a third number z.

Substituting a Numeral for a Symbol in an Algebraic Expression

Substitution of a numeral for a symbol in an algebraic expression, is the process whereby each *symbol* in an *algebraic expression* is replaced by a given *number* in order to simplify and determine the *particular numerical value of the expression*. When *substituting* a number for a symbol in an algebraic expression, we must pay particular attention so as not to *change* the *form* of the *expression*. Hence we *substitute* the number for the symbol *directly* into the *algebraic expression* and thus obtain an *arithmetic expression*, which we then *simplify* in order to obtain a *particular numerical value*.

The *rules* for the *multiplication* and *division* of the different combinations of *positive* and *negative numbers* are reinforced below:

$(+1) \times (+1) = +1$
$(-1) \times (-1) = +1$
Both *products* are *positive*.

$(+1) \times (-1) = -1$
$(-1) \times (+1) = -1$
Both *products* are *negative*.

$(+1) \times (+1) \times (+1) = (+1) \times (+1) = +1$
$(-1) \times (-1) \times (+1) = (+1) \times (+1) = +1$
Both *products* are *positive*.

$(-1) \times (-1) \times (-1) = (+1) \times (-1) = -1$
$(+1) \times (+1) \times (-1) = (+1) \times (-1) = -1$
Both *products* are *negative*.

From the *examples* above it can be *seen* that:

(i) When we *multiply positive numbers*, then *the product* is always *positive*.
(ii) When we *multiply* an *even number* of *negative numbers*, then the *product* is always *positive*.
(iii) When we *multiply* an *odd number* of *negative numbers*, then the *product* is always *negative*.

$$(+1) \div (+1) = \frac{+1}{+1} = +1$$
$$(-1) \div (-1) = \frac{-1}{-1} = +1$$
Both *quotients* are *positive*.

$$(+1) \div (-1) = \frac{+1}{-1} = -1$$
$$(-1) \div (+1) = \frac{-1}{+1} = -1$$
Both *quotients* are *negative*.

From the *examples* above it can be *seen* that:

(i) When we *divide numbers* with *like signs*, then the *quotient* is always *positive*.
(ii) When we *divide numbers* with *unlike signs* (that is, the representative sign for the numerator and for the denominator are unlike.), then the *quotient* is always *negative*.

The *meaning* of the *power of a number* is reinforced below:

5 to the *power* 2 = 5 *squared* = $5^2 = 5 \times 5 = 25$
2 to the *power* 3 = 2 *cubed* = $2^3 = 2 \times 2 \times 2 = 8$
3 to the *power* 4 = $3^4 = 3 \times 3 \times 3 \times 3 = 81$
2 to the *power* 5 = $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$.

a to the *power* 2 = a *squared* = $a^2 = a \times a$
b to the *power* 3 = b *cubed* = $b^3 = b \times b \times b$
m to the *power* 4 = $m^4 = m \times m \times m \times m$
n to the *power* 5 = $n^5 = n \times n \times n \times n \times n$.

$5a^2b = 5 \times a \times a \times b$
$3pq^2 = 3 \times p \times q \times q$
$$\begin{aligned} 4(mn)^2 &= 4 \times mn \times mn = 4 \times m \times n \times m \times n \\ &= 4 \times m \times m \times n \times n \\ &= 4m^2n^2 \end{aligned}$$
$7r^2s^3 = 7 \times r \times r \times s \times s \times s$.

Example 2

If $x = 2$, $y = -3$ and $z = 4$, calculate the value of each of the following algebraic expressions:

(a) $2x + y$ *(b)* $z - 2y$
(c) $5x - 2y + 3z$ *(d)* $\dfrac{8x + y - 2z}{2x - y + z}$

Solution

(a) Now $2x + y = 2 \times 2 + (-3) = 4 - 3 = 1$

(b) Now $z - 2y = 4 - 2 \times (-3) = 4 + 6 = 10$

(c) Now
$$\begin{aligned} 5x - 2y + 3z &= 5 \times 2 - 2 \times (-3) \\ &\quad + 3 \times 4 \\ &= 10 + 6 + 12 \\ &= 28 \end{aligned}$$

(d) Now
$$\begin{aligned} \frac{8x + y - 2z}{2x - y + z} &= \frac{8 \times 2 + (-3) - 2 \times 4}{2 \times 2 - (-3) + 4} \\ &= \frac{16 - 3 - 8}{4 + 3 + 4} \\ &= \frac{16 - 11}{11} \\ &= \frac{5}{11} \end{aligned}$$

Example 3

If $x = 2$, $y = -3$ and $z = 5$, determine the value of each of the following algebraic expressions:

(a) $5x^2z$

(b) $3xy^2 + 2x^3z$

(c) $9x^2z - 4xy^3$

(d) $\frac{5xy^2}{9z^2}$

Solution

(a) Now

$$\begin{aligned} 5x^2z &= 5 \times 2^2 \times 5 \\ &= 5 \times 2 \times 2 \times 5 \\ &= 5 \times 4 \times 5 \\ &= 100 \end{aligned}$$

(b) Now

$$\begin{aligned} &3xy^2 + 2x^3z \\ &= 3 \times 2 \times (-3)^2 + 2 \times 2^3 \times 5 \\ &= 3 \times 2 \times (-3) \times (-3) + 2 \times 2 \\ &\quad \times 2 \times 2 \times 5 \\ &= 3 \times 2 \times 9 + 2 \times 8 \times 5 \\ &= 54 + 80 \\ &= 134 \end{aligned}$$

Alternatively,

$$\begin{aligned} 3xy^2 + 2x^3z &= 3 \times 2 \times (-3)^2 + 2 \times 2^3 \times 5 \\ &= 6 \times 9 + 2 \times 8 \times 5 \\ &= 54 + 80 \\ &= 134 \end{aligned}$$

(c) Now

$$\begin{aligned} &9x^2z - 4xy^3 \\ &= 9 \times 2^2 \times 5 - 4 \times 2 \times (-3)^3 \\ &= 9 \times 2 \times 2 \times 5 - 4 \times 2 \times (-3) \\ &\quad \times (-3) \times (-3) \\ &= 9 \times 4 \times 5 + 4 \times 2 \times 27 \\ &= 180 + 216 \\ &= 396 \end{aligned}$$

Alternatively,

$$\begin{aligned} 9x^2z - 4xy^3 &= 9 \times 2^2 \times 5 - 4 \times 2 \times (-3)^3 \\ &= 9 \times 4 \times 5 - 4 \times 2 \times (-27) \\ &= 180 - 8 \times (-27) \\ &= 180 + 216 \\ &= 396 \end{aligned}$$

(d) Now

$$\begin{aligned} \frac{5xy^2}{9z^2} &= \frac{5 \times 2 \times (-3)^2}{9 \times 5^2} \\ &= \frac{5 \times 2 \times (-3) \times (-3)}{9 \times 5 \times 5} \\ &= \frac{\overset{1}{\cancel{5}} \times 2 \times \overset{1}{\cancel{9}}}{\underset{1}{\cancel{9}} \times \underset{1}{\cancel{5}} \times 5} \\ &= \frac{2}{5} \end{aligned}$$

Alternatively,

$$\begin{aligned} \frac{5xy^2}{9z^2} &= \frac{5 \times 2 \times (-3)^2}{9 \times 5^2} \\ &= \frac{\overset{1}{\cancel{5}} \times 2 \times \overset{1}{\cancel{9}}}{\underset{1}{\cancel{9}} \times \underset{5}{\cancel{25}}} \\ &= \frac{2}{5} \end{aligned}$$

Exercise 6b

If $x = 2$, $y = 3$ and $z = 4$, calculate the value of each of the following algebraic expressions:

1. $x + 5$
2. $y + 3$
3. $z + 2$
4. $x + 1$
5. $y + 4$
6. $z + 7$
7. $x - 1$
8. $y - 2$
9. $z - 2$
10. $9 - x$
11. $8 - y$
12. $7 - z$
13. $3xy$
14. $4xz$
15. $2yz$
16. $-2xy$
17. $-3xz$
18. $-5yz$
19. $x \div 2$
20. $y \div 3$
21. $z \div 2$
22. $16 \div x$
23. $15 \div y$
24. $24 \div z$
25. $xy \div 2$
26. $xz \div 4$
27. $yz \div 8$
28. $xyz \div 3$
29. $24 \div xyz$
30. $3x + 2y$
31. $2x + 3z$
32. $4y + 3z$
33. $3z - 4x$
34. $5z - 3y$
35. $4z - 5x$
36. $2x + 3y + z$
37. $3x + 2y + z$
38. $4x + 3y + 2z$
39. $4x + 3y - 2z$
40. $5x - 4y + 3z$
41. $5x + 2y - 3z$
42. $4x - 3y + 2z$
43. $7x - 2y - z$
44. $9x - 3y - 2z$
45. $3y + 2z - 5x$
46. $4y + 3z - 2x$
47. $5y + 4z - 3x$
48. $\frac{3x + 2y + z}{x + y + z}$
49. $\frac{4x - 3y + 2z}{2x + y - z}$
50. $\frac{5x + 3y - 2z}{3x - 2y + z}$

Given that $x = 2$, $y = -3$ and $z = 5$, calculate the value of each of the following algebraic expressions:

51. $2x + y$
52. $5x - 2y + 3z$
53. $\frac{8x + y - 2z}{2x - y + z}$

Given that $a = 2$, $b = 3$ and $c = 5$, determine the value of each of the following algebraic expressions:

54. $9c - 5b$
55. $5a + 3b - 2c$

56. $\dfrac{3a - 2b + 5c}{2a + b - c}$

57. Given that $x = 2$, $y = 3$ and $z = 5$, calculate the value of the algebraic expression:

$$\frac{4z + 3x - y}{2y - x + 2z}.$$

58. Given that $x = 3$, $y = 2$ and $z = 4$, determine the value of the algebraic expression:

$$\frac{3z + 2x + y}{4y - 2x + z}.$$

59. Given that $x = 4$, $y = -3$ and $z = 2$, calculate the value of $xy - 5(x - y) + 2z$.

60. Given that $x = 3$, $y = 4$ and $z = -5$, determine the value of $2xz - 3(y - x) + 4z$.

If $x = 2$, $y = -3$ and $z = 4$, determine the value of each of the following algebraic expressions:

61. x^2
62. y^2
63. z^2
64. x^3
65. y^3
66. z^3
67. $5x^2$
68. $3y^2$
69. $2z^2$
70. $4x^3$
71. $5y^3$
72. $2z^3$
73. xy^2
74. xy^3
75. x^2y
76. x^2z
77. xz^2
78. yz^2
79. y^2z^3
80. $(xy)^2$
81. $(xy)^3$
82. $(xz)^2$
83. $3x^2y$
84. $3xy^2$
85. $4x^3z$
86. $5x^2z^3$
87. $7x^3y$
88. $2x^3y + 3x^2z$
89. $3x^2y + 2xz^2$
90. $5x^2z + 3xz^2$
91. $2x^2z - 3y^2z$
92. $4xz - 3x^2y$
93. $4x^2y - 3xz$
94. $\dfrac{x^2y}{z}$
95. $\dfrac{xy^2}{z}$
96. $\dfrac{xz^2}{y}$
97. $\dfrac{x^2z}{y}$
98. $\dfrac{3xy^2}{2z}$
99. $\dfrac{4x^2z}{3y}$
100. $\dfrac{5x^3y}{2z^2}$

101. Given that $a = 2$, $b = 3$ and $c = 5$, determine the value of the algebraic expression:

$$\frac{3a^4}{c^2} - 6a.$$

102. Given that $a = 3$, $b = 2$ and $c = 1$, calculate the value of the algebraic expression:

$$4a^3 \div 3bc.$$

103. Given that $a = 2$, $b = 3$ and $c = 5$, calculate the value of the algebraic expression:

$$4c^3 \div 5ab^2.$$

104. Given that $p = 5$ and $q = -1$, determine the value of p^2q^3.

105. Given that $p = 2$ and $q = -1$, calculate the value of p^2q^3.

106. Given that $a = 3$, $b = 5$ and $c = 4$, determine the value of the algebraic expression:

$$6b^3c^2 \div 25a^4.$$

107. Given that $a = 4$, $b = -3$ and $c = 2$, calculate the value of $a^2(2b - c)$.

108. Given that $m = 5$ and $n = -2$, determine the value of m^2n^3.

Addition and Subtraction of Algebraic Terms

We can only *add* and *subtract like algebraic terms. Like algebraic terms* are defined as those terms which are represented by the *same algebraic symbol* regardless of the *magnitude* or *sign* of their *coefficients.*

Thus:

$3x$, $-4x$, $\frac{1}{2}x$, $-\frac{3}{5}x$, $0.6x$ and $-0.5x$

are all *like terms*, since they are all represented by the *same algebraic symbol* x.

Also $4x^2$, $-3x^2$, $\frac{3}{4}x^2$, $-\frac{1}{2}x^2$, $0.65x^2$ and $-0.7x^2$ are all *like terms*, since they are all represented by the *same algebraic symbol* x^2.

Unlike algebraic terms are defined as those terms which are represented by *different algebraic symbols.*

Thus:

x, x^2, x^3, x^4 and x^5 are *five different algebraic terms*, Hence, regardless of the *magnitude* of their *coefficients*, we *cannot add* or *subtract* such *terms* together.

Example 4

Simplify each of the following algebraic expressions:

(a) $7x + x + 12x$

(b) $-15x - 8x - x$

(c) $9x - 5x$

(d) $3x - 7x$

(e) $12x + 5x - 9x$

(f) $15x - 9x + x$

Solution

(a) Now $7x + x + 12x = (7 + 1 + 12)x = 20x$

(b) Now $-15x - 8x - x = (-15 - 8 - 1)x$
$= -24x$

(c) Now $9x - 5x = (9 - 5)x = 4x$

(d) Now $3x - 7x = (3 - 7)x = -4x$

(e) Now $12x + 5x - 9x = (12 + 5 - 9)x$
$= (17 - 9)x$
$= 8x$

(f) Now $15x - 9x + x = (15 - 9 + 1)x$
$= (15 + 1 - 9)x$
$= (16 - 9)x$
$= 7x$

From the above *examples* it can be *seen* that:

(i) The *addition* of *like algebraic terms* corresponds to the *addition* of their *coefficients*.

(ii) The *subtraction* of *like algebraic terms* corresponds to the *subtraction* of their *coefficients*.

In the above examples we were able to add and subtract *one set of like terms* in an *algebraic expression*. It is also possible to add and subtract *several sets of like terms* in an *algebraic expression*.

Example 5

Simplify each of the following algebraic expressions:

(a) $9x + 3y - 5z + 8y - 4x + 12z$

(b) $15x^2 - 7y^2 + 9y^2 - 8x^2$

(c) $3a^2b + 4ab^2 - a^2b - 3ab^2 + a^2b^2$

(d) $1.5p^2q - 0.3pq^3 + 4.1p^2q + 5.0pq^3$

Solution

(a) Now $9x + 3y - 5z + 8y - 4x + 12z$
$= 9x - 4x + 3y + 8y - 5z + 12z$
$= 9x - 4x + 3y + 8y + 12z - 5z$
$= (9 - 4)x + (3 + 8)y + (12 - 5)z$
$= 5x + 11y + 7z$

(b) Now $15x^2 - 7y^2 + 9y^2 - 8x^2$
$= 15x^2 - 8x^2 - 7y^2 + 9y^2$
$= 15x^2 - 8x^2 + 9y^2 - 7y^2$
$= (15 - 8)x^2 + (9 - 7)y^2$
$= 7x^2 + 2y^2$

(c) Now $3a^2b + 4ab^2 - a^2b - 3ab^2 + a^2b^2$
$= 3a^2b - a^2b + 4ab^2 - 3ab^2 + a^2b^2$
$= (3 - 1)a^2b + (4 - 3)ab^2 + a^2b^2$
$= 2a^2b + ab^2 + a^2b^2$

(d) Now $1.5p^2q - 0.3pq^3 + 4.1p^2q + 5.0pq^3$
$= 1.5p^2q + 4.1p^2q - 0.3pq^3 + 5.0pq^3$
$= 1.5\,p^2q + 4.1\,p^2q + 5.0\,pq^3 - 0.3\,pq^3$
$= (1.5 + 4.1)p^2q + (5.0 - 0.3)pq^3$
$= 5.6p^2q + 4.7pq^3$

Alternatively,
$1.5p^2q - 0.3pq^3 + 4.1p^2q + 5.0pq^3$
$= 1.5p^2q + 4.1p^2q + 5.0pq^3 - 0.3pq^3$
$= 5.6p^2q + 4.7pq^3$

From the above *examples* it can be *seen* that:

(i) We first *group* the *like terms* together in each *algebraic expression*.

(ii) We then *add and subtract* the *coefficients* of *each set* of *like terms*.

Exercise 6c

Simplify each of the following algebraic expressions:

1. $5x + 3x$
2. $9x + 2x$
3. $7y + 5y$
4. $9y + 6y$
5. $8a + 3a$
6. $7p + 9p$
7. $9x - 3x$
8. $8x - 5x$
9. $12x - 7x$
10. $7y - 3y$
11. $12r - 5r$
12. $15p - 8p$
13. $3x - 7x$
14. $5x - 9x$
15. $4y - 7y$
16. $2p - 9p$
17. $8r - 9r$
18. $7s - 8s$
19. $3x + 5x + 7x$
20. $4x + 7x + 3x$
21. $9x + 4x + 5x$
22. $8x + x + 3x$
23. $7x + 2x + 5x$
24. $6p + 3p + 5p$
25. $-8x - 3x - 4x$
26. $-7x - 4x - 9x$
27. $-4x - 7x - 3x$
28. $-5y - 3y - 2y$
29. $-6y - 8y - y$
30. $-4q - 3q - q$

31. $15x + 8x - 7x$

32. $12x + 5x - 9x$

33. $14x + 3x - 8x$

34. $11x + x - 10x$

35. $12y + 3y - 8y$

36. $17p + 3p - 12p$

37. $18x - 9x + x$

38. $17x - 8x + 3x$

39. $19x - 12x + 5x$

40. $21x - 14x + 3x$

41. $20x - 16x + x$

42. $24x - 13x + 2x$

43. $3x - 5x$

44. $7y - 3y$

45. $7x + 3x$

46. $p - (-r)$

47. $a + 2b - (-3c)$

48. $x - (-2y)$

49. $p + 3d + (-2c)$

50. $a + (-3b) + 4c$

51. $8x + 5y - 4z + 9y - 3x + 13z$

52. $12x + 7y - 5z + 8y - 4x + 11z$

53. $10x + 8y - 7z + 5y - 3x + 8z$

54. $15x + 9y - 8z + 6y - 5x + 7z$

55. $9x + 12y - 9z + 7y - 4x + 3z$

56. $11x + 9y - 8z + 4y - 7x + 7z$

57. $18x^2 - 9y^2 + 13y^2 - 6x^2$

58. $9x^2 - 8y^2 + 7y^2 - 3x^2$

59. $12x^2 - 5x^2 + 8y^2 - 4y^2$

60. $13x^2 + 8y^2 - 6x^2 - 5y^2$

61. $10x^2 - 9y^2 + 5y^2 - 6x^2$

62. $15x^2 - 6y^2 - 9x^2 + 3y^2$

63. $6a^2b + 3ab^2 - 2a^2b - 4ab^2 + 3a^2b^2$

64. $7a^2b + 5ab^2 - 4a^2b - 3ab^2 + 2a^2b^2$

65. $15x^2 + 13y^2 - 12x^2 - 9y^2 + 5x^3$

66. $12x^3 + 11y^3 - 10x^3 - 9y^3 + 3x^4$

67. $13p^2 + 11q^2 - 9p^2 + 3q^2 - 7pq^2$

68. $9r^2 + 3s^2 - 6r^2 - s^2 + 5r^2s$

69. $1.7p^2q - 1.5pq^3 + 3.1p^2q + 7.1pq^3$

70. $1.5r^2s - 2.1rs^2 + 3.4r^2s - 4.2rs^2$

71. $1.9x^2 + 3.5y^2 - 0.3x^2 - 1.4y^2$

72. $5.7p^2 + 3.9q^2 - 2.3p^2 - 1.5q^2$

73. $9.3x + 1.6y - 4.1x - 1.3y$

74. $5.4x - 1.4y - 3.1x + 3.2y$

75. Simplify the expression:
$$15xy - 7x - 3xy + 9x + 8$$

76. Simplify the expression:
$$3x^2 - 4xy + y + 2x^2 + 7y - 5 + 9xy$$

77. Simplify the expression:
$$3x^2y^2 - 5xy + 12xy + 8x^2y^2$$

78. Simplify the expression:
$$3.5x^3 - 2.3x^2 + 1.4x + 15.6 - 1.2x^3 \\ + 4.5x^2 - 0.3x - 3.2$$

79. Simplify the expression:
$$5x^2 - 3xy + y - 2x^2 + 6y - 4 + xy$$

80. Simplify:
$$2x - 5y + 3 - 8y + 7x - 4$$

81. Simplify:
$$ax + bx - cx$$

82. Simplify:
$$3x - y + 6y - 2x + 5$$

83. Simplify:
$$4x^2 - 6x - x^2 + 13x + 3y - 7$$

84. Simplify:
$$ax - bx^2 - cx + dx^2 + ey$$

Multiplication and Division of Algebraic Terms

It is necessary to reinforce the statement that the *commutative law* holds for the *multiplication* of *real numbers* and *algebraic terms*. Thus:

$$x \times y = y \times x = xy.$$

That is $xy = yx$.
Hence the *order* in which we *multiply algebraic terms* is *not* important.

Example 6

Simplify each of the following algebraic expressions:

(a) $3x \times 5y$

(b) $\frac{1}{2}p \times 8q$

(c) $5y \times (-3y)$

(d) $4p^2q^3 \times 3p^3q$

(e) $p^2q \times (-2pq) \times (-3p^2q^2)$

Solution

(a) Now $3x \times 5y = 3 \times 5 \times x \times y = 15xy$

(b) Now $\frac{1}{2}p \times 8q = \frac{1}{2} \times 8 \times p \times q = 4pq$

(c) Now $5y \times (-3y) = -3 \times 5 \times y \times y = -15y^2$

(d) Now
$$\begin{aligned} &4p^2q^3 \times 3p^3q \\ &= 4 \times 3 \times p^2 \times p^3 \times q^3 \times q \\ &= 4 \times 3 \times p \times p \times p \times p \times p \times q \times q \\ &\quad \times q \times q \\ &= 12p^5q^4 \end{aligned}$$

(e) Now
$$\begin{aligned} &p^2q \times (-2pq) \times (-3p^2q^2) \\ &= -2 \times (-3) \times p^2 \times p \times p^2 \times q \\ &\quad \times q \times q^2 \\ &= -2 \times (-3) \times p \times p \times p \times p \times p \\ &\quad \times q \times q \times q \times q \\ &= 6p^5q^4 \end{aligned}$$

From the above *examples* it can be *seen* that:

(i) In *multiplying algebraic terms* we *group like quantities* together.

(ii) The *magnitude* of the *resulting index* is dependent on the *number of times* that we *multiply* an *algebraic quantity.*

(iii) The relevent rule for the *multiplication of signs* is used in order to obtain the *sign* in the *final algebraic expression.*

It is necessary to reinforce the fact that, when we are *dividing algebraic expressions,* once we have a *product* in the *numerator* and a *product* in the *denominator*, then we can *cancel a quantity* in the *numerator* with a similar quantity in the *denominator*, if such *possibilities exist.*

Cancelling is *equivalent* to *dividing* both the *numerator* and the *denominator* by the *same quantity.*

Example 7

Simplify each of the following algebraic expressions:

(a) $x^2 \div x^2$ *(b)* $6a \div 2b$

(c) $15x^2y^3z \div 3xy^2$ *(d)* $18p^3q^4 \div (-3pq^2)$

Solution

(a) Now $x^2 \div x^2 = \dfrac{x^2}{x^2} = \dfrac{\overset{1}{\cancel{x}} \times \overset{1}{\cancel{x}}}{\underset{1}{\cancel{x}} \times \underset{1}{\cancel{x}}} = 1$

(b) Now $6a \div 2b = \dfrac{6a}{2b} = \dfrac{\overset{3}{\cancel{6}} \times a}{\underset{1}{\cancel{2}} \times b} = \dfrac{3a}{b} = 3\dfrac{a}{b}$

(c) Now
$$\begin{aligned} &15x^2y^3z \div 3xy^2 \\ &= \frac{15x^2y^3z}{3xy^2} \\ &= \frac{\overset{5}{\cancel{15}} \times \overset{1}{\cancel{x}} \times x \times \overset{1}{\cancel{y}} \times \overset{1}{\cancel{y}} \times y \times z}{\underset{1}{\cancel{3}} \times \underset{1}{\cancel{x}} \times \underset{1}{\cancel{y}} \times \underset{1}{\cancel{y}}} \\ &= 5xyz \end{aligned}$$

(d) Now
$$\begin{aligned} &18p^3q^4 \div (-3pq^2) \\ &= \frac{18p^3q^4}{-3pq^2} \\ &= \frac{\overset{6}{\cancel{18}} \times \overset{1}{\cancel{p}} \times p \times p \times \overset{1}{\cancel{q}} \times \overset{1}{\cancel{q}} \times q \times q}{-\underset{1}{\cancel{3}} \times \underset{1}{\cancel{p}} \times \underset{1}{\cancel{q}} \times \underset{1}{\cancel{q}}} \\ &= -6p^2q^2 \end{aligned}$$

Exercise 6d

Simplify each of the following algebraic expressions:

1. $3 \times 2x$

2. $5 \times 3y$

3. $7 \times 4p$

4. $3x \times 4$

5. $4y \times 7$

6. $5q \times 5$

7. $3x \times 2y$

8. $4y \times 3x$

9. $7p \times 4q$

10. $5x \times 4x$

11. $7y \times 3y$

12. $4p \times 9p$

13. $4x \times (-2)$

14. $5y \times (-3)$

15. $6p \times (-5)$

16. $-4 \times 3x$

17. $-7 \times 5y$

18. $-9 \times 3p$

19. $-3x \times (-2)$

20. $-7x \times (-5)$

21. $-8x \times (-7)$

22. $3x \times (-4y)$

23. $4y \times (-5x)$

24. $8p \times (-5q)$

25. $-2x \times 3y$

26. $-4x \times 7y$

27. $-10x \times 3y$

28. $-3x \times (-4y)$

29. $-8x \times (-5y)$

30. $-9p \times (-6q)$

31. $\frac{1}{2} \times 6y$

32. $\frac{1}{3} \times 9x$

33. $\frac{1}{4} \times 12p$

34. $\frac{1}{5} \times (-25x)$

35. $\frac{1}{9} \times (-36x)$

36. $\frac{1}{8} \times (-64q)$

37. $\frac{1}{2}x \times 12y$

38. $\frac{1}{3}p \times 18q$

39. $\frac{1}{8}q \times 32r$

40. $\frac{1}{9}x \times (-36y)$

41. $\frac{1}{8}x \times (-32y)$

42. $\frac{1}{12}p \times (-108q)$

43. $-\frac{1}{2}x \times (-16y)$

44. $-\frac{1}{3}x \times (-42y)$

45. $-\frac{1}{4}x \times (-56y)$

46. $\frac{1}{2}x \times 18x$

47. $\frac{1}{5}x \times 60x$

48. $\frac{1}{9}p \times 81p$

49. $-\frac{1}{4}x \times (-24x)$

50. $-\frac{1}{3}x \times (-30x)$

51. $-\frac{1}{7}x \times (-49x)$

52. $-\frac{1}{5}x \times 60x$

53. $-\frac{1}{8}x \times 64x$

54. $-\frac{1}{9}x \times 27x$

55. $\frac{1}{2}x \times (-44x)$

56. $\frac{1}{4}x \times (-64x)$

57. $\frac{1}{12}p \times (-144p)$

58. $p \times q \times 3r$

59. $2p \times 3q \times 5r$

60. $5x \times 2y \times 3z$

61. $-2x \times (-5y) \times (-4z)$

62. $-5p \times (-3q) \times (-2r)$

63. $-7r \times (-4t) \times (-3s)$

64. $-5x \times (-3z) \times 2y$

65. $-7p \times (-4r) \times 3q$

66. $-8x \times (-3y) \times 2z$

67. $4x \times (-5y) \times (-3z)$

68. $3x \times (-7y) \times (-4z)$

69. $5p \times (-3q) \times (-4r)$

70. $-7x \times 3y \times (-2z)$

71. $-9p \times 3r \times (-4q)$

72. $-7l \times 3n \times (-5m)$

73. $3x \times 2x \times 4y$

74. $5y \times 3y \times 2x$

75. $2p \times 5p \times 3r$

76. $4x \times 3y \times 5y$

77. $5p \times 4q \times 3q$

78. $3r \times 5s \times 2s$

79. $4x \times 3y \times 2x$

80. $5y \times 4z \times 3y$

81. $7p \times 3q \times 2p$

82. $-3x \times (-2x) \times 4y$

83. $-4x \times (-3x) \times 2y$

84. $-5y \times (-4y) \times 3z$

85. $-2x \times 3y \times (-4x)$

86. $-3p \times 4q \times (-2p)$

87. $-4r \times 3s \times (-2r)$

88. $5x \times (-3y) \times (-4y)$

89. $7p \times (-5q) \times (-3q)$

90. $8p \times (-2q) \times (-3p)$

91. $-3x \times (-5x) \times (-4y)$

92. $-4p \times (-7q) \times (-3q)$

93. $-4y \times (-3x) \times (-5x)$

94. $x \times x \times x$

95. $2x \times 3x \times 4x$

96. $3p \times 2p \times 4p$

97. $-3x \times (-5x) \times (-4x)$

98. $-6y \times (-2y) \times (-3y)$

99. $-3p \times (-4p) \times (-6p)$

100. $-4x \times (-5x) \times 7x$

101. $-9y \times (-5y) \times 2y$

102. $-8p \times (-3p) \times 4p$

103. $5x \times (-4x) \times (-7x)$

104. $3y \times (-5y) \times (-12y)$

105. $5p \times (-8p) \times (-9p)$

106. $(-3x) \times 2x \times (-5x)$

107. $(-4y) \times 3y \times (-9y)$

108. $(-8p) \times 4p \times (-7p)$

109. $5p^2q \times 4pq^3$

110. $4x^2y \times 9x^3y^2$

111. $7r^2s^2 \times 3rs^3$

112. $(-8x^3y^2) \times (-3x^2y)$

113. $(-7x^2y) \times (-5xy^3)$

114. $(-5x^3y) \times (-4xy^2)$

115. $(-7x^2y) \times 5x^2y$

116. $(-8xy^2) \times 9x^2y$

117. $(-9xy^3) \times 4x^2y$

118. $12xy^2 \times (-3x^2y)$

119. $9x^2y \times (-3xy^2)$

120. $8p^2q \times (-9pq^3)$

121. $3p^2q \times 2pq \times 5pq^3$

122. $4x^2y^2 \times 3xy \times 5xy^2$

123. $5x^3y \times 4x^2y \times 7xy^3$

124. $-5x^2y \times (-3xy) \times (-4xy^3)$

125. $-9xy^2 \times (-4x^2y) \times (-3xy)$

126. $-4p^2q \times (-3pq) \times (-5pq^3)$

127. $(-4x^2y) \times (-5xy) \times 3xy^2$

128. $(-7xy^3) \times (-4x^2y) \times 5xy$

129. $(-4p^2q) \times (-5pq^2) \times 8p^2q^3$

130. $(-4xy) \times 3x^2y \times (-5x^3y^2)$

131. $(-5x^3y) \times 4xy \times (-9x^2y^3)$

132. $(-8p^2q) \times 3pq^3 \times (-9p^2q^2)$

Simplify each of the following expressions:

133. $3q^3r^2 \times (-5qr)$

134. $5m^2n \times (-3mn) \times 2m^2n^2$

135. $2m^2n \times (-3mn) \times 5mn^2$

Simplify each of the following algebraic expressions:

136. $8a \div 2$

137. $78x \div 6$

138. $99y \div 9$

139. $12x \div (-3)$

140. $15y \div (-5)$

141. $18p \div (-6)$

142. $21a \div 3b$

143. $96x \div 12y$

144. $108p \div 9q$

145. $(-36x) \div (-12y)$

146. $(-54y) \div (-2a)$

147. $(-72p) \div (-12q)$

148. $(-84x) \div 12y$

149. $(-121p) \div 11q$

150. $(-76x) \div 4p$

151. $36x \div (-9y)$

152. $45p \div (-15q)$

153. $75r \div (-25s)$

154. $37x^3 \div 37x^3$

155. $-45x^4 \div (-45x^4)$

156. $39p^5 \div (-39p^5)$

157. $-47r^6 \div 47r^6$

158. $18x^3yz^2 \div 6xy^2$

159. $49xy^3z^2 \div 7x^2y$

160. $81p^3q^2r^4 \div 9pqr^2$

161. $(-121x^2y^3z) \div (-11x^4yz)$

162. $(-144p^4q^3) \div (-12p^2q)$

163. $(-169r^5s^4) \div (-13r^2s)$

164. $(-49x^3y^2z^4) \div 7xyz^2$

165. $(-100p^3q^2r^5) \div 10pr^3$

166. $(-225x^5y^2z^3) \div 15x^3yz$

167. $64x^3y^3z^4 \div (-8xy^2z^3)$

168. $144p^5q \div (-12p^3q^4)$

169. $169p^4q^3r \div (-13pqr)$

Simplify each of the following expressions:

170. $14x^3y^2z^4 \div 6xz^2$

171. $-9x^4y \div 3x^2y^3$

172. $9x^2y^3z^5 \div 6x^3yz^2$

173. $\dfrac{a^2b^3}{2ab}$

174. $\dfrac{5s^2t}{20st^3}$

Distributive Law

Sometimes it is necessary to *group terms together* when they *cannot* be *added or subtracted.* We achieve this *grouping* through the *distributive law* by using *brackets.* As a matter of fact, we apply the *distributive law* both to *insert or remove brackets* in algebraic expressions. The *distributive law* states that:

$$(a + b)x = x(a + b) = ax + bx,$$

where a and b are *real numbers* and x is a *variable.* In words, the *distributive law* states that *each term inside* the *brackets* is *multiplied* by the *term directly outside.*

Example 8

Remove the brackets using the distributive law in each of the following:

(a) $5(x + y)$

(b) $3(5x - 2y)$

(c) $-7(a + b)$

(d) $-8(3a - 2b)$

Solution

(a) Using the *distributive law*:
Then $5(x + y) = 5 \times x + 5 \times y = 5x + 5y$

(b) Using the *distributive law*:
Then $3(5x - 2y) = 3 \times 5x + 3 \times (-2y)$
$= 15x - 6y$

(c) Using the *distributive law*:
Then $-7(a + b) = -7 \times a - 7 \times b$
$= -7a - 7b$

(d) Using the *distributive law*:
Then $-8(3a - 2b) = -8 \times 3a - 8 \times (-2b)$
$= -24a + 16b$

From the above *examples* it can be *seen* that:

(i) If the *term outside* the brackets is *positive*, then the *signs* of the *terms inside* the *brackets* are *unchanged*, after the brackets are removed using the *distributive law.*

(ii) If the *term outside* the brackets is *negative*, then the *signs* of the *terms inside* the *brackets* are *all changed*, after the brackets are removed using the *distributive law.*

Example 9

Remove the brackets using the distributive law and simplify each of the following:

(a) $3(x + y) + 5(x + y)$

(b) $4(3x - 2y) + 3(4x - 3y)$

(c) $7(x + y) - 4(x - y)$

(d) $8(5x - 2y) - 3(4x - 3y)$

(e) $\frac{1}{2}(2x - 1) - \frac{1}{4}(x - 1)$

Solution

(a) Using the *distributive law*:

$$\begin{aligned} & 3(x + y) + 5(x + y) \\ &= 3 \times x + 3 \times y + 5 \times x + 5 \times y \\ &= 3x + 3y + 5x + 5y \\ &= 3x + 5x + 3y + 5y \\ &= 8x + 8y \end{aligned}$$

(b) Using the *distributive law*:

$$\begin{aligned} & 4(3x - 2y) + 3(4x - 3y) \\ &= 4 \times 3x + 4 \times (-2y) + 3 \times 4x \\ &\quad + 3 \times (-3y) \\ &= 12x - 8y + 12x - 9y \\ &= 12x + 12x - 8y - 9y \\ &= 24x - 17y \end{aligned}$$

(c) Using the *distributive law*:

$$\begin{aligned} & 7(x + y) - 4(x - y) \\ &= 7 \times x + 7 \times y - 4 \times x - 4 \times (-y) \\ &= 7x + 7y - 4x + 4y \\ &= 7x - 4x + 7y + 4y \\ &= 3x + 11y \end{aligned}$$

(d) Using the *distributive law*:

$$\begin{aligned} & 8(5x - 2y) - 3(4x - 3y) \\ &= 8 \times 5x + 8 \times (-2y) - 3 \times 4x \\ &\quad - 3 \times (-3y) \\ &= 40x - 16y - 12x + 9y \\ &= 40x - 12x - 16y + 9y \\ &= 28x - 7y \end{aligned}$$

(e) Using the *distributive law*:

$$\begin{aligned} & \frac{1}{2}(2x - 1) - \frac{1}{4}(x - 1) \\ &= \frac{1}{2} \times 2x + \frac{1}{2} \times (-1) \\ &\quad - \frac{1}{4} \times x - \frac{1}{4} \times (-1) \\ &= x - \frac{1}{2} - \frac{1}{4}x + \frac{1}{4} \\ &= x - \frac{1}{4}x - \frac{1}{2} + \frac{1}{4} \\ &= \frac{3}{4}x - \frac{1}{4} \end{aligned}$$

From the above *examples* it can be seen that:

(i) When we need to *remove two pairs of brackets*, then we have to use the *distributive law twice*.

(ii) After *removing* the *brackets* we then *group* all *like terms together*.

(iii) We then *simplify* each *set of like terms* by *adding* and/or *subtracting*.

Exercise 6e

Remove the brackets using the distributive law in each of the following:

1. $2(x + 3)$

2. $3(x + 4)$

3. $5(x + 7)$

4. $6(y + 3)$

5. $9(p + 5)$

6. $2(x - 3)$

7. $3(x - 5)$

8. $5(x - 7)$

9. $6(y - 4)$

10. $9(p - 3)$

11. $-3(x + 4)$

12. $-4(x + 5)$

13. $-8(x + 3)$

14. $-9(y + 5)$

15. $-10(p + 3)$

16. $-4(x - 3)$

17. $-5(x - 7)$

18. $-6(x - 9)$

19. $-8(y - 4)$

20. $-9(p - 3)$

21. $3(x + y)$

22. $8(x + y)$

23. $9(x + y)$

24. $7(p + q)$

25. $10(r + s)$

26. $4(x - y)$

27. $5(x - y)$

28. $8(x - y)$

29. $7(p - q)$

30. $12(r - s)$

31. $-5(x + y)$

32. $-7(x + y)$

33. $-8(x + y)$

34. $-9(p + q)$

35. $-15(r + s)$

36. $-3(x - y)$

37. $-4(x - y)$

38. $-5(x - y)$

39. $-6(p - q)$

40. $-13(r - s)$

41. $\frac{1}{2}(6x + 3)$

42. $\frac{1}{3}(9x + 6)$

43. $\frac{1}{5}(25x + 5)$

44. $\frac{1}{6}(36x + 42y)$

45. $\frac{1}{7}(49x + 7y)$
46. $\frac{1}{4}(4x - 8)$
47. $\frac{1}{8}(8x - 16)$
48. $\frac{1}{9}(81x - 9)$
49. $\frac{1}{10}(100x - 10y)$
50. $\frac{1}{12}(144x - 12y)$
51. $-\frac{1}{2}(4x - 6)$
52. $-\frac{1}{4}(16x - 4)$
53. $-\frac{1}{5}(25x - 5)$
54. $-\frac{1}{7}(49x - 7y)$
55. $-\frac{1}{8}(64x - 8y)$
56. $-\frac{1}{9}(81x + 9)$
57. $-\frac{1}{10}(100x + 10)$
58. $-\frac{1}{12}(144x + 12)$
59. $-\frac{1}{7}(49x + 7)$
60. $-\frac{1}{6}(36x + 12)$
61. $5(2x + 3)$
62. $7(3x + 4)$
63. $8(5x + 3)$
64. $9(7y + 4)$
65. $12(3p + 2)$
66. $3(2x - 3)$
67. $4(7x - 5)$
68. $5(8x - 7)$
69. $6(5y - 3)$
70. $7(9p - 5)$
71. $-5(3x + 2)$
72. $-6(5x + 3)$
73. $-8(7x + 5)$
74. $-9(3y + 5)$
75. $-10(5p + 7)$
76. $-6(7x - 3)$
77. $-7(3y - 4)$
78. $-8(4y - 5)$
79. $-9(5x - 7)$
80. $-10(3p - 4)$
81. $3(4x + 3y)$
82. $5(3x + 2y)$
83. $6(5x + 7y)$
84. $8(3x + 4y)$
85. $9(7x + 5y)$
86. $-3(5x + 2y)$
87. $-6(4x + 3y)$
88. $-7(3x + 5y)$
89. $-8(4x + 3y)$
90. $-9(5x + 8y)$
91. $-4(5x - 3y)$
92. $-5(3x - 5y)$
93. $-7(8x - 3y)$
94. $-8(7x - 2y)$
95. $-9(8p - 3q)$
96. $3(5x - 2y)$
97. $8(7x - 3y)$
98. $9(8x - 5y)$
99. $12(3p - 2q)$
100. $13(5r - 4s)$

Remove the brackets using the distributive law and simplify each of the following:

101. $3(x + y) + 5(x + y)$
102. $4(x + y) + 3(x + y)$
103. $5(x + y) + 7(x + y)$
104. $8(x + y) + 7(x + y)$
105. $9(p + q) + 3(p + q)$
106. $3(x - y) + 5(x - y)$
107. $4(x - y) + 3(x - y)$
108. $7(x - y) + 2(x - y)$
109. $8(p - q) + 3(p - q)$
110. $7(p - q) + 4(p - q)$
111. $8(x + y) - 4(x - y)$
112. $9(x + y) - 3(x - y)$
113. $7(x + y) - 4(x - y)$
114. $8(x + y) - 5(x - y)$
115. $6(p + q) - 4(p - q)$
116. $4(x - y) - 3(x - y)$
117. $5(x - y) - 4(x - y)$
118. $7(x - y) - 4(x - y)$
119. $8(p - q) - 3(p - q)$
120. $9(r - s) - 4(r - s)$
121. $3(4x + 3y) + 4(3x + 2y)$
122. $5(3x + 2y) + 3(2x + 5y)$
123. $7(3x + 4y) + 3(4x + 5y)$
124. $6(5x + 2y) + 3(4x + 3y)$
125. $8(7x + 3y) + 4(5x + 2y)$
126. $5(3x + 2y) - 2(3x + 2y)$
127. $6(5x + 2y) - 3(2x + 3y)$
128. $7(5x + 3y) - 4(3x + 2y)$
129. $8(4x + 3y) - 5(3x + 5y)$
130. $9(5x + 3y) - 5(4x + 7y)$
131. $3(2x + y) - 4(5x + 2y)$
132. $4(3x + 2y) - 5(3x + 4y)$
133. $7(4x + 3y) - 6(3x + 5y)$
134. $8(5x + 2y) - 7(5x + 3y)$
135. $9(7x + 4y) - 8(3x + 7y)$
136. $3(2x - 3y) - 4(3x + 2y)$
137. $5(4x - 3y) - 8(4x - 3y)$
138. $6(3x - 4y) - 9(2x - 3y)$

139. $8(4x - 3y) - 5(2x - 3y)$

140. $9(5x - 2y) - 6(3x - 2y)$

141. $\frac{1}{2}(4x + 1) + \frac{1}{3}(6x + 3)$

142. $\frac{1}{3}(9x + 1) + \frac{1}{6}(12x + 2)$

143. $\frac{1}{5}(5x + 1) + \frac{1}{10}(10x + 4)$

144. $\frac{1}{7}(49x + 7) + \frac{1}{8}(64x + 8)$

145. $\frac{1}{9}(81x + 9) + \frac{1}{7}(49x + 7)$

146. $\frac{1}{2}(6x + 1) - \frac{1}{5}(25x + 10)$

147. $\frac{1}{3}(9x + 1) - \frac{1}{6}(18x + 2)$

148. $\frac{1}{4}(8x + 1) - \frac{1}{8}(32x + 4)$

149. $\frac{1}{5}(25x + 1) - \frac{1}{10}(10x + 2)$

150. $\frac{1}{7}(14x + 7) - \frac{1}{9}(45x + 3)$

151. $\frac{1}{2}(2x + 1) - \frac{1}{4}(4x - 1)$

152. $\frac{1}{5}(5x + 1) - \frac{1}{8}(64x - 8)$

153. $\frac{1}{7}(14x + 1) - \frac{1}{14}(28x - 4)$

154. $\frac{1}{6}(12x + 3) - \frac{1}{12}(12x - 1)$

155. $\frac{1}{9}(9x + 1) - \frac{1}{3}(9x - 1)$

156. $\frac{1}{2}(4x - 1) - \frac{1}{4}(8x - 2)$

157. $\frac{1}{3}(6x - 1) - \frac{1}{6}(12x - 3)$

158. $\frac{1}{5}(25x - 1) - \frac{1}{10}(30x - 5)$

159. $\frac{1}{4}(8x - 3) - \frac{1}{8}(16x - 4)$

160. $\frac{1}{7}(49x - 5) - \frac{1}{14}(7x - 2)$

Simplify each of the following:

161. $3x + 2(x - y)$ **162.** $4(m + n) - 5(m - n)$

163. $x - 3(x - y)$ **164.** $4(p - r) - 2(p + r)$

Remove the brackets using the distributive law and simplify each of the following:

165. $3(4x + 5)$ **166.** $-5(4 - 3x)$

167. $4(3x + 1) - 3(2x - 1)$

168. $\frac{1}{2}(y - 3) + \frac{1}{3}(6x - 1)$

Simplify each of the following expressions:

169. $3(2a + 4) - 2(a - 1)$

170. $3(m + n) - 2(m - n)$

Using the distributive law, simplify each of the following:

171. $-3(5 - 2x)$ **172.** $\frac{2}{5}(y - 5) + \frac{1}{3}(x - 1)$

173. $2(x - 5) - 3(x + 4) + 5(x - 1)$

Expand and simplify each of the following:

174. $a(a - 2b) - b(a + 2b)$

175. $3x(x^2 - 1) + 2x^2(x + 4) - x(7 - x^2)$

Simplify:

176. $5x + 2y + 3(x - y + 1)$

177. $3(x + 2y) + 5x - (y + 7)$

Expand each of the following expressions:

178. $3(x + 1)$ **179.** $4(x - 1)$

180. $8r(3t - 2s)$ **181.** $3x(4y + z)$

Simplify each of the following:

182. $3(2x - y) - 4(x + 3y)$

183. $5(3x - y) - 4(2x - y)$

184. $5(3x - 2) - 3(2 - 5x)$

185. $5(3x + y) - 2(x - y)$

186. $5(6x - 3) + (x + 4)$

187. $3(5x - y) - 2(4x - y)$

Binary Operation

A binary operation is an operation which combines two numbers to produce a third number. In algebra we can use a *symbol* to *represent a binary operation*, other than the *four basic operations* defined previously in this chapter, and *perform computations* with them.

Example 10

An operation is defined by $x * y = 2x + 3y$.
Evaluate:
(a) $4 * 5$ *(b)* $(4 * 5) * 6$ *(c)* $3 * (4 * 5)$

Solution

(a) Given that $x * y = 2x + 3y$

then $4 * 5 = 2 \times 4 + 3 \times 5$; substituting $x = 4$ and $y = 5$.

$$= 8 + 15$$
$$= 23$$

(b) Now $(4 * 5) * 6 = 23 * 6$

$$= 2 \times 23 + 3 \times 6$$
$$= 46 + 18$$
$$= 64$$

(c) Now $3 * (4 * 5) = 3 * 23$

$$= 2 \times 3 + 3 \times 23$$
$$= 6 + 69$$
$$= 75$$

Example 11

If $a \circ b$ *means* $\frac{1}{4}\sqrt{a^2 - b^2}$, *determine the value of* $5 \circ 3$.

Solution

Given that $a \circ b = \frac{1}{4}\sqrt{a^2 - b^2}$

then $5 \circ 3 = \frac{1}{4}\sqrt{5^2 - 3^2}$; substituting $a = 5$ and $b = 3$.

$$= \frac{1}{4}\sqrt{25 - 9}$$
$$= \frac{1}{4}\sqrt{16}$$
$$= \frac{1}{4} \times 4$$
$$= 1$$

Exercise 6f

1. An operation is defined by $a * b = 3a - b$. Calculate the exact value of $2 * 3$.
2. Given that $p * q$ denotes $p + 2q$, evaluate $-5 * 3$.
3. If $m * n$ means $m + 7n$, determine the value of $5 * (1 * 3)$.
4. (a) Given $m * n$ denotes $2m + n$, evaluate:
   (i) $3 * 5$ (ii) $2 * (3 * 5)$
   (b) Determine the value of x such that:
   $x * 10 \approx 810$.
5. If $a * b = a^2 + b$, write the values of:
   (a) $3 * 5$ (b) $4 * (3 * 5)$
6. An operation is defined by $a \square b = \sqrt{5a - 3b}$. State the value of $4 \square 2$.
7. An operation is defined by $p \dagger q = 3p^2 - 2q$. Calculate the exact value of $5 \dagger 3$.
8. If $m \triangle n$ means $m^2 - 2mn + n^3$, determine the value of $3 \triangle 2$.
9. Given that $p \boxed{} q$ denotes $p^3 - 2pq + q^2$, calculate the exact value of $3 \boxed{} 2$.
10. If $m * n$ denotes $5m^3 - 2n^2$, state the exact value of $-1 * 4$.
11. If $m * n$ denotes $m + 5n$, evaluate $3 * (1 * 2)$.
12. If $m \square n$ denotes $2m^2 - 3n^3$, determine the exact value of $3 \square (-2)$.
13. An operation is defined by $p \circ q = p^2 + q^3$. State the exact value of $5 \circ (-4)$.
14. If $x \boxdot y = x + y^2$, write down the values of
    (a) $2 \boxdot 3$ (b) $2 \boxdot (2 \boxdot 3)$
15. An operation is defined by $p \dagger q = 4pq^2 - q$. Calculate the value of $2 \dagger (-3)$.

Factorization

We *factorize* an *algebraic expression* by expressing them as the *product* of some of their *factors*. When we use the *distributive law to insert brackets in an expression*, we are said to be *factorizing*.

Thus:

$$ax + bx = (a + b)x = x(a + b),$$

where x and $(a + b)$ are said to be *factors* of $ax + bx$.

Factorizing Using the Distributive Law

Given the algebraic expression $ax + ay$, then a is *common* to *both terms*.
Hence, by the *distributive law*:

$$ax + ay = a(x + y),$$

where a and $(x + y)$ are *factors* of $ax + ay$.

Example 12

Factorize each of the following algebraic expressions:

(a) $5x + 5y$ *(b)* $64a^2 - 8a$
(c) $25x - 10$ *(d)* $-49p^2 + 7p$
(e) $-64x - 16$ *(f)* $5wx + 10wy - 15wz.$

Solution

(a) Now $5x + 5y = 5(x + y) \leftarrow$

$\frac{5x}{5} = x$	$\frac{5y}{5} = y$

Since 5 is *common* to both $5x$ and $5y$.

(b) Now $64a^2 - 8a = 8a(8a - 1)$ $\leftarrow$

$\frac{64a^2}{8a} = 8a$	$\frac{8a}{8a} = 1$

Since $8a$ is *common* to both $64a^2$ and $8a$.

Alternatively, $64a^2 - 8a = 8a \times 8a + 8a \times (-1)$
$= 8a(8a - 1)$

(c) Now $25x - 10 = 5(5x - 2)$ $\leftarrow$

$\frac{25x}{5} = 5x$	$\frac{10}{5} = 2$

Since 5 is *common* to both $25x$ and 10.

Alternatively, $25x - 10 = 5 \times 5x + 5 \times (-2)$
$= 5(5x - 2)$

(d) Now $-49p^2 + 7p = -7p(7p - 1)$ $\leftarrow$

$\frac{49p^2}{7p} = 7p$	$\frac{7p}{7p} = 1$

Since $7p$ is *common* to both $49p^2$ and $7p$.

Alternatively, $-49p^2 + 7p = -7p \times 7p - 7p \times (-1)$
$= -7p(7p - 1).$

(e) Now $-64x - 16 = -16(4x + 1)$ $\leftarrow$

$\frac{64x}{16} = 4x$	$\frac{16}{16} = 1$

Since 16 is *common* to both $64x$ and 16.

Alternatively, $-64x - 16 = -16 \times 4x - 16 \times 1$
$= -16(4x + 1).$

(f) Now $5wx + 10wy - 15wz = 5w(x + 2y - 3z)$ $\leftarrow$

$\frac{5wx}{5w} = x$	$\frac{10wy}{5w} = 2y$	$\frac{15wz}{5w} = 3z$

Since $5w$ is *common* to $5wx$, $10wy$ and $15wz$.

Alternatively,

$$\begin{aligned} & 5wx + 10wy - 15wz \\ &= 5w \times x + 5w \times 2y + 5w \times (-3z) \\ &= 5w(x + 2y - 3z) \end{aligned}$$

Exercise 6g

Factorize each of the following algebraic expressions:

1. $6x + 6y$
2. $9x + 9y$
3. $mx + my$
4. $qx + qy$
5. $zx + zy$
6. $5x - 5y$
7. $7x - 7y$
8. $8x - 8y$
9. $mx - my$
10. $rx - ry$
11. $-3x + 3y$
12. $-7x + 7y$
13. $-8x + 8y$
14. $-px + py$
15. $-qx + qy$
16. $-5x - 5y$
17. $-6x - 6y$
18. $-7x - 7y$
19. $-px - py$
20. $-rx - ry$
21. $25a^2 + 5a$
22. $36p^2 + 6p$
23. $49r^2 + 7r$
24. $64x^2 + 8x$
25. $144a^2 + 12a$
26. $9p^2 - 3p$
27. $16r^2 - 4r$
28. $81x^2 - 9x$
29. $144y^2 - 12y$
30. $169x^2 - 13x$
31. $-25r^2 - 5r$
32. $-49y^2 - 7y$
33. $-64s^2 - 8s$
34. $-81x^2 - 9x$
35. $-121p^2 - 11p$
36. $-4x^2 + 2x$
37. $-9y^2 + 3y$
38. $-16r^2 + 4r$
39. $-25p^2 + 5p$
40. $-64s^2 + 8s$
41. $25x + 10$
42. $36x + 12$
43. $49x + 14$
44. $64x + 16$
45. $81y + 18$
46. $4x - 2$
47. $9x - 6$

48. $16x - 8$ **49.** $25y - 10$

50. $36p - 18$ **51.** $-49x - 21$

52. $-81x - 27$ **53.** $-100x - 30$

54. $-121x - 55$ **55.** $-144x - 36$

56. $-25x + 10$ **57.** $-36x + 18$

58. $-49x + 21$ **59.** $-64y + 24$

60. $-81p + 36$ **61.** $16x + 4$

62. $25y + 5$ **63.** $36p + 6$

64. $64q + 8$ **65.** $100r + 10$ **66.** $25x - 5$

67. $36y - 6$ **68.** $49p - 7$ **69.** $81y - 9$

70. $121r - 11$ **71.** $-25x - 5$ **72.** $-49y - 7$

73. $-64p - 8$ **74.** $-100r - 10$

75. $-144s - 12$ **76.** $-4x - 2$

77. $-9y - 3$ **78.** $-25p - 5$

79. $-36r - 6$ **80.** $-64s - 8$

81. $5wp + 15wq + 20wr$

82. $3rs + 9rt + 18ru$

83. $8pq + 16pr + 24ps$

84. $81ab + 27ac + 36ad$

85. $36pa + 72pb + 144pc$

86. $-5lx + 15ly + 25lz$

87. $-8pq + 24pr + 32ps$

88. $-9ab + 36ac + 45ad$

89. $-10ra + 25rb + 35rc$

90. $-12rt + 24ru + 36rv$

91. $-5ra - 10rb + 15rc$

92. $-6rx - 36ry + 18rz$

93. $-7px - 21py + 28pz$

94. $-8lx - 24ly + 32lz$

95. $-9kg - 27lg + 18mg$

96. $-3pa - 9pb - 18pc$

97. $-5lg - 25mg - 10ng$

98. $-7ra - 21sa - 14ta$

99. $-8xt - 16yt - 24zt$

100. $-81la - 36lb - 45lc$

Factorize each of the following:

101. $9x - 27y^3$ **102.** $5x + 5$

103. $18x - 6$ **104.** $3x^2 - 27x$

105. $4x^4 + 16x^2$ **106.** $20abc - 8bcd$

107. $4gh_1 - 4gh_2$ **108.** $\frac{4}{3}\pi r^3 - \frac{1}{3}\pi r^2 h$

109. $5\pi R^2 + 10\pi r^2$ **110.** $6x - 18y^2$

111. $7ac - 14ad$ **112.** $9y^2 - 6y + 3$

113. $\frac{1}{3}\pi r^2 h - \frac{4}{3}\pi r^3$

Highest Common Factor (H.C.F.)

The *highest common factor* (*H.C.F.*) of a *set of numbers* is the *greatest number* that *divides exactly* into *each of them.* Thus the *highest common factor* (*H.C.F.*) of 3, 18 and 27 is 3, because it is the *greatest number* that is a *factor* of *each* of the *numbers.*

The *highest common factor* (*H.C.F.*) of a *set of algebraic terms* is the *highest expression* that is a *factor* of *each* of the given *terms.* The *highest common factor* (*H.C.F.*) of a *set of algebraic terms* is obtained by simply taking the *lowest power* of *each quantity* that is *common* to *each* of the *terms* and *multiplying* them altogether. Thus the *H.C.F.* of x^2y^5, xy^3 and x^4y^2 is xy^2.

Example 13

Determine the H.C.F. of each of the following sets of algebraic terms:

(a) $a^2b^4c^3,\ a^3b^2c^4d^3,\ a^4b^3c^5d^4$

(b) $x^2y^3z^5,\ x^3y^3z^4,\ x^4y^4z^5$

(c) $10x^3y^2,\ 5x^2y^5,\ 15xy^3$

(d) $\frac{r^2s}{9p},\ \frac{r^3s^4}{3p^3},\ \frac{r^4s^5q}{27p^2}$

Solution

(a) The *H.C.F.* of $a^2b^4c^3$, $a^3b^2c^4d^3$ and $a^4b^3c^5d^4$

$= a^2b^2c^3$

Note that the *quantity d* is *not contained* in the first term $a^2b^4c^3$ and hence *cannot* be a *part* of the *H.C.F.*

(b) The *H.C.F.* of $x^2y^3z^5$, $x^3y^3z^4$ and $x^4y^4z^5$

$$= x^2y^3z^4$$

(c) The *H.C.F.* of $10x^3y^2$, $5x^2y^5$ and $15xy^3$

$$= 5xy^2$$

In an *algebraic term* the *coefficient* is regarded as the *numerical part* of the *term.*
In the example above
the *coefficient* of $10x^3y^2$ is 10
the *coefficient* of $5x^2y^5$ is 5
the *coefficient* of $15xy^3$ is 15.

> Note that in the above example, the *H.C.F.* of the *coefficients* is 5 and the *H.C.F.* of the *variables* is xy^2. Hence the *H.C.F.* of the *set of algebraic terms* is the *product* of 5 and xy^2, that is $5xy^2$.

(d) The *H.C.F.* of $\dfrac{r^2s}{9p}, \dfrac{r^3s^4}{3p^3}$, and $\dfrac{r^4s^5q}{27p^2}$

$$= \frac{r^2s}{3p} = \frac{1}{3}\frac{r^2s}{p}$$

In the above example, we first found the *H.C.F.* of the *numerator*, that is, r^2s, then we found the *H.C.F.* of the *denominator*, that is, $3p$. Hence the *H.C.F.* of the *set of algebraic terms* is the *quotient* $\dfrac{r^2s}{3p}$.

Exercise 6h

Determine the H.C.F. of each of the following sets of algebraic terms:

1. $a^3b^5c^4, a^2b^4c^5d^2, a^4b^3c^4d^3$

2. $p^3q^4r^2, p^4q^5, p^5q^3r^4$

3. $x^4y^5z^3, x^5y^4z^2, x^6y^3$

4. $l^5m^2n^4, l^4m^5n^3, l^6m^4n^5p^3$

5. $x^5y^3z^4, x^4y^5z^3, w^3x^6y^4$

6. $a^4b^3c^2, a^3b^4c^5, a^5b^5c^4$

7. $p^4q^3r^5, p^3q^4r^6, p^5q^5r^4$

8. $x^3y^4z^5, x^4y^5z^3, x^2y^3z^4$

9. $l^4m^3n^2, l^5m^4n^3, l^3m^2n^4$

10. $x^6y^5z^4, x^5y^3z^5, x^4y^4z^3$

11. $9a^2b^3c, 3ab^2c^3, 6a^3bc^2$

12. $15p^3q^2r^4, 20p^2q^3r^5, 25p^4qr^3$

13. $4x^3y^2z, 8x^2y^3z^2, 12x^4y^2z^3$

14. $8l^2m^3n^4, 24l^3m^2n^3, 16l^4m^4n^5$

15. $18x^4y^3z^5, 9x^3y^2z^4, 27x^2y^4z^3$

16. $\dfrac{a^2b^3}{5c^2}, \dfrac{a^3b}{10c^3}, \dfrac{a^4b^2}{15c^4}$

17. $\dfrac{3p^3q^3}{r^4}, \dfrac{18pq^2}{r^3}, \dfrac{9p^3q^2}{r^2}$

18. $\dfrac{2x^3y^2}{7z^4}, \dfrac{8x^2y^3}{7z^3}, \dfrac{4x^4y^2}{7z^2}$

19. $\dfrac{5l^3m^4}{7n^2}, \dfrac{10l^2m^3}{7n^2}, \dfrac{15l^4m^2}{7n^4}$

20. $\dfrac{21x^5y^4}{z^3}, \dfrac{7x^4y^3}{z^4}, \dfrac{14x^3y^2}{z^5}$

Factorizing Using the Highest Common Factor

In this *method* we first determine the *highest common factor* (*H.C.F.*) of the *algebraic expression.* We then determine the *second factor* by *dividing* each *algebraic term* by the *H.C.F.* and placing the *quotient* in *brackets*, due care being taken to obtain the *correct signs*. The *method* can be seen illustrated below.

Example 14

Factorize each of the following algebraic expressions:

(a) $10x^3y^2 + 5x^2y^5 - 15xy^3$

(b) $\dfrac{r^2s}{9p} - \dfrac{r^3s^4}{3p^3} + \dfrac{r^4s^5q}{27p^2}$

Solution

(a) The *H.C.F.* of $10x^3y^2$, $5x^2y^5$ and $15xy^3$ is $5xy^2$.

So $\quad 10x^3y^2 + 5x^2y^5 - 15xy^3$

$$= 5xy^2(2x^2 + xy^3 - 3y) \longleftarrow \begin{array}{|c|} \hline \dfrac{10x^3y^2}{5xy^2} = 2x^2 \\ \hline \dfrac{5x^2y^5}{5xy^2} = xy^3 \\ \hline \dfrac{15xy^3}{5xy^2} = 3y \\ \hline \end{array}$$

(b) The *H.C.F.* of $\frac{r^2s}{9p}, \frac{r^3s^4}{3p^3}$ and $\frac{r^4s^5q}{27p^2}$ is $\frac{r^2s}{3p}$.

So $\frac{r^2s}{9p} - \frac{r^3s^4}{3p^3} + \frac{r^4s^5q}{27p^2}$

$$= \frac{r^2s}{3p}\left(\frac{1}{3} - \frac{rs^3}{p^2} + \frac{r^2s^4q}{9p}\right)$$

$\frac{r^2s}{9p} \times \frac{3p}{r^2s} = \frac{1}{3}$
$\frac{r^3s^4}{3p^3} \times \frac{3p}{r^2s} = \frac{rs^3}{p^2}$
$\frac{r^4s^5q}{27p^2} \times \frac{3p}{r^2s} = \frac{r^2s^4q}{9p}$

Exercise 6i

Factorize each of the following algebraic expressions:

1. $15x^2y - 10xy^3$

2. $18r^2s^3 + 12r^3s^4$

3. $7p^3r^2 - 14pr^3$

4. $8l^2m^3 + 16lm^2$

5. $81x^4y^3 - 27x^3y^2$

6. $4a^2b + 2ab^2 - 8ab$

7. $2x^2 - 10x^2y + 8x^2y^2$

8. $10x^3y^2 - 5x^2y^3 + 15x^4y^5$

9. $18p^3q^2 + 27p^2q^3 - 36p^4q^3$

10. $21x^4y^5 - 7x^3y^2 + 14x^3y^4$

11. $\frac{p^3m^2}{7} + \frac{p^2m^3}{21} - \frac{p^4m}{14}$

12. $\frac{l^2m^2}{15} + \frac{l^2m}{10} - \frac{l^3m^2}{20}$

13. $\frac{9l^2m}{n} + \frac{3lm^2}{n^2} + \frac{6l^2m^2}{n^3}$

14. $\frac{l^2m^2}{25pn^2} - \frac{2l^3m^2}{5p^2n^3} + \frac{3l^2m^3}{10p^3n^2}$

15. $\frac{3m^4}{5p^2n^3} + \frac{9m^3}{10p^3n^2} - \frac{6m^2}{25p^4n^5}$

Factorize the following:

16. $49x^2 - 7x$

17. $18x^3y^2 - 6x^2y^2 + 3xy^5$

Factorizing by Grouping

In this *method* we are normally given *four algebraic terms* to *factorize*. We first *group* the *algebraic terms* in *pairs* so that *each pair of terms* has a *common factor*. The *common factor* is then used to *factorize each pair of terms*. A *common factor* is then found for the *pair of factorized terms* and then the *process* of *factorization* is *completed*. This *method* can be seen illustrated below.

Example 15

Factorize each of the following algebraic expressions:

(a) $px + py + qx + qy$

(b) $3ax - 6ay + bx - 2by$

(c) $4px - 4py - 3qx + 3qy$

(d) $mx + nx - my - ny$

(e) $lm(5x - 1) + 3pq(5x - 1)$

Solution

(a) Now

$px + py + qx + qy$

$= (px + py) + (qx + qy)$, *grouping in pairs.*

$= p(x + y) + q(x + y)$, *factorizing.*

$= (x + y)(p + q)$, *factorizing again*, since $(x + y)$ is *common* to *both terms.*

Alternative Method

(a) Now

$px + py + qx + qy$

$= px + qx + py + qy$, *rearranging the terms.*

$= (px + qx) + (py + qy)$, *grouping in pairs.*

$= (p + q)x + (p + q)y$, *factorizing.*

$= (p + q)(x + y)$, *factorizing again*, since $(p + q)$ is *common* to *both terms.*

Hence $(x + y)(p + q) = (p + q)(x + y)$.

(b) Now

$3ax - 6ay + bx - 2by$

$= (3ax - 6ay) + (bx - 2by)$, *grouping* in *pairs*.

$= 3a(x - 2y) + b(x - 2y)$, *factorizing*.

$= (x - 2y)(3a + b)$, *factorizing again*, since $(x - 2y)$ is *common* to *both terms*.

Alternative Method

(b) Now

$3ax - 6ay + bx - 2by$

$= 3ax + bx - 6ay - 2by$, *rearranging* the *terms*.

$= (3ax + bx) - (6ay - 2by)$, *grouping* in *pairs*.

$= (3a + b)x - (3a + b)2y$, *factorizing*.

$= (3a + b)(x - 2y)$, *factorizing again*, since $(3a + b)$ is *common* to *both terms*.

Hence $(x - 2y)(3a + b) = (3a + b)(x - 2y)$.

(c) Now

$4px - 4py - 3qx + 3qy$

$= (4px - 4py) - (3qx - 3qy)$, *grouping* in *pairs*.

$= 4p(x - y) - 3q(x - y)$, *factorizing*.

$= (x - y)(4p - 3q)$, *factorizing again*, since $(x - y)$ is *common* to *both terms*.

Alternative Method

(c) Now

$4px - 4py - 3qx + 3qy$

$= 4px - 3qx - 4py + 3qy$, *rearranging* the *terms*.

$= (4px - 3qx) - (4py - 3qy)$, *grouping* in *pairs*.

$= (4p - 3q)x - (4p - 3q)y$, *factorizing*.

$= (4p - 3q)(x - y)$, *factorizing again*, since $(4p - 3q)$ is *common* to *both terms*.

Hence $(x - y)(4p - 3q) = (4p - 3q)(x - y)$.

(d) Now

$mx + nx - my - ny$

$= (mx + nx) - (my + ny)$, *grouping* in *pairs*.

$= (m + n)x - (m + n)y$, *factorizing*.

$= (m + n)(x - y)$, *factorizing again*, since $(m + n)$ is *common* to *both terms*.

Alternative Method

(d) Now

$mx + nx - my - ny$

$= mx - my + nx - ny$, *rearranging* the *terms*.

$= (mx - my) + (nx - ny)$, *grouping* in *pairs*.

$= m(x - y) + n(x - y)$, *factorizing*.

$= (x - y)(m + n)$, *factorizing again*, since $(x - y)$ is *common* to *both terms*.

Hence $(m + n)(x - y) = (x - y)(m + n)$.

(e) Now

$lm(5x - 1) + 3pq(5x - 1)$, *factorization* was already carried out once.

$= (5x - 1)(lm + 3pq)$, *factorizing again*, since $(5x - 1)$ is *common* to *both terms*.

Exercise 6j

Factorize each of the following algebraic expressions:

1. $ax + ay + bx + by$

2. $mx + nx + my + ny$

3. $xp + yp + xq + yq$

4. $ax + 3bx + ay + 3by$

5. $ar + as + br + bs$

6. $2ax - 6ay + bx - 3by$

7. $2ax - ay + 2bx - by$

8. $mp - 4np + mq - 4nq$

9. $mx - 2nx + my - 2ny$

10. $3xr + 3xs - yr - ys$

11. $5ax - 5ay - 3bx + 3by$

12. $3px - 3py - 7qx + 7qy$

13. $4rx - 4ry - 3sx + 3sy$

14. $4ar - 4as - 5br + 5bs$

15. $7al - 7am - 3bl + 3bm$

16. $ax + bx - ay - by$

17. $3ax + 2bx - 3ay - 2by$

18. $5ax + 3bx - 5ay - 3by$

19. $4pr + 3qr - 4ps - 3qs$

20. $7pl + 3ql - 7pm - 3qm$

21. $2ab(3x + 1) + 3pq(3x + 1)$

22. $3lm(4x + 3) - 2ab(4x + 3)$

23. $4pq(5x - 3) - 3ab(5x - 3)$

24. $5lm(7x - 5) - 3pq(7x - 5)$

25. $3ab(4x - 7) - 2pq(4x - 7)$

Factorize completely each of the following algebraic expressions:

26. $4pqx - 4q - 3prx + 3r$

27. $(5x - y)(2x + 1) - 2x - 1$

28. $2bx + 3cy + 3cx + 2by$

29. $(5x - y)(3x + 1) - 2(5x - y)$

30. $(3x + y)(2x - 1) - (3x + y)$

31. $2ac + 4bc - a^2 - 2ab$

32. $2a^2 + ab - 4ac - 2bc$

33. $lp + mp - lq - mq$

34. $ap - 5bq - aq + 5bp$

35. $ar - as - br + bs$

36. $py + pz + y^2 + yz$

37. $10x - 2xy + 4y - 20$

38. $2pr - ps - 2qr + qs$

39. $4pr - 8ps + qr - 2qs$

40. $mn(5x + 1) - pq(5x + 1)$

41. $ax - 2bx + ay - 2by$

42. $xp + yp - xq - yq$

43. $mn(5x - 1) - pq(5x - 1)$

44. $ab + 2ac - bd - 2dc$

45. $(3x + y)(2x - 1) - 2x + 1$

46. $3a + at - 6p - 2pt$

Addition and Subtraction of Algebraic Fractions

The *method* used to *add* and *subtract algebraic fractions* is as follows:

(i) We first determine the *lowest common denominator* (*L.C.D.*), that is, the *lowest common multiple* of the *denominators* of the *algebraic fractions*.

(ii) We then *express* each *algebraic fraction* in terms of the *lowest common denominator* (*L.C.D.*)

(iii) The *distributive law* is then used to remove all *brackets* in the *numerator* if there are any.

(iv) *Like terms* in the *numerator* are *grouped together* and then *added* and *subtracted.*

(v) Finally, the *fraction* is reduced to its *lowest terms* if possible.

Example 16

Simplify each of the following algebraic expressions:

(a) $\frac{x}{3} + \frac{x}{5} + \frac{x}{10}$ (b) $\frac{3}{x} + \frac{4}{3x} - \frac{5}{4x}$

(c) $\frac{7}{9pq} + \frac{5}{18p}$ (d) $\frac{5x}{3y} - \frac{4y}{9x}$

(e) $5x - \frac{3y}{7z}$ (f) $\frac{8m - 3n}{5} - \frac{2m - 5n}{3}$

(g) $\frac{3(x + 4)}{7} - \frac{4(x + 1)}{5}$

(h) $\frac{1}{9}(5 - x) - \frac{1}{6}(4 + 9x)$

Solution

(a) The *L.C.M.* of the *denominators* 3, 5 and 10 is 30.

Thus $\frac{x}{3} + \frac{x}{5} + \frac{x}{10}$

$$= \frac{x \times 10 + x \times 6 + x \times 3}{30} \quad \boxed{\frac{30}{3} = 10} \quad \boxed{\frac{30}{5} = 6} \quad \boxed{\frac{30}{10} = 3}$$

$$= \frac{10x + 6x + 3x}{30}$$

$$= \frac{19x}{30}$$

$$= \frac{19}{30}x$$

(b) The *L.C.M.* of the *denominators* x, $3x$ and $4x$ is $12x$.

Thus $\frac{3}{x} + \frac{4}{3x} - \frac{5}{4x}$

$$= \frac{3 \times 12 + 4 \times 4 - 5 \times 3}{12x} \quad \boxed{\frac{12x}{x} = 12} \quad \boxed{\frac{12x}{3x} = 4} \quad \boxed{\frac{12x}{4x} = 3}$$

$$= \frac{36 + 16 - 15}{12x}$$

$$= \frac{36 + 1}{12x}$$

$$= \frac{37}{12x}$$

(c) The *L.C.M.* of the *denominators* $9pq$ and $18p$ is $18pq$.

Thus $\frac{7}{9pq} + \frac{5}{18p}$

$$= \frac{7 \times 2 + 5 \times q}{18pq} \quad \boxed{\frac{18pq}{9pq} = 2} \quad \boxed{\frac{18pq}{18p} = q}$$

$$= \frac{14 + 5q}{18pq}$$

(d) The *L.C.M.* of the *denominators* $3y$ and $9x$ is $9xy$.

Thus $\frac{5x}{3y} - \frac{4y}{9x}$

$$= \frac{5x \times 3x - 4y \times y}{9xy} \quad \boxed{\frac{9xy}{3y} = 3x} \quad \boxed{\frac{9xy}{9x} = y}$$

$$= \frac{15x^2 - 4y^2}{9xy}$$

(e) The *L.C.M.* of the *denominators* 1 and $7z$ is $7z$.

Thus $5x - \frac{3y}{7z}$

$$= \frac{5x}{1} - \frac{3y}{7z}$$

$$= \frac{5x \times 7z - 3y \times 1}{7z} \quad \boxed{\frac{7z}{1} = 7z} \quad \boxed{\frac{7z}{7z} = 1}$$

$$= \frac{35xz - 3y}{7z}$$

(f) The *L.C.M.* of the *denominators* 3 and 5 is 15.

Thus $\frac{8m - 3n}{5} - \frac{2m - 5n}{3}$

$$= \frac{(8m - 3n) \times 3 - (2m - 5n) \times 5}{15} \quad \boxed{\frac{15}{5} = 3} \quad \boxed{\frac{15}{3} = 5}$$

$$= \frac{3(8m - 3n) - 5(2m - 5n)}{15}$$

$$= \frac{24m - 9n - 10m + 25n}{15}$$, using the *distributive law.*

$$= \frac{24m - 10m + 25n - 9n}{15}$$, *grouping like terms.*

$$= \frac{14m + 16n}{15}$$

(g) The *L.C.M.* of the *denominators* 5 and 7 is 35.

Thus $\frac{3(x + 4)}{7} - \frac{4(x + 1)}{5}$

$$= \frac{3(x + 4) \times 5 - 4(x + 1) \times 7}{35} \quad \boxed{\frac{35}{7} = 5} \quad \boxed{\frac{35}{5} = 7}$$

$$= \frac{15(x + 4) - 28(x + 1)}{35}$$

$$= \frac{15x + 60 - 28x - 28}{35}$$, using the *distributive law.*

$$= \frac{15x - 28x + 60 - 28}{35}$$, *grouping like terms.*

$$= \frac{-13x + 32}{35}$$

$$= \frac{32 - 13x}{35}$$

(h) The *L.C.M.* of the *denominators* 6 and 9 is 18.

Thus $\frac{1}{9}(5 - x) - \frac{1}{6}(4 + 9x)$

$$= \frac{(5 - x) \times 2 - (4 + 9x) \times 3}{18} \quad \boxed{\frac{18}{9} = 2} \quad \boxed{\frac{18}{6} = 3}$$

$$= \frac{2(5 - x) - 3(4 + 9x)}{18}$$

$$= \frac{10 - 2x - 12 - 27x}{18}$$, using the *distributive law.*

$$= \frac{-2x - 27x + 10 - 12}{18}$$, *grouping like terms.*

$$= \frac{-29x - 2}{18}$$

Note that: $2\left(\frac{x + 5}{x - 3}\right) = \frac{2}{1}\left(\frac{x + 5}{x - 3}\right) = \frac{2x + 10}{x - 3}$.

Exercise 6k

Simplify each of the following algebraic expressions:

1. $\frac{x}{2}+\frac{x}{3}+\frac{x}{6}$

2. $\frac{x}{3}+\frac{x}{5}+\frac{x}{15}$

3. $\frac{x}{4}+\frac{x}{5}-\frac{x}{10}$

4. $\frac{x}{9}-\frac{x}{3}+\frac{x}{6}$

5. $\frac{x}{3}-\frac{x}{2}-\frac{x}{4}$

6. $\frac{4}{x}+\frac{3}{5x}+\frac{7}{10x}$

7. $\frac{3}{5x}+\frac{4}{7x}-\frac{2}{5x}$

8. $\frac{4}{9x}+\frac{5}{6x}-\frac{7}{18x}$

9. $\frac{7}{8x}-\frac{3}{4x}+\frac{5}{16x}$

10. $\frac{9}{5x}-\frac{4}{15x}-\frac{3}{10x}$

11. $\frac{7}{8pq}+\frac{5}{16p}$

12. $\frac{3}{5xy}+\frac{7}{10x}$

13. $\frac{4}{3pq}-\frac{3}{5p}$

14. $\frac{2}{7xy}-\frac{3}{8x}$

15. $\frac{9}{17xy}-\frac{8}{34x}$

16. $\frac{3x}{5y}+\frac{4y}{7x}$

17. $\frac{4x}{7y}+\frac{9y}{14x}$

18. $\frac{5x}{8y}-\frac{3y}{7x}$

19. $\frac{5x}{9y}-\frac{7y}{6x}$

20. $\frac{8x}{9y}+\frac{2y}{3x}$

21. $7x-\frac{3y}{5z}$

22. $8x+\frac{5y}{7z}$

23. $\frac{3x}{7y}+4z$

24. $\frac{8x}{9y}-7z$

25. $4x-\frac{3y}{8z}$

26. $\frac{7m-3n}{7}-\frac{2m-5n}{3}$

27. $\frac{8m-3n}{5}+\frac{3m-2n}{10}$

28. $\frac{4x-3y}{8}-\frac{3x-5y}{24}$

29. $\frac{5x-2y}{6}-\frac{4x+5y}{12}$

30. $\frac{6x+5y}{4}-\frac{3x+5y}{8}$

31. $\frac{3(x+2)}{5}+\frac{4(x+1)}{7}$

32. $\frac{4(x+3)}{9}-\frac{3(x-2)}{5}$

33. $\frac{2(3x+1)}{7}-\frac{4(2x+1)}{3}$

34. $\frac{5(2x+1)}{8}-\frac{3(2x-1)}{5}$

35. $\frac{3(x-4)}{8}-\frac{5(x-2)}{6}$

36. $\frac{1}{3}(5-x)+\frac{1}{4}(4+3x)$

37. $\frac{1}{5}(4-x)-\frac{1}{3}(5+x)$

38. $\frac{1}{4}(x-3)-\frac{1}{5}(x-2)$

39. $\frac{1}{2}(2x+3)-\frac{1}{3}(x-1)$

40. $\frac{1}{5}(3x+1)-\frac{1}{8}(4+x)$

Simplify each of the following:

41. $\frac{3}{5x}-\frac{2}{7x}$

42. $\frac{9}{5pq}+\frac{8}{15p}$

43. $\frac{1}{8}(5-x)+\frac{1}{6}(3+7x)$

44. $\frac{3(x-5)}{7}-\frac{4(x-1)}{3}$

45. $\frac{4}{5pq}+\frac{7}{15pq}$

46. $\frac{x-2}{5}+\frac{x+1}{4}$

47. $\frac{1}{8}(5x-4)-\frac{1}{3}(4x+1)$

48. $\frac{5(2x+1)}{2}+\frac{4(x-3)}{5}$

49. $\frac{4x}{3y}-\frac{5y}{6x}$

50. $\frac{5m-2n}{7}+\frac{4m+n}{3}$

51. $\frac{7m-2n}{5}-\frac{2m-9n}{3}$

52. $\frac{4m-3n}{5}-\frac{2m-5n}{3}$

53. $\frac{3x+1}{4}-\frac{x-5}{9}$

54. $\frac{2x-3}{5}+\frac{x+2}{3}$

Multiplication and Division of Algebraic Fractions

When *multiplying algebraic fractions*, we first *multiply* the *numerators together*, then we *multiply* the *denominators together* in order to form a *single algebraic fraction*. Once we have a *product* in the *numerator* and a *product* in the *denominator*, then we can *cancel factors* which are common to *both* the *numerator* and the *denominator*. *Cancelling* is *equivalent* to *dividing* both the *numerator* and the *denominator* by the *same quantity*.

Example 17

Simplify each of the following algebraic expressions:

(a) $\frac{a^2}{b^2c^3} \times \frac{b^3c^2}{a}$ *(b)* $\frac{4a^3}{5bc} \times \frac{b}{2a} \times \frac{5c^3}{ab}$

Solution

(a) Now

$$\frac{a^2}{b^2c^3} \times \frac{b^3c^2}{a} = \frac{a^2 \times b^3c^2}{b^2c^3 \times a}$$

$$= \frac{\overset{1}{\cancel{a}} \times a \times \overset{1}{\cancel{b}} \times \overset{1}{\cancel{b}} \times b \times \overset{1}{\cancel{c}} \times \overset{1}{\cancel{c}}}{\underset{1}{\cancel{b}} \times \underset{1}{\cancel{b}} \times \underset{1}{\cancel{c}} \times \underset{1}{\cancel{c}} \times c \times \underset{1}{\cancel{a}}}$$

$$= \frac{ab}{c}$$

(b) Now

$$\frac{4a^3}{5bc} \times \frac{b}{2a} \times \frac{5c^3}{ab}$$

$$= \frac{4a^3 \times b \times 5c^3}{5bc \times 2a \times ab}$$

$$= \frac{\overset{2}{\cancel{4}} \times \overset{1}{\cancel{a}} \times \overset{1}{\cancel{a}} \times a \times \overset{1}{\cancel{b}} \times \overset{1}{\cancel{5}} \times \overset{1}{\cancel{c}} \times c \times c}{\underset{1}{\cancel{5}} \times \underset{1}{\cancel{b}} \times \underset{1}{\cancel{c}} \times \underset{1}{\cancel{2}} \times \underset{1}{\cancel{a}} \times \underset{1}{\cancel{a}} \times b}$$

$$= \frac{2ac^2}{b}$$

$$= 2\frac{ac^2}{b}$$

When *dividing algebraic fractions*, we first *invert* (i.e. *upturn*) the *fraction* which is the *divisor* and *multiply* instead.

Example 18

Simplify each of the following algebraic expressions:

(a) $\frac{p^3x}{qr} \div \frac{py}{q^2r}$ *(b)* $\frac{3ab}{7rs} \div \frac{6a^3}{7r^2}$

Solution

(a) Now

$$\frac{p^3x}{qr} \div \frac{py}{q^2r}$$

$$= \frac{p^3x}{qr} \times \frac{q^2r}{py}$$

$$= \frac{p^3x \times q^2r}{qr \times py}$$

$$= \frac{\overset{1}{\cancel{p}} \times p \times p \times x \times \overset{1}{\cancel{q}} \times q \times \overset{1}{\cancel{r}}}{\underset{1}{\cancel{q}} \times \underset{1}{\cancel{r}} \times \underset{1}{\cancel{p}} \times y}$$

$$= \frac{p^2qx}{y}$$

(b) Now

$$\frac{3ab}{7rs} \div \frac{6a^3}{7r^2}$$

$$= \frac{3ab}{7rs} \times \frac{7r^2}{6a^3}$$

$$= \frac{3ab \times 7r^2}{7rs \times 6a^3}$$

$$= \frac{\overset{1}{\cancel{3}} \times \overset{1}{\cancel{a}} \times b \times \overset{1}{\cancel{7}} \times \overset{1}{\cancel{r}} \times r}{\underset{1}{\cancel{7}} \times \underset{1}{\cancel{r}} \times s \times \underset{2}{\cancel{6}} \times \underset{1}{\cancel{a}} \times a \times a}$$

$$= \frac{br}{2a^2s}$$

$$= \frac{1}{2}\,\frac{br}{a^2s}$$

Exercise 6l

Simplify each of the following algebraic expressions:

1. $\frac{ax^3}{by} \times \frac{b^3}{a^2x}$

2. $\frac{a^3}{b^3c^2} \times \frac{b^2c^3}{a^4}$

3. $\frac{p^2q}{r} \times \frac{r^2p}{q^3}$

4. $\frac{a^2x}{by^2} \times \frac{b^3y^3}{ax^2}$

5. $\frac{p^2q^3}{r} \times \frac{r^3}{pq^2}$

6. $\frac{9ab}{4mn} \times \frac{8m^2n}{3ab^3} \times \frac{a^2c}{b}$

7. $\frac{5pq}{r^2} \times \frac{qst}{3p} \times \frac{6rs}{q^2t}$

8. $\frac{3z^2y}{5ac^2} \times \frac{15a^3}{2zy^3} \times \frac{2c^4}{3y^2}$

9. $\frac{4a^2}{7b^2c} \times \frac{b^3}{8a} \times \frac{14ac^2}{b^2}$

10. $\frac{11x^2y}{5a^3c} \times \frac{15a^2b}{22xy^2} \times \frac{2xy^3}{3ab^2}$

11. $\frac{p^3x}{q^2r} \div \frac{py^2}{q^3r}$ **12.** $\frac{ab^2}{m^2n} \div \frac{ab}{m^3n^2}$

13. $\frac{a^3x^2}{b^2y} \div \frac{ac}{by^2}$ **14.** $\frac{p^3q}{r^2} \div \frac{p^2q^3}{r^3}$

15. $\frac{l^3mx}{pq} \div \frac{l^2m^2y}{p^2q}$ **16.** $\frac{9ab^2}{5m^2n} \div \frac{9ab}{10m^3n^3}$

17. $\frac{5a^2b}{6rs^2} \div \frac{5ab^3}{12r^2s}$ **18.** $\frac{4pq^2}{3rs} \div \frac{8p^2}{15r^2}$

19. $\frac{3p^3x}{8qr^2} \div \frac{p^2y}{16q^2r}$ **20.** $\frac{3l^3m^2x}{10p^2q^4} \div \frac{l^2my}{15pq^3}$

21. $\frac{25p^2q^3}{2rs^2} \div \frac{5pq^2}{6r^2s}$ **22.** $\frac{7a^2b}{9cd} \div \frac{14ab^3}{27c^4d^2}$

23. $\frac{15x^2y^3}{4z} \div \frac{5y^2}{8xz^2}$ **24.** $\frac{5z^3y^2}{6a^2c^3} \times \frac{3a^3}{2zy^3} \div \frac{15y}{3c^2}$

25. $\frac{5x^3y^2}{3a^2c^3} \times \frac{6a^4c^2}{5x^2y} \div \frac{9az^2}{4y}$

26. $\frac{3a^2}{4y} \times \frac{2y^2}{6a}$

Equations

The *symbol* (or *equals sign*) = is used to denote *equality* and it means *'is equal to'*. An *equation* is a *statement* using an *equals sign*. It is a *statement* of an *equality relationship* between *two quantities* or *expressions*. A *linear equation with one unknown* is an *equation* which can be *written* in the *form*:

$$ax + b = 0,$$

where a and b are *real numbers* and x is the *unknown value*.
For example: $3x + 5 = 8$ and $2x + 3 = x - 1$.
A linear equation can have only *one solution.*

An *equation* can be likened to a *pair of weighing pans* or a *see-saw* in *equilibrium* (or *balance*).

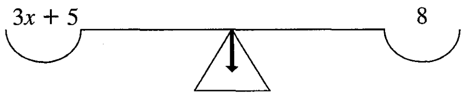


Fig. 6.1 Weighing pans

The *solution* of this *equation* is $x = 1$, since $3(1) + 5 = 3 + 5 = 8$.

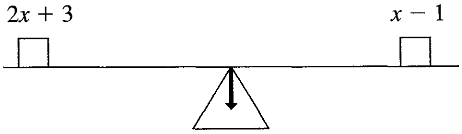


Fig. 6.2 See-saw

The *solution* of this *equation* is $x = -4$, since $2(-4) + 3 = -8 + 3 = -5$ and $-4 - 1 = -5$.

The *equilibrium* or *balance position* can only be maintained in the diagrams shown above if the *sum of the quantities* on the left-hand-side (*L.H.S.*) is *equal* to the *sum of the quantities* on the right-hand-side (*R.H.S.*).

We can *maintain* the *equality* or *balance* in an *equation* by:

(i) *Adding* the *same quantity* to *both sides* of the *equation.*
(ii) *Subtracting* the *same quantity* from *both sides* of the *equation.*
(iii) *Multiplying* both sides of the *equation* by the *same quantity.*
(iv) *Dividing* both sides of the *equation* by the *same quantity.*

Solution of a Linear Equation with One Unknown

In *solving a linear equation* with *one unknown* (i.e. a *simple equation*) we usually take all the *unknowns* to the left-hand-side (*L.H.S.*) and all the *numbers* to the right-hand-side (*R.H.S.*). The *various methods* used to *solve a linear equation with one unknown* can be seen below.

The general form of a linear equation with one unknown is:

$$ax + b = 0,$$

where a and b are constants (i.e. fixed real numbers) and x is the unknown value.

CASE 1: EQUATION CONTAINING ADDITION

Example 19

Solve the equation $x + 2 = 7$.

Solution

Given that $x + 2 = 7$

Subtracting 2 from *both sides*, we get

$$x + 2 - 2 = 7 - 2$$

i.e. $x = 5$

CHECK:

When $x = 5$

Then the *L.H.S.* $= x + 2 = 5 + 2 = 7$

And the *R.H.S.* $= 7$

Hence the *solution* $x = 5$ is *correct.*

Alternative Method

Given that $x + 2 = 7$

Then $x = 7 - 2$

So $x = 5$

From the above *example* it can be *seen* that, when a *value* is shown *added* on the *L.H.S.*, then we can *transfer* it to the *R.H.S.* by *changing* its *positive sign* to a *negative sign.*

CASE 2: EQUATION CONTAINING SUBTRACTION

Example 20

Solve the equation $x - 3 = 8$.

Solution

Given that $x - 3 = 8$

Adding 3 to *both sides*, we get

$$x - 3 + 3 = 8 + 3$$

i.e. $x = 11$

CHECK:

When $x = 11$

Then the *L.H.S.* $= x - 3 = 11 - 3 = 8$

And the *R.H.S.* $= 8$

Hence the *solution* $x = 11$ is *correct.*

Alternative Method

Given that $x - 3 = 8$

Then $x = 8 + 3$

So $x = 11$

From the above *example* it can be *seen*, that when a *value* is shown *subtracted* on the *L.H.S.*, then we can *transfer* it to the *R.H.S.* by *changing* its *negative sign* to a *positive sign.*

CASE 3: EQUATION CONTAINING MULTIPLICATION

Example 21

Solve the equation $5x = 35$.

Solution

Given that $5x = 35$

Dividing both sides by 5, we get

$$\frac{5x}{5} = \frac{35}{5}$$

i.e. $x = 7$

CHECK:

When $x = 7$

Then the *L.H.S.* $= 5x = 5 \times 7 = 35$

And the *R.H.S.* $= 35$

Hence the *solution* $x = 7$ is *correct.*

Alternative Method

Given that $5x = 35$

Then $x = \frac{35}{5}$

So $x = 7$

From the above *example* it can be *seen* that, when a *value* is shown *multiplying* on the *L.H.S.*, then we can *transfer* it to the *R.H.S.* by *cross-multiplying*. And when we *cross-multiply*, the *value* is *transferred* from the *numerator* to the *denominator.*

CASE 4: EQUATION CONTAINING DIVISION

Example 22

Solve the equation $\frac{x}{6} = 5$.

Solution

Given that $\frac{x}{6} = 5$

Multiplying both sides by 6, we get

$$\frac{x}{6} \times 6 = 5 \times 6$$

i.e. $x = 30$

CHECK:

When $x = 30$

Then the *L.H.S.* $= \frac{x}{6} = \frac{30}{6} = 5$

And the *R.H.S.* $= 5$

Hence the *solution* $x = 30$ is *correct.*

Alternative Method

Given that $\frac{x}{6} = 5$

Then $x = 5 \times 6$

So $x = 30$

From the above *example* it can be *seen* that, when a *value* is shown *dividing* on the *L.H.S.*, then we can *transfer* it to the *R.H.S.* by *cross-multiplying*. And when we *cross-multiply*, the *value* is *transferred* from the *denominator* to the numerator.

CASE 5: EQUATION CONTAINING THE UNKNOWN QUANTITY ON BOTH SIDES

When the equation contains the unknown quantity on both sides, then we group the unknowns on the L.H.S. and the constants on the R.H.S. and then solve.

Example 23

Solve the equation $5x - 3 = 2x + 9$.

Solution

Given that $5x - 3 = 2x + 9$

Subtracting $2x$ from *both sides*, we get

$$5x - 2x - 3 = 2x - 2x + 9$$

i.e. $3x - 3 = 9$

Adding 3 to *both sides*, we get

$$3x - 3 + 3 = 9 + 3$$

i.e. $3x = 12$

Dividing both sides by 3, we get

$$\frac{3x}{3} = \frac{12}{3}$$

i.e. $x = 4$

CHECK:

When $x = 4$

Then the *L.H.S.* $= 5x - 3 = 5 \times 4 - 3 = 20 - 3$
$= 17$

And the *R.H.S.* $= 2x + 9 = 2 \times 4 + 9 = 8 + 9$
$= 17$

Hence the *solution* $x = 4$ is *correct.*

Alternative Method

Given that $5x - 3 = 2x + 9$

Grouping like terms, we get

$$5x - 2x = 9 + 3$$

So $3x = 12$

i.e. $x = \frac{12}{3}$

$\therefore$ $x = 4$

CASE 6: EQUATION CONTAINING BRACKETS

When the *equation* contains *brackets*, we first use the *distributive law* to *remove* the *brackets* and then *solve*.

Example 24

Solve the equation $2(3x - 5) = 8$.

Solution

Given that $2(3x - 5) = 8$

Using the *distributive law*, we get

$$6x - 10 = 8$$

So $6x = 8 + 10$

i.e. $6x = 18$

$\therefore$ $x = \frac{18}{6}$

$\Rightarrow$ $x = 3$

CHECK:

When $x = 3$

Then the *L.H.S.* $= 2(3x - 5) = 2(3 \times 3 - 5)$

$= 2(9 - 5) = 2(4) = 8$

And the *R.H.S.* $= 8$

Hence the *solution* $x = 3$ is *correct.*

CASE 7: EQUATION CONTAINING FRACTIONS

When the *equation* contains *fractions*, we first find the *L.C.M.* of the *denominators*, then *multiply* each *term* by the *L.C.M.* and *solve*.

Example 25

Solve each of the following equations:

(a) $\frac{x}{5} - \frac{3}{7} = \frac{1}{5}$ (b) $\frac{7}{8}x - \frac{1}{4} = \frac{4}{5}x + \frac{11}{4}$

(c) $\frac{4x}{9} - \frac{3x+2}{3} = \frac{4}{9}$

Solution

(a) Given that $\frac{x}{5} - \frac{3}{7} = \frac{1}{5}$

The *L.C.M.* of the *denominators* 5 and 7 is 35. *Multiplying each term* by the *L.C.M.*, we get

$$\overset{7}{\cancel{35}}\left(\frac{x}{\underset{1}{\cancel{5}}}\right) - \overset{5}{\cancel{35}}\left(\frac{3}{\underset{1}{\cancel{7}}}\right) = \overset{7}{\cancel{35}}\left(\frac{1}{\underset{1}{\cancel{5}}}\right)$$

Then $7(x) - 5(3) = 7(1)$

So $7x - 15 = 7$

And $7x = 7 + 15$

i.e. $7x = 22$

$\therefore$ $x = \frac{22}{7}$

$\Rightarrow$ $x = 3\frac{1}{7}$

Hence the solution is $x = 3\frac{1}{7}$.

(b) Given that $\frac{7}{8}x - \frac{1}{4} = \frac{4}{5}x + \frac{11}{4}$

The *L.C.M.* of the *denominators* 4, 5 and 8 is 40. *Multiplying each term* by the *L.C.M.*, we get

$$\overset{5}{\cancel{40}}\left(\frac{7}{\underset{1}{\cancel{8}}}x\right) - \overset{10}{\cancel{40}}\left(\frac{1}{\underset{1}{\cancel{4}}}\right) = \overset{8}{\cancel{40}}\left(\frac{4}{\underset{1}{\cancel{5}}}x\right) + \overset{10}{\cancel{40}}\left(\frac{11}{\underset{1}{\cancel{4}}}\right)$$

Then $5(7x) - 10(1) = 8(4x) + 10(11)$

So $35x - 10 = 32x + 110$

And $35x - 32x = 110 + 10$

i.e. $3x = 120$

$\therefore$ $x = \frac{120}{3}$

$\Rightarrow$ $x = 40$

Hence the solution is $x = 40$.

(c) Given that $\frac{4x}{9} - \frac{3x+2}{3} = \frac{4}{9}$

The *L.C.M.* of the *denominators* 3 and 9 is 9. *Multiplying each term* by the *L.C.M.*, we get

$$\overset{1}{\cancel{9}}\left(\frac{4x}{\underset{1}{\cancel{9}}}\right) - \overset{3}{\cancel{9}}\left(\frac{3x+2}{\underset{1}{\cancel{3}}}\right) = \overset{1}{\cancel{9}}\left(\frac{4}{\underset{1}{\cancel{9}}}\right)$$

Then $1(4x) - 3(3x + 2) = 1(4)$

So $4x - 9x - 6 = 4$

And $-5x = 4 + 6$

i.e. $-5x = 10$

$\therefore$ $x = \frac{10}{-5}$

$\Rightarrow$ $x = -2$

Hence the solution is $x = -2$.

From the above *examples* it can be *seen* that:

(i) A *value* that is shown *added* on one side of an *equation* can be *transferred* to the other side of the *equation* by changing its *positive sign* (+) to a *negative sign* (−1).

(ii) A *value* that is shown *subtracted* on one side of an *equation* can be *transferred* to the other side of the *equation* by changing it *negative sign* (−) to a *positive sign* (+).

(iii) A *value* that is shown *multiplying* on one side of an equation can be *transferred* to the other side of the equation by *cross-multiplying* (↘ *or* ↙).

(iv) A *value* that is shown *dividing* on one side of an equation can be *transferred* to the other side of the equation by *cross-multiplying* (↗ *or* ↖).

(v) When we *cross-multiply*, we *transfer* from the *numerator to the denominator* (↘ *or* ↙) or from the *denominator to the numerator* (↖ *or* ↗).

Exercise 6m

Solve each of the following equations:

1. $x + 8 = 1$

2. $x + 5 = 9$

3. $7 + x = 16$

4. $x + 3 = 8$

5. $4 + x = 11$

6. $x + 4 = 9$

7. $x + 6 = 11$

8. $2 + y = 7$

9. $x - 2 = 5$

10. $9 = c - 2$

11. $y - 3 = 4$

12. $3 = b - 1$

13. $x - 3 = 12$

14. $18 - x = 11$

15. $x - 5 = 9$

16. $6 - x = 2$

17. $x - 2 = 6$

18. $3x = 12$

19. $5x = 45$

20. $9y = 81$

21. $7a = 42$

22. $12p = 96$

23. $5x = 20$

24. $2y = 9$

25. $\frac{x}{2} = 5$

26. $\frac{3x}{7} = \frac{1}{2}$

27. $\frac{5x}{2} = \frac{7}{9}$

28. $5x - 7 = 4$

29. $16 - 3x = 4 + x$

30. $6x - 3 = 15$

31. $5 = 3y + 2$

32. $4 + 3x = 20 - 5x$

33. $5 - 4x = 4x + 1$

34. $5x + 7 = 19 - x$

35. $5x - 2 = 13$

36. $2x + 3 = 6 - x$

37. $4x + 3 = 9 - 2x$

38. $3x - 22 = 3 + 4x$

39. $6x + 3 = 27$

40. $7x - 15 = 3x + 1$

41. $9 + 2x = 5x - 3 - x$

42. $5x + 2 = 7$

43. $6x + 5 = 3x + 11$

44. $4x - 3 = 5$

45. $3x - 4 = 2 - x$

46. $4x + 3 = x + 9$

47. $6x + 3 = 15$

48. $2x - 1 = 7$

49. $15 = 1 + 2y$

50. $13 = 24 - 3y$

51. $5x + 3 = 8$

52. $6x + 1 = 4x + 7$

53. $7x - 3 = -17$

54. $3y + 2 = 7 - 2y$

55. $3 - 2x = 9 - 5x$

56. $3x + 1 = 9 - x$

57. $6 - 5x = 4x - 3$

58. $5x + 3 = 13$

59. $6x + 4 = 3x + 10$

60. $3x - 4 = 6 - x$

61. $7x + 3 = 31$

62. $8x + 5 = 5x + 14$

63. $3x - 4 = 3 - x$

64. $6x = 2x - (x - 4)$

65. $4(2x + 3) = 9(3x - 5)$

66. $3(x + 4) = 5(x - 6) + 32$

67. $2x - (x + 4) = 0$

68. $3(2x - 1) - 2 = 2(x + 7) + 1$

69. $6 - 2(x - 3) = x - 3$

70. $4x - 2(1 + 3x) = 5 - 3(x + 2)$

71. $3(2x - 1) - 2 = 2(x + 7) + 1$

72. $3x - 2(4 - 5x) = 2(11 - x)$

73. $3 - 2(x - 2) = 8$

74. $4x = 2 - 3(x + 1)$

75. $7(5 - x) = 3(x - 5)$

76. $3x - 4(1 - 3x) = 2x - (x + 1)$

77. $8 - 5(2 - x) = 8$

78. $3 - 5(2x + 1) = 2x$

79. $7(x + 3) = 9(2x - 1) - 3$

80. $4(3x + 1) = 64$

81. $3(x - 2) - 4 = 2(x - 1) - 2$

82. $7x - 2(3 - x) = 12$

83. $7x - 2(3 + x) = 9$

84. $4x - (x - 1) = 22$

85. $5x - 3 = 2x + 15$

86. $3x - 2 = 5x - 32$

87. $\frac{3x}{4} = \frac{1}{8}$

88. $\frac{x}{4} + \frac{3}{4} = 1$

89. $\frac{5x}{7} - \frac{x}{14} = \frac{9}{7}$

90. $\frac{2}{3}x - \frac{1}{6} = \frac{5}{6}x - \frac{2}{3}$

91. $\frac{5}{12}x - \frac{3}{4} = \frac{7}{6} - \frac{2}{9}x$

92. $\frac{2x}{5} + \frac{x}{2} = 9$

93. $\frac{7}{8}x - \frac{1}{4} = \frac{4}{5}x + \frac{11}{4}$

94. $\frac{4}{5}x - \frac{3}{7} = \frac{4}{7}x + \frac{5}{7}$

95. $\frac{2x}{7} - \frac{x}{5} = \frac{12}{35}$

96. $5(1 - 2x) = 3 - 2(2 - x)$

97. $\frac{2x - 3}{7} = \frac{3x - 5}{10}$

98. $\frac{2x - 1}{5} = \frac{5x - 11}{4}$

99. $\frac{3x + 1}{5} = \frac{2(x - 1)}{3} - 9$

100. $\frac{5x - 2}{3} + \frac{3x + 2}{7} = 2x$

101. $\frac{4m - 3}{5} = \frac{5m + 2}{12}$

102. $\frac{2(x - 3)}{3} - \frac{x - 2}{4} = 1$

103. $\frac{3(x - 4)}{4} - \frac{x - 5}{2} = 1$

104. $\frac{5x - 7}{8} = \frac{3x + 1}{10}$

105. $\frac{2x - 1}{5} = \frac{5x - 11}{4}$

106. $x - 3 = \frac{2x + 3}{5}$

107. $\frac{3x}{5} = x - 6$

108. $5(3x - 1) = 4(3x - 2)$

109. $5x - 4(1 - 3x) = 4x - (x + 1)$

110. $\frac{3x - 1}{2} - \frac{x - 2}{3} = 6$

111. $\frac{p + 3}{5} = \frac{p - 2.4}{2}$

Inequations

The *inequality signs* $<$, $\leqslant$, $>$ *and* $\geqslant$ are used to represent *inequations*. They mean *'is less than'*, *'is less than or equal to'*, *'is greater than'* and *'is greater than or equal to'*, respectively. An *inequation* is a *statement* involving an *inequality*, that is, a statement showing that one quantity is not equal (in general) to another quantity. A *linear inequation with one unknown* is an *inequation* which can be *written* in the *form*:

$$ax + b < 0$$

or $$ax + b \leqslant 0$$

or $$ax + b > 0$$

or $$ax + b \geqslant 0,$$

where a and b are constants (i.e. fixed *real numbers*) and x is the *unknown value*.

For example: $4x + 3 < 7$ and $3x - 5 > 4$.

The *solution* of a *linear inequation* is a *range of values* and hence it is given as a *solution set.*

An *inequation* can be likened to a *pair of weighing pans* or a *see-saw out* of *equilibrium* (or *unbalanced*). However, in *some instances* the *weighing pans* or *see-saw* can be in *equilibrium* (or *balanced*) because of the *equal sign.*

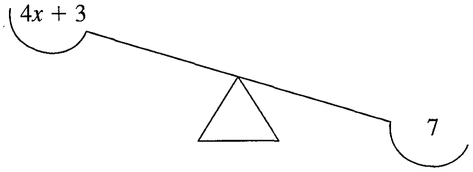


Fig. 6.3 Weighing pans

The *solution set* of this *inequation* is $\{x: x < 1\}$.

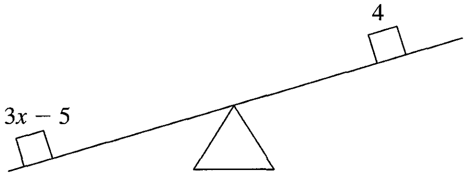


Fig. 6.4 See-saw

The *solution set* of this *inequation* is $\{x: x > 3\}$.

Inequalities remain *true* if:

(i) The *same quantity* is *added* to *both sides.*

(ii) The *same quantity* is *subtracted* from *both sides.*

(iii) *Both sides* are *multiplied* by the *same positive number.*

(iv) *Both sides* are *divided* by the *same positive number.*

However, if an *inequality* is *multiplied* or *divided* by a *negative number* then its *sense* is *changed.*

For example:

Given that $-4 < 2$ is *true.*

Multiplying both sides by -2, we get

$$-4 \times (-2) < 2 \times (-2)$$

$8 < -4$ is *incorrect*

However $8 > -4$ is *correct.*

Given that $8 > -4$ is *true.*

Dividing both sides by -4, we get

$$\frac{8}{-4} > \frac{-4}{-4}$$

$-2 > 1$ is *incorrect*

However $-2 < 1$ is *correct.*

Hence if we *multiply* or *divide* an *inequality* by a *negative number* then we must *reverse* the *inequality sign* in order to make the *statement true.*

Solution of a Linear Inequation with One Unknown

In the *solution* of *linear inequations* with *one unknown* (i.e. *simple inequations*) all the *rules* are *obeyed* as in the *solution* of *linear equations* with *one unknown*. However, the *solution* is now given in the *form* of a *solution set*, since it can take up a *range of values*. There is also *one exception*, which can be seen below.

CASE 1: INEQUATION CONTAINING ADDITION

Example 26

Solve the inequation $x + 2 \geqslant 7$.

Solution

Given that $\quad x + 2 \geqslant 7$

Subtracting 2 form *both sides*, we get

$$x \geqslant 7 - 2$$

i.e. $\quad x \geqslant 5$

Hence the *solution set* is $\{x: x \geqslant 5\}$.

We can represent the *solution set* on a *number line* or on a *graph* as shown below.

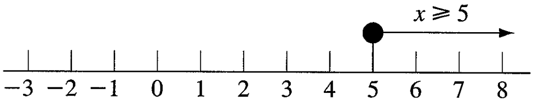


Fig. 6.5 Number line

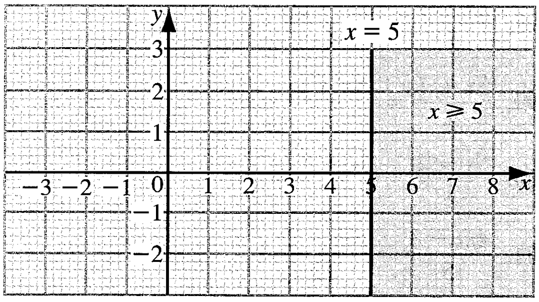


Fig. 6.6 Graph

CASE 2: INEQUATION CONTAINING SUBTRACTION

Example 27

Solve the inequation $x - 3 > 4$.

Solution

Given that $\quad x - 3 > 4$

Adding 3 to *both sides*, we get

$$x > 4 + 3$$

i.e. $\quad x > 7$

Hence the *solution set* is $\{x: x > 7\}$.

We can represent the *solution set* on a *number line* or on a *graph* as shown below.

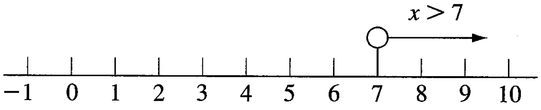


Fig. 6.7 Number line

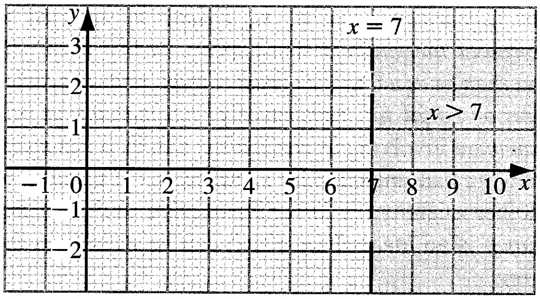


Fig. 6.8 Graph

CASE 3: INEQUATION CONTAINING MULTIPLICATION

Example 28

Solve the inequation $3x \leqslant 9$.

Solution

Given that $3x \leqslant 9$

Dividing both sides by 3, we get

$$x \leqslant \frac{9}{3}$$

i.e. $x \leqslant 3$

Hence the *solution set* is $\{x: x \leqslant 3\}$.

We can represent the *solution set* on a *number line* or on a *graph* as shown below.

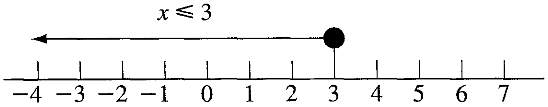


Fig. 6.9 Number line

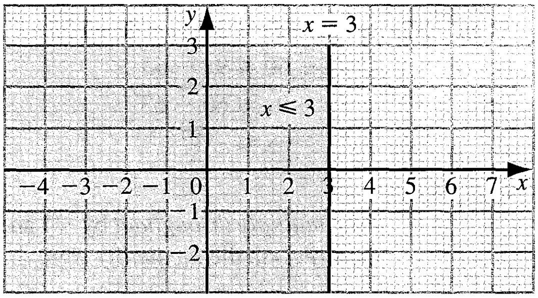


Fig. 6.10 Graph

CASE 4: INEQUATION CONTAINING DIVISION

Example 29

Solve the inequation $\frac{x}{2} < 2.5$

Solution

Given that $\frac{x}{2} < 2.5$

Multiplying both sides by 2, we get

$$x < 2.5 \times 2$$

i.e. $x < 5$

Hence the *solution set* is $\{x: x < 5\}$.

We can represent the *solution set* on a *number line* or on a *graph* as shown below.

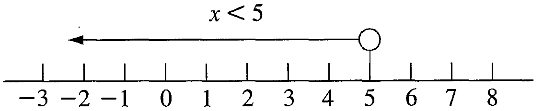


Fig. 6.11 Number line

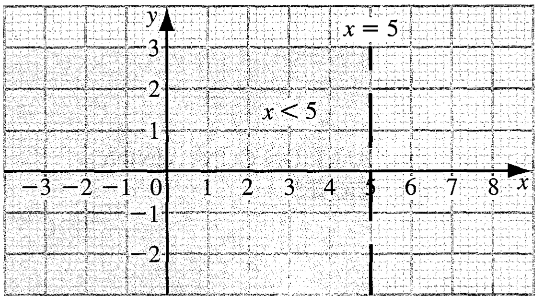


Fig. 6.12 Graph

CASE 5: INEQUATION CONTAINING THE UNKNOWN QUANTITY ON BOTH SIDES

Example 30

Solve the inequation $0.3x - 2 \leqslant 0.1x - 1.5$

Solution

Given that $0.3x - 2 \leqslant 0.1x - 1.5$

Taking all the *unknown quantities* to the *left-hand-side* and the *constants* to the *right-hand-side*, we get

$$0.3x - 0.1x \leqslant 2 - 1.5$$

So $0.2x \leqslant 0.5$

i.e. $x \leqslant \frac{0.5}{0.2}$

∴ $x \leqslant \frac{5}{2}$

⇒ $x \leqslant 2.5$

Hence the *solution set* is: $\{x: x \leqslant 2.5\}$.

We can represent the *solution set* on a *number line* or on a *graph* as shown below.

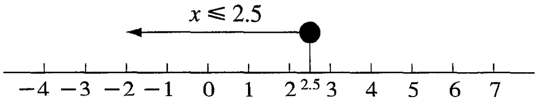


Fig. 6.13 Number line

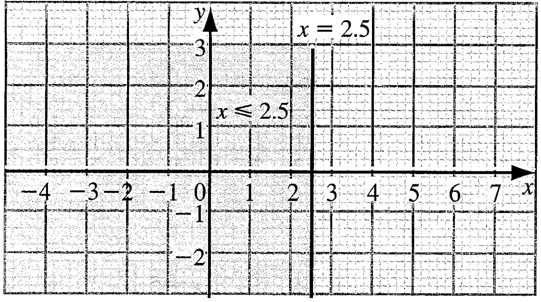


Fig. 6.14 Graph

CASE 6: INEQUATION CONTAINING BRACKETS

Example 31

Solve the inequation $3(2x - 1) > 6$.

Solution

Given that $3(2x - 1) > 6$

Using the *distributive law*, we get

$$6x - 3 > 6$$

So $6x > 6 + 3$

i.e. $6x > 9$

$\therefore$ $x > \frac{9}{6}$

$\Rightarrow$ $x > 1.5$

Hence the *solution set* is $\{x: x > 1.5\}$.

We can represent the *solution set* on a *number line* or on a *graph* as shown below.

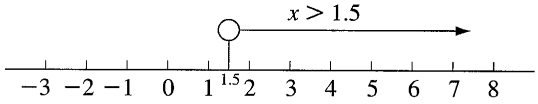


Fig. 6.15 Number line

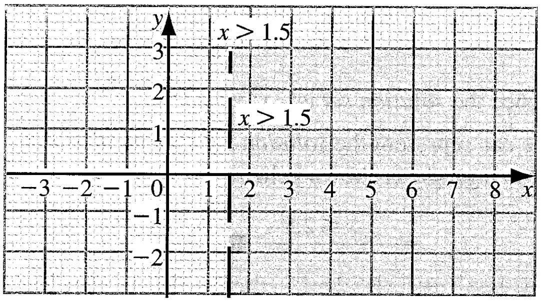


Fig. 6.16 Graph

CASE 7: INEQUATION CONTAINING FRACTIONS

Example 32

Solve each of the following inequations:

(a) $\frac{2}{3}x - 5 \leqslant \frac{3}{4}x - \frac{21}{4}$

(b) $\frac{2}{9}x + \frac{5}{2} \geqslant \frac{1}{3}x - \frac{1}{2}$

(c) $\frac{3}{8}x - \frac{1}{4} < \frac{3}{4}x + \frac{5}{8}$

Solution

(a) Given that $\frac{2}{3}x - 5 \leqslant \frac{3}{4}x - \frac{21}{4}$

The *L.C.M.* of the *denominators* 3 and 4 is 12. *Multiplying each term* by the *L.C.M.*, we get

$$\overset{4}{\cancel{12}}\left(\frac{2}{\underset{1}{\cancel{3}}}x\right) + 12\left(-5\right) \leqslant \overset{3}{\cancel{12}}\left(\frac{3}{\underset{1}{\cancel{4}}}x\right) + \overset{3}{\cancel{12}}\left(-\frac{21}{\underset{1}{\cancel{4}}}\right)$$

Then $4(2x) + 12(-5) \leqslant 3(3x) + 3(-21)$

So $8x - 60 \leqslant 9x - 63$

i.e. $8x - 9x \leqslant 60 - 63$

$\therefore$ $-x \leqslant -3$

Exception:

First *multiply throughout* by -1 and then *reverse* the *inequality sign.*

We get $x \geqslant 3$

Hence the *solution set* is $\{x: x \geqslant 3\}$.

Note that $x \leqslant \frac{-3}{-1}$

$\therefore$ $x \leqslant 3$ is an *incorrect solution.*

We can represent the *solution set* on a *number line* or on a *graph* as shown below.

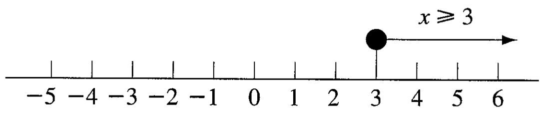


Fig. 6.17 Number line

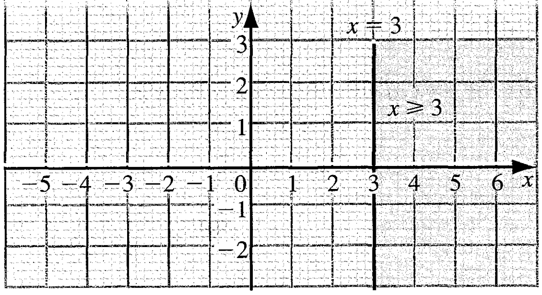


Fig. 6.18 Graph

(b) Given that $\frac{2}{9}x + \frac{5}{2} \geqslant \frac{1}{3}x - \frac{1}{2}$

The *L.C.M.* of the *denominators* 2, 3 and 9 is 18. *Multiplying each term* by the *L.C.M.*, we get

$$\overset{2}{\cancel{18}}\left(\frac{2}{\underset{1}{\cancel{9}}}x\right) + \overset{9}{\cancel{18}}\left(\frac{5}{\underset{1}{\cancel{2}}}\right) \geqslant \overset{6}{\cancel{18}}\left(\frac{1}{\underset{1}{\cancel{3}}}x\right) - \overset{9}{\cancel{18}}\left(\frac{1}{\underset{1}{\cancel{2}}}\right)$$

Then $2(2x) + 9(5) \geqslant 6(x) - 9(1)$

So $4x + 45 \geqslant 6x - 9$

i.e. $4x - 6x \geqslant -45 - 9$

$\therefore$ $-2x \geqslant -54$

Multiplying throughout by -1, we get

$$2x \leqslant 54$$

$\Rightarrow$ $x \leqslant \frac{54}{2}$

$\therefore$ $x \leqslant 27$

Hence the *solution set* is $\{x: x \leqslant 27\}$.

(c) Given that $\frac{3}{8}x - \frac{1}{4} < \frac{3}{4}x + \frac{5}{8}$

The *L.C.M.* of the *denominators* 4 and 8 is 8. *Multiplying each term* by the *L.C.M.*, we get

$$\overset{1}{\cancel{8}}\left(\frac{3}{\underset{1}{\cancel{8}}}x\right) - \overset{2}{\cancel{8}}\left(\frac{1}{\underset{1}{\cancel{4}}}\right) < \overset{2}{\cancel{8}}\left(\frac{3}{\underset{1}{\cancel{4}}}x\right) + \overset{1}{\cancel{8}}\left(\frac{5}{\underset{1}{\cancel{8}}}\right)$$

Then $1(3x) - 2(1) < 2(3x) + 1(5)$

So $3x - 2 < 6x + 5$

i.e. $3x - 6x < 5 + 2$

$\therefore$ $-3x < 7$

Multiplying throughout by -1, we get

$$3x > -7$$

$\Rightarrow$ $x > \frac{-7}{3}$

$\therefore$ $x > -\frac{7}{3}$

Hence the *solution set* is $\left\{x: x > -\frac{7}{3}\right\}$.

As can be *seen* from the above *examples*:

(i) The *circle* is *drawn shaded* in the *number line* and the *equality line* is *drawn unbroken* on the *graph*, when the *solution set* is *less than or equal to*, or *greater than or equal to*. The *shaded circle* and the *unbroken equality line* represent *'is equal to'*.

(ii) The *circle* is left *unshaded* in the *number line* and the *equality line* is *drawn broken* on the *graph*, when the *solution set* is *less than*, or *greater than*. The *unshaded circle* and the *broken equality line* represent *'is not equal to'*.

Exercise 6n

Solve each of the following inequations and represent the solution set on a number line or on a graph:

1. $x + 9 \geqslant 12$

2. $x + 5 < 8$

3. $x + 8 > 15$

4. $y + 3 \leqslant 12$

5. $y + 6 \geqslant 13$

6. $x - 5 \leqslant 8$

7. $x - 7 \geqslant 5$

8. $y - 4 < 3$

9. $y - 8 < 5$

10. $y - 9 > 8$

11. $7 < x + 3$

12. $3x \leqslant 12$

13. $5x \geqslant 45$

14. $9x < 27$

15. $3.5y > 7$

16. $1.9y \geqslant -9.5$

17. $\frac{x}{2} > 10$

18. $\frac{x}{3} \geqslant 4.5$

19. $\frac{x}{5} < -1.3$

20. $\frac{y}{4} \leqslant -1.2$

21. $\frac{y}{7} < 0.5$

22. $5x - 9 \leqslant 7x + 1$

23. $3x + 2 \leqslant 5x - 4$

24. $3x - 7 \geqslant 2x + 9$

25. $5x - 3 \leqslant 3x + 11$

26. $4x + 1 \leqslant 3x - 2$

27. $4(3x + 1) < 2(x - 3)$

28. $6(2x + 3) - 3(x - 2) > 6$

29. $4(3x - 1) \leqslant 20(x - 1)$

30. $5x - 4(3 + 2x) \geqslant 9$

31. $5(3x - 2) > 3(4x - 1)$

32. $2(x - 1) < 14$

33. $3(x + 4) - 5(x - 6) < 7$

34. $4(3x - 1) \leqslant 20(x - 1)$

35. $\frac{2+y}{3} \geqslant \frac{6}{7}$ **36.** $\frac{2}{9}x + \frac{5}{2} > \frac{1}{3}x - \frac{1}{2}$

37. $\frac{4x+1}{5} - \frac{x-3}{4} \geqslant x$

38. $\frac{3x+1}{4} < \frac{7x-1}{5}$

39. $\frac{5x-3}{9} < \frac{2x+1}{4}$ **40.** $\frac{x+3}{2} > \frac{2x-5}{5}$

Simultaneous Equations

Simultaneous equations are a system of *several equations* with *several unknowns*. Often the *equations* in this system are *linear*. For example:

$$5x + 3y = 21$$
$$2x + 7y = 20.$$

Simultaneous equations are *equations* that have the *same solutions*. The *equations* are all satisfied by the *same values* of the *unknown quantities*.

Simultaneous linear equations may be *solved algebraically* by:

(i) the *method* of *elimination*
(ii) the *method* of *substitution*.

Solution of Simultaneous Linear Equations

When we *solve two linear equations* with *two unknown values simultaneously*, then the *solutions* so obtained must satisfy *both equations* at the *same time*.

The Method of Elimination

In using the *method of elimination*, we make the *magnitude* of the *coefficients* of *one* of the *unknown values equal* in order to get rid of it (i.e. *eliminate it*). If the *signs* of the *equal coefficients* are *both* the *same* (i.e. *both positive* or *both negative*), then we *subtract one equation* from the *other*. *Otherwise* we *add* the *equations* (i.e. if one of the *equal coefficients* is *positive* and the other one is *negative*). All the *rules* for the *solution* of *linear equations* are *obeyed*.

Example 33

Solve the pair of simultaneous equations:

$$3x + y = 18$$
$$2x - y = 7.$$

Solution

Given that $3x + y = 18$ — ①
$2x - y = 7$ — ②

The *coefficients of* y are $+1$ and -1.
That is, the *coefficients of* y are *equal in magnitude*, but their *signs* are *different*.

Thus ① + ② gives us

$$3x + 2x + y - y = 18 + 7$$

So $5x = 25$

i.e. $x = \frac{25}{5} = 5$

We can now *substitute* $x = 5$ in any one of the *two equations* and *solve* for y.

Substituting $x = 5$ in ① gives us

$$3(5) + y = 18$$

So $15 + y = 18$

i.e. $y = 18 - 5 = 3$

Hence the *solutions* are:

$$x = 5 \text{ and } y = 3.$$

That is $x = 5$ when $y = 3$.

Example 34

Solve the simultaneous equations:

$$5x + 3y = 21$$
$$2x + 7y = 20.$$

Solution

Given that $5x + 3y = 21$ — ①
$2x + 7y = 20$ — ②

Then we can *make* the *coefficients of x equal* by *multiplying equation* ① by 2 and *equation* ② by 5.

Thus ① × 2 and ② × 5 gives us

$10x + 6y = 42$ — ③
$10x + 35y = 100$ — ④

The *coefficients of* x are *both* 10. That is, the *coefficients of* x are *equal in magnitude* and their *signs* are the *same* (i.e. *both* are *positive*). Therefore we need to *subtract one equation* from the *other*. It is conventient to *subtract* the *equations* in such a way so that the *coefficient of y* will be *positive* if possible.

Thus ④ − ③ gives us

$$10x - 10x + 35y - 6y = 100 - 42$$

So $29y = 58$

i.e. $y = \frac{58}{29} = 2$

We can now *substitute* $y = 2$ in any one of the *four equations* and *solve* for x.

Substituting $y = 2$ in ② gives us

$$2x + 7(2) = 20$$

So $2x + 14 = 20$

i.e. $2x = 20 - 14 = 6$

∴ $x = \frac{6}{2} = 3$

Hence the *solutions* are:

$$x = 3 \text{ and } y = 2.$$

That is $x = 3$ when $y = 2$.

Alternative Method

Given that $5x + 3y = 21$ — ①

$2x + 7y = 20$ — ②

Then we can *make* the *coefficients of y equal* by *multiplying equation* ① by 7 and *equation* ② by 3.

Thus ① × 7 and ② × 3 gives us

$35x + 21y = 147$ — ③

$6x + 21y = 60$ — ④

The *coefficients of y* are *both* 21. That is, the *coefficients of y* are *equal in magnitude* and their *signs* are the *same* (i.e. *both* are *positive*). Therefore we need to *subtract one equation* from the *other*. It is convenient to *subtract* the *equations* in such a way so that the *coefficient of x* will be *positive* if possible

Thus ③ − ④ gives us

$$35x - 6x + 21y - 21y = 147 - 60$$

So $29x = 87$

i.e. $x = \frac{87}{29} = 3$

We can now *substitute* $x = 3$ in any one of the *four equations* and *solve* for y.

Substituting $x = 3$ in ①, we get

$$5(3) + 3y = 21$$

So $15 + 3y = 21$

i.e. $3y = 21 - 15 = 6$

∴ $y = \frac{6}{3} = 2$

Hence the *solutions* are:

$$x = 3 \text{ and } y = 2.$$

That is $x = 3$ when $y = 2$.

Example 35

Solve the pair of simultaneous equations

$$7x - 2y = 3$$

$$5x + y = \frac{11}{4}.$$

Solution

Given that $7x - 2y = 3$ — ①

$5x + y = \frac{11}{4}$ — ②

Then we can *multiply* ② × 2 to *make* the *coefficients of y equal.*

Thus ② × 2 gives us

$10x + 2y = \frac{11}{2}$ — ③

The *coefficient of y* are −2 and +2. That is, the *coefficients of y* are *equal in magnitude*, but their *signs are different.*

Thus ① + ③ gives us

$$7x + 10x - 2y + 2y = 3 + \frac{11}{2}$$

So $17x = \frac{6 + 11}{2} = \frac{17}{2}$

i.e. $x = \frac{17}{2} \div 17$

$$= \frac{17}{2} \times \frac{1}{17}$$

∴ $x = \frac{1}{2}$

We can now *substitute* $x = \frac{1}{2}$ in any of the *three equations* and *solve* for y.

Substituting $x = \frac{1}{2}$ in ② gives us

$$5\left(\frac{1}{2}\right) + y = \frac{11}{4}$$

So $\frac{5}{2} + y = \frac{11}{4}$

i.e. $y = \frac{11}{4} - \frac{5}{2}$

$$= \frac{11 - 10}{4}$$

∴ $y = \frac{1}{4}$

Hence the *solutions* are:

$$x = \frac{1}{2} \text{ and } y = \frac{1}{4}.$$

That is $x = \frac{1}{2}$ when $y = \frac{1}{4}$.

Example 36

Solve the simultaneous equations

$$3x - 5y = -16$$
$$\frac{2}{3}x + \frac{4}{5}y = 6.$$

Solution

Given that $\quad 3x - 5y = -16 \quad$ — ①

$$\frac{2}{3}x + \frac{4}{5}y = 6 \quad \text{— ②}$$

The *L.C.M.* of the *denominators* 3 and 5 is 15.
Multiplying each term in ② by the *L.C.M.*, we get

$$\overset{5}{\cancel{15}}\left(\frac{2}{\underset{1}{\cancel{3}}}x\right) + \overset{3}{\cancel{15}}\left(\frac{4}{\underset{1}{\cancel{5}}}y\right) = 15(6)$$

i.e. $\quad 5(2x) + 3(4y) = 15(6)$

$\therefore \quad 10x + 12y = 90 \quad$ — ③

① × 10 and ③ × 3 gives us

$$30x - 50y = -160 \quad \text{— ④}$$
$$30x + 36y = 270 \quad \text{— ⑤}$$

⑤ − ④ gives us

$$30x - 30x + 36y + 50y = 270 + 160$$

i.e. $\quad 86y = 430$

$\therefore \quad y = \frac{430}{86} = 5$

Substituting $y = 5$ in ① gives us

$$3x - 5(5) = -16$$

So $\quad 3x - 25 = -16$

i.e. $\quad 3x = 25-16 = 9$

$\therefore \quad x = \frac{9}{3} = 3$

Hence the *solutions* are:

$$x = 3 \text{ and } y = 5.$$

That is $x = 3$ when $y = 5$.

The Method of Substitution

In the *method of substitution*, we use *one linear equation* to *substitute* in the *other linear equation* for *one* of the *two unknown values*. We then *solve* for that *value*. Then we can *substitute* the *determined value* in *one* of the *equations* and *solve* for the *second unknown value*.

Example 37

Solve the pair of simultaneous equations

$$5x + 3y = 31$$
$$2x + y = 12.$$

Solution

Given that $\quad 5x + 3y = 31 \quad$ — ①

$$2x + y = 12 \quad \text{— ②}$$

From ②, we get

$$y = 12 - 2x \quad \text{— ③}$$

Substituting $y = 12 - 2x$ in ①, we get

$$5x + 3(12 - 2x) = 31$$

Using the *distributive law*, we get

$$5x + 36 - 6x = 31$$

So $\quad 5x - 6x = 31 - 36$

i.e. $\quad -x = -5$

Multiplying throughout by -1, we get

$$x = 5$$

Substituting $x = 5$ in ③, we get

$$y = 12 - 2(5) = 12 - 10 = 2$$

Hence the *solutions* are:

$$x = 5 \text{ and } y = 2.$$

That is $x = 5$ when $y = 2$.

Exercise 6o

Solve each of the following pairs of simultaneous equations:

1. $5x + 2y = 29$, $\quad x - y = -4$

2. $2x - y = -1$, $\quad 3x - y = 2$

3. $2x + 3y = 1$, $\quad -x + 2y = -4$

4. $-4x - 6y = 7$, $\quad 4x + y = -2$

5. $5x + y = 16$, $\quad x - 2y = 1$

6. $x + y = 7$, $\quad 2x + y = 10$

7. $3x + y = 9$, $\quad 2x - y = 11$

8. $3y = x + 15$, $\quad y + 3x = 4$

9. $5x - y = 16$, $\quad -3x + y = 6$

10. $2x + y = 7$, $\quad -x + y = 1$

11. $x + y = 13$, $\quad 4x + y = 31$

12. $x + y = 144$, $\quad 2x - 3y = 63$

13. $-2x - 3y = \frac{7}{2}$, $\quad 4x + y = -2$

14. $-4x + 3y = 1$, $\quad 6x - y = 2$

Determine the solution of each of the following pairs of equations:

15. $5x + 2y = 16$
$-3x + 4y = -7$

16. $3x + 2y = 19$
$5x - 2y = 5$

17. $3x - 2y = -1$
$4x + 7y = 18$

18. $-x + 3y = 6$
$8x + 3y = 24$

19. $3x + 2y = 19$
$2x - y = 1$

20. $-5x + 2y = 24$
$-7x + 3y = 35$

21. $x + y = 7$
$2x - y = 5$

22. $3x + 5y = 21$
$2x + 3y = 13$

23. $x + 3y = 24$
$x - 2y = 4$

24. $3x + 2y = 5$
$4x - 3y = 18$

25. $7x + 2y = 17$
$2x - 2y = 1$

26. $7x - 2y = 19$
$3x + 5y = 14$

27. $3x + 4y = 27$
$5x - 2y = 19$

28. $2x - 3y = -15$
$5x + 2y = 29$

29. $5x + 2y = 16$
$-3x + 4y = 6$

30. $2x - 5y = 3$
$x - 3y = 1$

Solve each of the following pairs of simultaneous equations:

31. $4x + 3y = 17$
$5x - 2y = 4$

32. $9x + 5y = 15$
$3x - 2y = -6$

33. $3x - 2y = 7$
$-x + 3y = -7$

34. $3x - 5y = -13$
$-2x + 3y = 8$

35. $x - 5y = 2$
$-2x + 7y = -10$

36. $-3x + 2y = 4$
$x + 3y = 17$

37. $5x + 3y = 27$
$2x + 5y = 26$

38. $3x + 4y = 14$
$-2x + y = 9$

39. $5x + 2y = 137$
$4x + 3y = 160$

40. $2x + 3y = -8$
$5x - 2y = 18$

Solve each of the following pairs of equations:

41. $2x + 3y = 21.75$
$3x + 2y = 28.25$

42. $7x + 5y = 20.15$
$5x + 7y = 18.85$

43. $5x + 3y = 16.65$
$3x + 7y = 19.35$

44. $5x + 7y = 26.5$
$3x + 2y = 10.4$

45. $3x - 2y = 0$
$-7x + 5y = 0.25$

46. $2x - 3y = 0.5$
$5x + 4y = 18.5$

47. $5x + 4y = 19.75$
$x + 2y = 5.75$

48. $2x + 3y = 10.0$
$5x + 2y = 19.5$

49. $x + y = 3.75$
$2x + 3y = 9.00$

50. $3x + 4y = 13.40$
$4x + 3y = 14.95$

51. $3x + 4y = 795$
$x - y = 20$

52. $3x + 5y = 32.75$
$4x + 4y = 29.00$

53. $7x + 6y = 12.5$
$5x + 8y = 14.5$

54. $7x + 9y = 31.50$
$13x + 6y = 39.75$

55. $2x + 3y = 17.5$
$4x + 3y = 30.5$

Solve each of the following pairs of simultaneous equations:

56. $5x - 2y = -1$
$\frac{1}{5}x + \frac{1}{3}y = \frac{6}{5}$

57. $3x - 5y = -16$
$\frac{1}{3}x + \frac{2}{5}y = 3$

58. $\frac{2}{9}x + 3y = 1$
$2x - 9y = 5$

59. $2x + y - 1 = \frac{3}{4}$
$5x - y - \frac{1}{2} = \frac{5}{4}$

60. $3(x - 1) - 2(7y + 3) = 14$
$5(4x + 3) + 3(4y + 1) = 66$

61. $\frac{7x - 1}{3} - \frac{2y + 3}{5} = \frac{10}{3}$
$\frac{5x + 2}{4} + \frac{3y - 2}{5} = \frac{16}{5}$

62. The paths taken by two boys running to reach a bus is given by the simultaneous equations:

$$4x + 3y = 24$$
$$x - 2y = -5.$$

Determine the point (x, y) where they both finally reached the bus.

63. The positions of two roads are represented by the simultaneous equations

$$-x + 2y = -4$$
$$7x + 3y = 11.$$

Calculate the point (x, y) where the two roads intersect.

64. The positions of two railway lines are represented by the equations

$$2x + 3y = 5$$
$$-x + 2y = 8.$$

Determine the point (x, y) where the two lines meet.

65. The positions of two straight roads are represented by the equations

$$5x + 3y = 25$$
$$-3x + 5y = 19.$$

Solve the equation to find the position (x, y) where the two roads intersect.

Word Problem—Linear Equation

In this type of *problem* we have to *form* a *linear equation* with *one unknown* (i.e. a *simple equation*) from the *English statements* given. We then *solve* the *constructed linear equation* to determine the *magnitude* of the *unknown quantity*.

Example 38

When I think of a number, double it and subtract thirteen, I get 17.
What number did I think of?

Solution

Let the *number* I thought of be n

Then $n \times 2 - 13 = 17$

So $2n - 13 = 17$

i.e. $2n = 17 + 13 = 30$

$\therefore$ $n = \dfrac{30}{2} = 15$

Hence the *number* that I thought of was 15.

Example 39

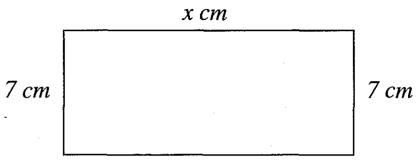


Fig. 6.19 Rectangle

The sides of a rectangle are x cm and 7 cm. Its perimeter is 40 cm.
Calculate the value of x.

Solution

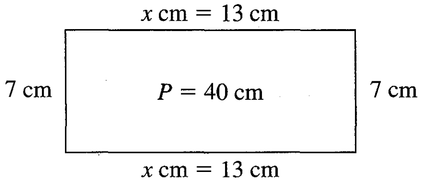


Fig. 6.20 Rectangle

The *perimeter* of the *rectangle*, $P = (x + x + 7 + 7)$ cm $= (2x + 14)$ cm

And the *perimeter* of the *rectangle*, $P = 40$ cm

Thus $2x + 14 = 40$

So $2x = 40 - 14 = 26$

i.e. $x = \dfrac{26}{2} = 13$

Hence the *value* of x is 13.

Example 40

The three angles of a triangle are (x − 20)°, (2x + 30)° and 20°. Determine the magnitude of each angle.

Solution

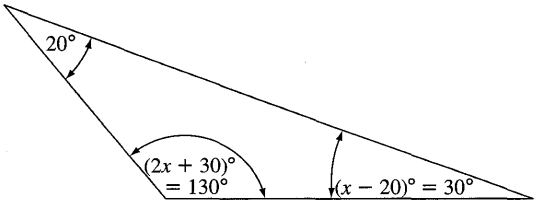


Fig. 6.21 Triangle

The *sum* of the *angles of the triangle*

$= (x - 20)° + (2x + 30)° + 20°$

$= (x + 2x + 30 + 20 - 20)°$

$= (3x + 30)°$

And the *sum* of the *angles of the triangle* = 180°

Thus $3x + 30 = 180$

So $3x = 180 - 30 = 150$

i.e. $x = \dfrac{150}{3} = 50$

$\therefore$ $(x - 20)° = (50 - 20)° = 30°$

And $(2x + 30)° = (2 \times 50 + 30)° = (100 + 30)° = 130°$

Hence the *magnitudes* of the *angles of the triangle* are 30°, 130° and 20°.

Example 41

(a) *Determine three consecutive even numbers whose sum is 42.*

(b) *State three consecutive odd numbers whose sum is 33.*

(c) Determine the number which when added to both the numerator and the denominator of the fraction $\frac{5}{8}$ gives a new fraction $\frac{3}{4}$.

Solution

(a) Let the *three consecutive even numbers be*

$n, n + 2$ and $n + 4$

So the *sum* of the *three consecutive even numbers*

$$= n + (n + 2) + (n + 4)$$
$$= n + n + 2 + n + 4$$
$$= n + n + n + 2 + 4$$
$$= 3n + 6$$

And the *sum* of the *three consecutive even numbers*

$$= 42$$

Thus $3n + 6 = 42$

So $3n = 42 - 6 = 36$

i.e. $n = \frac{36}{3} = 12$

$\therefore$ $n + 2 = 12 + 2 = 14$

And $n + 4 = 12 + 4 = 16$

Hence the *three consecutive even numbers* are 12, 14 and 16.

Alternative Method 1

(a) Let the *three consecutive even numbers be*

$n, n - 2$ and $n - 4$

So the *sum* of the *three consecutive even numbers*

$$= n + (n - 2) + (n - 4)$$
$$= n + n - 2 + n - 4$$
$$= n + n + n - 2 - 4$$
$$= 3n - 6$$

And the *sum* of the *three consecutive even numbers*

$$= 42$$

Thus $3n - 6 = 42$

So $3n = 42 + 6 = 48$

i.e. $n = \frac{48}{3} = 16$

$\therefore$ $n - 2 = 16 - 2 = 14$

And $n - 4 = 16 - 4 = 12$

Hence the *three consecutive even numbers* are 12, 14 and 16.

Alternative Method 2

(a) Let the *three consecutive even numbers be*

$n - 2, n$ and $n + 2$

So the *sum* of the *three consecutive even numbers*

$$= (n - 2) + n + (n + 2)$$
$$= n - 2 + n + n + 2$$
$$= n + n + n - 2 + 2$$
$$= 3n$$

And the *sum* of the *three consecutive even numbers*

$$= 42$$

Thus $3n = 42$

So $n = \frac{42}{3} = 14$

$\therefore$ $n - 2 = 14 - 2 = 12$

And $n + 2 = 14 + 2 = 16$

Hence the *three consecutive even numbers* are 12, 14 and 16.

(b) Let the *three consecutive odd numbers be*

$n, n + 2$ and $n + 4$

So the *sum* of the *three consecutive odd numbers*

$$= n + (n + 2) + (n + 4)$$
$$= n + n + 2 + n + 4$$
$$= n + n + n + 2 + 4$$
$$= 3n + 6$$

And the *sum* of the *three consecutive odd numbers*

$$= 33$$

Thus $3n + 6 = 33$

So $3n = 33 - 6 = 27$

i.e. $n = \frac{27}{3} = 9$

$\therefore$ $n + 2 = 9 + 2 = 11$

And $n + 4 = 9 + 4 = 13$

Hence the *three consecutive odd numbers* are 9, 11 and 13.

(c) Let the *number* that was *added* be *n*

Thus $\frac{5 + n}{8 + n} = \frac{3}{4}$

Cross-multiplying, we get

$$4(5 + n) = 3(8 + n)$$

Using the *distributive law*, we get

$$20 + 4n = 24 + 3n$$

Then $4n - 3n = 24 - 20$

So $n = 4$

Hence the *number* added to both the numerator and the denominator is 4.

Example 42

A housewife out shopping decides to buy a total of 8 fruits for her son at home. She wants to spend \$4.75 on oranges and apples. An orange costs \$0.50 and an apple costs \$0.75. Calculate the number of each fruit bought.

Solution

Let the *number of oranges bought* $= x$

So the *number of apples bought* $= 8 - x$

The *cost of x oranges* $= \$0.50 \times x$

$= \$0.50x$

And the *cost of the (8 − x) apples* $= \$0.75(8 - x)$

$= \$(6.00 - 0.75x)$

Thus the *total cost* of the 8 fruits $= \$(0.50x + 6.00 - 0.75x)$

$= \$(6.00 - 0.25x)$

And the *total cost* of the 8 fruits $= \$4.75$

So $4.75 = 6.00 - 0.25x$

i.e. $0.25x = 6.00 - 4.75 = 1.25$

$\therefore$ $x = \dfrac{1.25}{0.25} = \dfrac{125}{25} = 5$

Since $x = 5$

Then $8 - x = 8 - 5 = 3$

Hence the housewife *bought* 5 oranges and 3 apples.

Alternative Method

Let the *number of oranges bought* $= x$

And the *number of apples bought* $= y$

The *total number of fruits bought*:

$$x + y = 8 \qquad \text{—①}$$

The *total cost of the fruits*:

$$\$(0.50x + 0.75y) = \$4.75$$

So $0.50x + 0.75y = 4.75$ —②

Then ② × 2 gives us

$$x + 1.5y = 9.5 \qquad \text{—③}$$

And ③ − ① gives us

$$x - x + 1.5y - y = 9.5 - 8$$

So $0.5y = 1.5$

i.e. $y = \dfrac{1.5}{0.5} = \dfrac{15}{5} = 3$

From ① we get

$$x = 8 - y = 8 - 3 = 5.$$

Hence the housewife *bought* 5 oranges and 3 apples.

Exercise 6p

1. When I think of a number, double it, then add seven, I get 25. What number did I think of?
2. I think of a number, halve it and the result is 9. Determine the number that I thought of.
3. The length of a rectangle is 10 cm, which is $\frac{1}{3}$ of its perimeter. Evaluate its perimeter.
4. I think of a number, double it and the result is 9. Evaluate the number that I first thought of.
5. When I think of a number, double it, then add seven, I get 23. Determine the number that I first thought of.
6. When I think of a number, double it, then add five, I get 13. Determine the number that I thought of.
7. I think of a number, double it, then subtract three. The result is 12. What number did I think of?
8. I think of a number, triple it, then add three, I get 33. Evaluate the number that I thought of.
9. I think of a number. If I subtract 6 from it and multiply the difference by 4 the result is 36. Evaluate the number that I thought of.
10. When I think of a number and add 5, then the result is 25. From an equation and solve it to find the number that I thought of.
11. When I think of a number and halve it, then the result is 7. Form an equation and solve it to find the number that I thought of.

Form equations to represent the following statements and determine the unknown numbers:

12. I think of a number, add 7 and the result is 15.
13. I think of a number, subtract 5 and the result is 9.
14. If 6 is subtracted from a number then we get 4.
15. I think of a number, double it and the result is 15.
16. An unknown number multiplied by 8 gives 32.
17. I think of a number and add $\frac{1}{5}$ of it to $\frac{1}{2}$ of it. The result is 14. Determine the number that I thought of.

18. I think of a number and add $\frac{1}{4}$ of it to $\frac{3}{5}$ of it. The result is 34. Evaluate the number that I thought of.

19. The lengths of the three sides of a triangle are x cm, $2x$ cm and $3x$ cm. Its perimeter is 30 cm. Evaluate x.

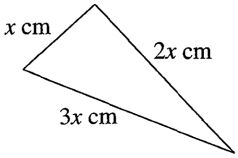


Fig. 6.22 Triangle

20. The sides of a rectangle are x cm and 5 cm. It perimeter is 29 cm. Calculate the value of x.

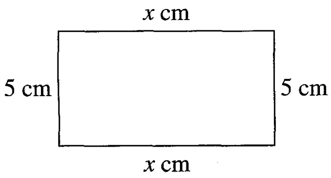


Fig. 6.23 Rectangle

21. The sides of a rectangle are x cm and 4 cm. Its perimeter is 46 cm. Determine the value of x.

22. The length of a rectangle is 5 cm more than its width. If its perimeter is 58 cm, calculate its dimensions.

23. The width of a rectangle is 7 cm less that its length. If its perimeter is 50 cm, calculate its dimensions.

24. The three angles of a triangle are $(x - 25)°$, $(2x + 40)°$ and 30°. Calculate the magnitude of each angle, given that the sum of the angles of a triangle is 180°.

25. The angles of a triangle are $(x - 5)°$, $(x + 15)°$ and $(2x + 10)°$. Given that the sum of the angles of a triangle is 180°, calculate the size of each angle.

26. The three angles of a triangle are $(2x + 5)°$, $(x - 10)°$ and 65°. Evaluate the magnitude of each angle.

27. Given that the angles of a triangle are $(2x + 20)°$, $(x + 25)°$ and $(2x - 15)°$, calculate the size of each angle.

28. Determine three consecutive even numbers whose sum is 60.

29. Determine three consecutive odd numbers whose sum is 57.

30. State the number which when added to both the numerator and the denominator of the fraction $\frac{2}{3}$ gives a new fraction $\frac{6}{7}$.

31. Determine three consecutive even numbers whose sum is 102.

32. Determine three consecutive odd numbers whose sum is 129.

33. Determine the number which when added to both the numerator and the denominator of the fraction $\frac{1}{2}$ gives a new fraction $\frac{3}{4}$.

34. Determine three consecutive even number whose sum is 138.

35. Determine three consecutive odd numbers whose sum is 123.

36. Determine the number which when added to both the numerator and the denominator of the fraction $\frac{3}{5}$ gives a new fraction $\frac{4}{5}$.

37. State the number which when subtracted from both the numerator and the denominator of the fraction $\frac{5}{9}$ gives a new fraction $\frac{1}{2}$.

38. State the number which when subtracted from both the numerator and the denominator of the fraction $\frac{8}{11}$ gives a new fraction $\frac{2}{3}$.

39. Kelly had 12 dollars and spent x dollars. Ami had 6 dollars and collected x dollars. The two girls then had the same amount of money. Form an equation and solve it to determine the value of x.

40. When shopping, Mrs. Van Damme spent $\$x$ in the first shop, twice that amount in the second shop, \$3 in the third shop and \$8 in the last shop. The total amount that she spent was \$26.

(a) Form an equation for the amount of money that Mrs. Van Damme spent.

(b) Solve the equation to determine the amount of money that she spent at the first shop.

41. Nine books are to be bought by a student. Some cost \$6 each and the remainder cost \$6.50 each. If the total amount spent was \$56, how many of each book are bought?

42. Fourteen articles are bought. Some cost $2.00 each and the remainder cost $2.25 each. If the total amount spent is $30, how many of each article are bought?

43. A man bought 18 fruits. Some cost $1.50 each and the remainder cost $2.00 each. He spent a total of $32.50. How many of each fruit did he buy?

44. A father wants to buy a total of five milk drinks for his son and spend $7.95. An eggnog costs $1.55 and a peanut punch costs $1.65. Determine the number of each milk drink bought.

45. Andrew has eight cassettes. Mary has x cassettes and Jim has twice as many as Andrew. Together they have four times as many as Mary has. Form an equation and determine how many cassettes Mary has.

46. (a) A box of mass 9 kg contains x articles each of mass 1.2 kg. Write down an expression for the total mass of the box and its contents.

(b) How many articles are there in the box if the total mass of the box and articles is 21 kg?

47. If four shirts and five jerseys cost $370, calculate the cost of a shirt given that the cost of a jersey is $30.

48. Mrs. Neils bought $155 in groceries. She paid her bill in $5 and $20 notes using a total of 13 notes. Calculate how many $20 notes were used.

49. The length of a rectangle is 3 m greater than its width. Determine its dimensions, if the perimeter of the rectangle is 26 m.

50. A woman had $200. She went to a meatshop, a bookstore and a drugstore. She spent four times as much money at the meatshop as she did at the drugstore. She spent $15 less at the bookstore than at the drugstore. She then had $5 left.

(a) Using $\$x$ to represent the amount she spent at the drugstore, express in algebraic terms
   (i) the amount she spent at the meatshop
   (ii) the amount she spent at the bookshop

(b) Obtain an equation for the total amount of money spent and hence calculate the amount she spent at the drugstore.

Word Problem—Linear Inequation

In this type of *problem* we have to form a *linear inequation* with *one unknown* (i.e. a *simple inequation*) from the *English statements* given. We then *solve* the *constructed linear inequation* to find the *solution set*. From the *solution set* we can then determine a *particular solution* to the given *problem.*

Example 43

The area of a rectangle must not be more than 126 cm². If the length of the rectangle is 18 cm, calculate the greatest possible value of its width.

Solution

The *area* of a *rectangle*, $A = lb$

Then $18 \times b \leqslant 126$

So $b \leqslant \dfrac{126}{18}$

$\therefore \quad b \leqslant 7$

So the *solution set* is: $\{b: b \leqslant 7\}$.

$\Rightarrow \quad b_{\max} = 7$ cm

Hence the *greatest possible value* of its *width*, b_{max} is 7 cm.

Example 44

The area of a triangle must not be more than 108 cm². If the length of the base of the triangle is 12 cm, determine the greatest possible value of its altitude.

Solution

The *area* of a *triangle*, $A = \dfrac{1}{2}bh$

Then $\dfrac{1}{2} \times 12 \times h \leqslant 108$

So $6h \leqslant 108$

i.e. $h \leqslant \dfrac{108}{6}$

$\therefore \quad h \leqslant 18$

So the *solution set* is $\{h: h \leqslant 18\}$.

$\Rightarrow \quad h_{\max} = 18$ cm

Hence the *greatest possible value* of its *altitude*, h_{max} is 18 cm.

Example 45

The width of a rectangular room is 5 metres shorter than its length.
If its perimeter must not exceed 38 metres, determine the greatest length the room can have.

Solution

Let the *length* of the *rectangular* room $= l$ metres
So the *width* of the *rectangular* room $= (l - 5)$ metres

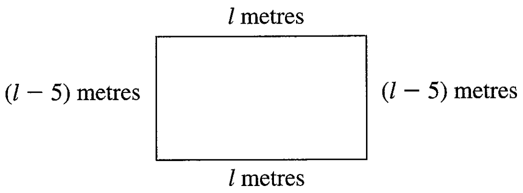


Fig. 6.24 Rectangle

The *perimeter* of the *rectangular room,*

$$\begin{aligned} P &= [l + l + (l - 5) + (l - 5)] \text{ metres} \\ &= (l + l + l - 5 + l - 5) \text{ metres} \\ &= (l + l + l + l - 5 - 5) \text{ metres} \\ &= (4l - 10) \text{ metres} \end{aligned}$$

Thus $4l - 10 \leqslant 38$

So $4l \leqslant 38 + 10$

i.e. $4l \leqslant 48$

$\therefore$ $l \leqslant \frac{48}{4}$

$\Rightarrow$ $l \leqslant 12$

So the *solution set* is $\{l: l \leqslant 12\}$.

$\Rightarrow$ $l_{\max} = 12$ metres

Hence the *greatest length* of the *rectangular room,* l_{max} is 12 metres.

Example 46

A mother has to buy 15 chocolates for Christmas for some children. Some cost \$4.50 each and the remainder cost \$7.50 each. Calculate the least number of chocolates she can buy at \$4.50 each if the total cost to her must not exceed \$85.50.

Solution

Let the *number* of chocolates bought at \$4.50 *each* $= x$

So the *number* of chocolates bought *at* \$7.50 *each* $= 15 - x$

The *total cost* of the chocolates bought at \$4.50 *each* $= \$4.50 \times x$
$= \$4.50x$

And the *total cost* of the chocolates bought at \$7.50 *each* $= \$7.50(15 - x)$
$= \$(112.50 - 7.50x)$

Thus $4.50x + 112.50 - 7.50x \leqslant 85.50$

So $4.50x - 7.50x \leqslant 85.50 - 112.50$

i.e. $-3x \leqslant -27$

Multiplying throughout by -1, we get

$3x \geqslant 27$

i.e. $x \geqslant \frac{27}{3}$

$\therefore$ $x \geqslant 9$

So the *solution set* is: $\{x: x \geqslant 9\}$.

$\Rightarrow$ $x_{\min} = 9$

Hence the *least number* of chocolates that can be bought at \$4.50 each is 9.

Exercise 6q

1. The area of a rectangle must not be more than 198 cm^2. If the length of the rectangle is 18 cm, calculate the greatest possible value of its width.
2. The area of a rectangle must not be more than 247 cm^2. If the width of the rectangle is 13 cm, determine the greatest possible value of its length.
3. The length of a rectangular field is 18 m. If its perimeter must not exceed 61 m, calculate the greatest width that the field can have.
4. The width of a rectangular field is 9.5 m. If its perimeter must not exceed 54 m, determine the greatest length that the field can have.
5. The area of triangle must not be more than 102 cm^2. If the length of the base of the triangle is 12 cm, calculate the greatest possible value of its altitude.
6. The area of a triangle must not be more than 132 cm^2. If the altitude of the triangle is 11 cm, calculate the greatest possible value of its base.

7. The base of a triangular field is 14.5 m. If its area must not exceed 130.5 m^2, determine the greatest altitude that the triangular field can have.

8. The altitude of a triangular field is 14.5 m. If its area must not exceed 174 m^2, calculate the greatest base length that the triangular field can have.

9. The width of a rectangular room is 7 m shorter than its length. If its perimeter must not exceed 38 m, calculate the greatest length that the room can have.

10. The length of a rectangular field is 6 m greater than its width. Calculate the greatest possible value of its width if the perimeter of the field must be less than or equal to 80 m.

11. The base of a triangle is 10 cm. Determine the greatest possible value of its altitude if the area of the triangle must be less than or equal to 125 cm.

12. The width of a rectangular room is 3 m shorter than its length. If its perimeter must not exceed 42 m, calculate the greatest length that the room can have.

13. Eleven books are to be bought for a library. Some cost \$3.50 each and the remainder cost \$5.00 each. What is the greatest number of books which can be bought at \$5.00 each if the total cost must not be more than \$49?

14. A mother has to buy 12 chocolates for Easter for some children. Some cost \$5.50 each and the remainder cost \$7.50 each. What is the greatest number of chocolates which she can buy at \$7.50 each if the total cost must not exceed \$74?

15. A mother has to buy 18 chocolates for Eid-ul-Fitr for some children. Some cost \$3.50 each and the remainder cost \$5.75 each. Calculate the least number of chocolates she can buy at \$3.50 each if the total cost to her must be less than or equal to \$76.50.

16. Fifteen books are to be bought for a school. Some cost \$8.50 each and the remainder cost \$9.50 each. Calculate the least number of books which can be bought at \$8.50 each if the total cost must not be more than \$135.50.

Word Problem—Simultaneous Linear Equations

In this type of *problem* we have to form a *pair* of *simultaneous linear equations* from the *English statements* given. We then *solve* the *constructed pair* of *simultaneous linear equations* in order to determine the *magnitude* of the *unknown values.*

Example 47

Romona bought 5 hamburgers and 3 pizzas for \$167.75. If however Romona had bought 4 hamburgers and 4 pizzas then she would have paid \$193. Calculate the price Romona paid per hamburger and per pizza, correct to the nearest cent.

Solution

Let the *price for a hamburger* $= \$h$

And the *price for a pizza* $= \$p$

So the *price* for 5 *hamburgers* $= \$h \times 5 = \$5h$

And the *price* for 3 *pizzas* $= \$p \times 3 = \$3p$

Hence the *total price* for 5 *hamburgers* and 3 *pizzas* is:

$$\$(5h + 3p) = \$167.75 \qquad — ①$$

Also the price for 4 *hamburgers* $= \$h \times 4 = \$4h$

And the *price* for 4 *pizzas* $= \$p \times 4 = \$4p$

Hence the *total price* for 4 *hamburgers* and 4 *pizzas* is:

$$\$(4h + 4p) = \$193 \qquad — ②$$

So the *constructed pair* of *simultaneous linear equations* is:

$$5h + 3p = 167.75 \qquad — ①$$
$$4h + 4p = 193 \qquad — ②$$

Now ① × 4 and ② × 3 gives us

$$20h + 12p = 671 \qquad — ③$$
$$12h + 12p = 579 \qquad — ④$$

And ③ − ④ gives us

$$20h - 12h + 12p - 12p = 671 - 579$$

i.e. $$8h = 92$$

$\therefore$ $$h = \frac{92}{8} = 11.5$$

So the *price for a hamburger* is \$11.50.

Substituting $h = 11.5$ in ①, we get

$$5 \times 11.5 + 3p = 167.75$$

i.e. $57.5 + 3p = 167.75$

∴ $3p = 167.75 - 57.5$

$= 110.25$

⇒ $p = \dfrac{110.25}{3} = 36.75$

So the *price for a pizza* is \$36.75.
Hence the *price for a hamburger* is \$11.50 and the *price for a pizza* is \$36.75.

Example 48

A boy bought 3 rotis and 4 pies from a shop and received \$7.50 change from \$40. If he had bought 4 rotis and 3 pies from the same shop, then he would have received \$0.75 change instead.

(a) State a pair of simultaneous equations that represents the information given above.
USE: The cost per roti = \$r
The cost per pie = \$p.

(b) Calculate the cost of:
(i) a roti
(ii) a pie.

Solution

(a) Let the *cost per roti* $= \$r$
And the *cost per pie* $= \$p$
So the *cost* for 3 *rotis* $= \$r \times 3 = \$3r$
And the *cost* for 4 *pies* $= \$p \times 4 = \$4p$
Hence the *total cost* for 3 *rotis* and 4 *pies* is:

$\$(3r + 4p) = \$(40 - 7.50)$

i.e. $\$(3r + 4p) = \$32.50$ — ①

Also the *cost* for 4 *rotis* $= \$r \times 4 = \$4r$
And the *cost* for 3 *pies* $= \$p \times 3 = \$3p$
Hence the *total cost* for 4 *rotis* and 3 *pies* is:

$\$(4r + 3p) = \$(40 - 0.75)$

i.e. $\$(4r + 3p) = \$39.25$ — ②

Hence the pair of *simultaneous equations* that represents the information given above is:

$3r + 4p = 32.50$ — ①
$4r + 3p = 39.25$ — ②

(b) (i) The *constructed pair* of *simultaneous linear equations* is:

$3r + 4p = 32.50$ — ①
$4r + 3p = 39.25$ — ②

Now ① × 3 and ② × 4 gives us

$9r + 12p = 97.5$ — ③
$16r + 12p = 157$ — ④

And ④ − ③ gives us

$16r - 9r + 12p - 12p = 157 - 97.5$

i.e. $7r = 59.5$

∴ $r = \dfrac{59.5}{7} = 8.5$

So the *cost per roti* is \$8.50.

(ii) *Substituting* $r = 8.5$ in ② we get

$4 \times 8.5 + 3p = 39.25$

i.e. $34 + 3p = 39.25$

∴ $3p = 39.25 - 34 = 5.25$

⇒ $p = \dfrac{5.25}{3} = 1.75$

So the *cost per pie* is \$1.75,
Hence the *cost per roti* is \$8.50 and the *cost per pie* is \$1.75.

Example 49

A woman bought s shirts at \$20 each and j jerseys at \$15 each at a total cost of \$155. If, however, she had bought half as many shirts and twice as many jerseys, then the total cost would have been \$190.
Evaluate s and j.

Solution

The *total cost* of *s shirts* and *j jerseys*

$= \$(20 \times s + 15 \times j)$
$= \$(20s + 15j)$

Thus the *equation* is:

$\$(20s + 15j) = \$155$ — ①

The *total cost* of $\frac{1}{2}$ *s shirts* and 2 *j jerseys*

$= \$(20 \times \frac{1}{2}s + 15 \times 2j)$
$= \$(10s + 30j)$

Thus the *equation* is:

$\$(10s + 30j) = \$190$ — ②

The *constructed pair* of *simultaneous linear equations* is:

$20s + 15j = 155$ — ①
$10s + 30j = 190$ — ②

Now ② ÷ 2 gives us

$5s + 15j = 95$ — ③

And ① − 3 gives us

$20s - 5s + 15j - 15j = 155 - 95$

i.e. $15s = 60$

$\therefore$ $s = \frac{60}{15} = 4$

Substituting $s = 4$ in ③ gives us

$$5 \times 4 + 15j = 95$$

i.e. $20 + 15j = 95$

$\therefore$ $15j = 95 - 20 = 75$

$\Rightarrow$ $j = \frac{75}{15} = 5$

Hence $s = 4$ and $j = 5$.

Exercise 6r

1. Ria bought 3 hot dogs and 5 hamburgers for $32.75. If, however, Ria had bought 4 hot dogs and 4 hamburgers she would have paid $29.00. Calculate the price Ria paid per hot dog and per hamburger, to the nearest cent.

2. Mrs. Monty bought 10 chickens and 4 ducks for $274. If, however, she had bought 4 chickens and 3 ducks the total cost would have been $160. Write down two equations in c and d to represent the information given above. Hence solve the equations to determine the cost of a chicken and the cost of a duck.

3. Nine books are to be bought by a student. Some cost $6 each and the remainder cost $6.50 each. If the total amount spent was $56, how many books at each price are bought?

4. Mrs. Naidu bought $155 in groceries. She paid her bill with $5 and $20 notes using a total of 13 notes. Calculate how many of each type of note were used.

5. A student went to a bookstore and bought x books costing $5.50 each and y books costing $8.50 each. She spent $53.00 and bought a total of 8 books. Determine the number of books bought at each price.

6. A father wants to buy a total of 5 milk drinks for his son and spend $7.95. An eggnog costs $1.55 and a peanut punch costs $1.65. Calculate the number of each type of milk drink bought.

7. A housewife out shopping decides to buy a total of 8 fruits for her son at home. She wants to spend $4.75 on oranges and apples. An orange costs $0.50 each and an apple costs $0.75 each. Calculate the number of each type of fruit bought.

8. A teacher has to buy 6 snacks. Snack A costs $4.75 and Snack B costs $6.25. If he spends $34.50, determine the number of each type of snack bought.

9. A girl went to a supermarket and bought 5 packs of grape fruit juice and 4 packs of orange juice at a total cost of $19.75. If she had bought 3 packs of grape fruit juice and 6 packs of orange juice instead, the total cost would have been $17.25. Calculate:

(a) the cost per pack of grapefruit juice

(b) the cost per pack of orange juice.

10. A girl bought 2 rotis and 3 pies for $10.00. If she had bought 5 rotis and 2 pies she would have paid $19.50.

(a) Write two equations to represent the information given above.

(b) Calculate the cost per roti.

(c) Determine the cost per pie.

11. At a market, 7 mangoes and 6 pears cost $12.50; and 5 mangoes and 8 pears cost $14.50. Let $$m$ represent the cost of one mango and $$p$ represent the cost of one pear, hence write down a pair of simultaneous equations to represent the information above. Hence, determine:

(a) the cost of a mango

(b) the cost of a pear.

12. A mother bought 5 packs of chocolate milk and 3 packs of strawberry milk from a grocery and received $3.35 change from a $20 bill. If she had bought 3 packs of chocolate milk and 7 packs of strawberry milk from the same grocery, then she would have received $0.65 change instead.

(a) State a pair of simultaneous equations that represents the information given above.
USE: The cost per pack of chocolate milk = $$c$
The cost per pack of strawberry milk = $$s$.

(b) Calculate the cost of:
  (i) a pack of chocolate milk
  (ii) a pack of strawberry milk.

13. An uncle bought 7 packs of milk and 5 packs of orange juice for $20.15. If he had bought 5 packs of milk and 7 packs of orange juice at the same grocery, then the cost would have been $18.85.

(a) Using m to represent the cost in dollars per pack of milk and j to represent the cost in dollars per pack of orange juice, write down a pair of simultaneous equations to represent the information given above.

(b) Calculate:
  (i) the cost per pack of milk
  (ii) the cost per pack of orange juice.

14. An aunt bought 3 bottles of coconut water and 4 bottles of watermelon juice for $13.40. If she had bought 4 bottles of coconut water and 3 bottles of watermelon juice at the same grocery, then the cost would have been $14.95.

(a) Using c to represent the cost in dollars per bottles of coconut water and w to represent the cost in dollars per bottle of watermelon juice, write a pair of simultaneous equations to represent the information given above.

(b) Hence, determine:
  (i) the cost per bottle of coconut water
  (ii) the cost per bottle of watermelon juice.

15. A student bought a hot dog and a bottle of cane juice for $3.75. If she had bought two hot dogs and three bottles of cane juice she would had paid $9.00. Calculate the price of:

(a) a hot dog and

(b) a bottle of cane juice.

16. The cost of two rotis and three patties is $17.50, while the cost of four rotis and three patties is $30.50. Form a pair of simultaneous equations and solve them to determine:

(a) the cost of a roti

(b) the cost of a pattie.

17. At a grocery 7 packs of milk and 9 bottles of water cost $31.50, while 13 packs of milk and 6 bottles of water cost $39.75.

(a) Using m to represent the cost (in dollars) of one pack of milk and w to represent the cost (in dollars) of one bottle of water, write down a pair of simultaneous equations to represent the information above.

(b) Hence, determine:
  (i) the cost of a pack of milk
  (ii) the cost of a bottle of water.

18. At a grocery, 5 packs of X and 7 packs of Y cost $26.50. 9 packs of X and 6 packs of Y also cost $31.20.

(a) Using x to represent the cost (in dollars) of one pack of X and y to represent the cost (in dollars) of one pack of Y, write down a pair of simultaneous equations to represent the information above.

(b) Hence, determine:
  (i) the cost of a pack of X
  (ii) the cost of a pack of Y.

19. The cost of two rotis and three bottled drinks is $21.75. If I had bought three rotis and two bottled drinks instead, then I would have had to pay $28.25. Calculate the cost of:

(a) a roti

(b) a bottled drink.

20. Mrs. Khan bought 3 dresses and 4 shirts for $795. The cost of a shirt is $20 less than the cost of a dress. Write down two equations in d (cost in dollars per dress) and s (cost in dollars per shirt). Solve the equations to determine the amount of money Mrs. Khan paid for a dress and a shirt.

21. A store clerk sold 25 Mathematics books and 10 English books for a total of $855. If she had sold 10 Mathematics books and 40 English books, she would have got $135 more. Calculate the price of each type of book.

22. The sum of two numbers is 144. Double the first number minus thrice the second number is equal to 63. Determine the two numbers.

23. A man went to a post office to buy some stamps. If he bought x, 50 cents stamps and y, 25 cents stamps, the total cost would have been $3.50. If, however, he bought twice as many 50 cents stamps and half as many 25 cents stamps, then the total cost would have been $4.75. Evaluate x and y.

24. If 3 is added both to the numerator and the denominator of a fraction, the result is equivalent to $\frac{4}{5}$. If 2 is subtracted from both the numerator and denominator of the original fraction, the new result is equivalzent to $\frac{3}{5}$. Determine the original fraction.

25. A girl bought s skirts at $30 each and j jerseys at $15 each at a total cost of $285. If, however, she had bought half as many shirts and twice as many jerseys, then the total cost would have been $210. Evaluate s and j.

26. A woman bought s shirts at \$56.50 each and d dresses at \$95.60 each at a total cost of \$417.20. If she had bought half as many shirts and twice as many dresses, then the total cost would have been \$495.40. Evaluate s and d.

27.

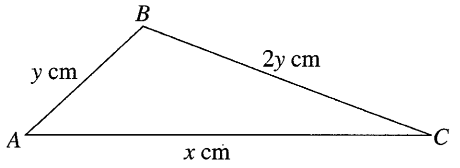


Fig. 6.25 Triangle

The perimeter of a triangle ABC is 24 cm. AC is 4 cm longer than BC. Evaluate x and y.

Positive Integral Index

We know that 2^5 means $2 \times 2 \times 2 \times 2 \times 2$. And we know that 2^5 is read as '2 *to the power* 5'. We call the *number* 2 the *base* and the *number* 5 the *index* (or *power* or *exponent*).

When a *number* or an *algebraic quantity* is *multiplied* by *itself repeatedly*, then it can be *expressed* as a base to a *power*. Thus $a \times a \times a \times \ldots$ *to the* n^{th} *term* $= a^n$, $a \neq 0$, where a is called the *base* and n is the *index* (or *power* or *exponent*).

Example 50

Express each of the following products in index form:

(a) $3 \times 3 \times 3 \times 3$

(b) $a \times a \times a \times a \times a$

(c) $q \times q \times q \times q \times q \times q \times q$

(d) $r \times r \times r \ldots$ *to the* m^{th} *term*

(e) $x \times x \times x \ldots$ *to the* z^{th} *term.*

Solution

(a) Now $3 \times 3 \times 3 \times 3 = 3^4$

(b) Now $a \times a \times a \times a \times a = a^5$

(c) Now $q \times q \times q \times q \times q \times q \times q = q^7$

(d) Now $r \times r \times r \ldots$ to the m^{th} term $= r^m$

(e) Now $x \times x \times x \ldots$ to the z^{th} term $= x^z$.

Multiplication

Now $5^3 \times 5^4 = (5 \times 5 \times 5) \times (5 \times 5 \times 5 \times 5)$
$= 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$
$= 5^7$

And $5^{3+4} = 5^7$

Thus $5^3 \times 5^4 = 5^{3+4} = 5^7$.

Also $a^4 \times a^5 = (a \times a \times a \times a)$
$\times (a \times a \times a \times a \times a)$
$= a \times a \times a \times a \times a \times a \times a$
$\times a \times a$
$= a^9$

And $a^{4+5} = a^9$

Thus $a^4 \times a^5 = a^{4+5} = a^9$.

Hence in *general*

$$a^m \times a^n = a^{m+n}.$$

That is, when we *multiply quantities* with the *same base*, we *add* their *indices.*

Example 51

Express each of the following products in index form:

(a) $2^4 \times 2^3$

(b) $x^3 \times x^5$

(c) $x^3 \times y^2 \times y^4 \times x^2$

(d) $p^2 \times q^3 \times r \times q^4 \times p^4 \times r^3$

Solution

(a) Now $2^4 \times 2^3 = 2^{4+3} = 2^7$

(b) Now $x^3 \times x^5 = x^{3+5} = x^8$

(c) Now $x^3 \times y^2 \times y^4 \times x^2 = x^{3+2}y^{2+4} = x^5y^6$

(d) Now $p^2 \times q^3 \times r \times q^4 \times p^4 \times r^3$
$= p^{2+4}q^{3+4}r^{1+3}$
$= p^6q^7r^4$

Division

Now $2^8 \div 2^5$

$$= \frac{\overset{1}{\cancel{2}} \times \overset{1}{\cancel{2}} \times \overset{1}{\cancel{2}} \times \overset{1}{\cancel{2}} \times \overset{1}{\cancel{2}} \times 2 \times 2 \times 2}{\underset{1}{\cancel{2}} \times \underset{1}{\cancel{2}} \times \underset{1}{\cancel{2}} \times \underset{1}{\cancel{2}} \times \underset{1}{\cancel{2}}}$$

$= 2^3$

And $\quad 2^{8-5} = 2^3$

Thus $\quad 2^8 \div 2^5 = 2^{8-5} = 2^3$.

Also $\quad m^7 \div m^3$

$$= \frac{\overset{1}{\cancel{m}} \times \overset{1}{\cancel{m}} \times \overset{1}{\cancel{m}} \times m \times m \times m \times m}{\underset{1}{\cancel{m}} \times \underset{1}{\cancel{m}} \times \underset{1}{\cancel{m}}} = m^4$$

And $\quad m^{7-3} = m^4$

Thus $\quad m^7 \div m^3 = m^{7-3} = m^4$.

Hence in *general*

$$a^m \div a^n = a^{m-n}.$$

That is, when we *divide quantities* with the *same base*, we *subtract* the *denominator index* from the *numerator index.*

Example 52

Express each of the following quotients in index form:

(a) $5^9 \div 5^8$ *(b)* $a^{12} \div a^5$

(c) $x^4y^3 \div xy^2$ *(d)* $p^5q^3r^4 \div p^3q^2r$

Solution

(a) Now $5^9 \div 5^8 = 5^{9-8} = 5^1 = 5$

(b) Now $a^{12} \div a^5 = a^{12-5} = a^7$

(c) Now $x^4y^3 \div xy^2 = x^{4-1}y^{3-2} = x^3y^1 = x^3y$

(d) Now $p^5q^3r^4 \div p^3q^2r = p^{5-3}q^{3-2}r^{4-1}$
$= p^2q^1r^3$
$= p^2qr^3$

Power to a Power

Now
$$\begin{aligned}(2^3)^4 &= (2^3) \times (2^3) \times (2^3) \times (2^3)\\ &= (2 \times 2 \times 2) \times (2 \times 2 \times 2)\\ &\quad \times (2 \times 2 \times 2) \times (2 \times 2 \times 2)\\ &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2\\ &\quad \times 2 \times 2 \times 2 \times 2\\ &= 2^{12}\end{aligned}$$

And $\quad 2^{3\times4} = 2^{12}$

Thus $\quad (2^3)^4 = 2^{3\times4} = 2^{12}$.

Also
$$\begin{aligned}(p^3)^2 &= (p^3) \times (p^3)\\ &= (p \times p \times p) \times (p \times p \times p)\\ &= p \times p \times p \times p \times p \times p\\ &= p^6\end{aligned}$$

And $\quad p^{3\times2} = p^6$

Thus $\quad (p^3)^2 = p^{3\times2} = p^6$.

Hence in *general*

$$(a^m)^n = a^{m\times n} = a^{mn}.$$

That is, when we *raise the power of a quantity* to a *power*, we *multiply* the *indices*.

Example 53

Simplify each of the following expressions leaving your answer in index form:

(a) $(5^2)^7$ *(b)* $(x^4)^3$

(c) $(x^3y^2)^4$ *(d)* $\left(\frac{5m^2}{3n^3}\right)^3$

Solution

(a) Now $(5^2)^7 = 5^{2\times7} = 5^{14}$

(b) Now $(x^4)^3 = x^{4\times3} = x^{12}$

(c) Now $(x^3y^2)^4 = x^{3\times4}y^{2\times4} = x^{12}y^8$

(d) Now $\left(\frac{5m^2}{3n^3}\right)^3 = \frac{5^{1\times3}m^{2\times3}}{3^{1\times3}n^{3\times3}} = \frac{5^3m^6}{3^3n^9} = \frac{125m^6}{27n^9}$

Zero Index

Now $\quad 5^2 \div 5^2 = \frac{\overset{1}{\cancel{5}} \times \overset{1}{\cancel{5}}}{\underset{1}{\cancel{5}} \times \underset{1}{\cancel{5}}} = 1$

And $\quad 5^2 \div 5^2 = 5^{2-2} = 5^0$

Thus $\quad 5^0 = 1$.

Also $\quad p^3 \div p^3 = \frac{\overset{1}{\cancel{p}} \times \overset{1}{\cancel{p}} \times \overset{1}{\cancel{p}}}{\underset{1}{\cancel{p}} \times \underset{1}{\cancel{p}} \times \underset{1}{\cancel{p}}} = 1$

And $\quad p^3 \div p^3 = p^{3-3} = p^0$

Thus $\quad p^0 = 1$.

Hence in *general*

$$a^0 = 1.$$

That is, *any quantity raised* to the *zero index* is *equal* to *one*.

Example 54

Determine the value of each of the following:

(a) 7^0 *(b)* 19^0

(c) x^0 *(d)* z^0

Solution

(a) Now $7^0 = 1$

(b) Now $19^0 = 1$

(c) Now $x^0 = 1$

(d) Now $z^0 = 1$

Negative Index

Now $5^3 \div 5^5 = \dfrac{\overset{1}{\cancel{5}} \times \overset{1}{\cancel{5}} \times \overset{1}{\cancel{5}}}{\underset{1}{\cancel{5}} \times \underset{1}{\cancel{5}} \times \underset{1}{\cancel{5}} \times 5 \times 5} = \dfrac{1}{5^2}$

And $5^3 \div 5^5 = 5^{3-5} = 5^{-2}$

Thus $5^{-2} = \dfrac{1}{5^2}$.

Also $q^4 \div q^7 = \dfrac{\overset{1}{\cancel{q}} \times \overset{1}{\cancel{q}} \times \overset{1}{\cancel{q}} \times \overset{1}{\cancel{q}}}{\underset{1}{\cancel{q}} \times \underset{1}{\cancel{q}} \times \underset{1}{\cancel{q}} \times \underset{1}{\cancel{q}} \times q \times q \times q}$

$= \dfrac{1}{q^3}$

And $q^4 \div q^7 = q^{4-7} = q^{-3}$

Thus $q^{-3} = \dfrac{1}{q^3}$.

Hence in *general*

$$a^{-m} = \frac{1}{a^m}.$$

That is, a *quantity* with a *negative index* is the *inverse* (or *reciprocal*) of the *quantity* with a *positive index* of the *same magnitude*.

Example 55

(a) Rewrite each of the following expressions using positive index only:

(i) 4^{-3} *(ii)* x^{-5}

(b) Rewrite each of the following expressions using negative index only:

(i) $\dfrac{1}{3^5}$ *(ii)* $\dfrac{1}{x^4}$

Solution

(a) (i) Now $4^{-3} = \dfrac{1}{4^3}$ (*positive index*)

(ii) Now $x^{-5} = \dfrac{1}{x^5}$ (*positive index*)

(b) (i) Now $\dfrac{1}{3^5} = 3^{-5}$ (*negative index*)

(ii) Now $\dfrac{1}{x^4} = x^{-4}$ (*negative index*)

Fractional (Rational) Index

Now $4^{\frac{1}{2}} = (2^2)^{\frac{1}{2}} = 2^{2\times\frac{1}{2}} = 2^1 = 2$

And $\sqrt{4} = \sqrt{2^2} = 2$

Thus $4^{\frac{1}{2}} = \sqrt{4}$

Hence $4^{\frac{1}{2}}$ is the *square root* of *4*.

Also $8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2^{3\times\frac{1}{3}} = 2^1 = 2$

And $\sqrt[3]{8} = \sqrt[3]{2^3} = 2$

Thus $8^{\frac{1}{3}} = \sqrt[3]{8}$

Hence $8^{\frac{1}{3}}$ is the *cube root* of 8.

Now $8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^{3\times\frac{2}{3}} = 2^2 = 4$

And $\sqrt[3]{8^2} = \sqrt[3]{64} = \sqrt[3]{4^3} = 4$

Thus $8^{\frac{2}{3}} = \sqrt[3]{8^2}$

Hence $8^{\frac{2}{3}}$ is the *cube root* of the *square* of 8.

Now $81^{\frac{3}{4}} = (3^4)^{\frac{3}{4}} = 3^{4\times\frac{3}{4}} = 3^3 = 27$

And $\sqrt[4]{81^3} = \sqrt[4]{(3^4)^3} = \sqrt[4]{3^{4\times 3}} = 3^3 = 27$

Thus $81^{\frac{3}{4}} = \sqrt[4]{81^3}$

Hence $81^{\frac{3}{4}}$ is the *fourth root* of the *cube* of 81.

NOTE: The *even root of a number* can be either *positive* or *negative*. For example:

$$\sqrt{25} = \pm 5 \text{ and } \sqrt[4]{81} = \pm 3.$$

However we are only taking the *positive root* in this chapter.

Hence in *general* $a^{\frac{1}{n}} = \sqrt[n]{a}$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

That is, in a *quantity* with a *fractional* (or *rational index*), the *denominator* is the *root* and the *numerator* is the *power* to which the *quantity* is to be *raised*.

Example 56

Determine the value of each of the following:

(a) $25^{\frac{1}{2}}$ *(b)* $125^{\frac{1}{3}}$

Solution

(a) Now $25^{\frac{1}{2}} = (5^2)^{\frac{1}{2}} = 5^{2\times\frac{1}{2}} = 5$

Alternatively $25^{\frac{1}{2}} = \sqrt{25} = \sqrt{5^2} = 5$

(b) Now $125^{\frac{1}{3}} = (5^3)^{\frac{1}{3}} = 5^{3\times\frac{1}{3}} = 5$

Alternatively $125^{\frac{1}{3}} = \sqrt[3]{125} = \sqrt[3]{5^3} = 5$

Example 57

Determine the value of each of the following:

(a) $32^{\frac{2}{5}}$ *(b)* $27^{\frac{2}{3}}$

Solution

(a) Now $32^{\frac{2}{5}} = (2^5)^{\frac{2}{5}} = 2^{5\times\frac{2}{5}} = 2^2 = 4$

Alternatively $32^{\frac{2}{5}} = \sqrt[5]{32^2} = \sqrt[5]{(2^5)^2}$

$= \sqrt[5]{2^{5\times2}} = 2^2 = 4$

(b) Now $27^{\frac{2}{3}} = (3^3)^{\frac{2}{3}} = 3^{3\times\frac{2}{3}} = 3^2 = 9$

Alternatively $27^{\frac{2}{3}} = \sqrt[3]{27^2} = \sqrt[3]{(3^3)^2}$

$= \sqrt[3]{3^{3\times2}} = 3^2 = 9$

Exercise 6s

Express each of the following products in index form:

1. $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

2. $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$

3. $5 \times 5 \times 5 \times 5 \times 5 \times 5$

4. $9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9$

5. $7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7$

6. $x \times x \times x \times x \times x$

7. $y \times y \times y \times y \times y \times y \times y$

8. $a \times a \times a \times \ldots$, to the p^{th} term

9. $m \times m \times m \times \ldots$, to the q^{th} term

10. $z \times z \times z \times \ldots$, to the n^{th} term.

Express each of the following products in index form:

11. $2^5 \times 2^3$

12. $3^4 \times 3^5$

13. $5^6 \times 5^3$

14. $7^3 \times 7^4$

15. $8^5 \times 8^4$

16. $x^2 \times x^4$

17. $y^3 \times y^5$

18. $z^4 \times z^5$

19. $p^7 \times p^3$

20. $q^5 \times q^8$

21. $x^2 \times y^3 \times x^5 \times y^2$

22. $x^4 \times y^2 \times y^3 \times x^7$

23. $p^3 \times q^2 \times p^8 \times q^5$

24. $p^5 \times q^3 \times p^2 \times q^6$

25. $a^3 \times b^2 \times b^3 \times a^4$

26. $p^3 \times q^2 \times r \times p^2 \times q \times r^3$

27. $p^2 \times q^3 \times r^4 \times q^2 \times p \times r$

28. $x^4 \times y^3 \times z^2 \times x^3 \times y^2 \times z$

29. $x^2 \times y^3 \times z^4 \times z^3 \times y^2 \times x^7$

30. $a^5 \times b^2 \times c^3 \times a \times c^2 \times b^3$

Express each of the following quotients in index form:

31. $6^7 \div 6^4$

32. $7^5 \div 7^2$

33. $8^9 \div 8^7$

34. $9^{12} \div 9^7$

35. $10^{13} \div 10^8$

36. $a^9 \div a^5$

37. $x^7 \div x^3$

38. $m^8 \div m^5$

39. $p^{12} \div p^9$

40. $q^{15} \div q^{13}$

41. $x^5y^2 \div x^3y$

42. $x^7y^3 \div x^4y$

43. $p^3q^4 \div pq^3$

44. $r^5s^4 \div r^2s^3$

45. $m^6n^5 \div m^3n^2$

46. $p^7q^5r^3 \div p^4q^2r$

47. $p^8q^7r^4 \div p^5q^2r^3$

48. $x^7y^5z^3 \div x^5y^3z$

49. $l^3m^5n^2 \div lm^3n$

50. $a^5b^4c^3 \div a^2b^3c$

Simplify each of the following expressions leaving each answer in index form:

51. $(5^3)^4$

52. $(6^5)^2$

53. $(8^4)^5$

54. $(7^5)^3$

55. $(9^2)^8$

56. $(x^3)^2$

57. $(p^4)^5$

58. $(m^5)^5$

59. $(n^7)^2$

60. $(r^8)^3$

61. $(x^2y^3)^3$

62. $(x^3y^2)^5$

63. $(p^4q^3)^4$

64. $(p^5q^2)^6$

65. $(a^3b^5)^7$

66. $\left(\dfrac{2m^3}{3n^2}\right)^3$

67. $\left(\dfrac{3x^5}{5y^3}\right)^2$

68. $\left(\dfrac{4p^7}{9q^5}\right)^3$

69. $\left(\dfrac{2r^5}{5s^3}\right)^4$

70. $\left(\dfrac{5l^4}{7m^3}\right)^2$

Determine the value of each of the following:

71. 9^0 **72.** 15^0 **73.** 127^0 **74.** p^0 **75.** s^0

Write each of the following expressions using positive indices only:

76. 5^{-4} **77.** 7^{-5} **78.** x^{-3} **79.** y^{-7} **80.** a^{-9}

Write each of the following expressions using negative indices only:

81. $\frac{1}{8^3}$ **82.** $\frac{1}{10^5}$ **83.** $\frac{1}{x^4}$ **84.** $\frac{1}{a^7}$ **85.** $\frac{1}{m^9}$

Determine the value of each of the following:

86. $1^{\frac{1}{2}}$ **87.** $16^{\frac{1}{2}}$ **88.** $49^{\frac{1}{2}}$ **89.** $81^{\frac{1}{2}}$

90. $144^{\frac{1}{2}}$ **91.** $1^{\frac{1}{3}}$ **92.** $27^{\frac{1}{3}}$ **93.** $64^{\frac{1}{3}}$

94. $216^{\frac{1}{3}}$ **95.** $343^{\frac{1}{3}}$ **96.** $64^{\frac{2}{3}}$ **97.** $81^{\frac{3}{2}}$

98. $125^{\frac{2}{3}}$ **99.** $8^{\frac{4}{3}}$ **100.** $100^{\frac{3}{2}}$ **101.** $8^{\frac{5}{3}}$

102. $32^{\frac{3}{5}}$ **103.** $81^{\frac{5}{4}}$ **104.** $16^{\frac{3}{4}}$ **105.** $512^{\frac{2}{3}}$

Laws of Indices

The *laws of indices* are summarised below:

(1) $a^m \times a^n = a^{m+n}$

(2) $a^m \div a^n = a^{m-n}$

(3) $(a^m)^n = a^{mn}$

(4) $a^0 = 1$

(5) $a^{-m} = \frac{1}{a^m}$

(6) $a^{\frac{1}{n}} = \sqrt[n]{a}$

(7) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

where $a \neq 0$.

Example 58

Simplify each of the following expressions:

(a) $2x^3 \times 3x^2$ *(b)* $3x^2 \times 4xy$

(c) $7a^5 \div 3a^2$ *(d)* $(5x^2y^3)^4$

(e) $\sqrt{4x^6}$

Solution

(a) Now $2x^3 \times 3x^2 = 6x^{3+2} = 6x^5$

(b) Now $3x^2 \times 4xy = 12x^{2+1}y = 12x^3y$

(c) Now $7a^5 \div 3a^2 = \frac{7a^5}{3a^2} = \frac{7}{3}a^{5-2} = \frac{7}{3}a^3$

(d) Now
$$\begin{aligned}(5x^2y^3)^4 &= 5^{1\times4}x^{2\times4}y^{3\times4}\\ &= 5^4x^8y^{12}\\ &= 625x^8y^{12}\end{aligned}$$

(e) Now $\sqrt{4x^6} = \sqrt{2^2x^6} = 2x^3$

Example 59

Simplify each of the following expressions:

(a) $(x^{\frac{1}{6}})^3 \times \sqrt{x^5}$ *(b)* $3x^{-3}y^{\frac{1}{3}} \times 2x^2y^{\frac{2}{3}}$

(c) $(16x^4)^{-\frac{1}{2}}$ *(d)* $3a^{-\frac{1}{2}}(a^{\frac{3}{2}} - a^{-\frac{1}{4}})$

Solution

(a) Now
$$\begin{aligned}(x^{\frac{1}{6}})^3 \times \sqrt{x^5} &= x^{\frac{1}{6}\times3} \times x^{\frac{5}{2}}\\ &= x^{\frac{1}{2}} \times x^{\frac{5}{2}}\\ &= x^{\frac{1}{2}+\frac{5}{2}}\\ &= x^{\frac{1+5}{2}}\\ &= x^{\frac{6}{2}}\\ &= x^3\end{aligned}$$

(b) Now
$$\begin{aligned}&3x^{-3}y^{\frac{1}{3}} \times 2x^2y^{\frac{2}{3}}\\ &= 6x^{-3+2}y^{\frac{1}{3}+\frac{2}{3}}\\ &= 6x^{-1}y^1\\ &= 6x^{-1}y\\ &= 6\frac{y}{x}\end{aligned}$$
(if need to write using *positive indices* only)

(c) Now
$$\begin{aligned}(16x^4)^{-\frac{1}{2}} &= 16^{1\times(-\frac{1}{2})}x^{4\times(-\frac{1}{2})}\\ &= 16^{-\frac{1}{2}}x^{-2}\\ &= (4^2)^{-\frac{1}{2}}x^{-2}\\ &= 4^{2\times(-\frac{1}{2})}x^{-2}\\ &= 4^{-1}x^{-2}\\ &= \frac{1}{4x^2}\end{aligned}$$
(if need to write using *positive indices* only)

Alternatively $(16x^4)^{-\frac{1}{2}} = \dfrac{1}{(16x^4)^{\frac{1}{2}}}$

$$= \frac{1}{16^{1\times\frac{1}{2}}x^{4\times\frac{1}{2}}}$$

$$= \frac{1}{16^{\frac{1}{2}}x^2}$$

$$= \frac{1}{(4^2)^{\frac{1}{2}}x^2}$$

$$= \frac{1}{4^{2\times\frac{1}{2}}x^2}$$

$$= \frac{1}{4x^2}$$

(d) Now $3a^{-\frac{1}{2}}(a^{\frac{3}{2}} - a^{-\frac{1}{4}})$

$= 3a^{-\frac{1}{2}} \times a^{\frac{3}{2}} + 3a^{-\frac{1}{2}} \times (-a^{-\frac{1}{4}})$ (using the *distributive law*)

$= 3a^{-\frac{1}{2}+\frac{3}{2}} - 3a^{-\frac{1}{2}+(-\frac{1}{4})}$

$= 3a^{\frac{-1+3}{2}} - 3a^{\frac{-2-1}{4}}$

$= 3a^{\frac{2}{2}} - 3a^{\frac{-3}{4}}$

$= 3a^1 - 3a^{-\frac{3}{4}}$

$= 3a - \dfrac{3}{a^{\frac{3}{4}}}$ (if need to write using *positive indices* only)

Example 60

Determine the value of each of the following expressions:

(a) $3^4 \times 3^{-5} \times 3^2$

(b) $\dfrac{2^5 \times 2^3 \times 2^4}{2^7 \times 2^3}$

(c) $81^{\frac{1}{2}} \times 27^{\frac{1}{3}} \times 16^{\frac{1}{4}}$

(d) $49^{\frac{1}{2}} \times 27^{-\frac{1}{3}}$

(e) $\sqrt[3]{8} \times \sqrt[5]{32}$

Solution

(a) Now $3^4 \times 3^{-5} \times 3^2 = 3^{4+(-5)+2} = 3^{6-5} = 3^1$
$= 3$

(b) Now $\dfrac{2^5 \times 2^3 \times 2^4}{2^7 \times 2^3} = \dfrac{2^{5+3+4}}{2^{7+3}} = \dfrac{2^{12}}{2^{10}} = 2^{12-10}$
$= 2^2 = 4$

(c) Now $81^{\frac{1}{2}} \times 27^{\frac{1}{3}} \times 16^{\frac{1}{4}} = (9^2)^{\frac{1}{2}} \times (3^3)^{\frac{1}{3}} \times (2^4)^{\frac{1}{4}}$

$= 9^{2\times\frac{1}{2}} \times 3^{3\times\frac{1}{3}} \times 2^{4\times\frac{1}{4}}$

$= 9^1 \times 3^1 \times 2^1$

$= 9 \times 3 \times 2$

$= 54$

(d) Now $49^{\frac{1}{2}} \times 27^{-\frac{1}{3}} = (7^2)^{\frac{1}{2}} \times (3^3)^{-\frac{1}{3}}$

$= 7^{2\times\frac{1}{2}} \times 3^{3\times(-\frac{1}{3})}$

$= 7^1 \times 3^{-1}$

$= \dfrac{7}{3}$

Alternatively $49^{\frac{1}{2}} \times 27^{-\frac{1}{3}} = \dfrac{49^{\frac{1}{2}}}{27^{\frac{1}{3}}}$

$$= \frac{(7^2)^{\frac{1}{2}}}{(3^3)^{\frac{1}{3}}}$$

$$= \frac{7^{2\times\frac{1}{2}}}{3^{3\times\frac{1}{3}}}$$

$$= \frac{7^1}{3^1}$$

$$= \frac{7}{3}$$

(e) Now $\sqrt[3]{8} \times \sqrt[5]{32} = \sqrt[3]{2^3} \times \sqrt[5]{2^5}$

$= 2 \times 2$

$= 4$

Example 61

Determine the value of each of the following expressions:

(a) 2^{-3}

(b) $(3^2)^{-3}$

(c) $27^{-\frac{1}{3}}$

(d) $125^{-\frac{2}{3}}$

(e) $\left(\dfrac{81}{16}\right)^{-\frac{3}{4}}$

Solution

(a) Now $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

(b) Now $(3^2)^{-3} = \dfrac{1}{(3^2)^3} = \dfrac{1}{3^{2\times 3}} = \dfrac{1}{3^6} = \dfrac{1}{729}$

(c) Now $27^{-\frac{1}{3}} = \dfrac{1}{(3^3)^{\frac{1}{3}}} = \dfrac{1}{3^{3\times\frac{1}{3}}} = \dfrac{1}{3^1} = \dfrac{1}{3}$

(d) Now $125^{-\frac{2}{3}} = \dfrac{1}{125^{\frac{2}{3}}} = \dfrac{1}{(5^3)^{\frac{2}{3}}} = \dfrac{1}{5^{3\times\frac{2}{3}}} = \dfrac{1}{5^2}$

$= \dfrac{1}{25}$

(e) Now $\left(\dfrac{81}{16}\right)^{-\frac{3}{4}} = \dfrac{81^{-\frac{3}{4}}}{16^{-\frac{3}{4}}} = \dfrac{16^{\frac{3}{4}}}{81^{\frac{3}{4}}}$

$= \dfrac{(2^4)^{\frac{3}{4}}}{(3^4)^{\frac{3}{4}}} = \dfrac{2^{4\times\frac{3}{4}}}{3^{4\times\frac{3}{4}}} = \dfrac{2^3}{3^3} = \dfrac{8}{27}$

Alternatively $\left(\frac{81}{16}\right)^{-\frac{3}{4}} = \frac{1}{\left(\frac{81}{16}\right)^{\frac{3}{4}}}$

$$= \frac{1}{\left(\frac{3^4}{2^4}\right)^{\frac{3}{4}}}$$

$$= \frac{1}{\frac{3^{4\times\frac{3}{4}}}{2^{4\times\frac{3}{4}}}}$$

$$= \frac{1}{\frac{3^3}{2^3}}$$

$$= \frac{1}{\frac{27}{8}}$$

$$= \frac{8}{27}$$

Example 62

Determine the value of y^{12} *when* $y = 7^{-\frac{1}{6}}$.

Solution

Given that $y = 7^{-\frac{1}{6}}$

Then $y^{12} = (7^{-\frac{1}{6}})^{12}$

$$= 7^{-\frac{1}{6}\times 12}$$

$$= 7^{-2}$$

$$= \frac{1}{7^2}$$

$$= \frac{1}{49}$$

Exercise 6t

Simplify each of the following expressions:

1. $2x^2 \times 3x^3$
2. $3x^3 \times 4x^5$
3. $4x^5 \times 3x^2$
4. $5x^4 \times 2x^3$
5. $3x^4 \times 5x^5$
6. $2x^3 \times 3xy^2$
7. $3x^2 \times 4x^2y$
8. $5x^4 \times 3xy^3$
9. $4x^3 \times 3x^2y^4$
10. $7x \times 4x^4y^3$
11. $2r^5 \times 3r^2 \times 4r^3$
12. $3a^4 \times a \times 5a^3$
13. $3x^2y \times 2xy^3 \times 4xy$
14. $2x^3y \times 4x^2y \times xy^2$
15. $5x^2y \times 3xy \times 4x^2y$
16. $5x^7 \div 2x^3$
17. $\frac{3a^5 \times 4a^2}{2a^3}$
18. $\frac{6a^5 \times 5a^3}{3a^6}$
19. $5a^4 \div (a^2 \times 3a)$
20. $\frac{14a^5}{2a^3 \times 7a^4}$
21. $(5x^2y)^3$
22. $(3x^3y^2)^4$
23. $(4x^2y^3)^3$
24. $\left(\frac{4a^2b^3}{3c}\right)^2$
25. $\left(\frac{3x^3y^2}{5z}\right)^3$
26. $\sqrt{9x^4}$
27. $\sqrt{16y^6}$
28. $\sqrt[3]{125x^3y^6}$
29. $\sqrt[4]{81x^8y^4}$
30. $\sqrt[5]{32x^5y^{15}}$

Simplify each of the following expressions:

31. $(x^{\frac{1}{2}})^7 \times \sqrt{x^9}$
32. $(x^{\frac{1}{3}})^5 \times \sqrt[3]{x^4}$
33. $(x^{\frac{1}{4}})^9 \times \sqrt[4]{x^7}$
34. $(x^{\frac{1}{5}})^9 \times \sqrt[5]{x^6}$
35. $(x^{\frac{1}{6}})^9 \times \sqrt[6]{x^9}$
36. $3x^{-4}y^{\frac{2}{3}} \times 5x^5y^{\frac{1}{3}}$
37. $5x^{-3}y^{\frac{3}{5}} \times 3x^5y^{\frac{2}{5}}$
38. $4x^3y^{\frac{3}{4}} \times 3x^{-2}y^{\frac{1}{4}}$
39. $4x^{\frac{1}{3}}y^{-3} \times 2x^{-\frac{4}{3}}y^5$
40. $7x^{\frac{2}{5}}y^{-5} \times 3x^{-\frac{7}{5}}y^3$
41. $(3p^4)^{-3}$
42. $(4x^5)^{-2}$
43. $(125x^6)^{-\frac{1}{3}}$
44. $(16x^4)^{-\frac{1}{2}}$
45. $(27x^{12})^{-\frac{1}{3}}$
46. $5a(a^{-1} - a)$
47. $7a^{-2}(a^2 + \frac{a}{2})$
48. $5a^{-\frac{1}{2}}(a^{\frac{9}{2}} - a^{-\frac{5}{2}})$
49. $8a^{-\frac{1}{2}}(a^{\frac{9}{2}} - a^{\frac{5}{2}})$
50. $7a^{-\frac{1}{3}}(a^{\frac{4}{3}} - a^{-\frac{5}{3}})$

Determine the value of each of the following expressions:

51. $2^5 \times 2^{-3}$
52. $3^4 \times 3^{-5}$
53. $2^3 \times 2^{-5} \times 2^4$
54. $5^4 \times 5^{-7} \times 5$
55. $8^5 \times 8^{-6} \times 8^3$
56. $\frac{2^3 \times 2^4}{2^2}$
57. $\frac{5^4 \times 5^3}{5^6}$
58. $\frac{6^5 \times 6^3 \times 6^2}{6^7 \times 6^4}$
59. $\frac{8^3 \times 8^2 \times 8^5}{8^7 \times 8^4}$
60. $\frac{9^4 \times 9^5}{9^2 \times 9 \times 9^4}$
61. $81^{\frac{1}{2}} \times 27^{\frac{1}{3}}$
62. $81^{\frac{1}{4}} \times 49^{\frac{1}{2}}$
63. $16^{\frac{1}{4}} \times 25^{\frac{1}{2}} \times 27^{\frac{1}{3}}$
64. $125^{\frac{1}{3}} \times 16^{\frac{1}{2}} \times 81^{\frac{1}{4}}$
65. $64^{\frac{1}{2}} \times 64^{\frac{1}{3}} \times 64^{\frac{1}{6}}$
66. $64^{\frac{1}{2}} \times 27^{-\frac{1}{3}}$
67. $25^{\frac{1}{2}} \times 64^{-\frac{1}{3}}$
68. $81^{-\frac{1}{2}} \times 125^{\frac{1}{3}}$
69. $32^{-\frac{1}{5}} \times 27^{\frac{1}{3}}$
70. $625^{-\frac{1}{2}} \times 49^{\frac{1}{2}}$
71. $\sqrt[3]{125} \times \sqrt[6]{64}$
72. $\sqrt{49} \times \sqrt[3]{125}$

73. $\sqrt[3]{27} \times \sqrt[5]{32}$

74. $\sqrt[4]{81} \times \sqrt[3]{64}$

75. $\sqrt[3]{125} \times \sqrt[3]{343}$

Determine the value of each of the following expressions:

76. 7^{-2}

77. 2^{-5}

78. 3^{-4}

79. 5^{-3}

80. 8^{-3}

81. $(2^3)^{-2}$

82. $(4^3)^{-2}$

83. $(5^2)^{-1}$

84. $(6^1)^{-3}$

85. $(3^2)^{-2}$

86. $4^{-\frac{1}{2}}$

87. $64^{-\frac{1}{3}}$

88. $81^{-\frac{1}{4}}$

89. $243^{-\frac{1}{5}}$

90. $64^{-\frac{1}{6}}$

91. $27^{-\frac{2}{3}}$

92. $25^{-\frac{3}{2}}$

93. $64^{-\frac{5}{6}}$

94. $81^{-\frac{3}{4}}$

95. $343^{-\frac{2}{3}}$

96. $\left(\frac{49}{25}\right)^{-\frac{1}{2}}$

97. $\left(\frac{27}{8}\right)^{-\frac{2}{3}}$

98. $\left(\frac{125}{64}\right)^{-\frac{2}{3}}$

99. $\left(\frac{16}{81}\right)^{-\frac{3}{4}}$

100. $\left(\frac{256}{625}\right)^{-\frac{3}{4}}$

101. Determine the value of x^6 when $x = 5^{-\frac{1}{3}}$.

102. Calculate the value of x^{10} when $x = 8^{-\frac{1}{5}}$.

103. Determine the value of y^{12} when $y = 10^{-\frac{1}{6}}$.

104. Calculate the value of y^{14} when $y = 13^{-\frac{1}{7}}$.

105. Calculate the value of y^4 when $y = 12^{-\frac{3}{4}}$.

Solution of an Equation Where the Unknown Quantity is in the Index

If $a^m = a^n$,
then $m = n$ since the *bases are equal.*

We use this fact to *solve equations* where the *unknown quantity* is in the *index* and their *bases* are equal or their bases can be *equalised.*

Example 63

Solve the equations:

(a) $64^x = 16$

(b) $625^{p-2} = 5^{1+p}$

(c) $(5^x)(25^{2x+1}) = 625$

(d) $9^{2x} = \frac{1}{81}$

Solution

(a) Given that $64^x = 16$

Then $(4^3)^x = 4^2$

So $4^{3x} = 4^2$

Thus $3x = 2$

$\therefore$ $x = \frac{2}{3}$

Hence x is $\frac{2}{3}$.

(b) Given that $625^{p-2} = 5^{1+p}$

Then $(5^4)^{p-2} = 5^{1+p}$

So $5^{4(p-2)} = 5^{1+p}$

Thus $4(p - 2) = 1 + p$

Using the *distributive law*, we get

$4p - 8 = 1 + p$

Then $4p - p = 1 + 8$

So $3p = 9$

i.e. $p = \frac{9}{3} = 3$

Hence p is 3.

(c) Given that $(5^x)(25^{2x+1}) = 625$

Then $(5^x)[(5^2)^{2x+1}] = 5^4$

So $(5^x)[5^{2(2x+1)}] = 5^4$

i.e. $(5^x)(5^{4x+2}) = 5^4$

$\therefore$ $5^{x+4x+2} = 5^4$

$\Rightarrow$ $5^{5x+2} = 5^4$

Thus $5x + 2 = 4$

So $5x = 4 - 2 = 2$

i.e. $x = \frac{2}{5} = 0.4$

Hence x is 0.4.

(d) Given that $9^{2x} = \frac{1}{81}$

Then $(3^2)^{2x} = \frac{1}{3^4}$

So $3^{4x} = 3^{-4}$

Thus $4x = -4$

$\therefore$ $x = \frac{-4}{4} = -1$

Hence x is -1.

Alternatively $9^{2x} = \frac{1}{81}$

So $9^{2x} = \frac{1}{9^2}$

i.e. $9^{2x} = 9^{-2}$

Thus $2x = -2$

$\therefore$ $x = \frac{-2}{2} = -1$

Exercise 6u

Solve the following equations:

1. $2^{3x} = 128$ **2.** $3^{2x} = 243$ **3.** $5^{4x} = 625$

4. $7^{3x} = 2401$ **5.** $2^{5x} = 1024$

6. If $64^{3p-2} = 4^{2p+5}$, evaluate p.

7. If $27^{q+2} = 3^{5-q}$, calculate the value of q.

8. Given that $243^{3-p} = 3^{p-3}$, evaluate the magnitude of p.

9. Given that $2401^{r-5} = 7^{3(r-2)}$, calculate and state the value of r.

10. Solve $729^{s-5} = 3^{2(1-s)}$ for s.

11. Solve the equation $6 \times 6^{5x} = 36 \times 6^{x-7}$.

12. Determine the value of x for which $(3^{2x})(9^{x-1}) = 27$.

13. Solve $(4^{x})(8^{2x+1}) = 64$ for x.

14. Given that $(5^{2x})(25^{3x-2}) = 625$, determine the value of x.

15. If $a^{3p+5} = a^{p-2}$, evaluate p.

16. Solve $(m^{2x})(m^{x-1}) = m^{2x+5}$ for x.

Solve:

17. $8^{2x} = \dfrac{1}{64}$ **18.** $8^{3x} = \dfrac{1}{256}$

19. $9^{3x} = \dfrac{1}{243}$ **20.** $25^{2x} = \dfrac{1}{125}$

21. $49^{3x} = \dfrac{1}{343}$

Standard Form (or Scientific Notation)

A *number* is said to be written in *standard form* (or *scientific notation*) when it is in the form $A \times 10^{n}$, where $1 \leqslant A < 10$ and n is an *integer*, that is $n \in Z$.

Example 64

Express each of the following numbers in standard form correct to 3 significant figures:

(a) 1964

(b) 768 500

(c) 0.063 47

(d) 0.000 4376

Solution

(a) Now
$$\begin{aligned} 1964 &= 1.964 \times 1000 \\ &= 1.964 \times 10^{3} \\ &= 1.96 \times 10^{3} \text{ (in \textit{standard form correct} to 3 \textit{s.f.})} \end{aligned}$$

(b) Now
$$\begin{aligned} 768\,500 &= 7.685 \times 100\,000 \\ &= 7.685 \times 10^{5} \\ &= 7.69 \times 10^{5} \text{ (in \textit{standard form correct} to 3 \textit{s.f.})} \end{aligned}$$

(c) Now
$$\begin{aligned} 0.063\,47 = \frac{6.347}{100} &= 6.347 \times 10^{-2} \\ &= 6.35 \times 10^{-2} \text{ (in \textit{standard form correct} to 3 \textit{s.f.})} \end{aligned}$$

(d) Now
$$\begin{aligned} 0.000\,437\,6 = \frac{4.376}{10\,000} &= 4.376 \times 10^{-4} \\ &= 4.38 \times 10^{-4} \text{ (in \textit{standard form correct} to 3 \textit{s.f.})} \end{aligned}$$

From the above *examples* it can be *seen* that the *number* between 1 *and* 10, that is *A*, is corrected to the given number of *significant figures*.

Example 65

(a) Express $\sqrt{25\,600}$ in standard form.

(b) Evaluate $(1.44 \times 10^{4})^{\frac{1}{2}}$, giving your answer in standard form.

(c) Calculate $\sqrt{0.0009 \times 10^{-6}}$, stating your answer in standard form.

(d) Evaluate $\sqrt{\dfrac{0.09}{225}}$, giving your answer in standard form.

Solution

(a) Now $\sqrt{25\,600} = \sqrt{256 \times 100}$

$$\begin{aligned} &= \sqrt{16^2 \times 10^2} \\ &= 16 \times 10 \\ &= 1.6 \times 10 \times 10 \\ &= 1.6 \times 10^{1+1} \\ &= 1.6 \times 10^2 \text{ (in } \textit{standard form}\text{)} \end{aligned}$$

(b) Now $(1.44 \times 10^4)^{\frac{1}{2}} = (1.2^2 \times 10^4)^{\frac{1}{2}}$

$$\begin{aligned} &= 1.2^{2\times\frac{1}{2}} \times 10^{4\times\frac{1}{2}} \\ &= 1.2^1 \times 10^2 \\ &= 1.2 \times 10^2 \text{ (in } \textit{standard form}\text{)} \end{aligned}$$

(c) Now $\sqrt{0.0009 \times 10^{-6}} = \sqrt{9 \times 10^{-4} \times 10^{-6}}$

$$\begin{aligned} &= \sqrt{9 \times 10^{-4+(-6)}} \\ &= \sqrt{9 \times 10^{-4-6}} \\ &= \sqrt{3^2 \times 10^{-10}} \\ &= 3 \times 10^{-5} \\ &\text{(in } \textit{standard form}\text{)} \end{aligned}$$

(d) Now $\sqrt{\dfrac{0.09}{225}} = \sqrt{\dfrac{0.3^2}{15^2}}$

$$\begin{aligned} &= \frac{0.3}{15} \\ &= 0.02 \\ &= 2.0 \times 10^{-2} \text{ (in } \textit{standard form}\text{)} \end{aligned}$$

From the above *examples* it can be *seen* that, a *number* whose *square root* is to be found, must be *written* as a *square* (*times* a power of 10, whose index is ± a multiple *of* 2, where necessary) before its *square root* can be *found.*

Example 66

(a) Express $\sqrt[3]{27\,000}$ in standard form.

(b) Evaluate $(2.7 \times 10^7)^{\frac{1}{3}}$, giving your answer in standard form.

(c) Calculate $\sqrt[3]{0.0008 \times 10^{-5}}$, stating your answer in standard form.

(d) Evaluate $\sqrt[3]{\dfrac{0.008}{125}}$, giving your answer in standard form.

▼ **Solution**

(a) Now $\sqrt[3]{27\,000} = \sqrt[3]{27 \times 10^3}$

$$\begin{aligned} &= \sqrt[3]{3^3 \times 10^3} \\ &= 3 \times 10^1 \text{ (in } \textit{standard form}\text{)} \end{aligned}$$

(b) Now $(2.7 \times 10^7)^{\frac{1}{3}} = \sqrt[3]{27 \times 10^{-1} \times 10^7}$

$$\begin{aligned} &= \sqrt[3]{27 \times 10^{-1+7}} \\ &= \sqrt[3]{3^3 \times 10^6} \\ &= 3 \times 10^2 \\ &\text{(in } \textit{standard form}\text{)} \end{aligned}$$

(c) Now $\sqrt[3]{0.0008 \times 10^{-5}} = \sqrt[3]{8 \times 10^{-4} \times 10^{-5}}$

$$\begin{aligned} &= \sqrt[3]{8 \times 10^{-4+(-5)}} \\ &= \sqrt[3]{8 \times 10^{-4-5}} \\ &= \sqrt[3]{2^3 \times 10^{-9}} \\ &= 2 \times 10^{-3} \\ &\text{(in } \textit{standard form}\text{)} \end{aligned}$$

(d) Now $\sqrt[3]{\dfrac{0.008}{125}} = \sqrt[3]{\dfrac{8 \times 10^{-3}}{125}}$

$$\begin{aligned} &= \sqrt[3]{\frac{2^3 \times 10^{-3}}{5^3}} \\ &= \frac{2 \times 10^{-1}}{5} \\ &= 0.4 \times 10^{-1} \\ &= 4.0 \times 10^{-1} \times 10^{-1} \\ &= 4.0 \times 10^{-2} \\ &\text{(in } \textit{standard form}\text{)} \end{aligned}$$

From the above *examples* it can be *seen* that, any *number* whose *cube root* is to be found, must be *written* as a *cube* (*times* a power of 10, whose index is ± a multiple of 3, where necessary) before its *cube root* can be *found.*

Example 67

Calculate the exact value of each of the following expressions, giving each answer in standard form:

(a) $3.5 \times 10^4 + 2.1 \times 10^3$

(b) $3.4 \times 10^3 - 5.1 \times 10^2$

(c) $(1.3 \times 10^3)^2$

(d) $\left(\dfrac{1.5 \times 10^3}{0.3}\right)^2$

▼ **Solution**

(a) Now

$$\begin{aligned} & 3.5 \times 10^4 + 2.1 \times 10^3 \\ &= 3.5 \times 10^4 + 0.21 \times 10^4 \\ &= (3.5 + 0.21) \times 10^4 \\ &= 3.71 \times 10^4 \end{aligned}$$

(in *standard form*)

(b) Now

$$\begin{aligned} & 3.4 \times 10^3 - 5.1 \times 10^2 \\ &= 3.4 \times 10^3 - 0.51 \times 10^3 \\ &= (3.4 - 0.51) \times 10^3 \\ &= 2.89 \times 10^3 \end{aligned}$$

(in *standard form*)

(c) Now

$$\begin{aligned} & (1.3 \times 10^3)^2 \\ &= 1.3^{1\times2} \times 10^{3\times2} \\ &= 1.3^2 \times 10^6 \\ &= 1.69 \times 10^6 \end{aligned}$$

(in *standard form*)

(d) Now

$$\begin{aligned} \left(\frac{1.5 \times 10^3}{0.3}\right)^2 &= \left(\frac{15 \times 10^3}{3}\right)^2 \\ &= (5 \times 10^3)^2 \\ &= 5^{1\times2} \times 10^{3\times2} \\ &= 5^2 \times 10^6 \\ &= 25 \times 10^6 \\ &= 2.5 \times 10 \times 10^6 \\ &= 2.5 \times 10^{1+6} \\ &= 2.5 \times 10^7 \end{aligned}$$

(in *standard form*)

Alternatively

$$\begin{aligned} \left(\frac{1.5 \times 10^3}{0.3}\right)^2 &= \frac{1.5^{1\times2} \times 10^{3\times2}}{0.3^{1\times2}} \\ &= \frac{1.5^2 \times 10^6}{0.3^2} \\ &= \frac{2.25 \times 10^6}{0.09} \\ &= \frac{225 \times 10^6}{9} \\ &= 25 \times 10^6 \\ &= 2.5 \times 10 \times 10^6 \\ &= 2.5 \times 10^{1+6} \\ &= 2.5 \times 10^7 \end{aligned}$$

(in *standard form*)

From the above *examples* it can be *seen* that, when we are *adding* or *subtracting numbers* written in *standard form*, then we must *convert each number* to the *highest power* of 10 *given*, (so that they have the same power of 10) before we can proceed to *add* or *subtract* them.

Exercise 6V

Express each of the following numbers in standard form correct to 3 significant figures:

1. 147
2. 253
3. 768
4. 8 250
5. 9 485
6. 75 360
7. 124 000
8. 847 300
9. 9 457 000
10. 76 800 000
11. 0.043 12
12. 0.007 834
13. 0.004 853
14. 0.000 761 2
15. 0.000 487 1
16. 0.000 032 46
17. 0.000 018 74
18. 0.000 003 123
19. 0.000 004 897
20. 0.000 000 184 8

Express each of the following numbers in standard form:

21. $\sqrt{12\,100}$
22. $\sqrt{14\,400}$
23. $\sqrt{22\,500}$
24. $\sqrt{28\,900}$
25. $\sqrt{36\,100}$

Evaluate each of the following expressions, giving each answer in standard form:

26. $(3.24 \times 10^4)^{\frac{1}{2}}$
27. $(5.29 \times 10^6)^{\frac{1}{2}}$
28. $(6.25 \times 10^8)^{\frac{1}{2}}$
29. $(28.9 \times 10^5)^{\frac{1}{2}}$
30. $(57.6 \times 10^7)^{\frac{1}{2}}$

Simplify each of the following expressions, stating each answers in standard form:

31. $\sqrt{0.0004 \times 10^{-6}}$
32. $\sqrt{0.0025 \times 10^{-8}}$
33. $\sqrt{0.0144 \times 10^{-6}}$
34. $\sqrt{0.0225 \times 10^{-8}}$
35. $\sqrt{0.0441 \times 10^{-12}}$

Evaluate each of the following expressions, giving each answer in standard form:

36. $\sqrt{\frac{0.04}{64}}$
37. $\sqrt{\frac{0.09}{144}}$
38. $\sqrt{\frac{0.81}{225}}$
39. $\sqrt{\frac{0.0169}{676}}$
40. $\sqrt{\frac{0.0196}{256}}$

Express each of the following expressions in standard form:

41. $\sqrt[3]{64\,000}$
42. $\sqrt[3]{125\,000}$
43. $\sqrt[3]{216\,000}$
44. $\sqrt[3]{8\,000\,000}$
45. $\sqrt[3]{27\,000\,000}$

Evaluate each of the following expressions, giving each answers in standard form:

46. $(6.4 \times 10^7)^{\frac{1}{3}}$ **47.** $(1.25 \times 10^8)^{\frac{1}{3}}$

48. $(3.43 \times 10^8)^{\frac{1}{3}}$ **49.** $(5.12 \times 10^8)^{\frac{1}{3}}$

50. $(7.29 \times 10^8)^{\frac{1}{3}}$

Express each of the following numbers, in standard form:

51. $\sqrt[3]{0.0027 \times 10^{-5}}$ **52.** $\sqrt[3]{0.0125 \times 10^{-5}}$

53. $\sqrt[3]{0.0216 \times 10^{-5}}$ **54.** $\sqrt[3]{0.00008 \times 10^{-7}}$

55. $\sqrt[3]{0.00064 \times 10^{-7}}$

Evaluate each of the following expressions, giving each answers in standard form:

56. $\sqrt[3]{\frac{0.027}{64}}$ **57.** $\sqrt[3]{\frac{0.064}{125}}$

58. $\sqrt[3]{\frac{0.125}{512}}$ **59.** $\sqrt[3]{\frac{216}{0.000125}}$

60. $\sqrt[3]{\frac{729}{0.000064}}$

Calculate the exact value of each of the following, expressing each answers in standard form:

61. $4.5 \times 10^4 + 3.2 \times 10^3$

62. $6.1 \times 10^5 + 4.7 \times 10^4$

63. $5.3 \times 10^6 + 8.2 \times 10^5$

64. $7.8 \times 10^7 + 5.9 \times 10^6$

65. $8.1 \times 10^8 + 9.4 \times 10^7$

Calculate the exact value of each of the following, giving each answers in standard form:

66. $4.7 \times 10^3 - 8.3 \times 10^2$

67. $5.8 \times 10^4 - 9.5 \times 10^3$

68. $6.5 \times 10^5 - 5.3 \times 10^4$

69. $7.8 \times 10^6 - 6.5 \times 10^5$

70. $9.4 \times 10^7 - 8.1 \times 10^6$

Calculate the exact value of each of the following, stating each answer in standard form:

71. $(1.2 \times 10^3)^2$ **72.** $(1.5 \times 10^4)^2$

73. $(1.7 \times 10^5)^2$ **74.** $(1.9 \times 10^6)^2$

75. $(2.0 \times 10^7)^2$

Calculate the exact value of the following, leaving each answers in standard form:

76. $\left(\frac{1.8 \times 10^3}{0.3}\right)^2$ **77.** $\left(\frac{2.0 \times 10^4}{0.4}\right)^2$

78. $\left(\frac{2.25 \times 10^5}{0.15}\right)^2$ **79.** $\left(\frac{2.1 \times 10^6}{0.03}\right)^2$

80. $\left(\frac{8.1 \times 10^7}{0.27}\right)^2$

Logarithm

Before the advent of calculators, *logarithms* were very useful since they *converted multiplication* and *division* problems into *addition* and *subtraction* problems, respectively. This saved much time and tedious numerical calculations, since it is easier to add or subtract than to multiply or divide.

The *logarithm of a number* to a given *base* is defined as the *power* to which the *base* must be *raised* in order to give that *number*.

Thus $\quad number = base^{logarithm}$

At this level we use *common logarithms*, that is, *logarithms to base* 10. The *logarithm of a number to base* 10 is abbreviated as log_{10} or *lg*. The *logarithm* of a *negative number does not exist*, that is, it has *no meaning*. The *logarithm* of a *positive number* to *base* 10 is its *power of* 10. Thus:

If $\quad y = 10^x$
then $\quad log_{10}y = lg\, y = x$

Since $\quad 1 = 10^0$
then $\quad log_{10}1 = 0$

Since $\quad 10 = 10^1$
then $\quad log_{10}10 = 1$

Since $\quad 100 = 10^2$
then $\quad log_{10}100 = 2$

Since $\quad 1000 = 10^3$
then $\quad log_{10}1000 = 3$ et cetera.

The *logarithm of any number A* between 1 *and* 10, that is, $1 \leqslant A < 10$, can be found *directly* from the *logarithm table*.

Example 68

Find the logarithm of each of the following numbers:
(a) 3.65 *(b) 475* *(c) 0.0478*

Solution

(a) Now $log_{10}3.65 = 0.562$ (from the *logarithm table*)

(b) Now
$$\begin{aligned} 475 &= 4.75 \times 10^2 \\ &= 10^? \times 10^2 \\ &= 10^{0.677} \times 10^2 \text{ (from the } \textit{logarithm table}\text{)} \\ &= 10^{2+0.677} \\ &= 10^{2.677} \end{aligned}$$

So $log_{10}475 = 2.677$

Since $log_{10}4.75 = 0.677$ (from the *logarithm table*)

(c) Now
$$\begin{aligned} 0.0478 &= 4.78 \times 10^{-2} \\ &= 10^? \times 10^{-2} \\ &= 10^{0.679} \times 10^{-2} \text{ (from the } \textit{logarithm table}\text{)} \\ &= 10^{-2+0.679} \\ &= 10^{\bar{2}.679} \end{aligned}$$

So $log_{10}0.0478 = \bar{2}.679$

Since $log_{10}4.78 = 0.679$ (from the *logarithm table*)

Note that we wrote the *negative power of* 10, -2, as $\bar{2}$, called *bar* 2.
And $\bar{2}.679 = -2 + 0.679 = -1.321$

In finding the *logarithm of a number* that is *greater than* 10 or less than 1, we first write it in *standard form* (or *scientific notation*), that is, as $A \times 10^n$.

In this method, n is the whole *number part* called the *characteristic*.

We *find* the value of $log_{10}A$ from the *logarithm table*, which is the *decimal part* called the *mantissa*.

Thus:

Characteristic — $\enclose{circle}{2}.\boxed{677}$ — Mantissa

Characteristic — $\enclose{circle}{\bar{2}}.\boxed{679}$ — Mantissa.

The *table* below is very helpful in understanding how to find the *logarithm of a number*.

Table 6.1

Number	*Standard form*	*Logarithm*
945 000	9.45×10^5	5.975
94 500	9.45×10^4	4.975
9450	9.45×10^3	3.975
945	9.45×10^2	2.975
94.5	9.45×10^1	1.975
9.45	9.45×10^0	0.975
0.945	9.45×10^{-1}	$\bar{1}.975$
0.0945	9.45×10^{-2}	$\bar{2}.975$
0.009 45	9.45×10^{-3}	$\bar{3}.975$
0.000 945	9.45×10^{-4}	$\bar{4}.975$

Note that $log_{10}A = log_{10}9.45 = 0.975$, from the *logarithm table.*

Antilogarithm

The *antilogarithm of a number* is the *converse* of its *logarithmic value.* It is used specifically for *calculation* purposes when *base* 10 *logarithms* are employed. The *antilogarithm* of a *number to base* 10 is abbreviated as *antilog*$_{10}$ (or *antilg*).

If $y = 10^x$
then $\log_{10}y = x$
and $antilog_{10}x = y$

Since $log_{10}1 = 0$
then $antilog_{10}0 = 1$

Since $log_{10}10 = 1$
then $antilog_{10}1 = 10$

Since $log_{10}100 = 2$
then $antilog_{10}2 = 100$

Since $log_{10}1\,000 = 3$
then $antilog_{10}3 = 1\,000$ et cetera.

When we have a *logarithmic value* and we want to *find* the *original number*, then we use the *antilogarithm table*. We find the *antilogarithm* of the *mantissa* — this gives us a *number* A, such that $1 \leqslant A < 10$. And the *characteristic* is the *power of* 10. This *operation* is the *reverse of finding* the *logarithm of a number*.

Example 69

Find the antilogarithm of each of the following numbers:

(a) 0.362 *(b) 2.843* *(c) $\bar{3}.587$*

Solution

(a) Now $antilog_{10}0.362$
$= 2.30$ (from the *antilogarithm table*)

(b) Now $antilog_{10}2.843$
$= (\text{antilog}_{10}0.843) \times 10^2$
$= 6.97 \times 100$ (from the *antilogarithm table*)
$= 697$

(c) Now $antilog_{10}\bar{3}.587$
$= (\text{antilog}_{10}0.587) \times 10^{-3}$
$= 3.86 \times \frac{1}{1000}$ (from the *antilogarithm table*)
$= 0.003\,86$

Table 6.2 indicates the principles involved in *finding a number* using the *antilogarithm.*

Table 6.2

Logarithm	*Number*
5.738	$5.47 \times 10^5 = 547\,000$
4.738	$5.47 \times 10^4 = 54\,700$
3.738	$5.47 \times 10^3 = 5470$
2.738	$5.47 \times 10^2 = 547$
1.738	$5.47 \times 10^1 = 54.7$
0.738	$5.47 \times 10^0 = 5.47$
$\bar{1}.738$	$5.47 \times 10^{-1} = 0.547$
$\bar{2}.738$	$5.47 \times 10^{-2} = 0.0547$
$\bar{3}.738$	$5.47 \times 10^{-3} = 0.005\,47$
$\bar{4}.738$	$5.47 \times 10^{-4} = 0.000\,547$

Note that $A = antilog_{10}0.738 = 5.47$, from the *antilogarithm table.*

Exercise 6w

Find the logarithm of each of the following numbers:

1. 1.23
2. 4.71
3. 34.5
4. 57.8
5. 479
6. 536
7. 6850
8. 9530
9. 12 500
10. 84 300
11. 347 000
12. 731 000
13. 0.347
14. 0.832
15. 0.0768
16. 0.0934
17. 0.007 42
18. 0.008 31
19. 0.000 387
20. 0.000 815

Evaluate the antilogarithm of each of the following numbers:

21. 0.345
22. 0.981
23. 1.347
24. 1.863
25. 2.714
26. 2.915
27. 3.471
28. 3.862
29. 4.639
30. 4.937
31. 5.832
32. 5.647
33. $\bar{1}.378$
34. $\bar{1}.483$
35. $\bar{2}.234$
36. $\bar{2}.418$
37. $\bar{3}.674$
38. $\bar{3}.936$
39. $\bar{4}.317$
40. $\bar{4}.615$

Logarithm Theory

As was stated earlier—*logarithms* allow us to *add or subtract* instead of *multiplying or dividing*. The *rules* governing the *logarithm theory* are *defined* below.

Multiplication

If $k = xy$
then $log_{10}k = log_{10}xy = log_{10}x + log_{10}y.$

That is, when we take the *logarithm of a product*, we *add* the *logarithmic values.*

Note that $log_{10}x + log_{10}y = log_{10}xy.$

This *formula* tells us how to *add logarithms.*

For example:

$$\log_{10}2 + \log_{10}50 = \log_{10}(2 \times 50)$$
$$= \log_{10} 100$$
$$= \log_{10} 10^2$$
$$= 2$$

Example 70

Use logarithms to find the value of each of the following products:

(a) 21.7×0.0385

(b) $54.3 \times 0.00871 \times 134$

Solution

(a) Now $\quad log_{10}(21.7 \times 0.0385)$

$= \log 21.7 + \log 0.0385$

$= 1.336 + \bar{2}.585$

$= \bar{1}.921$ ← $\boxed{\begin{aligned} 1 + \bar{2} &= 1 + (-2) \\ &= 1 - 2 \\ &= -1 \end{aligned}}$

So $\quad 21.7 \times 0.0385$

$= \text{antilog}_{10}\bar{1}.921$

$= 8.34 \times 10^{-1}$

$= 0.834$

It might be more *convenient* to use a *table* as shown below in order to *solve* the *problem* given.

Table 6.3

Number	*Logarithm*
× 21.7 0.0385	+ 1.336 $\bar{2}$.585
$8.34 \times 10^{-1} = 0.834$	$\bar{1}$.921

$\boxed{\begin{aligned} 1 + \bar{2} &= 1 + (-2) \\ &= 1 - 2 \\ &= -1 \end{aligned}}$

(b) Now $\quad log_{10}(54.3 \times 0.00871 \times 134)$

$= \log_{10}54.3 + \log_{10}0.00871$
$+ \log_{10}134$

$= 1.735 + 3.940 + 2.127$

$= 1.802$

So $\quad 54.3 \times 0.00871 \times 134$

$= \text{antilog}_{10}1.802$

$= 6.34 \times 10^1$

$= 63.4$

Alternatively, we can use a *table* as shown below.

Table 6.4

Number	*Logarithm*
54.3 × 0.00871 × 134	1.735 + $\bar{3}$.940 + 2.127
$6.34 \times 10^1 = 63.4$	1.802

Division

If $\quad k = \dfrac{x}{y}$

then $\quad log_{10}k = log_{10}\dfrac{x}{y} = log_{10}x - log_{10}y.$

That is, when we take the *logarithm of a quotient*, we *subtract* the *logarithm* of the *denominator* from the *logarithm* of the *numerator.*

Note that $\quad log_{10}x - log_{10}y = log\dfrac{x}{y}.$

This *formula* tells us how to *subtract logarithms.*

For example: $\quad \log_{10}90 - \log_{10}9 = \log_{10}\dfrac{90}{9}$

$= \log_{10}10$

$= 1$

Example 71

Use logarithms to find the value of each of the following quotients:

(a) $\dfrac{23.5}{0.0458}$ $\qquad$ *(b)* $\dfrac{27.5 \times 0.518}{0.00763 \times 147}$

Solution

(a) Now $\quad log_{10}\left(\dfrac{23.5}{0.0458}\right)$

$= \log_{10}23.5 - \log_{10}0.0458$

$= 1.371 - \bar{2}.661$

$= 2.710$ ← $\boxed{-\bar{2} = -(-2) = 2}$

So $\quad \left(\dfrac{23.5}{0.0458}\right)$

$= \text{antilog}_{10}2.710$

$= 5.13 \times 10^2$

$= 513$

Alternatively, we can use a *table* as shown below:

Table 6.5

Number	*Logarithm*
23.5 ÷ 0.0458	1.371 − $\bar{2}$.661
$5.13 \times 10^2 = 513$	2.710

$\boxed{-\bar{2} = -(-2) = 2}$

(b) Now $log_{10}\left(\dfrac{27.5 \times 0.518}{0.007\,63 \times 147}\right)$

$$= \log_{10}(27.5 \times 0.518) - \log_{10}(0.007\,63 \times 147)$$
$$= (1.439 + \bar{1}.714) - (\bar{3}.883 + 2.167)$$
$$= 1.153 - 0.050 \longleftarrow \boxed{3 + \bar{3} = 3 - 3 = 0}$$
$$= 1.103$$

So $\dfrac{27.5 \times 0.518}{0.007\,63 \times 147} = \text{antilog}_{10}1.103$

$$= 1.27 \times 10^1$$
$$= 12.7$$

Alternatively, we can use a *table* as shown below:

Table 6.6

Number	*Logarithm*
× 27.5 0.518	+ 1.439 $\bar{1}$.714
Numerator	1.153
× 0.007 63 147	+ $\bar{3}$.383 2.167
Denominator	0.050
÷ Numerator Denominator	− 1.153 0.050
1.27 × 10¹ = 12.7	1.103

Power

If $k = x^m$

then $log_{10}k = log_{10}x^m = mlog_{10}x.$

That is, when we take the *logarithm of a power*, we *multiply* the *logarithm* of its *base* by the *power*.

Example 72

Use logarithms to find the value of each of the following expressions:

(a) $(0.527)^3$ *(b) $(3.41)^4$*

Solution

(a) Now $log_{10}(0.527)^3$

$$= 3\log_{10}0.527$$
$$= 3 \times \bar{1}.722$$
$$= \bar{1}.166 \longleftarrow \boxed{\begin{array}{l} 3 + \bar{1} = 3 \times (-1) = -3 \\ \hline -3 + 2 = -1 = \bar{1} \end{array}}$$

So $(0.527)^3$

$$= \text{antilog}_{10}\bar{1}.166$$
$$= 1.47 \times 10^{-1}$$
$$= 0.147$$

Alternatively, we can use a *table* as shown below:

Table 6.7

Number	*Operation*	*Logarithm*
$(0.527)^3$	3 × $\bar{1}$.722	$\bar{1}$.166
$1.47 \times 10^{-1} = 0.147$		$\bar{1}$.166

(b) Now $log_{10}(3.41)^4 = 4\log_{10}3.41$

$$= 4 \times 0.533$$
$$= 2.132$$

So $(3.41)^4 = \text{antilog}_{10}2.132$

$$= 1.36 \times 10^2$$
$$= 136$$

Alternatively, we can use a *table* as shown below:

Table 6.8

Number	*Operation*	*Logarithm*
$(3.41)^4$	4 × 0.533	2.132
$1.36 \times 10^2 = 136$		2.132

Root

If $k = x^{\frac{1}{n}} = \sqrt[n]{x}$

then $log_{10}k = log_{10}x^{\frac{1}{n}} = \dfrac{1}{n}log_{10}x = \dfrac{log_{10}x}{n}.$

That is, when we take the *logarithm of a root*, we *multiply* the *logarithm* of its *base* by the *reciprocal of the root*. Or we *divide* the *logarithm* of its *base* by the *root*.

We should also be able to *see* that:

If $k = x^{\frac{m}{n}}$ *(fractional index)*

then $log_{10}k = log_{10}x^{\frac{m}{n}} = \dfrac{m}{n}log_{10}x.$

This *rule* actually *combines* the *rules* for the *logarithm of a power* and the *logarithm of a root*.

Example 73

Use logarithms to find the value of $\sqrt[3]{15.7}$

Solution

Now $\log_{10}\sqrt[3]{15.7} = \dfrac{\log_{10}15.7}{3} \doteq \dfrac{1.196}{3} = 0.399$

So $\quad \sqrt[3]{15.7} = \text{antilog}_{10}0.399 = 2.51$

Alternatively, we can use a *table* as shown below.

Table 6.9

Number	*Operation*	*Logarithm*
$\sqrt[3]{15.7}$	$\frac{1}{3}(1.196) = \frac{1.196}{3}$	0.399
$2.51 \times 10^0 = 2.51$		0.399

Exercise 6x

Use logarithms to find the value of each of the following expressions:

1. 25.3×8.41 **2.** 34.8×12.7

3. 39.6×0.435 **4.** 84.7×0.00365

5. $47.3 \times 0.0478 \times 125$

6. $74.8 \times 0.000341 \times 247$

7. $127 \times 0.00876 \times 25.9$

8. $348 \times 0.000125 \times 37.6$

9. $24.7 \div 0.0341$ **10.** $47.8 \div 0.00374$

11. $125 \div 0.0475$ **12.** $248 \div 0.00124$

13. $\dfrac{29.4 \times 0.914}{0.0765 \times 12.6}$ **14.** $\dfrac{47.5 \times 0.0348}{0.00465 \times 23.4}$

15. $\dfrac{37.6 \times 250}{2.73 \times 45.1}$ **16.** $\dfrac{54.8 \times 345}{64.1 \times 12.4}$

Evaluate each of the following expressions using logarithms:

17. $(0.632)^3$ **18.** $(0.471)^4$

19. $(0.843)^5$ **20.** $(0.372)^6$

21. $(3.21)^3$ **22.** $(4.73)^4$

23. $(1.34)^5$ **24.** $(2.74)^6$

25. $\sqrt{0.736}$ **26.** $\sqrt{0.563}$

27. $\sqrt{147}$ **28.** $\sqrt[3]{0.00138}$

29. $\sqrt[3]{0.00475}$ **30.** $\sqrt[3]{17.4}$

31. $\sqrt[3]{0.00847}$ **32.** $\sqrt[3]{25.7}$

33. $\sqrt[4]{148}$ **34.** $\sqrt[4]{528}$

Calculate using logarithms:

35. $\dfrac{335 \times (27.1)^3}{1430}$ **36.** $\dfrac{144 \times (34.7)^4}{2880}$

Solution of an Equation Using Logarithms

In *equations* where the *unknown quantity* is in the *index* and the *bases cannot readily be equalised*, then we *write* the *equation* in *logarithmic form* and *solve* for the *unknown quantity*.

Example 74

Solve each of the following equations:

(a) $2^x = 7$ *(b)* $5 \times 3^{2x+1} = 37$

Solution

(a) Given that $\quad 2^x = 7$

Then taking *logs*, we get

$$lg2^x = lg7$$

So $\quad x\,lg2 = lg7$

i.e. $\quad x \times 0.301 = 0.845$

$\therefore \quad x = \dfrac{0.845}{0.301} = 2.81$ (*correct* to 3 *s.f.*)

Hence x is 2.81.

(b) Given that $\quad 5 \times 3^{2x+1} = 37$

Then taking logs, we get

$$lg\,5 \times 3^{2x+1} = lg37$$

So $\quad lg5 + (2x+1)lg3 = lg37$

i.e. $0.699 + (2x+1) \times 0.447 = 1.568$

$\therefore \quad (2x+1) \times 0.477 = 1.568 - 0.699 = 0.869$

And $\quad 2x + 1 = \dfrac{0.869}{0.477} = 1.82$

So $\qquad 2x = 1.82 - 1 = 0.82$

$\therefore \qquad x = \dfrac{0.82}{2} = 0.41$

Hence x is 0.41.

Example 75

Solve each of the following equations:

(a) $lg\,16.2 + y = lg\,64.8$

(b) $lg\,x^2 - lg\,100 = 1$

Solution

(a) Given that $\quad lg\,16.2 + y = lg\,64.8$

Then $\quad y = lg\,64.8 - lg\,16.2$

$= lg\left(\dfrac{64.8}{16.2}\right)$

$= lg\,4$

So $\quad y = 0.602$

Alternatively $\quad lg\,16.2 + y = lg\,64.8$

So $\quad 1.210 + y = 1.812$

i.e. $\quad y = 1.812 - 1.210$

$\therefore \quad y = 0.602$

Hence y is 0.602

(b) Given that $\quad lg\,x^2 - lg\,100 = 1$

Then $\quad lg\,x^2 - lg\,10^2 = 1$

So $\quad 2\,lg\,x - 2lg\,10 = 1$

i.e. $\quad 2\,lg\,x - 2 = 1$

$\therefore \quad 2\,lg\,x = 1 + 2 = 3$

And $\quad lg\,x = \dfrac{3}{2} = 1.5$

$\Rightarrow \quad x = \text{antilg } 1.5$

$= 3.16 \times 10^1$

$\therefore \quad x = 31.6$

Hence x is 31.6

Exercise 6y

Solve each of the following equations:

1. $3^x = 19$

2. $5^{2x} = 13$

3. $7^{3x} = 28$

4. $9^{2x+1} = 18$

5. $10^{3x+1} = 29$

6. $2 \times 3x = 55$

7. $3 \times 5^{2x} = 138$

8. $5 \times 3^{2x+1} = 148$

9. $7 \times 4^{3x-1} = 153$

10. $8 \times 7^{5x-2} = 175$

Solve each of the following equations:

11. $lg\,15.3 + y = lg\,91.8$

12. $lg\,18.5 + 2y = lg\,74$

13. $lg\,24.7 + 3y = lg\,49.4$

14. $lg\,35.2 - y = lg\,52.8$

15. $lg\,43.5 - 2y = lg\,87$

16. $lg\,x^2 - lg\,1000 = 1$

17. $lg\,x^2 - lg\,10000 = 1$

18. $lg\,x^2 + lg\,1000 = 5$

19. $lg\,x^2 + lg\,10000 = 7$

20. $lg\,x^3 - lg\,10000 = 8$

C.X.C. Past Paper Questions

The following supplementary questions were taken from C.X.C. Past Papers.

Exercise 6z

1. Two buckets were bought at a price of x dollars each, and a third bucket for 3 dollars less than twice the price of one of the first two buckets.

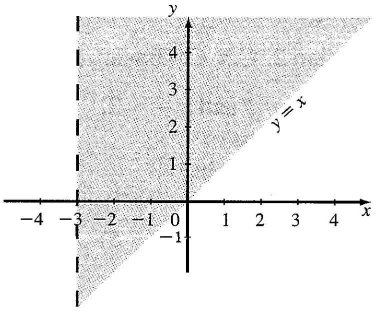
(a) Write down an expression for the total cost of the three buckets.

(b) If the total cost of the buckets was less than \$34 write an inequality in x and solve it.

(c) If x is a whole number, state the maximum cost of each of the buckets.

Question 5. C.X.C. (Basic). June 1979.

2.



(a) Write the inequalities which define the shaded region in the diagram above.

(b) Sketch a diagram and shade the region for which $y > x$ and $1 \leqslant y \leqslant 4$.

Name 3 points, whose coordinates are integers, which lie in this region.

Question 6. C.X.C. (Basic). June 1979.

3. Find the range of values of m for which

$$4(m + 2) \geqslant 25 - 3(m + 1)$$

Question 7(i). C.X.C. (Basic). June 1982.

4. On Monday, Allan bought 3 ice-cream cones and 2 buns for \$2.99. On Tuesday he bought 4 ice-cream cones and 1 bun for \$2.72. The price was the same for each item on both days.

Using c cents to represent the price of an ice-cream cone and b cents to represent the price of a bun,

(a) write down two equations in b and c to represent Allan's purchases on Monday and Tuesday, and

(b) use these equations to calculate the cost of an ice-cream cone and the cost of a bun.

Question 7. C.X.C. (Basic). June 1983.

5. A man had \$100. He went to a meatshop, a bookshop and a drugstore. He spent three times as much money at the meatshop as he did at the drugstore. He spent \$12 less at the bookstore than at the drugstore. He then had \$37 left.

(a) Using \$$x$ to represent the amount he spent at the drugstore, express in algebraic terms
  (i) the amount he spent at the meatshop
  (ii) the amount he spent at the bookstore.

(b) Obtain an equation for the total amount of money spent and hence calculate the amount he spent at the drugstore.

Question 4. C.X.C. (Basic). June 1985.

6. Solve the equation

$$\frac{3x - 1}{3} - \frac{2x + 5}{5} = x$$

Question 2. C.X.C. (Basic). June 1986.

7. (a) Given that $m = 3$ and $n = -2$, calculate the value of

$$2m^2 - 3n^3$$

(b) Simplify

$$2(x + 3y) + 3x - (y + 5)$$

(c) Express as a simple fraction

$$\frac{5x - 3}{9} - \frac{2x + 1}{4}$$

Question 2. C.X.C. (Basic). June 1990.

8. A patron can pay to see a show either by paying \$8 for a ticket in advance or by paying \$10 at the door. Total receipts for the show amounted to \$9 072. This consisted of cash from 324 tickets paid for in advance as well as cash collected at the door. Expenses for the show totalled \$3 850. A government tax of 12% is payable on the gross profits obtained.

Calculate

(a) the number of persons who paid at the door

(b) the amount paid for government tax

(c) the net profit after the tax was paid.

Question 4. C.X.C. (Basic). June 1990.

9. The cost of a table and four chairs is \$292. The cost of two tables and five chairs is \$482.

Using x to represent the cost, in dollars, of a table and y to represent the cost of a chair,

(i) write two algebraic equations to represent the information above

(ii) solve the equations and hence, determine the cost of a table and the cost of a chair.

Question 4(b). C.X.C. (Basic). June 1992.

10. (a) Simplify:

$$4c^2 \times 3c^3.$$

(b) If $a * b = a - 2b$, evaluate $5 * 2$.

(c) Factorize completely $6x + 9x^2$.

(d) Simplify:
  (i) $4x - 2(x - 4)$
  (ii) $\frac{a}{2} + \frac{a - 1}{3}$.

Question 2. C.X.C. (Basic). June 1993.

11. A racket costs \$12.00 more than a bat. The cost of two rackets and three bats is \$619.

Using x to represent the cost, in dollars, of a bat,

(i) write an algebraic expression for the cost of a racket

(ii) write am algebraic equation to represent the total cost of the two rackets and three bats

(iii) solve the equation and hence, determine the cost of a racket.

Question 9(a). C.X.C. (Basic). June 1993.

CHAPTER 7

Relations, Functions and Graphs I

This chapter will teach you

- about the Cartesian plane and the scales used
- to plot simple linear graphs
- about inequalities and how to represent an inequality on a number line and on a graph
- to define relations and functions
- about direct variation and inverse variation
- to draw graphs for linear and quadratic functions
- about straight, parallel and perpendicular lines.

The Cartesian Plane

The *Cartesian plane* contains two straight lines *intersecting* at right angles.

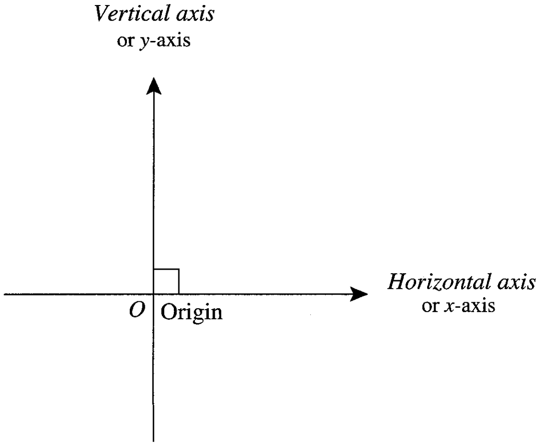


Fig. 7.1 The Cartesian plane

The horizontal line is called the *horizontal axis* or usually, the *x-axis*. The vertical line is called the *vertical axis* or usually, the *y-axis*.

The *point of intersection* of these *two axes* is called the *origin* and is denoted by *O* (0, 0).

The *Cartesian plane* is used to plot *points* and hence draw graphs, using a system of *rectangular coordinates*. In this system of rectangular coordinates, the *origin O* is taken as the *point of reference*. The *x-coordinate* (or *abscissa*) is *positive* to the right of the origin, and *negative* to the left of the origin. The *y-coordinate* (or *ordinate*) is *positive* above the origin, and *negative* below the origin. Further, each axis is like a *number line*.

Each *point P* can be uniquely defined by stating a *horizontal coordinate* and a *vertical coordinate*, or usually, an *x-coordinate* and a *y-coordinate*. We say that the *point P* is *P* (*horizontal coordinate*, *vertical coordinate*), or *P* (*x-coordinate*, *y-coordinate*). That is, the *coordinates of P* are (*horizontal coordinate, vertical coordinate*) or (*x-coordinate, y-coordinate*).

It is also worth noting that, the *set of x-coordinates* is also called the *domain* or the *object set*. And the *set of y-coordinates* is also called the *co-domain*, the *range* or the *image set*.

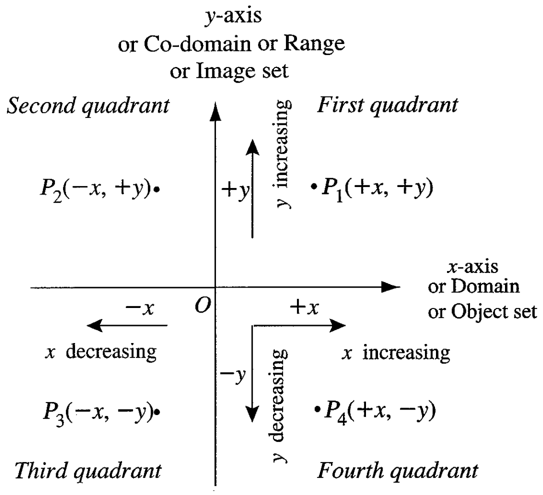


Fig. 7.2 The Cartesian plane

The four unique *points*, in the four different *quadrants in* the diagram above, indicate how we *plot* points in the rectangular *Cartesian plane*.

Scale

In *plotting points* on the *Cartesian plane*, suitable *scales* must be chosen or given to be used. The *same scale* can be used for *both axes*. Or *one scale* can be used for the *horizontal axis*, and *another scale* used for the *vertical axis*. However, *two different scales* cannot be used for the *same axis*; say, the horizontal axis. *Graph paper* is normally used to *plot points accurately*. And the *position of a point* is usually indicated on a graph by ⊙ or ×.

Example 1

Using a scale of 1 cm to represent 1 unit on each axis, plot the following points:

A(3, 5), B(−3, 3), C(−2, −3) and D(2, −2).

Solution

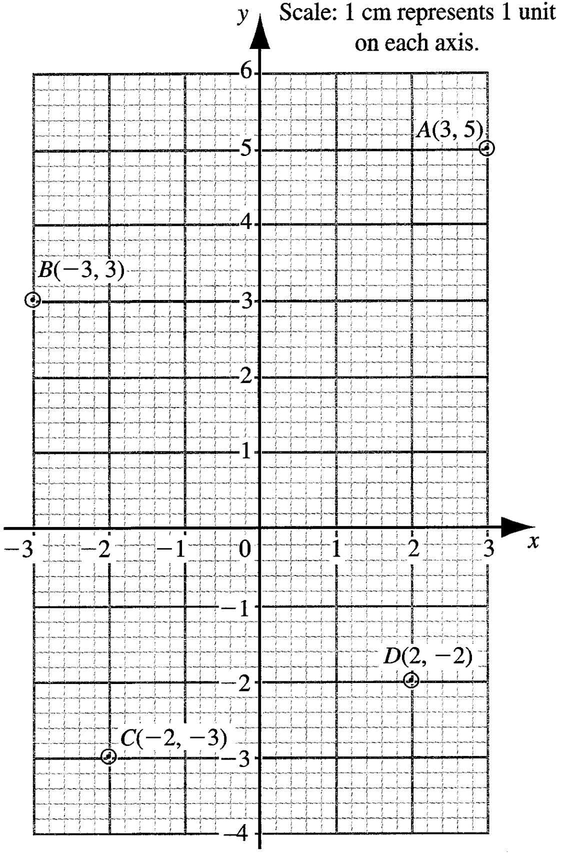


Fig. 7.3 The Cartesian plane

Above can be *seen* the *graph* with the given *points plotted*, using the given *scale*.

Example 2

Using a scale of 1 cm to represent 1 unit on the x-axis, and 2 cm to represent 1 unit on the y-axis, plot the following points:

P(2, 2), Q(−3, 1), R(−2, −1.5) and S(1, −2.5).

Solution

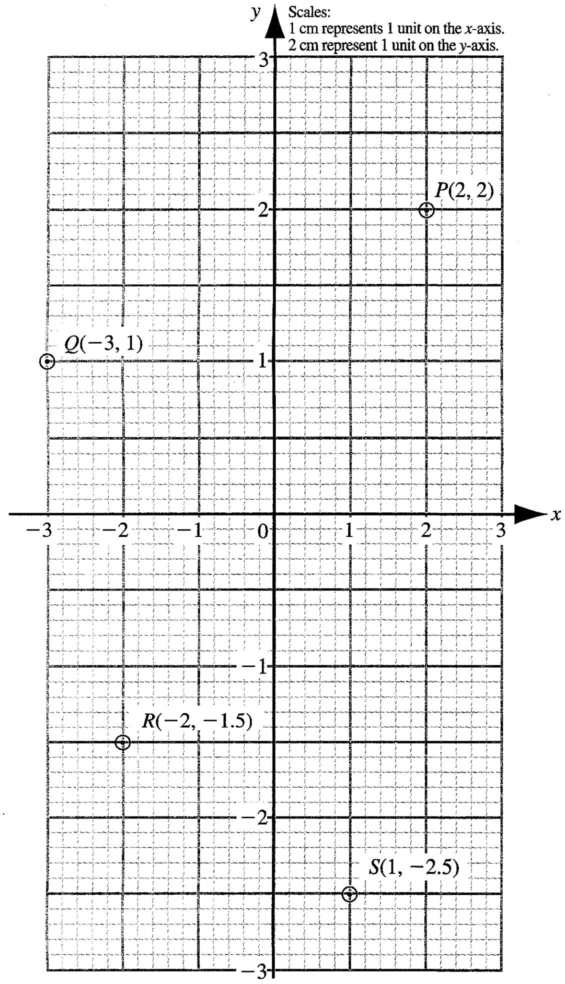


Fig. 7.4 The Cartesian plane

Above can be *seen* the *graph* with the given *points plotted*, using the given *scales*.

Drawing a Diagram

Once *points* have been *plotted* on graph paper, they can be *joined* in a given direction in order to form a *specific shape*.

Example 3

Plot the points K(−2, −3), L(−2, 2), M(3, 2) and N(3, −3) on graph paper, using a scale of 1 cm to represent 1 unit on both axes.

- *(a) Join the points in alphabetical order, then complete the plane figure.*
- *(b) What type of quadrilateral is KLMN?*
- *(c) Draw the diagonals KM and LN to intersect at C. State the coordinates of C.*
- *(d) Measure and state the lengths of the diagonals KM and LN.*
- *(e) Measure and state the magnitude of the angles at the point of intersection of the diagonals C.*
- *(f) Measure and state the length of each of the four sides of the quadrilateral KLMN.*
- *(g) Measure and state the length for the altitude h of the quadrilateral KLMN.*
- *(h) Hence, calculate the*
  - *(i) perimeter of the quadrilateral KLMN*
  - *(ii) area of the quadrilateral KLMN.*

Solution

(a)

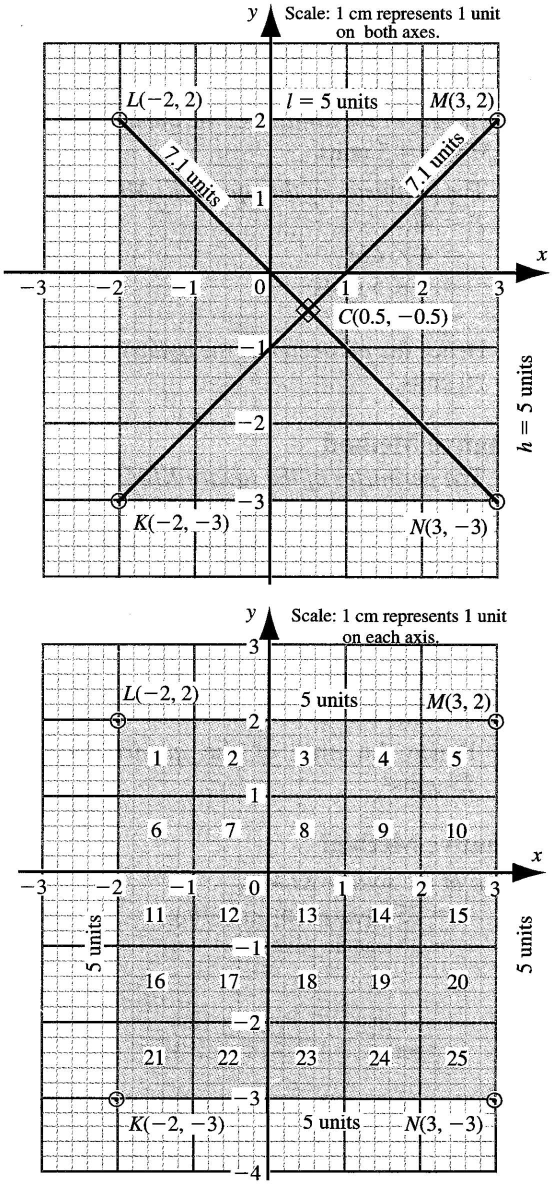


Fig. 7.5 Square

(b) The *quadrilateral KLMN* is a *square*.

(c) The *coordinates of C* are (0.5, −0.5).

(d) The *length of the diagonal* $KM = 7.1$ units.
The *length of the diagonal* $LN = 7.1$ units.

(e) The *magnitude of the angles* at the *point of intersection* of the *diagonals* $C = 90°$.
i.e. $K\hat{C}L = L\hat{C}M = M\hat{C}N = K\hat{C}N = 90°$.
Hence the *angle* at the *point of intersection* of the *diagonals* is 90°.

(f) The *length* of each of the four sides of the *quadrilateral* $KLMN = 5$ units.
i.e. $KL = NM = LM = KN = 5$ units.

Note that $LM = KN = 3 - (-2) = 3 + 2$
$= 5$ units.

And $KL = NM = 2 - (-3) = 2 + 3 = 5$ units.
Hence the *length* of each of the four sides of the *quadrilateral* is 5 units.

(g) The *length of the altitude* of the *quadrilateral KLMN*, $h = 5$ units.

(h) (i) The *perimeter of the square KLMN*.

$$\begin{aligned} P &= 4l \\ &= 4 \times LM \\ &= 4 \times 5 \text{ units} \\ &= 20 \text{ units.} \end{aligned}$$

Hence the *perimeter* of the *quadrilateral* is 20 units.

Alternative Method

(i) The *perimeter of the square KLMN*,

$$\begin{aligned} P &= KL + LM + NM + KN \\ &= (5 + 5 + 5 + 5) \text{ units} = 20 \text{ units.} \end{aligned}$$

(ii) The *area of the square KLMN*,

$$\begin{aligned} A &= l^2 \\ &= LM^2 \\ &= (5 \text{ units})^2 \\ &= 25 \text{ units}^2 \end{aligned}$$

Hence the *area* of the *quadrilateral* is 25 units^2.

Alternative Method

The *area of the square KLMN*,

$A = 25$ squares (by *counting*)
$= 25$ square units.

Exercise 7a

1. Plot the graph of the following points on graph paper, using 1 cm to represent 1 unit, and join the points to make the figures *ABCD* and *PQR*. Hence determine the area of the shape produced in each case.

(a) $A(-2, 0)$, $B(4, 0)$, $C(5, 3)$ and $D(-1, 3)$.

(b) $P(2, 1)$, $Q(4, 1)$ and $R(6, 5)$.

2. Using a scale of 1 cm to represent 1 unit, mark the points $A(1, 5)$, $B(-1, -1)$, and $C(5, -1)$ on graph paper. Join the points to make the figure *ABC*, and describe *ABC*.

3. Using a scale of 1 cm to represent 1 unit, mark the points $A(-4, 3)$, $B(5, 3)$, $C(6, -2)$ and $D(-3, -2)$ on graph paper. Join the points to make the figure *ABCD*, and describe *ABCD*.

4. $P(-2, 6)$, $Q(9, 6)$, $R(7, -1)$ and $S(-4, -1)$ are the vertices of a quadrilateral.

(a) Draw the quadrilateral on graph paper.

(b) Name the type of quadrilateral.

(c) Write down the sides which are equal in length.

(d) Write down which sides are parallel.

(e) Measure the angles of the quadrilateral. Write down which, if any, of the angles are equal.

5. (a) Draw a set of axes of your own. Give them scales from 0 to 10. Mark the following points and label each point with its own letter: $A(2, 9)$, $B(8, 9)$, $C(8, 1)$ and $D(2, 1)$.

(b) Join *A* to *B*, *B* to *C*, *C* to *D* and *D* to *A*. What is the shape of the figure *ABCD*?

(c) Draw the diagonals of the figure. Measure and state the lengths of the diagonals.

(d) Measure and state the sizes of the angles where the diagonals intersect.

(e) If *X* is the mid-point of *AB*, state its co-ordinates.

(f) If *Y* is the mid-point of *BC*, state its co-ordinates.

6. (a) The points *A*, *B*, *C*, *D*, *E* and *F* are all on the same straight line. Mark the points on graph paper:
$A(-4, -9)$, $B(-2, -5)$, $C(0, -1)$, $D(2, 3)$, $E(4, 7)$ and $F(6, 11)$. Draw a straight line from *A* to *F*.

(b) *H*, *I*, *J*, *K*, *L*, *M* and *N* are further points on the same line. Fill in the missing co-ordinates:
$H(-5, ?)$, $I(-3, ?)$, $J(1, ?)$, $K(3, ?)$, $L(5, ?)$, $M(?, 13)$ and $N(a, ?)$

7. Mark the following points on your own set of axes:
$A(3, 6)$, $B(8, 6)$, $C(8, 2)$ and $D(3, 2)$.
What is the name of the shape of the figure $ABCD$?

8. Mark the points $P(1, 5)$, $Q(-1, -1)$, and $R(5, -1)$. What is the name of the type of figure represented by PQR?

9. Mark the points J, K, L and M and join them to form the quadrilateral $JKLM$. $J(6, -2)$, $K(2, 4)$, $L(-3, 4)$ and $M(0, -2)$.
- (a) What type of quadrilateral is $JKLM$?
- (b) Join J to L and K to M. These are the diagonals of the quadrilateral. Mark the point where the diagonals cross as E.
- (c) Measure the diagonals. Are they the same length?
- (d) Is E the midpoint of either, or both, of the diagonals?
- (e) Measure the four angles at E. Do the diagonals cross at right angles?

10. $A(4, 4)$, $B(4, -4)$ and $C(-4, -4)$ are the vertices of a triangle.
- (a) Draw the triangle on graph paper.
- (b) Name the type of triangle.
- (c) Write down which sides are equal, if any exist.
- (d) Measure and state the sizes of the angles of the triangle.
- (e) Find and mark the mid-point of AC. State the coordinates of the mid-point.

11. Determine from a graph, the area of the following shape:
$A(-4, 0)$, $B(3, 0)$, $C(5, 2)$ and $D(-2, 2)$.

12. Determine by drawing a graph, the area of the following shape:
$P(2, 0)$, $Q(7, 0)$ and $R(7, 5)$.

13. Plot the graph of the following points on graph paper, using 1 cm to represent 1 unit. Hence find the area of the plane figure PQR formed.
$P(2, 1)$, $Q(4, 1)$ and $R(6, 5)$.

14. (a) Plot the points $A(6, 5)$, $B(10, 5)$, $C(14, 3)$ and $D(4, 3)$ on graph paper, using 1 cm to represent 1 unit on each axis.
- (b) State the name of the plane figure $ABCD$ formed.
- (c) Calculate the area of the plane figure $ABCD$.

15. Mark the points $A(-6, 0)$, $B(-2, -4)$, $C(2, 0)$ and $D(-2, 4)$ on graph paper, using 1 cm to represent 1 unit on both axes. Join the points in alphabetical order and then close the figure.
- (a) What type of quadrilateral is $ABCD$?
- (b) Measure and state the lengths of the diagonals AC and BD.

16. Mark the points $P(0, 2)$, $Q(8, 2)$, $R(6, -2)$ and $S(-2, -2)$ on graph paper, using 1 cm to represent 1 unit on each axis. Join the points in alphabetical order and then close the figure.
- (a) What type of quadrilateral is $PQRS$?
- (b) Measure and state the lengths of the diagonals PR and QS.

17. Mark the points $J(-4, 4)$, $K(-6, -2)$, $L(10, -2)$ and $M(4, 4)$ on graph paper, using 1 cm to represent 1 unit on both the x-axis and the y-axis. Join the points in alphabetical order and then close the figure.
- (a) What type of quadrilateral is $JKLM$?
- (b) Measure and state the lengths of JK and LM.

Simple Linear Graphs

In using the *Cartesian plane* to draw *graphs*, the *origin O* is taken as the *point of reference* and designated the value (0, 0). That is, at the *origin O*, $x = 0$ and $y = 0$. This implies that the *equation* of the *x-axis* is $y = 0$. And the *equation* of the *y-axis* is $x = 0$. This fact can be seen indicated in the graph on the next page.

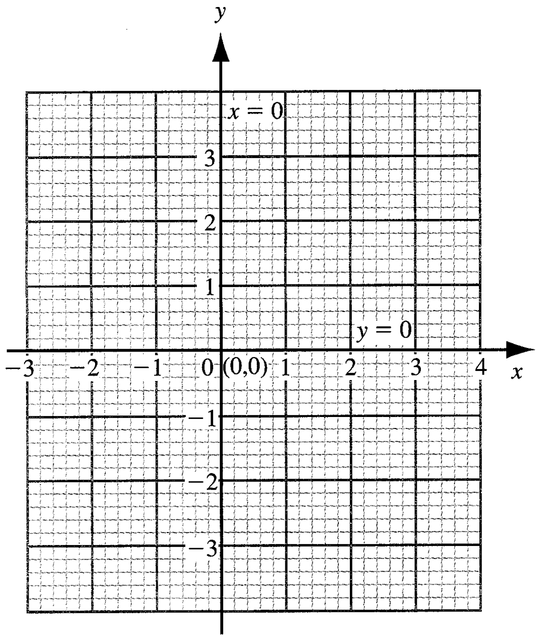


Fig. 7.6 The Cartesian plane

GRAPH OF THE FORM $x = p$, WHERE
p IS A REAL NUMBER, THAT IS,
$p \in R$.

A *Graph* of the form $x = p$, where $p \in R$ is a *straight line parallel* to the *y-axis.*

A relation is a connection between pairs of coordinates that obey a rule.

Example 4

Draw the graphs of the following relations on the same graph paper, using suitable scales:

(a) $x = -2.5$ *(b)* $x = -1$

(c) $x = 1.5$ *(d)* $x = 3$

Solution

The *graphs* of the given *relations* can be seen *drawn* below.

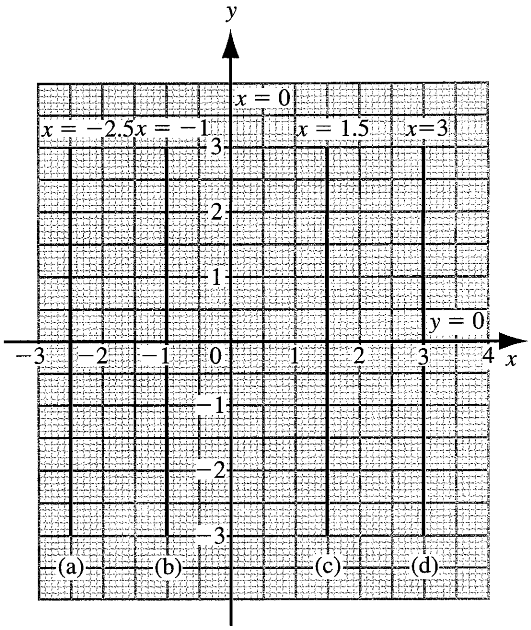


Fig. 7.7 Parallel lines

GRAPH OF THE FORM $y = q$, WHERE
q IS A REAL NUMBER, THAT IS,
$q \in R$.

A *Graph* of the form $y = q$, where $q \in R$ is a *straight line parallel* to the *x-axis.* And $y = f(x) = q$, $q \in R$ is called the *constant function.*

$y = f(x)$ is a notation which indicates that a function of x is produced, when x-coordinates are paired with y-coordinates.

Example 5

Draw the graphs of the following constant functions on the same graph paper, using suitable scales:

(a) $y = 2.5$ *(b)* $y = 1$

(c) $y = -1.5$ *(d)* $y = -3$

Solution

The *graphs* of the given *functions* can be seen *drawn* below.

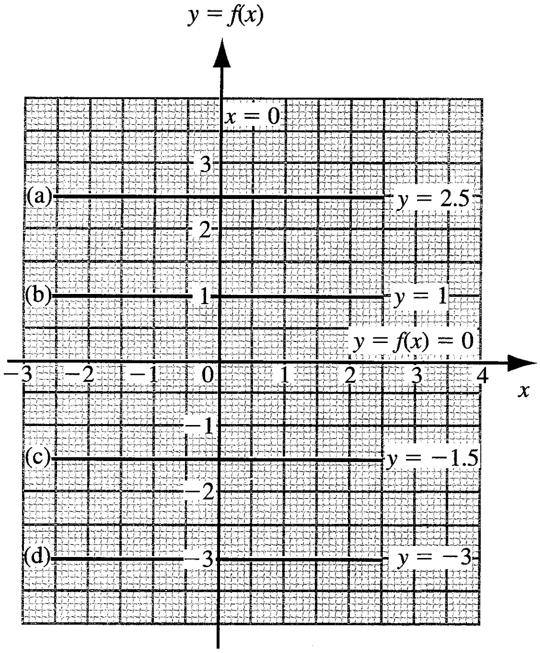


Fig. 7.8 Parallel lines

Exercise 7b

1. Draw the graphs of the following relations, on the same graph paper, using suitable scales:

   (a) $x = 2$ (c) $x = -1$

   (b) $x = 3.5$ (d) $x = -2.5$

2. Draw the graphs of the following functions, on the same graph paper, using suitable scales:

   (a) $y = 2.5$ (c) $y = -3.5$

   (b) $y = 4$ (d) $y = -2$

3. Using a scale of 2 cm to represent 1 unit, draw the graphs of the following relations, on the same graph paper with the same axes:

   (a) $x = 1.6$ (c) $x = -3.8$

   (b) $x = 4.7$ (d) $x = -1.4$

4. Using a scale of 2 cm to represent 1 unit, draw the graphs of the following functions, on the same graph paper with the same axes:

   (a) $y = -2.9$ (c) $y = 2.1$

   (b) $y = -4.3$ (d) $y = 3.2$

5. Draw the graphs of the following relations, using the same scales and axes:

   (a) $x = 4.5$ (c) $y = 3.9$

   (b) $x = -3.2$ (d) $y = -2.4$

6. Draw the graphs of the following relations, using the same scales and axes:

   (a) $y = -3.7$ (c) $x = -4.2$

   (b) $y = 2.8$ (d) $x = 5.6$

7. Draw the graphs of the following constant functions, using the same scales and axes:

   (a) $f(x) = -1.0$ (c) $f(x) = 5.0$

   (b) $f(x) = -3.0$ (d) $f(x) = 2.0$

8. Draw the graphs of the following constant functions, using the same scales and axes:

   (a) $f(x) = 2.4$ (c) $f(x) = 1.8$

   (b) $f(x) = -3.9$ (d) $f(x) = 2.7$

9. Using a scale of 2 cm to represent 1 unit, draw the graphs of the following functions, on the same graph paper:

   (a) $f(x) = -3.55$ (c) $f(x) = 3.45$

   (b) $f(x) = -2.65$ (d) $f(x) = 5.75$

10. Using a scale of 2 cm to represent 1 unit, draw the graphs of the following relations, on the same graph paper:

    (a) $f(x) = -4.55$ (c) $f(x) = 2.95$

    (b) $f(x) = -1.65$ (d) $f(x) = 3.85$

Inequations (or Inequalities)

An *inequation* (or *inequality*) is a mathematical statement showing that one quantity is *not equal* to another quantity, that is, one quantity is *greater or less* than another quantity.

The *four mathematical signs* that are used to represent *inequalities* are:

(i) $<$ which means '*is less than*'
(ii) $\leqslant$ which means '*is less than or equal to*'
(iii) $>$ which means '*is greater than*'
(iv) $\geqslant$ which means '*is greater than or equal to*'

In everyday life we use *inequalities* readily. For example: Ronald is taller than Janet. The car was travelling with a speed of at least 100 km per hour before the accident.

Example 6

Express each of the following English statements as an inequality statement with symbols:

(a) x is less than 17
(b) y is greater than −5
(c) p is not more than 8
(d) q is at least 13.

Solution

(a) The *inequality statement* is $x < 17$
(b) The *inequality statement* is $y > -5$
(c) The *inequality statement* is $p \leqslant 8$
(d) The *inequality statement* is $q \geqslant 13$

Exercise 7c

Express each of the following English statements as an inequality statement with symbols:

1. x is less than 21.

2. y is more than 57.

3. p is not more than q.

4. r is at least s.

5. a is less than b.

6. b is less than or equal to 103.

7. r is greater than j.

8. N is greater than or equal to M.

9. Z is greater than zero.

10. T is at least 33 °C.

11. x is less than y cm.

12. t is less than −7 °C.

13. W is no more than 1.2 m.

14. x is at least 1 m.

15. p is less than twice q.

16. r is greater than thrice s.

17. x is at least double y.

18. l is no more than triple m.

19. Twice a is greater than b.

20. Double p is at least triple q.

Example 7

Express each of the following English statements as an inequality statement using your own symbols:

(a) More than 20 000 people attended the cultural show.
(b) Less than 6 500 workers protested.
(c) John had at least \$25.00 in his pocket.
(d) Christine ran not more than 5 km today.

Solution

(a) Let s = the *number of people* who attended the cultural show.
Then the *inequality statement* is:

$$s > 20\,000$$

(b) Let p = the *number of workers* who protested.
Then the *inequality statement* is:

$$p < 6\,500$$

(c) Let a = the *amount of money* in dollars John had in his pocket.
Then the *inequality statement* is:

$$a \geqslant 25.00$$

(d) Let r = the *distance* Christine ran in kilometres today.
Then the *inequality statement* is:

$$r \leqslant 5$$

Exercise 7d

Express each of the following English statements as an inequality statement using your own symbols:

1. Frank will be away from home for more than 3 years.

2. The train was travelling at more than 150 km/h before the accident.

3. The audience at the show was less than the 12 000 expected.

4. The number of students that can travel in a 25-seater maxi taxi.

5. The number of kick boxers taking part in the contest were not more than 13.

6. The number of people involved in the accident were at least 9.

7. After the budget the cost of living will rise by at least 35%.

8. Iman is no more than 1.2 m in height.

9. One United States dollar is worth at least five dollars and seventy-six cents in Trinidad and Tobago currency.

10. The standard of living will drop by less than 15%.

11. The temperature of a human being should normally not be more than 38 °C.

12. The mean mark in the Mathematics test was less than 64%.

13. The price of new car is at least six times the cost of a new personal computer.

14. The amount spent on wages is at least twice the amount spent on raw materials.

15. The increase in the price of gas will not be more than 25%.

16. Frank's age is less than half of his father's age.

17. The grocery bill today is more than $1\frac{1}{2}$ times the last grocery bill.

18. Maria's height is less than a quarter of her mother's height.

19. The audience at the cultural show was larger than the expected 22 500.

20. The cost of living today is at least $1\frac{1}{2}$ times what is was last year.

Example 8

Express each of the following sentences as a single inequality statement using your own symbols:

(a) The mass of the parcel is between 2.5 kg and 3.5 kg exclusive.

(b) The cost of a ticket to the pageant is between $20.00 and $75.00 inclusive.

(c) The increase in salary will be greater than $125 but no more than $150.

(d) The cost of posting the parcel is at least $2.25, but less than $2.75.

Solution

(a) Let m = the *mass* of the parcel in kg.
Then the *inequality statement* is:

$$2.5 < m < 3.5$$

Exclusive means that the *two extreme values are not included.*

(b) Let p = the *cost* of a ticket in dollars for the pageant.
Then the *inequality statement* is:

$$20.00 \leqslant p \leqslant 75.00$$

Inclusive means that the *two extreme values are included.*

(c) Let x = the *increase* in salary in dollars.

Then $x > 125 \Rightarrow 125 < x$

And $x \leqslant 150.$

Thus we have $125 < x$ and $x \leqslant 150$.

Combining the two separate inequalities, we get the *inequality statement* as:

$$125 < x \leqslant 150$$

Note that the sign ⇒ means '*implies*'.

(d) Let p = the *cost* of posting the parcel in dollars.

Then $p \geqslant 2.25 \Rightarrow 2.25 \leqslant p.$

And $p < 2.75.$

Thus we have $2.25 \leqslant p$ and $p < 2.75$.

Combining the two seperate inequalities, we get the *inequality statement* as:

$$2.25 \leqslant p < 2.75$$

Exercise 7e

Express each of the following sentences as a single inequality statement using your own symbols:

1. The temperature on Monday was between 30 °C and 34 °C inclusive.

2. The price of a ticket to the cinema is between $6.50 and $8.50 inclusive.

3. The mass of the computer is between 3.4 kg and 4.7 kg inclusive.

4. The temperature of the sick child was between 99 °C and 102 °C exclusive.

5. The length of the road is between 15.7 km and 18.5 km exclusive.

6. The mass of the tablets is between 125 mg and 137 mg exclusive.

7. The wind travelled faster than 25 km/h but did not exceed 40 km/h.

8. The decrease in salary was greater than $175 but no more than $250.

9. The increase of a person's mass after eating is more than 1.5 kg but cannot exceed 4 kg.

10. The cost of registering a letter is at least \$2.25, but less than \$3.25.

11. A monthly electricity bill is normally not less than \$75.00, but certainly less than \$124.00.

12. A postman delivers at least 350 letters, but less than 975 letters per day.

13. Amanda's mass varies between 49.5 kg and 54.3 kg each day.

14. The speed of a car on a journey varies between 40 km/h and 160 km/h.

15. My son can lift weights of mass at least 40 kg but not more than 75 kg.

Representing an Inequality on a Number Line

A *number line* is similar to the *x*-axis which is drawn on the Cartesian plane. All *inequalities* can be represented on a *number line.*

In the case of an *inequality*, the solution is in the form of *a solution set* or a '*range of values*' within which possible answers lie to a given problem. For example: the *answer* may be only whole numbers or positive integers or negative integers. Hence we are able to *quantify* the *answer*.

Example 9

Represent each of the following inequalities on a number line and state the solution set:

(a) $x > 4$ *(b)* $x \leqslant 1$

(c) $-1 \leqslant x \leqslant 3$ *(d)* $-2 < x < 4$

(e) $-1.5 < x \leqslant 2.5$ *(f)* $2 \leqslant x < 4.5$

Solution

(a)

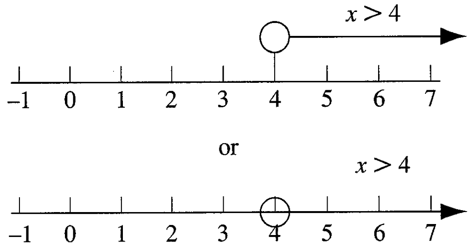


Fig. 7.9 Number lines

The *inequality* $x > 4$ can be seen represented on the *number line* shown previously.

The *solution set* is $\{x: x > 4\}$.

The symbol $\{x: \ldots\}$ means '*the set of all x such that*'.

The circle is *unshaded* to indicate that *x is not equal to* 4, $x \neq 4$. And the *arrow* points in the *direction of the solution.*

The sign $\neq$ means '*is not equal to*'.

(b)

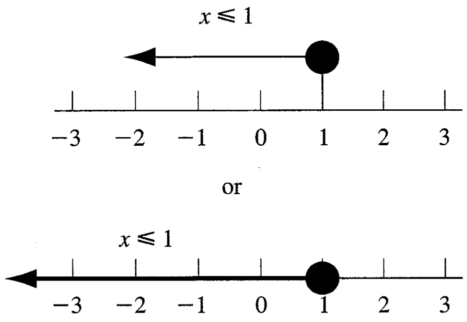


Fig. 7.10 Number lines

The *inequality* $x \leqslant 1$ can be seen represented on the *number line* shown above.

The *solution set* is $\{x: x \leqslant 1\}$.

The circle is *shaded* to indicate that *x is equal to* 1, $x = 1$. And the *arrow* points in the *direction within which the solution lies.*

(c)

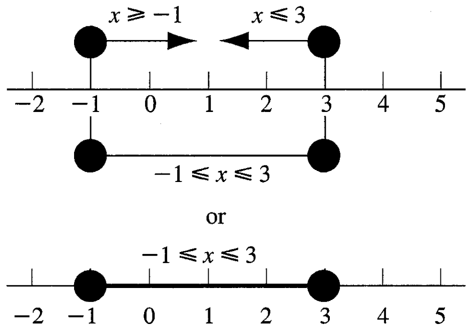


Fig. 7.11 Number lines

The *inequality* $-1 \leqslant x \leqslant 3$ can be seen represented on the *number line* shown above.

The *solution set* is $\{x: -1 \leqslant x \leqslant 3\}$.

Note that $x \geqslant -1$ and $x \leqslant 3 \Leftrightarrow -1 \leqslant x \leqslant 3$.

The sign $\Leftrightarrow$ means '*is equivalent to*' or '*cross implies*'.

(d)

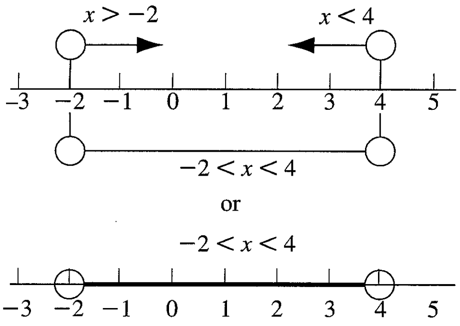


Fig. 7.12 Number lines

The *inequality* $-2 < x < 4$ can be seen represented on the *number line* shown above.

The *solution set* is $\{x: -2 < x < 4\}$.

Note that $x > -2$ and $x < 4 \Leftrightarrow -2 < x < 4$.

(e)

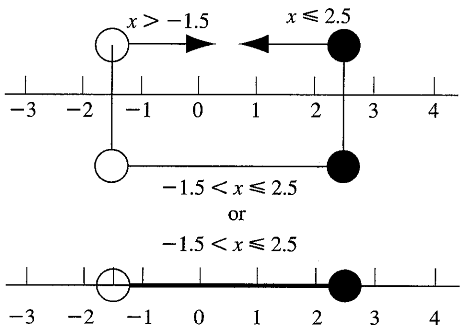


Fig. 7.13 Number lines

The *inequality* $-1.5 < x \leqslant 2.5$ can be seen represented on the *number line* shown above.

The *solution set* is $\{x: -1.5 < x \leqslant 2.5\}$.

(f)

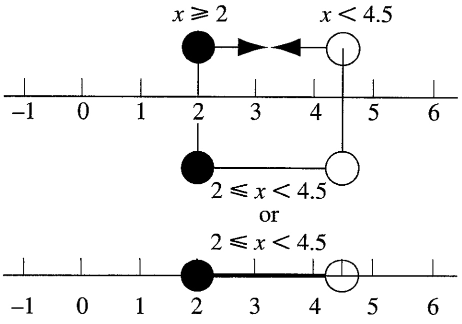


Fig. 7.14 Number lines

The *inequality* $2 \leqslant x < 4.5$ can be seen represented on the *number line* shown above.

The *solution set* is $\{x: 2 \leqslant x < 4.5\}$.

Exercise 7f

Represent each of the following inequalities on a number line and state the solution set:

1. $x \geqslant 3$
2. $x \geqslant 2.5$
3. $x \geqslant 0$
4. $x \geqslant -2$
5. $x \geqslant -3.5$
6. $x < 4$
7. $x < 3.5$
8. $x < 0$
9. $x < -1.5$
10. $x < -4.5$
11. $-1 \leqslant x \leqslant 2$
12. $1 \leqslant x \leqslant 4$
13. $0 \leqslant x \leqslant 3.5$
14. $-5 \leqslant x \leqslant -1$
15. $-5.5 \leqslant x \leqslant -0.5$
16. $1 < x < 5$
17. $0 < x < 4.5$
18. $-1 < x < 3$
19. $-2.5 < x < 2$
20. $-3.5 < x < -0.5$
21. $1 < x < 4$
22. $0 < x < 5$
23. $-1.5 < x < 3$
24. $-2.5 < x < 1$
25. $-4.5 < x < -1$
26. $1 \leqslant x < 5$
27. $0 \leqslant x < 4$
28. $-0.5 \leqslant x < 3$
29. $-2.5 \leqslant x < 2.5$
30. $-4.5 \leqslant x < 0.5$

Representing an Inequality on a Graph

An *inequality* can be represented by a *region* on a graph.

Example 10

Represent each of the following inequalities on a graph and state the solution set:

(a) $x \geqslant 3$
(b) $y < 1.5$
(c) $-1 \leqslant x \leqslant 4$
(d) $-2 < y < 3$
(e) $0.5 < x \leqslant 4.5$
(f) $0 \leqslant y < 3.5$

Solution

(a)

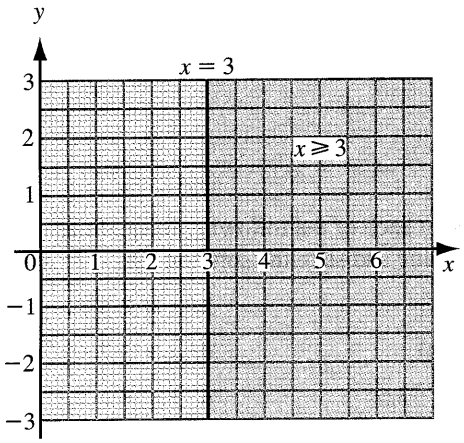


Fig. 7.15 Inequality

The *inequality* $x \geqslant 3$ can be seen represented on the *graph* shown above.
The *straight line* representing $x = 3$ is drawn *unbroken* to indicate that $x = 3$ is a *part* of the *inequality* $x \geqslant 3$.
And the *shaded region* represents $x > 3$.
Thus the *unbroken straight line* $x = 3$ and the *shaded region* $x > 3$ together represent the *inequality* $x \geqslant 3$.
Hence the *solution set* is $\{x: x \geqslant 3\}$.
The symbol $\{x: \ldots\}$ means '*the set of all x such that*'.

(b)

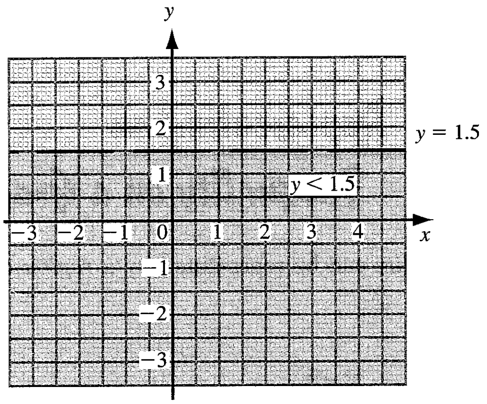


Fig. 7.16 Inequality

The *inequality* $y < 1.5$ can be seen represented on the *graph* shown above.
The *straight line* representing $y = 1.5$ is drawn *broken* to indicate that $y = 1.5$ is *not* a *part* of the *inequality* $y < 1.5$.
Thus the *shaded region only* represents the *inequality* $y < 1.5$.
Hence the *solution set* is $\{y: y < 1.5\}$.
The symbol $\{y: \ldots\}$ means '*the set of all y such that*'.

(c)

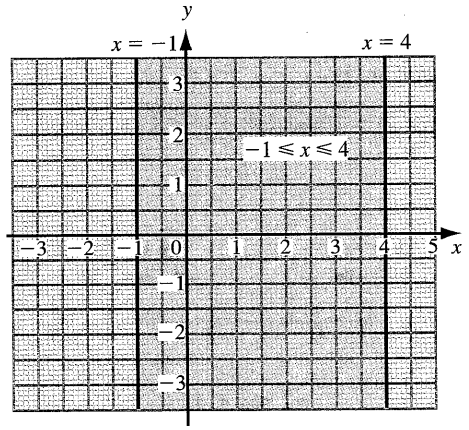


Fig. 7.17 Inequality

The *inequality* $-1 \leqslant x \leqslant 4$ can be seen represented on the *graph* shown above.
The *straight lines* representing $x = -1$ and $x = 4$ are drawn *unbroken* to indicate that $x = -1$ and $x = 4$ are a *part* of the *inequality* $-1 \leqslant x \leqslant 4$.
And the *shaded region* represents $-1 < x < 4$.
Thus the *unbroken straight lines* $x = -1$ and $x = 4$ and the *shaded region* $-1 < x < 4$ together represent the *inequality* $-1 \leqslant x \leqslant 4$.
Hence the *solution set* is $\{x: -1 \leqslant x \leqslant 4\}$.

(d)

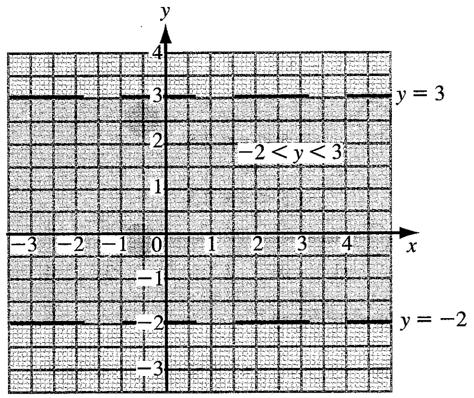


Fig. 7.18 Inequality

The *inequality* $-2 < y < 3$ can be seen represented on the *graph* shown above.
The *straight lines* representing $y = -2$ and $y = 3$ are drawn *broken* to indicate that $y = -2$ and $y = 3$ are *not a part* of the *inequality* $-2 < y < 3$.
Thus the *shaded region only* represents the *inequality* $-2 < y < 3$.
Hence the *solution set* is $\{y: -2 < y < 3\}$.

(e)

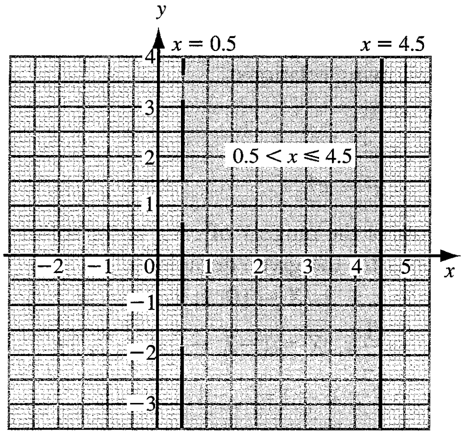


Fig. 7.19 Inequality

The *inequality* $0.5 < x < 4.5$ can be seen represented on the *graph* shown above.
The *shaded region* and the *line* $x = 4.5$ represent the *inequality* $0.5 < x \leqslant 4.5$.
Hence the *solution set* is $\{x: 0.5 < x \leqslant 4.5\}$.

(f)

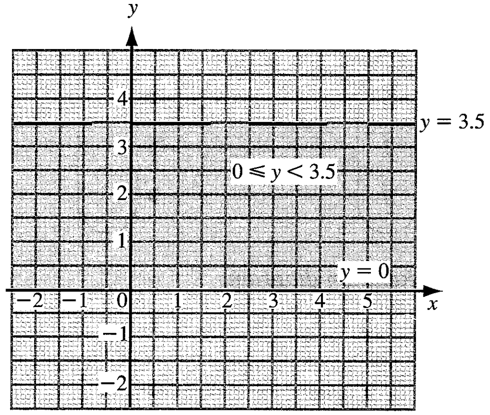


Fig. 7.20 Inequality

The *inequality* $0 \leqslant y < 3.5$ can be seen represented on the *graph* shown above.
The *shaded region* and the *line* $y = 0$ represent the *inequality* $0 \leqslant y < 3.5$.
Hence the *solution set* is $\{y: 0 \leqslant y < 3.5\}$.

Exercise 7g

Represent each of the following inequalities on a graph and state the solution set:

1. $x \geqslant 0$
2. $x \geqslant 4$
3. $x \geqslant -2.5$
4. $y \leqslant 0$
5. $y \leqslant 3$
6. $y \leqslant -4.5$
7. $x < 2$
8. $x < 3.5$
9. $x < -1$
10. $y < 1$
11. $y < 2.5$
12. $y < -2.5$
13. $1 \leqslant x \leqslant 3$
14. $0 \leqslant x \leqslant 2.5$
15. $-3 \leqslant x \leqslant 1.5$
16. $1 \leqslant y \leqslant 3$
17. $0 \leqslant y \leqslant 2.5$
18. $-3 \leqslant y \leqslant 1.5$
19. $1 < y < 5$
20. $0.5 < y < 3.5$
21. $-3 < y < -0.5$
22. $1 < x < 3$
23. $-0.5 < x < 2.5$
24. $-3 < x < 1.5$
25. $0 < x \leqslant 3$
26. $-1 < x \leqslant 2.5$
27. $-2.5 < x \leqslant 3.5$
28. $0 < y \leqslant 3$
29. $-2 < y \leqslant 3$
30. $-3.5 < y \leqslant 1$
31. $0 \leqslant x < 4$
32. $-0.5 \leqslant x < 3$
33. $-3.5 \leqslant x < 2$
34. $0 \leqslant y < 3$
35. $-1.5 \leqslant y < 2$
36. $-2.5 \leqslant y < 3$

Relation

A *relation* is defined as a *set of ordered pairs* that obeys a *particular rule.*

Example 11

If $x \rightarrow 3x + 5$ and $\{x: -2 \leqslant x \leqslant +2\}$, then we have the following *table of values* for the *relation* $y = 3x + 5$.

Table 7.1 Table of values

x	-2	-1	0	$+1$	$+2$
$3x$ $+5$	-6 $+5$	-3 $+5$	0 $+5$	$+3$ $+5$	$+6$ $+5$
y	-1	2	5	8	11

The notation $x \rightarrow 3x + 5$ means '*x is mapped onto* $3x + 5$'. And we can write the *set of ordered pairs* as: $\{(-2, -1), (-1, 2), (0, 5), (1, 8), (2, 11)\}$.
The *set* of *first* values in the *set of ordered pairs*, we call the *domain elements*, i.e. $\{-2, -1, 0, 1, 2\}$. These are the *x values* (or set of *x-coordinates*).
The *set* of *second* values in the *set of ordered pairs*, we call the *range elements*, i.e. $\{-1, 2, 5, 8, 11\}$.

These are the *y values* (or set of *y-coordinates*). The *relation* $x \rightarrow 3x + 5$ (i.e. $y = 3x + 5$) can be shown on a *graph* as seen below.

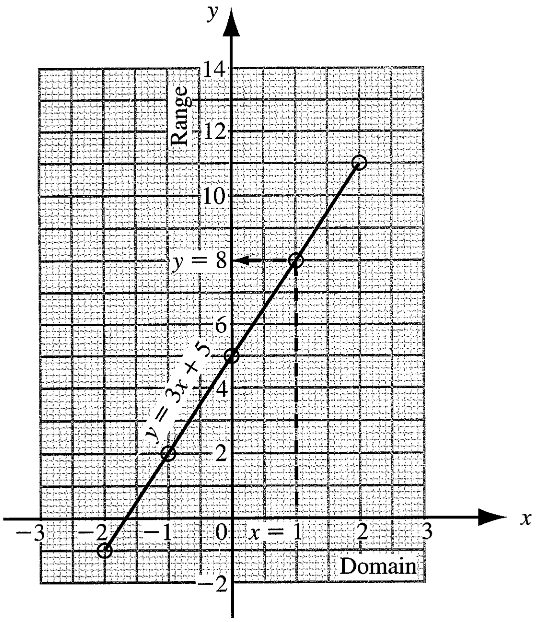


Fig. 7.21 One-to-one relation

The *relation* $x \rightarrow 3x + 5$ can also be shown on an *arrow diagram* (or *relation diagram* or *mapping diagram*) as seen in the following diagram.

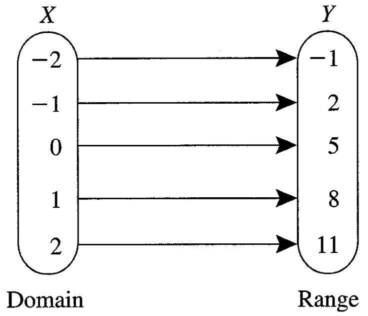


Fig. 7.22 One-to-one relation

This is an example of a *one-to-one relation.*

Exercise 7h

1. (a) Draw a relation diagram for the relation defined as follows:

Relation	Domain
$x \rightarrow 2x + 1$	$\{-3, -2, -1, 0, 1, 3\}$

   (b) State the type of relation diagram obtained.

2. With a domain $\{-3, -2, -1, 0, 1, 2, 3\}$, what would be the range corresponding to each of the following relations:
   (a) $x \rightarrow 3x$
   (b) $x \rightarrow x^2$
   (c) $x \rightarrow x^2 - x$?

3. Determine the range Y of each of the following relations for the given domain X:
   (a) $x \rightarrow x + 3,\ X = \{0, 1, 2, 3\}$
   (b) $x \rightarrow \frac{1}{2}x,\ X = \{-2, -1, 0, 1, 2, 3\}$
   (c) $x \rightarrow 2^x,\ X = \{-2, -1, 0, 1, 2\}$.

4. Copy and complete the diagram below, for each of the following relations:
   (a) $x \rightarrow 2x + 3$
   (b) $x \rightarrow 5x$
   (c) $x \rightarrow 3x - 1$

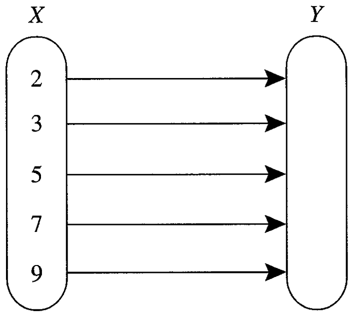


Fig. 7.23 Relation diagram

5.

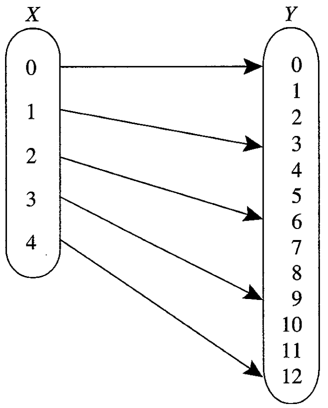


Fig. 7.24 Relation diagram

State the relation that gives the relation diagram shown above.

6. If a relation from X to Y is defined by $x \rightarrow x^2 - 2x + 1$, complete the relation diagram shown below.

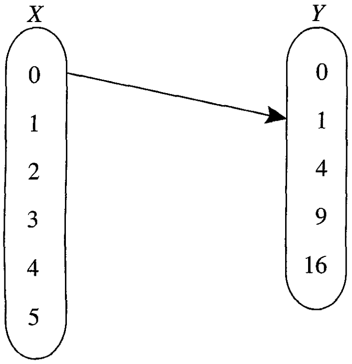


Fig. 7.25 Relation diagram

7.

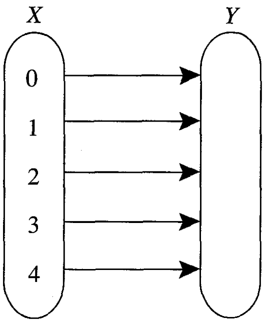


Fig. 7.26 Relation diagram

Given the relation $x \rightarrow 2^x$, complete the relation diagram above.

8.

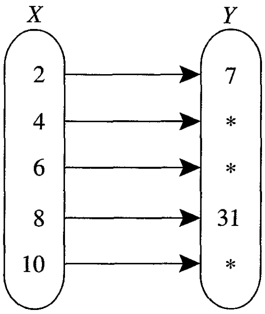


Fig. 7.27 Relation diagram

The diagram above shows a linear relation between the domain X and the range Y. Determine:

(a) the relation between an element x in X and the corresponding element in Y

(b) the numbers marked*.

9. For the relation $x \rightarrow x^3$:

(a) What is 2 mapped onto?

(b) What is 3 mapped onto?

10. For the relation $x \rightarrow 2 - x$:

(a) What is 7 mapped onto?

(b) What is 9 mapped onto?

11. Given the relation:
$\{(F, S): S = 0.5F + 3, F \in N\}$,
where $N = \{$natural numbers from 1 to 10$\}$,
list the set of ordered pairs.

12. Given the relation:
$\{(x, y): y = 3x + 0.5, x \in W\}$,
where $W = \{$whole numbers less than 7$\}$,
list the set of ordered pairs.

Functions

A *function* is defined as a *relation* in which each element in the *domain* is mapped onto one and only one element in the *range*.

Example

CASE 1: ONE-TO-ONE RELATION

Now $x \rightarrow 3x + 5$ is a *one-to-one relation* as shown in Fig. 7.21 and Fig. 7.22.
Further, the *one-to-one relation* is a *function*. We write the *linear function* as:

$$f: x \rightarrow 3x + 5$$

or $$f(x) = 3x + 5$$

or $$\begin{array}{ccc} \textit{Domain} & \xrightarrow{f} & \textit{Range} \\ x & & 3x + 5 \end{array}$$

and $$y = f(x).$$

All *one-to-one relations* are *functions*.

Note that $f: x \rightarrow 3x + 5$ means '*the function of x is* $3x + 5$'.

CASE 2: MANY-TO-ONE RELATION

Consider $x \rightarrow 3x^2 + 2x - 5$ and $\{x: -2 \leqslant x \leqslant 2\}$. Then we have the following *table of values* for the *relation* $y = 3x^2 + 2x - 5$.

Table 7.2 Table of values

x	-2	-1	0	$+1$	$+2$
x^2	4	1	0	1	4

Table continued

Table 7.2 Continued

$3x^2$	12	3	0	3	12
$+2x$	−4	−2	0	+2	+4
−5	−5	−5	−5	−5	−5
y	3	−4	−5	0	11

The *relation* $x \to 3x^2 + 2x - 5$, for $-2 \leqslant x \leqslant 2$ can be shown on a *graph* as seen in the diagram that follows.

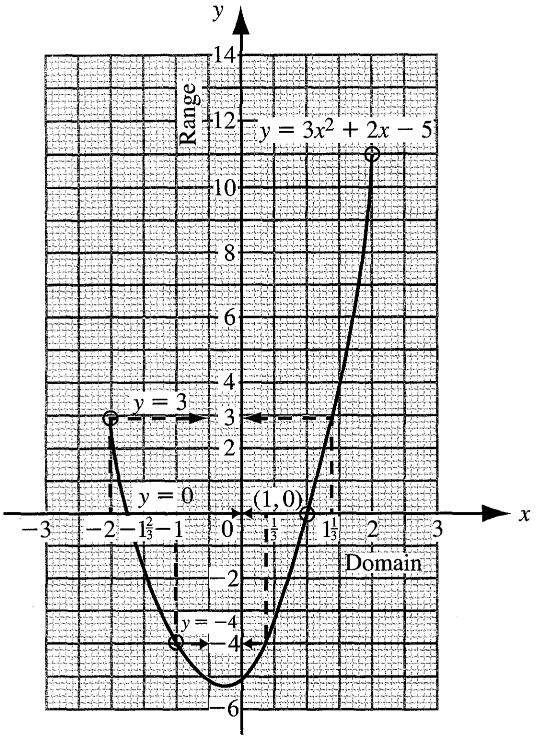


Fig. 7.28 Many-to-one relation

When $y = 3$, then $x = -2$ or $1\frac{1}{3}$. Similarly, for *each value of y*, we get *two corresponding values for x*. Therefore, $x \to 3x^2 + 2x - 5$ is a *many-to-one relation* as seen in the diagram below.

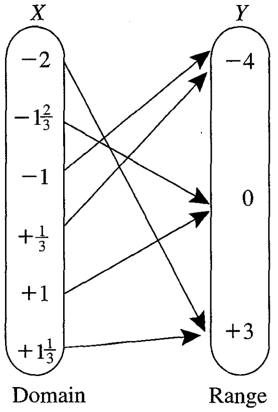


Fig. 7.29 Many-to-one relation

Each value of x is mapped onto one and only one value of y. Hence the *many-to-one relation* is a *function.*

We write the quadratic function as:

f: $x \to 3x^2 + 2x - 5$

or $f(x) = 3x^2 + 2x - 5$

or $\text{Domain} \xrightarrow{f} \text{Range}$

$x \qquad 3x^2 + 2x - 5$

and $y = f(x)$.

Note that f: $x \to 3x^2 + 2x - 5$ means, '*the function of x is* $3x^2 + 2x - 5$'.

All *many-to-one relations* are *functions*.

A Relation but not a Function

No *one-to-many relation* is a *function.*

Example 13

Consider the *relation* $x \to \pm\sqrt{x}$, $x \geqslant 0$, and the following *table of values*:

Table 7.3 Table of values

x	0	1	4	9	16	25	36
$y = \pm\sqrt{x}$	0	±1	±2	±3	±4	±5	±6

We can represent this *relation* on a *graph* and *mapping diagram* as seen below.

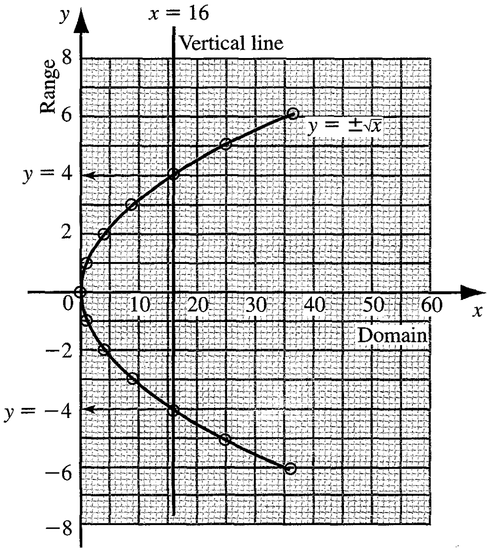


Fig. 7.30 One-to-many relation

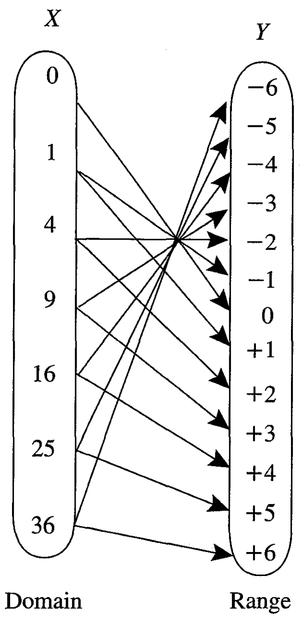


Fig. 7.31 One-to-many relation

The *relation* is a *one-to-many relation*. This is an example of a *relation* that is *not a function*, since one domain element is paired with more than one range element.

If we draw a *vertical line* through the graph, then for *each x value* we get *two corresponding y values.* This implies that the relation is *not a function*. No *one-to-many relation* is a *function*.

Image of x

For a *function f* and any *element x* in the *domain of f*, the *image of x* is denoted by $f(x)$, where x and $y = f(x)$ are *variables*. We can use the *function notation* to determine the *image of x* (i.e. the *value of* $f(x)$) for a given *value of x*.

Example 14

(a) If $f(x) = 2x^2 - 3x + 1$, determine the value of each of the following:

(i) $f(-3)$

(ii) $f(0)$

(iii) $f(2)$

Solution

(i) Given $f(x) = 2x^2 - 3x + 1$

Then
$$\begin{aligned} f(-3) &= 2(-3)^2 - 3(-3) + 1 \\ &= 2(9) + 9 + 1 \\ &= 18 + 10 \\ &= 28 \end{aligned}$$

Hence $f(-3)$ is 28.

(ii) Now
$$\begin{aligned} f(0) &= 2(0)^2 - 3(0) + 1 \\ &= 0 - 0 + 1 \\ &= 1 \end{aligned}$$

Hence $f(0)$ is 1.

(iii) Now
$$\begin{aligned} f(2) &= 2(2)^2 - 3(2) + 1 \\ &= 2(4) - 6 + 1 \\ &= 8 - 5 \\ &= 3 \end{aligned}$$

Hence $f(2)$ is 3.

(b) If $g: x \to 3x - 1$, calculate the value of each of the following:

(i) $g(2)$

(ii) $g(-3)$.

(i) Given $g: x \to 3x - 1$

Then
$$\begin{aligned} g(2) &= 3(2) - 1 \\ &= 6 - 1 \\ &= 5 \end{aligned}$$

Hence $g(2)$ is 5.

(ii) Now
$$\begin{aligned} g(-3) &= 3(-3) - 1 \\ &= -9 - 1 \\ &= -10 \end{aligned}$$

Hence $g(-3)$ is -10.

(c) If $h(x) = x^2 - 1$, for what values of x is $h(x) = 8$?

Given $h(x) = x^2 - 1$

and that $h(x) = 8$

Then $x^2 - 1 = 8$

So $x^2 = 1 + 8 = 9$

i.e. $x = \pm\sqrt{9} = \pm 3$

Hence $h(x) = 8$ when $x = \pm 3$.

(d) If $k(x) = \dfrac{3x + 5}{2x - 1}$, state the real value of x which cannot be in the domain of $k(x)$.

Given $k(x) = \dfrac{3x + 5}{2x - 1}$

Then $2x - 1 \neq 0$ (*division* by *zero* is *meaningless*)

So $2x \neq 1$

i.e. $x \neq \frac{1}{2}$

Hence $x = \frac{1}{2}$ is the *real value of* x which *cannot* be in the *domain* of $k(x)$.

Exercise 7i

1. (a) Draw a relation diagram for the relation defined as follows:

Relation	Domain
$x \to 3x^2$	$\{-3, -2, -1, 0, 1, 2\}$

   (b) Name the type of relation diagram obtained.

2. (a) Draw a relation diagram for the function $f: x \to 2(3x - 1)$ where $x \in \{1, 2, 3, 4, 5\}$.
   (b) Name the type of relation diagram obtained.

3. (a) The function g is defined by: $g(x) = x^2 + 1$ where $x \in \{-3, -2, -1, 0, 1, 2, 3, 4\}$.
   Write the value of each of the following:
   (i) $g(-3)$ (ii) $g(-1)$
   (iii) $g(0)$ (iv) $g(2)$
   (b) For what values of x is $g(x) = 5$?
   (c) For what values of x is $g(x) = 17$?

4. (a) Draw a relation diagram for the relation defined as follows:

Relation	Domain
$x \to 2x^2$	$\{-3, -2, -1, 0, 1, 2, 3\}$

   (b) State the type of relation diagram obtained.
   (c) Is the relation also a function?

5. (a) Draw a relation diagram for the function f over the given domain if:
   $f: x \to 3(x - 1)$ where $x \in \{1, 3, 6, 9\}$.
   (b) State the type of relation diagram obtained.

6. The function g is defined by $g: x \to 5x + 2$, where $x \in$ {whole numbers less than 8}.
   (a) What number in the image set is 2 map to?
   (b) What is the image of 4?
   (c) What number in the domain is mapped to 17?
   (d) What number has 37 as its image?

7. The function f is defined by $f: x \to x^2 + 5$, where $x \in \{1, 2, 3, 4\}$.
   Do you agree that $f(3) = 14$?
   State the value of each of the following:
   (a) $f(1)$ (b) $f(2)$ (c) $f(4)$
   For what value of x is $f(x) = 21$?

8. Evaluate x, if $g(x) = 26$ and $g: x \to 3(x - 1) + 2$.

9. If $k: x \to \frac{1}{x} - 3$ evaluate:
   (a) $k\left(\frac{1}{2}\right)$ (b) $k(-1)$ (c) $k\left(\frac{1}{3}\right)$ (d) $k(-2)$

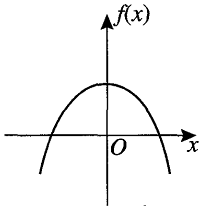
10. Given the relation $\{(F, S): S = 0.5F + 1.5, F \in N\}$, where N = {natural numbers less than 10}, state the set of ordered pairs.

11. Given the relation $x \to 3x - 1$.
   (a) Draw a relation diagram of the relation for $0 \leqslant x \leqslant 5$.
   (b) Draw a graph of the relation for $0 \leqslant x \leqslant 5$.
   (c) State whether the relation is a function or not. Give a reason for your answer.

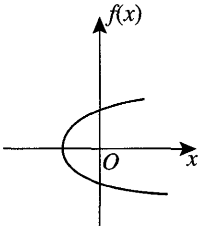
12. If $f: x \to \frac{1}{2}(4x - 1)$, calculate the values of:
   (a) $f(3)$ (b) $f(-2)$ (c) $f(0)$.

13. State which of the following relations represents a function.

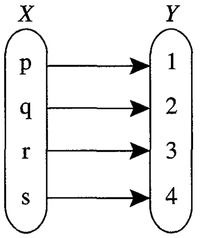
(a)



(b)



(c)



(d)

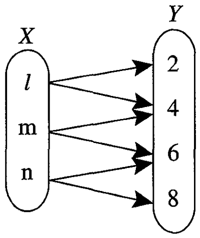


Fig. 7.32 Relations

14. Given the function $f: x \to 3x^2 + 2x - 1$.
   (a) Evaluate:
   (i) $f(0)$ (ii) $f(2)$ and (iii) $f(-1)$.

(b) State what type of graph you would expect to obtain, if a graph of the function $f: x \rightarrow 3x^2 + 2x - 1$ was drawn on graph paper.

(c) State whether the function has a maximum or minimum value. Give a reason for your answer.

15. Given $g(x) = \frac{3x - 1}{x + 2}$ and $h(x) = \frac{7x + 1}{5x - 2}$.

Evaluate:

(i) $g(3)$ (ii) $h(2)$.

16. State which of the following relations represents a function:

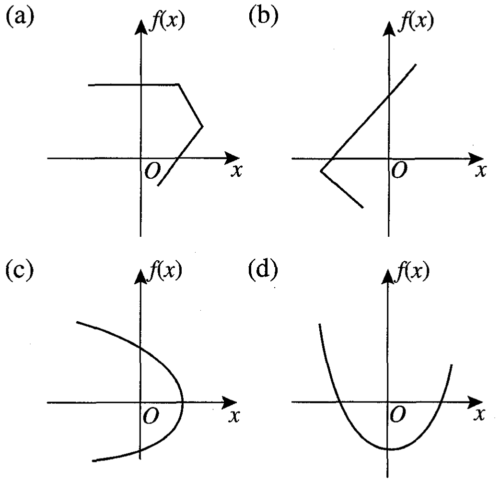


Fig. 7.33 Relations

17.

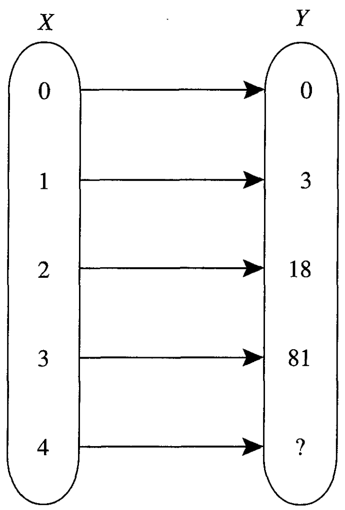


Fig. 7.34 Relation

(a) Write an equation in x and y to represent the relation shown by the mapping in the diagram above.

(b) Calculate the missing value of y.

18. Below are three sets of codes, P, Q and R in which letters and numbers are related.

$P = \{(p, 15), (q, 3), (r, 9), (q, 4), (s, 11), (t, 15)\}$,

$Q = \{(p, 5), (q, 13), (s, 5), (r, 7), (t, 5), (u, 4)\}$,

$R = \{(p, 5), (q, 6), (p, 0), (r, 6), (s, 5), (r, 8)\}$.

Draw an arrow diagram to represent each relation, and state whether each of the following statements is true or false:

(a) P is a function.

(b) Q is a function.

(c) R is a function.

Graph of the Function $f: x \rightarrow ax$

The *function* $f: x \rightarrow ax$ is called the *linear function*. The *graph* of the *function* $f: x \rightarrow ax$ is a *straight line* passing through the *origin*. The *slope of the line varies* as the value of the *constant* a, which is a *real number*, that is, $a \in R$.

The codomain is the set of elements that the mapping is going to.

The range of a mapping are all those elements in the codomain which are actually used in the mapping.

Example 15

Draw the graph of the linear function:

(a) $f: x \rightarrow -2x$ *(b)* $f: x \rightarrow -x$

(c) $f: x \rightarrow -\frac{1}{2}x$ *(d)* $f: x \rightarrow \frac{1}{2}x$

(e) $f: x \rightarrow x$ *(f)* $f: x \rightarrow 2x$

for $-2 \leqslant x \leqslant 6$, *on the same graph paper, using the same scales and axes.*

Solution

From the information given, we have the following *tables of values*, which are then used to *draw* the *graphs*.

Table 7.4 Tables of values

x	-2	2	4	6
$f(x) = 2x$	-4	4	8	12
$f(x) = -2x$	4	-4	-8	-12

x	-2	2	4	6
$f(x) = \frac{1}{2}x$	-1	1	2	3
$f(x) = -\frac{1}{2}x$	1	-1	-2	-3

x	-2	2	4	6
$f(x) = x$	-2	2	4	6
$f(x) = -x$	2	-2	-4	-6

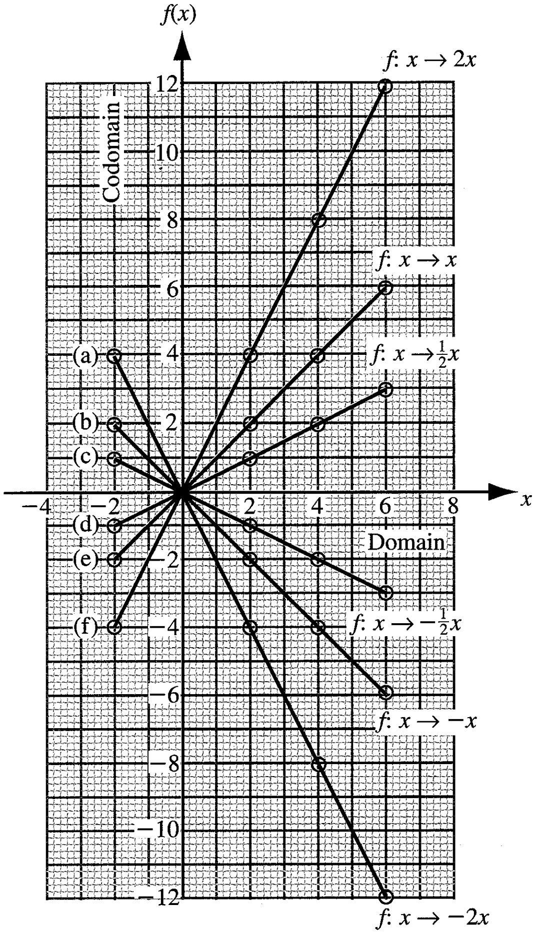


Fig. 7.35 Linear functions

In the diagram above, the *graphs* of $f: x \to -2x$, $f: x \to -x$, $f: x \to -\frac{1}{2}x, f: x \to \frac{1}{2}x, f: x \to x$ and $f: x \to 2x$ for the *domain* $-2 \leqslant x \leqslant 6$ have been *plotted*. It can be seen that the *graphs* all passed through the *origin*, and that the *slope* of each *straight line* varied as the value of the *coefficient of x*. Whether the line was sloping / or \, depended on whether the coefficient of x was positive or negative.

Graph of the Function $f: x \to ax^2$

The *function* $f: x \to ax^2$ is called the *quadratic function*. The *graph* of the *function* $f: x \to ax^2$ is a *smooth curve* passing through the *origin*, which is *symmetrical* about the *y-axis*. This *smooth curve* is called a *parabola*. The *width of the curve varies* as the value of the *constant a*, which is a *real number*, that is, $a \in R$.

Example 16

Draw the graph of the quadratic function:

(a) $f: x \to -2x^2$ (b) $f: x \to -x^2$

(c) $f: x \to -\frac{1}{2}x^2$ (d) $f: x \to \frac{1}{2}x^2$

(e) $f: x \to x^2$ (f) $f: x \to 2x^2$

for $-3 \leqslant x \leqslant 3$, *on the same graph paper, using the same scales and axes.*

Solution

From the information given, we have the following tables of values, which are then used to draw the graphs.

Table 7.5 Tables of values

x	-3	-2	-1	0	1	2	3
$f(x) = x^2$	9	4	1	0	1	4	9
$f(x) = -x^2$	-9	-4	-1	0	-1	-4	-9
$f(x) = 2x^2$	18	8	2	0	2	8	18
$f(x) = -2x^2$	-18	-8	-2	0	-2	-8	-18
$f: x = \frac{1}{2}x^2$	4.5	2	0.5	0	0.5	2	4.5
$f: x = -\frac{1}{2}x^2$	-4.5	-2	-0.5	0	-0.5	-2	-4.5

In the following diagram, the *graphs* of $f: x \to -2x^2, f: x \to -x^2, f: x \to -\frac{1}{2}x^2, f: x \to \frac{1}{2}x^2$, $f: x \to x$ and $f: x \to 2x^2$

for the *domain* $-3 \leqslant x \leqslant 3$ have been *plotted.* It can be seen that the *graphs* all passed through the *origin*, and that the *width of the curve* varied as the value of the *coefficient of* x^2. Whether the curve was *inverted* or not, depended on whether the *coefficient of* x^2 was *negative* or *positive*.

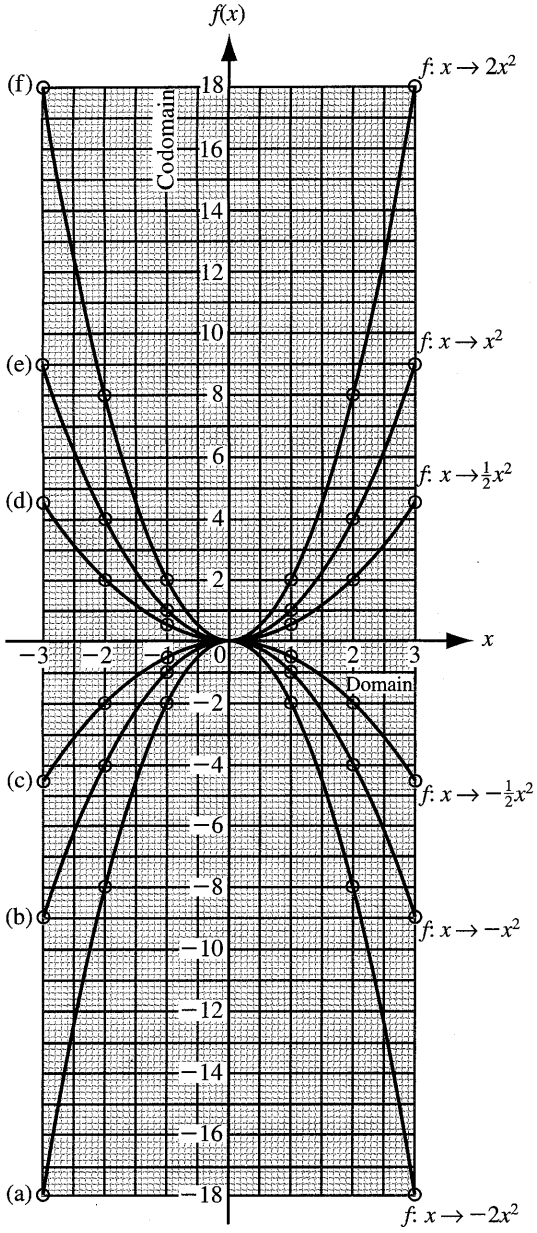


Fig. 7.36 Quadratic functions

Exercise 7j

1. Draw the graph of the linear function:
   (a) $f: x \to 2x$ (b) $f: x \to -2x$
   for the domain $-2 \leqslant x \leqslant 4$.
2. Draw the graph of the linear function:
   (a) $f: x \to \frac{1}{2}x$ (b) $f: x \to -\frac{1}{2}x$
   for the domain $-1 \leqslant x \leqslant 5$.
3. Draw the graph of the linear function:
   (a) $f: x \to 5x$ (b) $f: x \to -5x$
   for the domain $-2 \leqslant x \leqslant 6$.
4. Draw the graph of the linear function:
   (a) $f: x \to \frac{3}{4}x$ (b) $f: x \to -\frac{3}{4}x$
   for the domain $-2 \leqslant x \leqslant 8$.
5. Draw the graph of the quadratic function:
   (a) $f: x \to 2x^2$ (b) $f: x \to -2x^2$
   for the domain $-4 \leqslant x \leqslant 4$.
6. Draw the graph of the quadratic function:
   (a) $f: x \to 4x^2$ (b) $f: x \to -4x^2$
   for the domain $-3 \leqslant x \leqslant 3$.
7. Draw the graph of the quadratic function:
   (a) $f: x \to 3x^2$ (b) $f: x \to -3x^2$
   for the domain $-5 \leqslant x \leqslant 5$.
8. Draw the graph of the quadratic function:
   (a) $f: x \to \frac{1}{2}x^2$ (b) $f: x \to -\frac{1}{2}x^2$
   for the domain $-4 \leqslant x \leqslant 4$.
9. Draw the graph of the quadratic function:
   (a) $f: x \to \frac{4}{5}x^2$ (b) $f: x \to -\frac{4}{5}x^2$
   for the domain $-5 \leqslant x \leqslant 5$.
10. Draw the graph of the quadratic function:
    (a) $f: x \to \frac{2}{3}x^2$ (b) $f: x \to -\frac{2}{3}x^2$
    for the domain $-9 \leqslant x \leqslant 9$.

Direct Variation

If y is *directly proportional to* x, then we can write,

$$y \propto x\ .$$

We say that y *varies directly as* x. That is, when x *increases*, then y will also *increase* by a constant multiplier. And when x *decreases*, then y will also *decrease* by a constant multiplier. This implies that a *graph* of y *versus* x is a *straight line* passing through the *origin*.

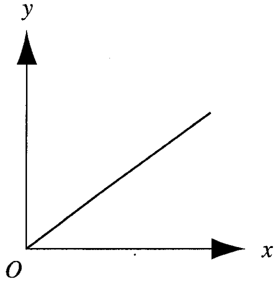


Fig. 7.37 Direct proportion graph

Further, we can write,

$$y = ax$$

where a = the *constant of proportion.*

In everyday situations we come across many examples of *direct variation.* For example:

(a) If the cost of a book is \$25, then the cost of x books is \$25x. Hence we say that the *total cost* of the books is *directly proportional* to x.

(b) The *area of a circle* is *directly proportional* to its *radius squared.* Since $A = \pi r^2$, then $A \propto r^2$, where π is the *constant of proportion.*

(c) The *volume of a sphere* is *directly proportional* to its *radius cubed.* Since $V = \frac{4}{3}\pi r^3$, then $V \propto r^3$, where $\frac{4}{3}\pi$ is the *constant of proportion.*

Example 17

If y is directly proportional to x, and y = 10 when x = 2.5, determine the value of:

(a) the constant of proportion, a

(b) x when y = 20

(c) y when x = 1.25

Solution

(a) Since $y \propto x$

Then $y = ax$

Given that $y = 10$ when $x = 2.5$

Then $10 = a(2.5)$

So the *constant,* $a = \frac{10}{2.5} = 4$

Thus $y = ax = 4x$

Hence the *constant of proportion, a* is 4.

(b) When $y = 20$

Then $y = 4x$ becomes

$20 = 4x$

So $x = \frac{20}{4} = 5$

Hence $x = 5$ when $y = 20$.

(c) When $x = 1.25$

Then $y = 4x = 4 \times 1.25 = 5$

Hence $x = 1.25$ when $y = 5$.

Let us look at the *results*:

Table 7.6 Table of results

x	1.25	2.5	5
y	5	10	20

It can be *seen* that:

(a) When x is *doubled*, then y is *doubled.* So the constant multiplier is 2.

(b) When x is *halved*, then y is *halved.* So the constant multiplier is $\frac{1}{2}$.

If y is *directly proportional* to x^2, then we can write,

$$y \propto x^2.$$

We say that y *varies directly* as x^2. This implies that a *graph* of y *against* x^2 is a *straight line* passing through the *origin.*

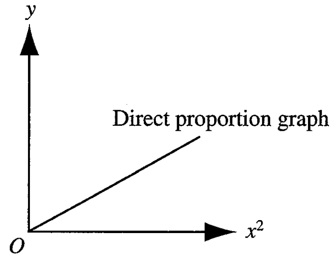


Fig. 7.38 Graph of y against x^2

Further, we can write,

$$y = ax^2$$

where a = the *constant of proportion.*

If however, we plot y *against* x, then we get a *curve* when $x \geq 0$.

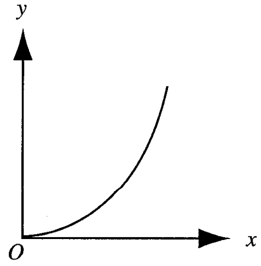


Fig. 7.39 Graph of y against x

Example 18

Given that $y = 3x^2$,

(a) plot a graph of y against x for $0 \leqslant x \leqslant 5$

(b) plot a graph of y against x^2 for $0 \leqslant x \leqslant 5$.

Solution

Table 7.7 Table of values

(a)

x	0	1	2	3	4	5
x^2	0	1	4	9	16	25
$y = 3x^2$	0	3	12	27	48	75

The *table of values* above, was then used to *plot* the *graph* of *y against x* for the *domain* $0 \leqslant x \leqslant 5$.

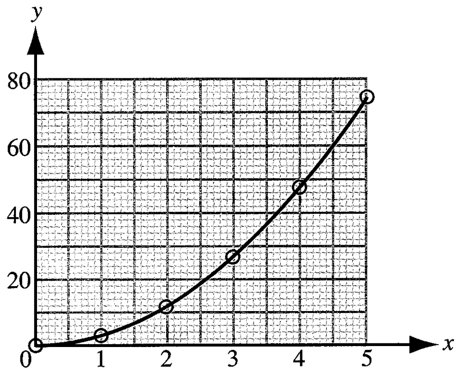


Fig. 7.40 Graph of y against x

Table 7.8 Table of values

(b)

x^2	0	1	4	9	16	25
$y = 3x^2$	0	3	12	27	48	75

The *table of values* above, was then use to *plot* the *graph* of *y against* x^2 for the *domain* $0 \leqslant x \leqslant 5$.

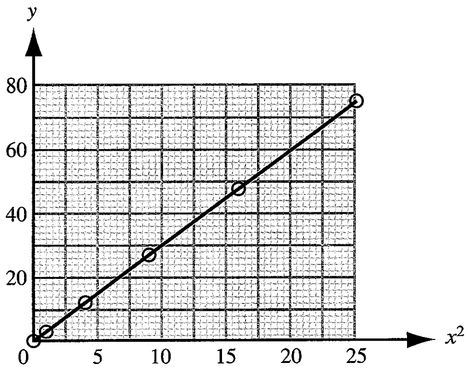


Fig. 7.41 Graph of y against x^2

Example 19

The area of a circle varies directly as the square of its radius. If the area of a circle of radius 7 cm is 154 cm², calculate:

(a) the area of a circle of radius 14 cm

(b) the radius of a circle of area 38.5 cm².

Solution

(a) Since $A \propto r^2$

Then $A = ar^2$

Given that $A = 154$ cm² when $r = 7$ cm

Then $154 = a(7)^2 = 49a$

So the *constant*, $a = \frac{154}{49} = \frac{22}{7}$

And $A = \frac{22}{7}(14)^2$

$= \frac{22}{7} \times 14 \times 14$

$= 616$ cm²

Hence the *area of the circle* of *radius* 14 cm is 616 cm².

(b) When $A = 38.5$ cm² and $a = \frac{22}{7}$

Then $A = ar^2$ becomes

$$38.5 = \frac{22}{7}r^2$$

So $r^2 = 38.5 \times \frac{7}{22} = 12.25$

i.e. $r = \sqrt{12.25} = 3.5$ cm

Hence the *radius of the circle* of *area* 38.5 cm² is 3.5 cm.

Let us look at the *results*:

Table 7.9 Table of results

r^2	12.25	49	196
A	38.5	154	616

It can be *seen* that:

(a) When r^2 is *multiplied* by 4, then A is *multiplied* by 4.

(b) When r^2 *divided* by 4, then A is *divided* by 4.

or

When r^2 is multiplied by $\frac{1}{4}$, then A is multiplied by $\frac{1}{4}$.

Inverse Variation

If y is *directly proportional to* $\frac{1}{x}$, then we can write,

$$y \propto \frac{1}{x}.$$

The *inverse* of x is $\frac{1}{x}$.

So we can say that, if y is *inversely proportional* to x, then we can write,

$$y \propto \frac{1}{x}.$$

We say that y *varies inversely* as x. That is, when x *increases* then y will *decrease*. And when y *decreases*, then x will *increase*. Thus a *graph* of y *versus* $\frac{1}{x}$ is a *straight line* passing through the *origin*.

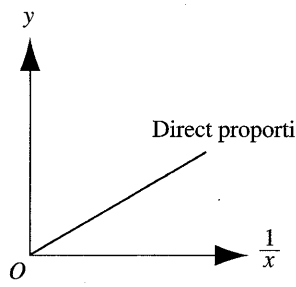


Fig. 7.42 Graph of y against $\frac{1}{x}$

Further, we can write,

$$y = \frac{a}{x} = ax^{-1}$$

where $a =$ the *constant of proportion*.

If however, we plot y *against* x, then we get a *curve* when $x \geqslant 0$.

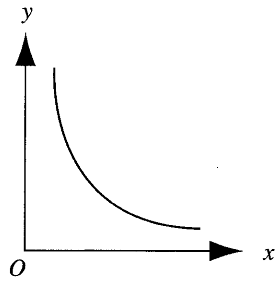


Fig. 7.43 Graph of y against x

In everyday situations we come across many examples of *inverse variation*. For example:

(a) In circular motion, the *acceleration* is *inversely proportional* to the *radius of the circle* described, provided the *velocity* is *constant*.

Since $a = \frac{v^2}{r}$, then $a \propto \frac{1}{r}$, where v^2 is the *constant of proportion*.

(b) The *current* conducted in a wire is *inversely proportional* to the *resistance* of the wire, provided the *voltage applied* is *constant*.

Since $I = \frac{V}{R}$ then $I \propto \frac{1}{R}$,

where V is the *constant of proportion*.

(c) If 8 men can do a piece of work in 6 days, then 4 men working at the *same rate* will take 12 days. The *time taken* to do the piece of work is *inversely proportional* to the *number of men* on the job.

Example 20

If y is inversely proportional to x, and y = 3 when x = 6, determine the value of:

(a) the constant of proportion, a

(b) x when y = 6

(c) y when x = 1.5.

Solution

(a) Since $y \propto \frac{1}{x}$

then $y = \frac{a}{x}$

Given that $y = 3$ when $x = 6$

then $3 = \frac{a}{6}$

So the *constant*, $a = 3 \times 6 = 18$

Thus $y = \frac{a}{x} = \frac{18}{x}$

Hence the *constant of proportion*, a is 18.

(b) When $y = 6$

then $y = \frac{18}{x}$ becomes

$6 = \frac{18}{x}$

So $x = \frac{18}{6} = 3$

Hence $x = 3$ when $y = 6$.

(c) When $x = 1.5$

Then $y = \frac{18}{x} = \frac{18}{1.5} = 12$

Hence $x = 1.5$ when $y = 12$.

Let us look at the *results*:

Table 7.10 Table of results

x	1.5	3	6
y	12	6	3

It can be *seen* that:

(a) When x is *doubled*, then y is *halved*.

(b) When x is *halved*, then y is *doubled*.

Example 21

Given that $y = \frac{3}{x}$,

(a) *plot a graph of* y *against* x *for* $1 \leqslant x \leqslant 9$

(b) *plot a graph of* y *against* $\frac{1}{x}$ *for* $1 \leqslant x \leqslant 9$.

Solution

Table 7.11 Table of values

(a)

x	1	3	6	9
$y = \frac{3}{x}$	3	1	$\frac{1}{2}$	$\frac{1}{3}$

The *table of values* above, was then used to plot the *graph* of *y against x* for the *domain* $1 \leqslant x \leqslant 9$.

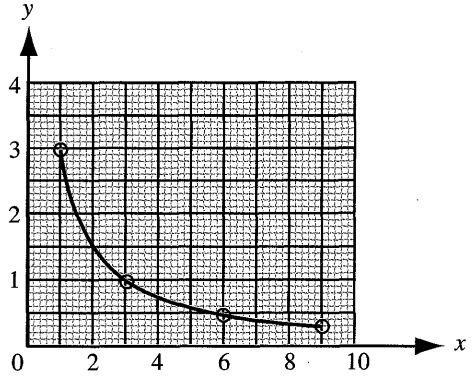


Fig. 7.44 Graph of y against x

Table 7.12 Table of values

(b)

$\frac{1}{x}$	1	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{9}$
$y = \frac{3}{x}$	3	1	$\frac{1}{2}$	$\frac{1}{3}$

The *table of values* above, was then used to plot the *graph* of *y against* $\frac{1}{x}$ for the *domain* $1 \leqslant x \leqslant 9$.

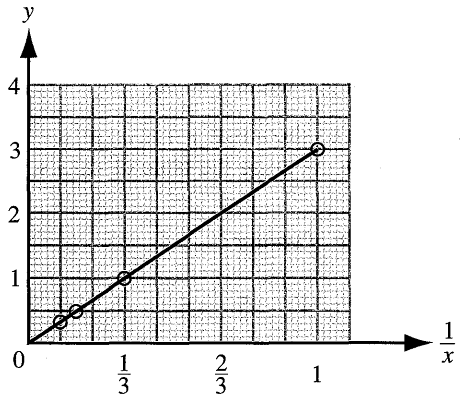


Fig. 7.45 Graph of y against $\frac{1}{x}$

Exercise 7k

1. Express the following with an equals sign and a constant of proportionality:
   (a) y varies directly as the square of x.
   (b) y varies directly as the cube of x.
   (c) y varies directly as the square root of x.
   (d) y varies directly as the cube root of x.

2. Express the following with an equals sign and a constant:
   (a) y varies inversely as the square of x.
   (b) y varies inversely as the cube of x.
   (c) y varies inversely as the square root of x.
   (d) y varies inversely as the cube root of x.

3. If $y = 2$ when $x = 4$, write down the value of y when $x = 6$ for the following:
   (a) y varies directly as the square of x.
   (b) y varies inversely as the square root of x.

4. (a) If y is proportional to the square root of x, and $y = 20$ when $x = 5$, evaluate y when $x = 9$.
   (b) Indicate how a straight line graph could be obtained for this relation.

5. (a) If y is inversely proportional to the cube of x and $y = 8$ when $x = 3$, evaluate y when $x = 6$.

(b) Indicate how a straight line graph could be obtained for this relation.

6. y is directly proportional to x. When $x = 15$, $y = 10$. What is the value of y when $x = 21$?

7. y is inversely proportional to x^2. When $x = 4$, $y = 5$. What is the positive value of x when y is 20?

8. y is directly proportional to x. When $x = 12$ then $y = 36$. What is y when $x = 24$?

9. Draw a graph to show that y is inversely proportional to x^2 using the table of values given below.

Table 7.13 Table of values

x	2	3	4	5	6	7	8	9
y	7.5	3.3	1.9	1.2	0.8	0.6	0.5	0.4

General Form of the Linear Function

The *general form* of the *linear function* is:

$f: x \rightarrow ax + c$

or $f(x) = ax + c$

or $y = ax + c$

or $\{(x, y): y = ax + c\}$,

where a = the *coefficient of* x,

c = the *constant term*,

x = the *independent variable*

and y = the *dependent variable*.

Further a and c are *real numbers*, that is, $a, c \in R$.

Of course, $y = ax + c = mx + c$.

That is, $a = m$ = the *gradient* of the straight *line* representing the *linear function*.

And c = the *intercept* of the straight line on the $f(x)$-*axis* (or y-*axis*).

Note that $\{(x, y): \ldots\}$ means '*The set of all ordered pairs x and y such that*'.

Graph of the Linear Function

One *method* of drawing the *graph* of the *linear function* $f: x \rightarrow ax + c$, is to use a *table of values* to calculate a *set of ordered pairs* (x, y), from which a graph of y *against* x (or y *versus* x) can be drawn, using *graph paper* and suitable *scales*. This *method* is illustrated in Example 22, seen below. It should be noted that the *minimum number of points* that is sufficient to draw an *accurate linear graph is* 3.

Example 22

Draw the graph of the linear function:

(a) $f(x) = 2x - 1$ *(b)* $f(x) = -2x + 1$

for the domain $-3 \leqslant x \leqslant 3$, *using two different sheets of graph paper.*

Solution

(a) The *table of values* representing the *linear function* $f(x) = 2x - 1$, for the *domain* $-3 \leqslant x \leqslant 3$, can be seen *constructed* below.

Table 7.14 Table of values

x	-3	-2	-1	0	1	2	3
$2x$	-6	-4	-2	0	2	4	6
-1	-1	-1	-1	-1	-1	-1	-1
$f(x) = 2x - 1$	-7	-5	-3	-1	1	3	5

Using the *table of values* above, the *graph* of the *linear function*, for the given *domain*, was then *drawn* on graph paper.

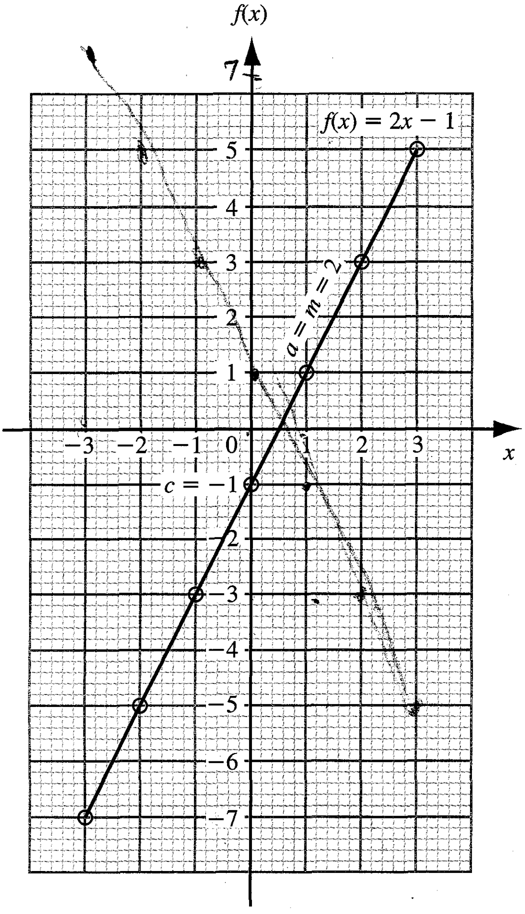


Fig. 7.46 Straight line

(b) The *table of values* representing the *linear function* $f(x) = -2x + 1$, for the *domain* $-3 \leqslant x \leqslant 3$, can be seen *constructed* below.

Table 7.15 Table of values

x	-3	-2	-1	0	1	2	3
$-2x$	6	4	2	0	-2	-4	-6
$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
$f(x) = -2x + 1$	7	5	3	1	-1	-3	-5

Using the *table of values* above, the *graph* of the *linear function*, for the given *domain*, was then *drawn* on graph paper.

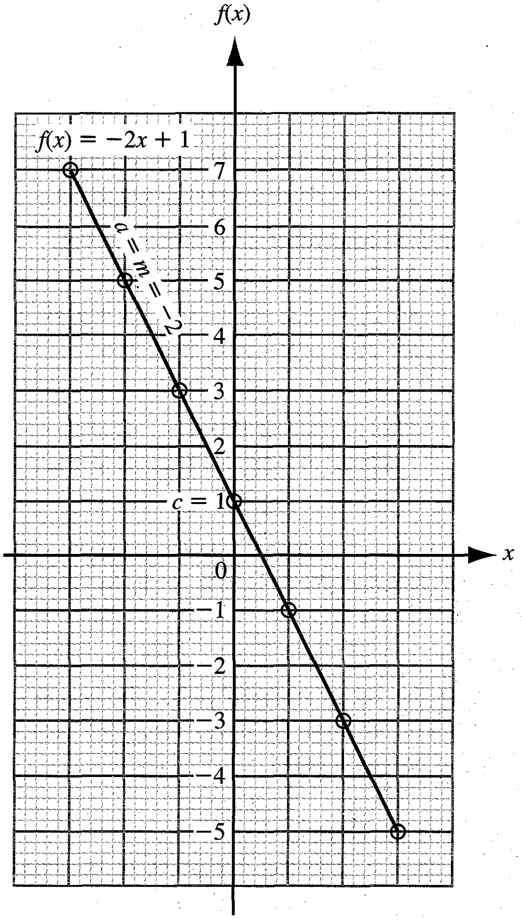


Fig. 7.47 Straight line

Alternative Method 1

A *second method* of drawing the *graph* of the *linear function* $f\colon x \rightarrow ax + c$, is to *substitute* values of x from the given *domain* in the *equation* $f(x) = ax + c$, and then *calculate* the *particular* value for $f(x)$. The required *set of ordered pairs* will then be obtained.

This *method* can be seen illustrated below.

(a) Given the *linear function* $f(x) = 2x - 1$.

Then $f(-3) = 2(-3) - 1 = -6 - 1 = -7$

$f(-2) = 2(-2) - 1 = -4 - 1 = -5$

$f(-1) = 2(-1) - 1 = -2 - 1 = -3$

$f(0) = 2(0) - 1 = 0 - 1 = -1$

$f(1) = 2(1) - 1 = 2 - 1 = 1$

$f(2) = 2(2) - 1 = 4 - 1 = 3$

and $f(3) = 2(3) - 1 = 6 - 1 = 5$

So the *set of ordered pairs* representing the *linear function* $f(x) = 2x - 1$, for the *domain* $-3 \leqslant x \leqslant 3$ is $\{(-3, -7), (-2, -5), (-1, -3), (0, -1), (1, 1), (2, 3), (3, 5)\}$.

The graph of the *linear function* for the given *domain* can then be *drawn* on graph paper.

(b) Given the linear function $f(x) = -2x + 1$.

$$\begin{aligned} \text{Then}\quad f(-3) &= -2(-3) + 1 = 6 + 1 = 7 \\ f(-2) &= -2(-2) + 1 = 4 + 1 = 5 \\ f(-1) &= -2(-1) + 1 = 2 + 1 = 3 \\ f(0) &= -2(0) + 1 = 0 + 1 = 1 \\ f(1) &= -2(1) + 1 = -2 + 1 = -1 \\ f(2) &= -2(2) + 1 = -4 + 1 = -3 \\ \text{and}\quad f(3) &= -2(3) + 1 = -6 + 1 = -5 \end{aligned}$$

So the *set of ordered pairs* representing the *linear function* $f(x) = -2x + 1$, for the domain $-3 \leqslant x \leqslant 3$ is $\{(-3, 7), (-2, 5), (-1, 3), (0, 1), (1, -1), (2, -3), (3, -5)\}$.

Using the *set of ordered pairs*, the *graph* of the *linear function*, for the *domain*, can then be *drawn* on graph paper.

Alternative Method 2

A *third method* of drawing the *graph* of the *linear function* $f: x \rightarrow ax + c$, is to use the *intercepts* on the x-axis and the y-axis. We know that the *equation* of the *y-axis* is $x = 0$, and the *equation* of the *x-axis* is $y = 0$. So we can *substitute* $x = 0$ and then $y = 0$ in the equation $y = ax + c$, and hence *calculate* the *intercepts* of the straight line on the *y-axis* and the *x-axis* respectively.

The *disadvantage* of this *method* is the fact that there are no *checks* and *balances*. We know that *any two points* will give us a *straight line*. So if *one* or *both* of the *intercepts* on the axes are calculated incorrectly, we will *still* get a *straight line*.

This *method* can be seen illustrated below.

(a) Given the *linear function* $f(x) = 2x - 1$.

When $x = 0$

Then $y = 2(0) - 1 = 0 - 1 = -1$

So the *point of intersection* on the y-axis is $(0, -1)$.

And when $y = 0$

Then $0 = 2x - 1$

So $2x = 1$

i.e. $x = \frac{1}{2} = 0.5$

So the *point of intersection* on the x-axis is $(0.5, 0)$.

Using these *two points*, the *graph* of the *linear function* $f(x) = 2x - 1$, for the *domain* $-3 \leqslant x \leqslant 3$, was then *drawn* on graph paper.

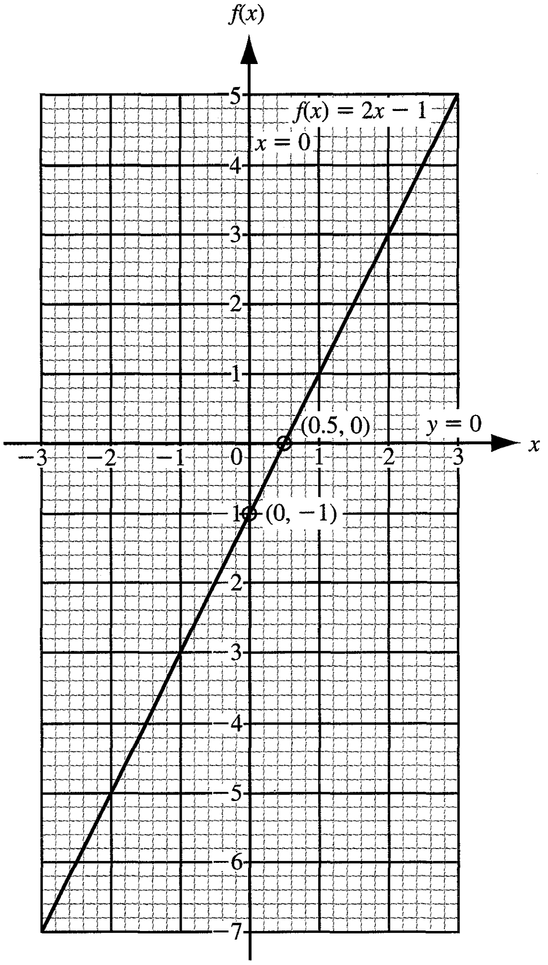


Fig. 7.48 Straight line

(b) Given the *linear function* $f(x) = -2x + 1$.

When $x = 0$

Then $y = -2(0) + 1 = 0 + 1 = 1$

So the *point of intersection* on the y-axis is $(0, 1)$.

And when $y = 0$

Then $0 = -2x + 1$

So $2x = 1$

i.e. $x = \frac{1}{2} = 0.5$

So the *point of intersection* on the x-axis is $(0.5, 0)$.

Using these *two points*, the *graph* of the *linear function* $f(x) = -2x + 1$, for the *domain* $-3 \leqslant x \leqslant 3$, was then *drawn* on graph paper.

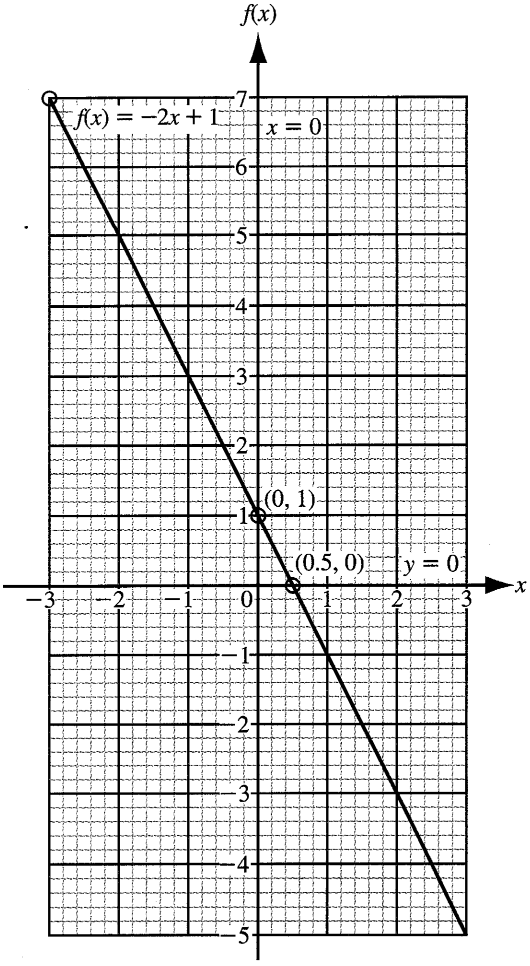


Fig. 7.49 Straight line

Example 23

Determine the set of ordered pairs (x, y) for the equation $2y + 3x = 6$ when $-2 \leqslant x \leqslant 6$, in order to plot a graph.

Solution

Before we can find the *set of ordered pairs* for the equation $2y + 3x = 6$, we need to make y the *subject of the equation*, that is, to write y in *terms* of x.

Given that $\quad 2y + 3x = 6$

Then $\quad 2y = 6 - 3x$

So $\quad y = \dfrac{6 - 3x}{2}$

i.e. $\quad y = 3 - \dfrac{3}{2}x$

The *set of ordered pairs* (x, y) for the *linear function* with the given *domain*, can then be obtained from either of the *two tables of values* below.

Table 7.16 Tables of values

x	−2	−1	0	1	2	3	4	5	6
6	6	6	6	6	6	6	6	6	6
$-3x$	+6	+3	0	−3	−6	−9	−12	−15	−18
$6 - 3x$	12	9	6	3	0	−3	−6	−9	−12
$y = \dfrac{6 - 3x}{2}$	6	$4\frac{1}{2}$	3	$1\frac{1}{2}$	0	$-1\frac{1}{2}$	−3	$-4\frac{1}{2}$	−6
	6	4.5	3	1.5	0	−1.5	−3	−4.5	−6

or

x	−2	−1	0	1	2	3	4	5	6
3	3	3	3	3	3	3	3	3	3
$-\dfrac{3}{2}x$	+3	$+1\frac{1}{2}$	0	$-1\frac{1}{2}$	−3	$-4\frac{1}{2}$	−6	$-7\frac{1}{2}$	−9
$y = 3 - \dfrac{3}{2}x$	6	$4\frac{1}{2}$	3	$1\frac{1}{2}$	0	$-1\frac{1}{2}$	−3	$-4\frac{1}{2}$	−6
	6	4.5	3	1.5	0	−1.5	−3	−4.5	−6

So the *set of ordered pairs* is {(−2, 6), (−1, 4.5), (0, 3), (1, 1.5), (2, 0), (3, −1.5), (4, −3), (5, −4.5), (6, −6)}.

Exercise 71

1. Draw the graph of each of the following linear functions for the domain $-5 \leqslant x \leqslant 5$ on graph paper.
   (a) $f: x \to 3x + 1$ (b) $f: x \to -3x - 1$
   (c) $f: x \to 3x + 1$ (d) $f: x \to -3x - 1$
2. Draw the graph of each of the following linear functions for the domain $-4 \leqslant x \leqslant 6$ on graph paper.
   (a) $f(x) = 5x - 2$ (b) $f(x) = 5x + 2$
   (c) $f(x) = -5x - 2$ (d) $f(x) = -5x + 2$
3. Draw the graph of each of the following linear equations for the domain $-3 \leqslant x \leqslant 5$ on graph paper.
   (a) $y = 6x + 5$ (b) $y = 6x - 5$
   (c) $y = -6x + 5$ (d) $y = -6x - 5$

4. Draw the graph of each of the following linear equations for the domain $-2 \leqslant x \leqslant 4$ on graph paper.

(a) $y = 4x - 3$ (b) $y = -4x - 3$
(c) $y = -4x + 3$ (d) $y = 4x + 3$

5. Draw the graph of each of the following functions using the domain $-3 \leqslant x \leqslant 6$.

(a) $f(x) = 7x + 1$ (b) $f: x \rightarrow -7x - 1$
(c) $y = -7x + 1$ (d) $\{(x, y): y = 7x - 1\}$

6. Draw the graph of each of the following linear functions using the domain $1 \leqslant x \leqslant 7$.

(a) $y = 8x + 5$ (b) $f(x) = -8x - 5$
(c) $f: x \rightarrow 8x - 5$ (d) $\{(x, y): y = -8x - 5\}$

7. Draw the graph of each of the following relations for the domain $3 \leqslant x \leqslant 10$.

(a) $\{(x, y): y = 2x + \frac{1}{2}\}$ (b) $f: x \rightarrow -2x - \frac{1}{2}$
(c) $y = -2x + \frac{1}{2}$ (d) $f(x) = 2x - \frac{1}{2}$

8. Draw the graph of each of the following relations for the domain $2 \leqslant x \leqslant 8$.

(a) $y = 9x + \frac{1}{4}$ (c) $f: x \rightarrow -9x + \frac{1}{4}$
(b) $f(x) = 9x - \frac{1}{4}$ (d) $\left\{(x, y): y = -9x - \frac{1}{4}\right\}$

9. Draw the graph of each of the following linear functions for the domain $0 \leqslant x \leqslant 8$.

(a) $y = \frac{2x + 3}{2}$

(b) $f: x \rightarrow \frac{2x - 3}{2}$

(c) $\left\{(x, y): y = \frac{-2x + 3}{2}\right\}$

(d) $f(x) = \frac{-2x - 3}{2}$

10. Draw the graph that represent each of the following linear functions for the domain $1 \leqslant x \leqslant 9$.

(a) $y = \frac{-5x + 1}{2}$

(b) $\left\{(x, y): \frac{5x + 1}{2}\right\}$

(c) $f(x) = \frac{-5x - 1}{2}$

(d) $f: x \rightarrow \frac{5x - 1}{2}$

Length of a Straight Line

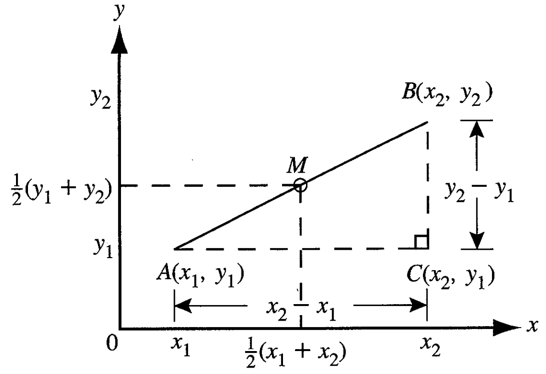


Fig. 7.50 Right-angled triangle.

Choose the *two end points* $A(x_1, y_1)$ and $B(x_2, y_2)$ of the *straight line AB*.
Then the *length of the straight line*,

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

according to *Pythagoras' theorem.*

Mid-Point of a Straight Line

The *coordinates* of the *mid-point of the straight line AB*,

$$M = \left(x_1 + \frac{1}{2}[x_2 - x_1],\ y_1 + \frac{1}{2}[y_2 - y_1]\right)$$
$$= \left(x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_1,\ y_1 + \frac{1}{2}y_2 - \frac{1}{2}y_1\right)$$
$$= \left(\frac{1}{2}x_1 + \frac{1}{2}x_2,\ \frac{1}{2}y_1 + \frac{1}{2}y_2\right)$$

i.e. $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Gradient of a Straight Line

The *gradient of a straight line AB*,

$$m = \frac{\text{The vertical rise}}{\text{The horizontal shift}} = \frac{BC}{AC}$$

i.e. $$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The *gradient* is *positive* when the line is *sloping* like this / and *negative* when the line is *sloping* like this \.

Example 24

Given the points P(3, −5) and Q(−9, 10), determine: (a) the length of PQ
(b) the mid-point of PQ
(c) the gradient of PQ

Solution

(a) The *length* of *PQ*

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-9 - 3)^2 + (10 - [-5])^2}$$
$$= \sqrt{(-12)^2 + (10 + 5)^2}$$
$$= \sqrt{(-12)^2 + (15)^2}$$
$$= \sqrt{144 + 225}$$
$$= \sqrt{369}$$
$$= 19.2 \text{ units}$$

Hence the *length* of *PQ* is 19.2 units.

(b) The *mid-point* of *PQ*,

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{3+(-9)}{2}, \frac{-5+10}{2}\right)$$
$$= \left(\frac{3 - 9}{2}, \frac{-5 + 10}{2}\right)$$
$$= \left(\frac{-6}{2}, \frac{5}{2}\right)$$
$$= (-3, 2.5)$$

Hence the *mid-point* of *PQ* is $M(-3, 2.5)$.

(c) The *gradient* of *PQ*, $m = \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{10 - (-5)}{-9 - 3}$$
$$= \frac{10 + 5}{-12}$$
$$= \frac{15}{-12}$$
$$= -\frac{5}{4}$$
$$= -1\frac{1}{4}$$
$$= -1.25$$

Hence the *gradient* of *PQ* is −1.25.

Alternative Method 1

The *alternative method* of *solving* this *problem* is a *graphical method.*

The *points* $P(3, -5)$ and $Q(-9, 10)$ are *plotted* on graph paper.

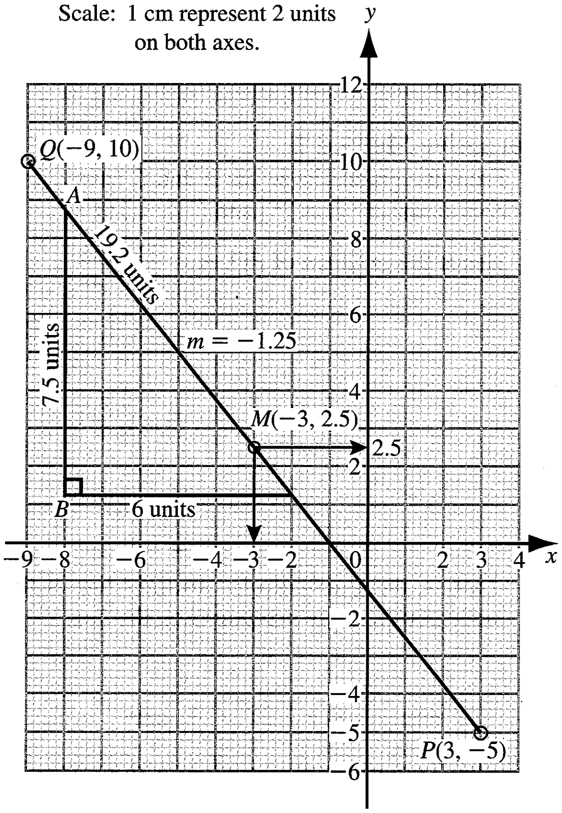


Fig. 7.51 Straight line

(a) *Draw* a *straight line* from P to Q.

By *measurement*,

the *length* of PQ $= 9.6$ cm

$= 9.6 \times 2$ units

$= 19.2$ units

(b) The horizontal shift defining PQ, range from -9 to 3, giving a mid-point with coordinate $x = -3$.

The vertical rise defining PQ, range from -5 to 10, giving a mid-point with coordinate $y = 2.5$.

Hence, the *mid-point* of PQ, $M = (-3, 2.5)$.

(c) Complete a suitable *right-angled triangle*, with PQ or part of PQ as the *hypotenuse*. For example, the *right-angled triangle ABC*.

Then the *gradient* of PQ,

$$\begin{aligned} m &= \frac{\textit{The vertical rise}}{\textit{The horizontal shift}} \\ &= \frac{AB}{BC} \\ &= -\frac{7.5 \cancel{\text{ units}}}{6 \cancel{\text{ units}}} \\ &= -1.25 \end{aligned}$$

Note that the *gradient* must be *negative* because of this *particular slope*.

Alternative Method 2

(c) From the graph, the *points A* and *C* have *coordinates* $(-8, 8.75)$ and $(-2, 1.25)$.

Hence the *gradient* of PQ, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$\begin{aligned} &= \frac{8.75 - 1.25}{-8 - (-2)} \\ &= \frac{7.5}{-8 + 2} \\ &= \frac{7.5}{-6} \\ &= -1.25 \end{aligned}$$

That is, the *gradient of a straight line* is the *same* between *any two points* on the *line*.

Exercise 7m

1. Using 2 cm to 1 unit on each axis, draw axes which range from 0 to 6 for x and from 0 to 10 for y. Plot the points

   $A(1, 2)$, $B(3, 6)$, and $C(5, 10)$.

   Determine the gradient of (a) AB and (b) BC.

2. Use the graph drawn for Question 1.

   (a) Determine the length of:

   (i) AB (ii) BC

   (b) State the mid-point of:

   (i) AB (ii) BC.

3. Plot the points $P(-3, 2)$, $Q(5, 6)$ and $R(4, -1)$ on graph paper.

   Hence determine the length of:

   (a) PQ (b) PR.

4. Use the graph drawn for Question 3.

   (a) Determine the mid-point of:

   (i) PQ (ii) PR

   (b) Calculate the gradient of:

   (i) PQ (ii) PR.

5. Given the points $A(-2, -4)$ and $B(5, -7)$,

   (a) determine the length of the line AB

   (b) calculate the gradient of the line AB

   (c) state the mid-point of the line AB.

6. Calculate the gradient of the line joining the pair of points $P(-3, 4)$ and $Q(5, -2)$.

7. Plot the points $P(-3, -4)$ and $Q(3, 7)$.

   Determine:

   (a) the length of the straight line PQ

   (b) the mid-point of PQ

   (c) the gradient of the straight line PQ.

8. Plot the points $A(8, 5)$ and $B(-4, 2)$.

   Calculate:

   (a) the length of the straight line AB

   (b) the mid-point of AB

   (c) the gradient of the straight line AB.

9. Plot the points $L(5, -6)$ and $M(-3, 5)$.

   Determine:

   (a) the length of the straight line LM

   (b) the mid-point of LM

   (c) the gradient of the straight line LM.

10. Plot the points $A(-5, -4)$, $B(2, 1)$, $C(4, 7)$ and $D(-3, 2)$. Join the points in alphabetical order to form the quadrilateral $ABCD$.

    (a) Name the type of quadrilateral.

    (b) Calculate the length of AB.

    (c) State the coordinates of the mid-point of AB.

    (d) Determine the magnitude of the gradient of AB.

Equation of a Straight Line

The *general equation* of a *straight line* is of the *form*:

$$y = mx + c\ ,$$

where m = the *gradient of the line*, which is the *coefficient of x*,

c = the *intercept on the y-axis*,

x = the *independent variable*

and y = the *dependent variable*.

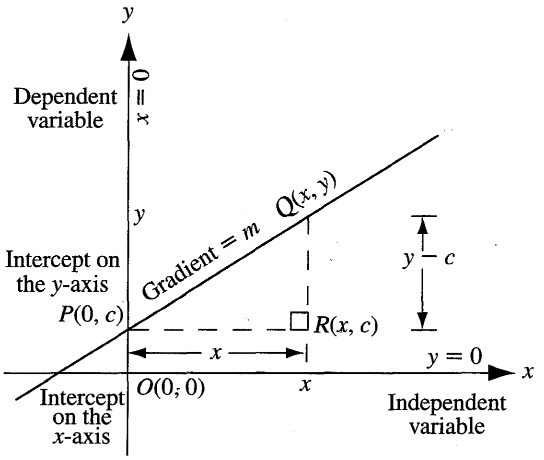


Fig. 7.52 Straight line

From the *diagram* above:

The *gradient* of PQ, $m = \dfrac{y - c}{x}$

$\therefore$ $mx = y - c$

Hence $y = mx + c$ is the *equation of the straight line PQ*.

If we want to determine the *particular equation* of a *straight line*, we can choose *any two points* on the *line* and *substitute* for x and y in $y = mx + c$, and then *solve* the two equations *simultaneously* for m and c.

Example 25

Two points, $A(-2, -4)$ and $B(4, 2)$ lie on a straight line L_1. Determine the particular equation of the line L_1.

Solution

Using the *equation of a straight line*:

$$y = mx + c$$

Then $-4 = m(-2) + c = -2m + c$

i.e. $-2m + c = -4$ —— ①

Also $2 = m(4) + c = 4m + c$

i.e. $4m + c = 2$ —— ②

② − ①, gives us,

$4m + 2m + c - c = 2 + 4$

i.e. $6m = 6$

$\therefore$ $m = \dfrac{6}{6} = 1$

Substituting $m = 1$ in ①, gives us,

$-2(1) + c = -4$

So $-2 + c = -4$

i.e. $c = -4 + 2 = -2$

Hence the *particular equation* of the line L_1 is:

$$y = x - 2.$$

Alternative Method 1

In this method, we first calculate the *gradient of the line*, using *two points* on the line. Then using the *gradient* and a *point* on the line, we *substitute* for m, x and y in $y = mx + c$, and *solve* for c.

Use the *points* $A(-2, -4)$ and $B(4, 2)$.

Then the *gradient* of the line L_1, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{-4 - 2}{-2 - 4}$

$= \dfrac{-6}{-6}$

$= 1$

Using the *gradient* $m = 1$, the *point* $B(4, 2)$ and the *equation of a straight line*:

xy

$$y = mx + c$$

Then $2 = 1(4) + c = 4 + c$

So $c = 2 - 4 = -2$

Or

Using the *gradient* $m = 1$, the *point* $A(-2, -4)$ and the *equation of a straight line*:

$$y = mx + c$$

Then $-4 = 1(-2) + c = -2 + c$

So $c = -4 + 2 = -2$

Hence the *particular equation* of the line L_1 is:

$$y = x - 2.$$

Exercise 7n

1. The straight line $y = mx + c$ passes through the points $(-2, -3)$ and $(4, 9)$. Determine the values of m and c and hence write the particular equation that represents the straight line.

2. Calculate the values of m and c if the straight line $y = mx + c$ passes through the points $(-3, -2)$ and $(1, 6)$. Hence state the particular equation for the straight line.

3. Determine the values of m and c if the straight line $y = mx + c$ passes through the points $(-4, 3)$ and $(1, 5)$. Hence write the particular equation for the straight line.

4. (a) Using a scale of 1 cm to represent 1 unit on each axis, plot on graph paper the points $P(3, -1)$ and $Q(-3, 5)$.
   (b) Calculate the gradient of PQ.
   (c) Determine the point where PQ meets the y-axis.
   (d) State the equation of PQ in the form $y = mx + c$.

5. The coordinates of A and B are $(4, 7)$ and $(6, 3)$, respectively. X is the mid-point of AB.
   (a) Calculate:
      (i) the length of AB
      (ii) the gradient of AB
      (iii) the coordinates of X
      (iv) the intercept of AB on the y-axis.
   (b) Hence, state the particular equation of the straight line AB.

6. The coordinates of P and Q are $(-4, -7)$ and $(6, 3)$, respectively. X is the mid-point of PQ.
   (a) Calculate:
      (i) the length of PQ
      (ii) the gradient of PQ
      (iii) the coordinates of X.
   (b) (i) From the graph, find the intercept on the y-axis.
      (ii) Hence, write the particular equation of the straight line PQ.

7. Plot the points $L(3, 6)$ and $M(9, 8)$ on graph paper.
   (a) From the graph, determine:
      (i) the gradient of LM
      (ii) the intercept of LM produced on the y-axis.
   (b) Hence, state the particular equation of the straight line passing through the given points.

8. Using a graphical method, determine the particular equation of the line passing through the points $R(-4, -6)$ and $S(1, -2)$.

9. Using a graphical method, determine the particular equation of the line passing through $X(-12, -4)$ and $Y(6, 8)$.

10. Using a graphical method, determine the particular equation of the line passing through $M(-5, 2)$ and $N(10, -4)$.

11. (a) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, plot on graph paper the points $L(4, 10)$ and $M(2, 7)$.
    (b) Join LM and calculate the gradient of the line LM.
    (c) Produce LM to intersect the y-axis at N. Hence, state the coordinates of N and write down the equation of the line LM.
    (d) Calculate the mid-point of the line LM.

Alternative Method 2

In this *method*, using the *gradient* and a *point*, we *substitute* the known quantities in the *formula* $y - y_1 = m(x - x_1)$.

Given the *gradient* $m = 1$ and the *point* $B(4, 2)$.

Then $\quad y - y_1 = m(x - x_1)$

So $\quad y - 2 = 1(x - 4) = x - 4$

i.e. $\quad y = x - 4 + 2 = x - 2$

Hence the *particular equation* of the line L_1 is:

$$y = x - 2.$$

Or

Given the *gradient* $m = 1$ and the *point* $A(-2, -4)$.

Then $\quad y - y_1 = m(x - x_1)$

So $\quad y - (-4) = 1(x - [-2])$

i.e. $\quad y + 4 = x + 2$

$\therefore \quad y = x + 2 - 4 = x - 2$

Hence the *particular equation* of the line L_1 is

$$y = x - 2.$$

Alternative Method 3

This *alternative method* of *solving* the *problem* is a *graphical method.*

The *points* $A(-2, -4)$ and $B(4, 2)$ are *plotted* on graph paper. Or the graph of the line is given.

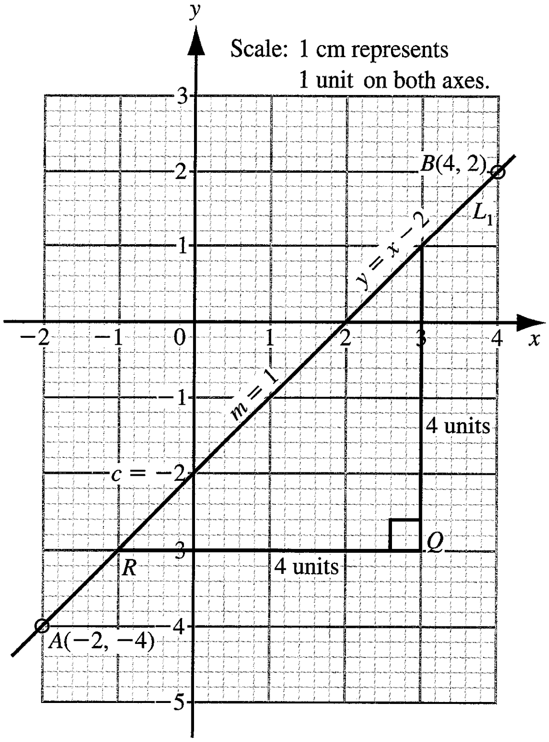


Fig. 7.53 Straight line

From the graph:
The *intercept* of L_1 on the *y-axis*, $c = -2$.

And the *gradient* of L_1,
$$m = \frac{\text{The vertical rise}}{\text{The horizontal shift}} = \frac{PQ}{QR} = \frac{4\ \cancel{\text{units}}}{4\ \cancel{\text{units}}} = 1$$

Note that the *gradient* must be *positive* because of this *particular slope.*

Hence, the *particular equation* of the straight line L_1 is:
$$y = x - 2.$$

Exercise 7o

1. (a) Determine the values of m and c if the straight line $y = mx + c$ passes through the point (3, 5) and has a gradient −4.
(b) State the particular equation of the straight line.

2. (a) Calculate the values of m and c if the straight line $y = mx + c$ passes through the point (−3, 5) and has a gradient of 4.
(b) State the particular equation of the straight line.

3. The end-points of a straight line are $P(-3, 7)$ and $Q(5, 9)$.
(a) Evaluate:
(i) the length of PQ
(ii) the mid-point of PQ
(iii) the gradient of PQ.
(b) Write down the particular equation of the straight line PQ.

4. The end-points of a straight line are $P(-1, 1)$ and $Q(-5, 2)$.
(a) Determine:
(i) the length of PQ
(ii) the mid-point of PQ
(iii) the gradient of PQ.
(b) Hence, state the particular equation of the straight line PQ.

5. The coordinates of P and Q are (−2, −3) and (4, 5), respectively. X is the mid-point of PQ.
(a) Calculate:
(i) the length of PQ
(ii) the gradient of PQ
(iii) the coordinates of X
(iv) the intercept of PQ on the y-axis.
(b) Hence, state the particular equation of the line PQ.

6. The coordinates of A and B are (2, 5) and (6, 3), respectively. X is the mid-point of AB.
(a) Calculate:
(i) the length of AB
(ii) the gradient of AB
(iii) the coordinates of X.
(b) Determine the equation of the perpendicular bisector of AB.

7. (a) Using a scale of 1 cm to represent 1 unit on each axis, plot on graph paper the points $A(-3, 2)$ and $B(3, -2)$
(b) Calculate the gradient of AB.
(c) Calculate the point where AB meets the y-axis.
(d) Write down the equation of AB in the form $y = mx + c$.
(e) Calculate:
(i) the length of AB
(ii) the mid-point of AB.

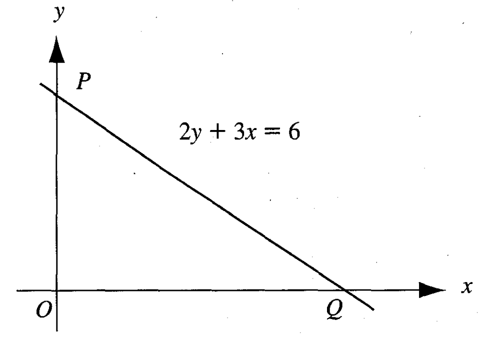


Fig. 7.54 Straight line

8. In the figure above, $2y + 3x = 6$ is a straight line.
   (a) Calculate the coordinates of P.
   (b) Determine the gradient of the straight line.
   (c) State the coordinates of Q.

9. Using a graphical method, determine the equation of the straight line passing through the points $P(-4, 2)$ and $Q(10, 16)$.

10. Using a graphical method, determine the particular equation of the line passing through the points $K(-6, 4)$ and $L(4, -2)$.

11. (a) Using 1 cm to represent 1 unit on each axis, plot the points $(-2, -5)$, $(-1, -3)$, $(3, 5)$ and $(4, 7)$ on graph paper. Draw a straight line passing through the points.
   (i) Evaluate the gradient of the line.
   (ii) Determine the y-intercept.
   (iii) Hence, state the equation of the straight line.
   (b) Use your graph to find the value of y when x is.
   (i) -1.5 (ii) 0 (iii) 2

12. (a) Draw the graph of the straight line $y = -3x - 2$ for x values $-2, 0, 3$. Use your graph to determine the value of x when y is
   (i) 5 (ii) -1 (iii) -4
   (b) State for the straight line:
   (i) its gradient
   (ii) its intercept on the y-axis.

13. Write down the (x, y) equation of a line which passes through the point $(0, 4)$ and has a gradient of 3.

14. State the (x, y) equation of a line which passes through the point $(1.5, 4.5)$ and has a gradient of -2.5.

Parallel Lines

Two straight lines are said to be *parallel*, if they have equal *gradients*.

Example 26

Prove that the two straight lines are parallel:
$y - 7x = 9$ and $2y = 14x - 5$.

Solution

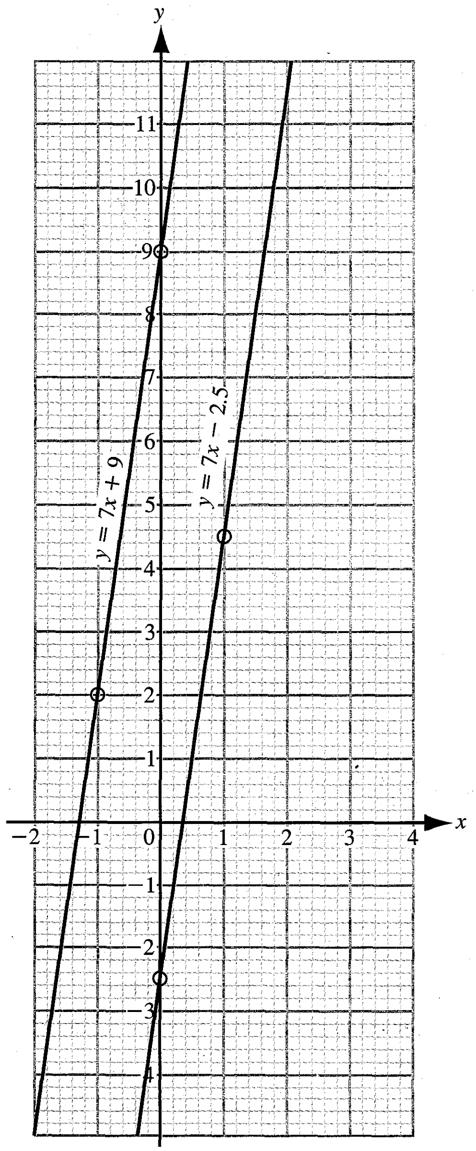


Fig. 7.55 Parallel lines

Given $y - 7x = 9$

Then $y = 7x + 9 \quad \Rightarrow m_1 = 7$

Given $2y = 14x - 5$

Then $y = \dfrac{14x - 5}{2}$

So $y = 7x - 2.5 \quad \Rightarrow m_2 = 7$

Thus $m_1 = m_2 = 7$

Hence the *two straight lines* are *parallel.*

Alternatively, a *graphical method* can be used to determine the *gradients* of the *lines.*

Perpendicular Lines

Two straight lines are said to be *perpendicular*, if the *product of their gradients* is *equal* to *negative one.*

Example 27

Prove that the two straight lines are perpendicular: 3x + 2y = 7 and 3y − 2x = 5.

Solution

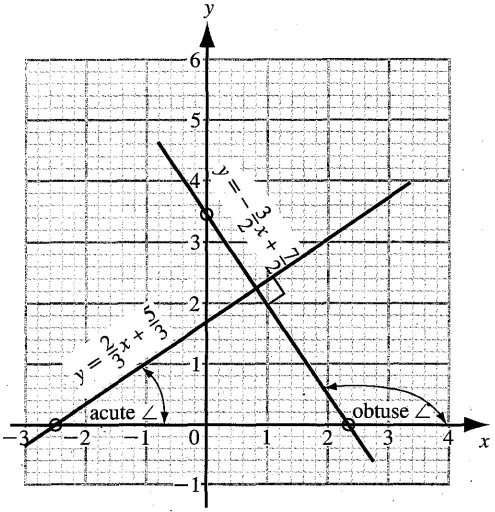


Fig. 7.56 Perpendicular lines

Given $3x + 2y = 7$

Then $2y = -3x + 7$

So $y = \dfrac{-3x + 7}{2}$

i.e. $y = -\dfrac{3}{2}x + \dfrac{7}{2} \quad \Rightarrow m_1 = -\dfrac{3}{2}$

Given $3y - 2x = 5$

Then $3y = 2x + 5$

So $y = \dfrac{2x + 5}{3}$

i.e. $y = \dfrac{2}{3}x + \dfrac{5}{3} \quad \Rightarrow m_2 = \dfrac{2}{3}$

Thus $m_1 \times m_2 = -\dfrac{3}{2} \times \dfrac{2}{3} = -1$

Hence the *two straight lines* are *perpendicular.*

> Note that when *m* is *positive*, the line makes an *acute angle* with the *x-axis*. And when *m* is *negative*, the line makes an *obtuse angle* with the *x-axis*. All *angles* being measured *counterclockwise* from the *positive x-axis.*

Exercise 7p

1. State if the following pairs of lines are:

(i) parallel,
(ii) perpendicular to each other, or
(iii) neither parallel nor perpendicular to each other.

(a) $y = 7x - 2$ and $y - 7x = 5$

(b) $2y + 4x = 5$ and $y - \dfrac{1}{2}x = 4$.

Prove your answer in each case.

2. State if the following lines are:

(i) parallel,
(ii) perpendicular to each other, or
(iii) neither parallel nor perpendicular to each other.

$2y = 6x - 5$

$3y = -x + 7.$

Prove your answer in each case.

3. Draw the graphs of the linear equations $y = -3x + 1$ and $y = -3x - 2$ on the same graph paper with the same scales and axes. Prove that the two straight lines are either parallel or perpendicular.

4. Given the linear equations:

$2y = 3x - 8$ —— ①

$3y = 6 - 2x$ —— ②

$2y - 3x = -4$ —— ③

write each of the three equations in the form $y = mx + c$.

Hence state:

(i) which pair/s of straight lines are parallel

(ii) which pair/s of straight lines are perpendicular.

Prove your answer in each case.

5. State which of the following pairs of lines are:

(i) parallel,

(ii) perpendicular to each other, or

(iii) neither parallel nor perpendicular to each other.

(a) $3y = 5x + 2, 5y + 3x = 4$

(b) $2y = 3x - 4, 4y + 8 = 6x$

(c) $5y + 8 = 6x, 5y = 4x - 3$

Prove your answer in each case.

6.

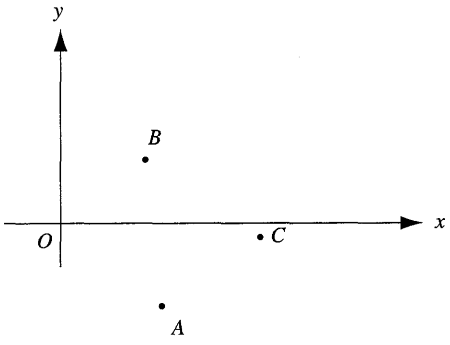


Fig. 7.57 Quadrilateral

The points A, B and C have coordinates $(4, -5)$, $(3, 4)$ and (p, q) respectively, as shown in the diagram above.

(a) Determine the length of AB.

(b) Calculate the values of p and q if $OACB$ is a parallelogram.

(c) Evaluate the mid-point of BC.

7. A quadrilateral $ABCD$ is formed by joining the points whose coordinates are $A(-1, -4)$, $B(0, 3)$, $C(3, 4)$ and $D(8, -1)$.

(a) Calculate the length of AC.

(b) Show that BD is perpendicular to A.

(c) Prove that $ABCD$ is a trapezium.

8. The coordinates of A and B are $(2, 5)$ and $(6, 3)$ respectively. X is the mid-point of AB.

(a) Calculate:

(i) the length of AB

(ii) the gradient of AB

(iii) the coordinates of X.

(b) Determine the gradient of the perpendicular bisector of AB.

9.

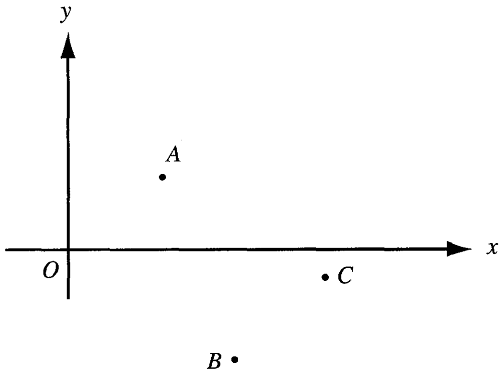


Fig. 7.58 Quadrilateral

In the diagram above, the points A, B and C have coordinates $(4, 3)$, $(6, -4)$ and (a, b), respectively.

(a) Determine the length of AB.

(b) If $OABC$ is a parallelogram, find the values of a and b.

10. Prove whether the following lines are

(i) parallel,

(ii) perpendicular to each other, or

(iii) neither parallel nor perpendicular to each other.

$15y - 6x = 5$

$5y = 2x - 5.$

11. Given the points $A(-1, -3)$ and $B(5, 2)$.

(a) Calculate:

(i) the length of the straight line AB

(ii) the mid-point of the straight line AB

(iii) the gradient of the straight line AB

(iv) the intercept on the y-axis

(v) the intercept on the x-axis

(vi) the equation of the line AB.

(b) Determine the equation of the perpendicular bisector of AB and state the coordinates of the point at which the perpendicular bisector meets the y-axis.

12. State the equation of the straight line through $(2, 3)$ parallel to $5x - 2y - 1 = 0$.

13. State the gradient of each of the lines $y = 2x + 5$, $2y = 4x - 9$, $3y = -6x + 7$ and $y = \frac{1}{2}x + 1$. Hence, determine which pair/s of lines are parallel.

Point of Intersection

A *point of intersection* is a point where at least two lines or two curves or a line and a curve meet.

Example 28

Draw the graphs for the relations:

$x = 2$

and $y = 3$

on the same graph paper, using the same scales and axes. Hence, find the point of intersection of the relations $x = 2$ *and* $y = 3$.

Solution

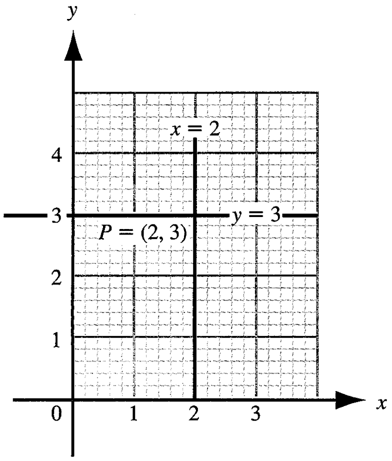


Fig. 7.59 Straight lines

From the *graphs*:

The *point of intersection* of $x = 2$ and $y = 3$ is $P(2, 3)$.

Solution of a Simple Equation by the Method of intersecting Graphs

In this *method*, we first have to draw the *graphs* representing a *linear function* $f(x) = ax + c$ and a *constant function* $f(x) = q$ or the *relation* $x = p$, on the same graph paper, using the same scales and axes. The *solution* is then given by the *point of intersection* of the two straight lines.

It should be noted that *three points* are the *minimum* number of points that is sufficient, to obtain an *accurate straight line graph.*

Example 29

Using a graphical method, solve each of the following pairs of equations:

(a) *(i)* $y = 3x - 2$
$y = 4$
(ii) $y = 3x - 2$
$y = -8$
(b) *(i)* $2y + 3x = 5$
$x = 3$
(ii) $2y + 3x = 5$
$x = -2$

Solution

Table 7.17 Table of values

(a)

x	-3	0	3
$3x$	-9	0	9
-2	-2	-2	-2
$y = 3x - 2$	-11	-2	7

Above can be seen the *table of values,* for the *equation* $y = 3x - 2$, for the *domain* $-3 \leqslant x \leqslant 3$.

Using the *table of values*, the *graph* representing the *linear function* $y = 3x - 2$, was *drawn* on graph paper.

The *graphs* representing the *constant functions* $y = 4$ and $y = -8$, were also *drawn* on the same graph paper, using the same scales and axes.

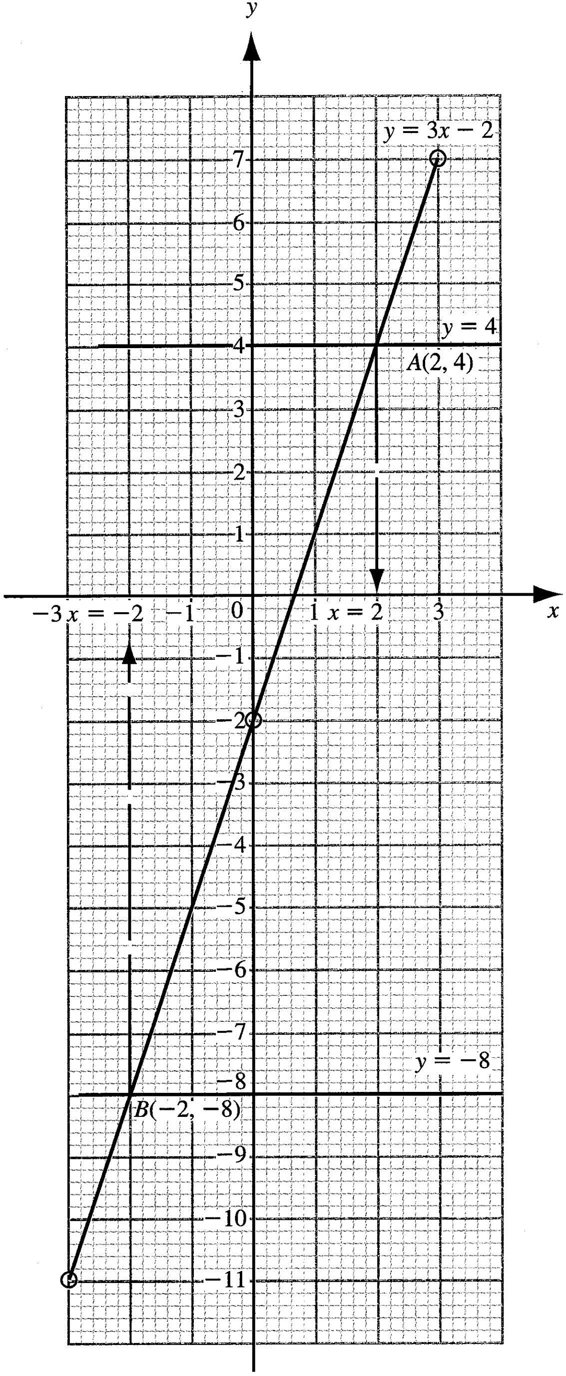


Fig. 7.60 Straight lines

From the graphs:

(i) The *point of intersection* of
$y = 3x - 2$ and $y = 4$ is $A(2, 4)$.
Hence, the *solution* is:
$x = 2$.

(ii) The *point of intersection* of
$y = 3x - 2$ and $y = -8$ is $B(-2, -8)$.
Hence, the *solution* is:
$x = -2$.

(b) Given that $2y + 3x = 5$
Then $2y = 5 - 3x$
So $y = \frac{5 - 3x}{2}$
y is now the *subject of the equation.*

Table 7.18 Table of values

x	-3	0	2
5	5	5	5
$-3x$	$+9$	0	-6
$5 - 3x$	14	5	-1
$y = \frac{5 - 3x}{2}$	7	2.5	-0.5

Above can be seen the *table of values*, for the *equation*
$2y + 3x = 5$ or $y = \frac{5 - 3x}{2}$, for the *domain* $-3 \leqslant x \leqslant 2$.

Using the *table of values*, the *graph* representing the *linear function* $2y + 3x = 5$, was *drawn* on graph paper.

The *graphs* representing the *relations* $x = 3$ and $x = -2$ were also *drawn* on the same graph paper, using the same scales and axes.

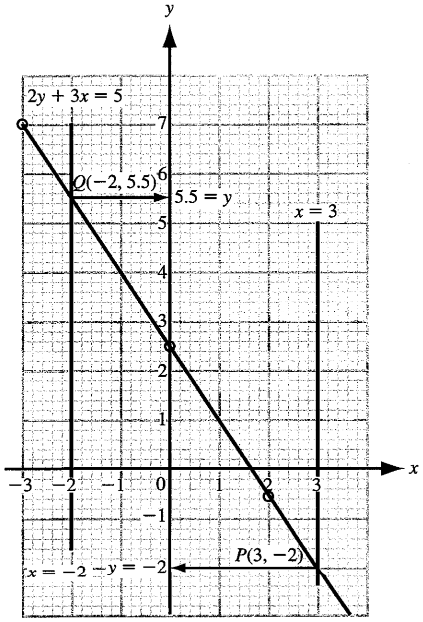


Fig. 7.61 Straight lines

From the graphs:

(i) The *point of intersection* of

$2y + 3x = 5$ and $x = 3$ is $P(3, -2)$.

Hence, the *solution* is:

$y = -2$.

(ii) The *point of intersection* of

$2y + 3x = 5$ and $x = -2$ is $Q(-2, 5.5)$.

Hence, the *solution* is:

$y = 5.5$.

Exercise 7q

1. (a) Draw the graph of the equation $y = -3x + 1$ for the x values -2, 0 and 2. Use your graph to find the value of y when x is

(i) -1 (ii) 1

(b) What is the value of the gradient of the straight line?

(c) State the intercept on the y-axis.

2. (a) Draw the graph of the equation $y = \frac{1}{2}x - 1$ for x values, -2, 1 and 4. Use your graph to find the value of x when y is

(i) -1 (ii) $\frac{1}{2}$

(b) What is the value of the gradient of the straight line?

(c) State the intercept on the x-axis.

3. (a) Draw the graph of the equation $y = 2x - 1$, for $-2 \leqslant x \leqslant 3$.

(b) State the value of the gradient.

(c) What is the magnitude of the intercept on the y-axis?

(d) Determine the value of x when $y = 3$.

4. (a) Draw the graph of the simple equation $y = 4x - 3$, for $-2 \leqslant x \leqslant 3$.

(b) State the value of the gradient.

(c) What is the magnitude of the intercept on the y-axis? Indicate this value on your graph.

(d) State the value of x when $y = 0.5$.

5. Solve the following equations using a graphical method:

(a) $y = 6x - 3$, $y = 15$ (b) $y = 5x + 2$, $y = 7$

(c) $y = 4x - 3$, $y = 5$ (d) $y = 6x + 3$, $y = 27$

6. Using a graphical method, solve the following equations:

(a) $y = 6x - 3$, $y = 15$ (b) $y = 5x + 3$, $y = 8$

(c) $y = 2x - 3$, $y = 12$ (d) $y = 7x - 3$, $y = -17$

7. Plot the following pairs of graphs and hence solve the equations stated:

(a) $y = 5x$, $y = 20$ (b) $y = 6x + 3$, $y = 15$

(c) $y = 2x - 1$, $y = 7$ (d) $y = 7x + 3$, $y = 31$

8. Plot the following pairs of graphs and hence solve the equations stated:

(a) $y = \frac{3}{7}x$, $y = \frac{1}{2}$ (b) $y = 4(3x + 1)$, $y = 64$

(c) $y = \frac{3}{4}x$, $y = \frac{1}{8}$ (d) $y = 3 - 5(2x + 1)$, $y = -2$

9. Plot the following graphs:

(a) $y = 3 - 2(x - 8)$, $y = 8$ (c) $y - 3x = 4$, $y = -2$

(b) $y = 2 - 3(x + 1)$, $y = -1$ (d) $y = 5 - 4x$, $y = 2$

Hence solve the equations given.

10. Plot the following graphs:

(a) $2y = 3x + 5$, $y = 2$ (c) $4y + 1 = 5x$, $4y = 19$

(b) $3y - 2x = -7$, $3y = -10$ (d) $3y + x = 2$, $6y = 7$

Hence solve the equations given.

Solution of Simultaneous Linear Equations by the Method of Intersecting Graphs

In this *method*, we first have to draw the *graphs* representing the *two linear equations* on the same graph paper, using the same scales and axes. The *solution* is then given by the *point of intersection* of the two *straight lines*.

Example 30

Using a graphical method, solve the pair of simultaneous equations:

$2x + 5y = 18$ ①

$3x - 2y = -11$ ②

Solution

Now $2x + 5y = 18$

So $5y = 18 - 2x$

$\therefore \quad y = \dfrac{18 - 2x}{5}$

y is now the *subject of the equation.*

Now $3x - 2y = -11$

So $2y = 3x + 11$

$\therefore \quad y = \dfrac{3x + 11}{2}$

y is now the *subject of the equation.*

Since both $y = \dfrac{18 - 2x}{5}$ and $y = \dfrac{3x + 11}{2}$ will give *straight line graphs*, we need only *choose three values for x* in the *table of values* in order to *draw* the *graphs.*

Table 7.19 Tables of values

x	-4	0	3
18	18	18	18
$-2x$	$+8$	0	-6
$18 - 2x$	26	18	12
$y = \dfrac{18 - 2x}{5}$	5.2	3.6	2.4

x	-4	0	3
$3x$	-12	0	9
$+11$	$+11$	$+11$	$+11$
$3x + 11$	-1	11	20
$y = \dfrac{3x + 11}{2}$	-0.5	5.5	10

Above are the *tables of values*, for $y = \dfrac{18 - 2x}{5}$ and $y = \dfrac{3x + 11}{2}$, for the *domain* $-4 \leqslant x \leqslant 3$.

Using the *tables of values*, the *graphs* were then *drawn* on graph paper.

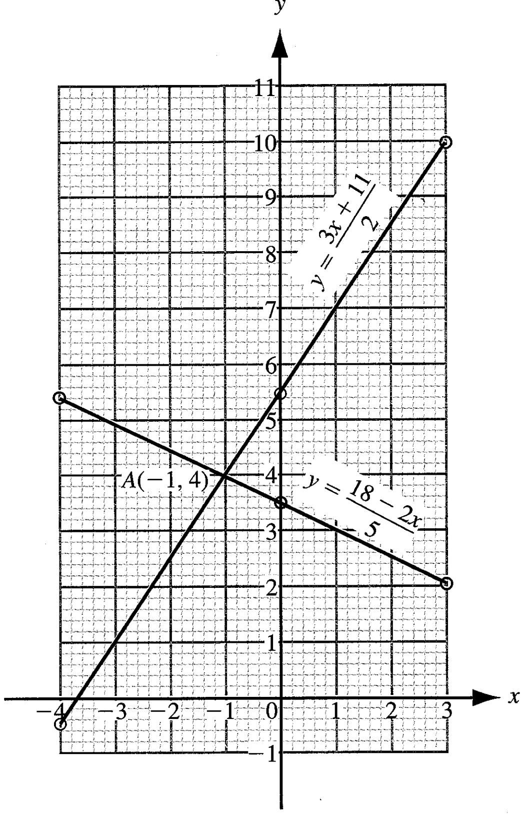


Fig. 7.62 Straight lines

From the graphs:
The *point of intersection* of
$2x + 5y = 18$
and $3x - 2y = -11$ is $A(-1, 4)$.
Hence, the *solution* is:
$x = -1$ and $y = 4$.

Exercise 7r

1. Using a graphical method, solve the following pairs of simultaneous equations:
   - (a) $y = 7 - x$, $y = 10 - 2x$
   - (b) $y = 7 - x$, $y = 2x - 5$
   - (c) $2y = 19 - 3x$, $2y = 5(x - 1)$
   - (d) $y = 3x - 22$, $y = 3 + 4x$
2. Plot the following graphs and hence solve each pair of simultaneous equations:
   - (a) $4x + 3y = 17$, $5x - 2y = 4$
   - (b) $-5x + 2y = 24$, $-7x + 3y = 35$
   - (c) $5x + 3y = 16.65$, $3x + 7y = 19.35$
   - (d) $2x + 3y = 10.0$, $5x + 2y = 19.5$

3. Draw the following graphs and hence solve the pairs of simultaneous equations:

(a) $3x + 2y = 19$
$5x - 2y = 5$

(b) $y = \frac{2(x-3)}{3}$
$y = \frac{x-2}{4} + 1$

(c) $-4x + 3y = 1$
$6x - y = 2$

(d) $y = \frac{21 - 3x}{5}$
$y = \frac{13 - 2x}{3}$

4. Solve the following pairs of simultaneous equations using a graphical method:

(a) $y = 4x + 3$
$y = 9 - 2x$

(b) $y = 6 - 2(x - 3)$
$y = x - 3$

(c) $y = -2x - 2$
$y = 5 - 3(x + 2)$

(d) $y = 7 - x$
$y = 10 - 2x$

5. Use a graphical method to solve the following equations:

(a) $4x + 3 = 9 - 2x$
(b) $6 - 2(x - 3) = x - 3$
(c) $3x + 2 = 7 - 2x$
(d) $y = 2 - 3(x + 1)$
$y = 4x$
(e) $7(5 - x) = 3(x - 5)$

6. Using a graphical method, solve each pair of the following simultaneous equations:

(a) $5x + 2y = 16$
$-3x + 4y = 6$

(b) $x + y = 7$
$2x + y = 10$

(c) $3x - 2y = -1$
$4x + 7y = 18$

(d) $x + y = 3.75$
$2x + 3y = 9.00$

7. Plot the following graphs and hence solve each pair of simultaneous equations:

(a) $-x + 3y = 6$
$8x + 3y = 24$

(b) $5x + y = 16$
$x - 2y = 1$

(c) $3x - 2y = 7$
$-x + 3y = -7$

(d) $3x - 5y = -13$
$-2x + 3y = 8$

8. Using graphs, solve the following equations:

(a) $3x - 2 = 5x - 32$
(b) $\frac{4}{5}x - \frac{3}{7} = \frac{4}{7}x + \frac{5}{7}$
(c) $3x - 5y + 16 = 0$
$\frac{2}{3}x + \frac{4}{5}y - 6 = 0$
(d) $5x - 3 = 2x + 15$

9. Use a graphical method to solve the following equations:

(a) $\frac{7}{8}x - \frac{1}{4} = \frac{4}{5}x + \frac{11}{4}$

(b) $5x + y = 16$
$x - 2y = 1$

(c) $y = \frac{19 - 3x}{2}$
$y = 2x - 1$

(d) $y = \frac{1 - 2x}{3}$
$y = \frac{x - 4}{2}$

10. Plot each of the following pairs of graphs:

(a) $2x - 5y = 3$
$x - 3y = 1$

(b) $2x + 3y = 1$
$-x + 2y = -4$

(c) $y = \frac{16 - 5x}{2}$
$y = \frac{3x + 6}{4}$

(d) $y = 7 - 2x$
$y = x + 1$

Hence, solve each pair of equations.

11. Solve each of the following pairs of simultaneous equations using graphs:

(a) $3x - 5y = -13$
$-2x + 3y = 8$

(b) $3x - 2y = 7$
$-x + 3y = -7$

(c) $x + y = 13$
$4x + y = 31$

(d) $x + y = 144$
$2x + 3y = 63$

12. Using graphs, solve the following pairs of simultaneous equations:

(a) $5x + 2y = 137$
$4x + 3y = 160$

(b) $f(x) = \frac{2x - 3}{7}$
$f(x) = \frac{3x - 5}{10}$

(c) $f(x) = 9 - 3x$
$f(x) = 2x - 11$

(d) $f(x) = \frac{5x + 24}{2}$
$f(x) = \frac{7x + 35}{3}$

13. Plot the following graphs:

(a) $f: x \rightarrow \frac{2x - 3}{7}$
$f: x \rightarrow \frac{x - 1}{3}$

(b) $f: x \rightarrow x$
$f: x \rightarrow \frac{7x + 0.5}{5}$

(c) $f: x \rightarrow \frac{-3.5 - 2x}{3}$
$f: x \rightarrow -4x - 2$

(d) $f: x \rightarrow \frac{29 - 5x}{2}$
$f: x \rightarrow x + 4$

Hence, solve each pair of simultaneous equations.

14. Plot the graphs of the following functions:

(a) $f: x \rightarrow \frac{3x + 1}{5}$
$f: x \rightarrow \frac{2(x - 1)}{3} - 9$

(b) $f: x \rightarrow \frac{15 - 9x}{5}$
$f: x \rightarrow \frac{3x + 6}{2}$

(c) $f: x \to \dfrac{x+6}{3}$ (d) $f: x \to \dfrac{4x+1}{3}$

$f: x \to \dfrac{24-8x}{3}$ $f: x \to 6x - 2$

Hence, solve each pair of simultaneous equations.

15. Use a graphical method to solve each of the following pairs of simultaneous equations:

(a) $2x - 3y = -15$ (b) $3x + 4y = 27$
$5x + 2y = 29$ $5x - 2y = 19$

(c) $7x - 2y = 19$ (d) $7x + 6y = 12.5$
$3x + 5y = 14$ $5x + 8y = 14.5$

16. Solve each of the following pairs of simultaneous equations using graphs:

(a) $4x + 6y = -7$ (b) $-4x + 3y = 1$
$4x + y = -2$ $6x - y = 2$

(c) $y = \dfrac{16 - 5x}{2}$ (d) $6y = -4x - 7$
$4y = 3x - 7$ $2 + y = -4x$

17. Using a graphical method, determine the solution of the simultaneous equations:

$$5x + 2y = 29$$
$$x - y = -4$$

Use a scale of 1 cm to represent 1 unit on each axis.

18. Solve the following simultaneous equations graphically, taking 2 cm to represent 1 unit.

$x + y = 7$ $0 \leqslant x \leqslant 5$
$y = x + 3$ $0 \leqslant y \leqslant 9$

19. Solve the following simultaneous equations graphically, taking 2 cm to represent 1 unit.

$3x + 2y = 6$ $0 \leqslant x \leqslant 4$
$2x - 2y = -1$ $-3 \leqslant y \leqslant 5$

20. Solve the following equations graphically. In each case draw axes for x and y and use values in the ranges indicated, taking 2 cm to represent 1 unit.

$x + y = 6$ $0 \leqslant x \leqslant 6$
$y = 3 + x$ $0 \leqslant y \leqslant 6$

21. Using a graphical method, solve the simultaneous equations:

$2x - 3y = 0.5$ —— ①
$5x + 4y = 18.5$ —— ②

Use the domain $0 \leqslant x \leqslant 4$.

22. Using a graphical method, solve the simultaneous equations:

$$3x - 2y = 0$$
$$-7x + 5y = 0.25$$

23. Find the point of intersection of the following pair of straight lines using a graphical method:

$$3y = x + 15$$
$$y + 3x = 4$$

24. Solve the following equation using graphs:

$$\frac{4x-3}{5} = \frac{5x+2}{12}$$

25. Solve the equation

$$\frac{2x-1}{5} = \frac{5x-11}{4}$$

using a graphical method.

Graphs of Linear Inequalities

A *linear relationship* between two variables x and y, is one that can always be represented graphically by a *straight line* which can be written in the form $y = mx + c$.

A *linear inequality* can be written in one of the *forms*:

(i) $y < mx + c$ (ii) $y \leqslant mx + c$
(iii) $y > mx + c$ (iv) $y \geqslant mx + c$.

Each linear inequality can be represented on a graph, by a region which is either shaded or unshaded.

Example 31

Illustrate on a graph the region that represents each of the following linear inequalities and state its solution set:

(a) $y > 3x - 1$ *(b)* $y \leqslant -4x + 3$.

Solution

(a) We first need to *draw* up a *table of values* for the *equation* $y = 3x - 1$.

Table 7.20 Table of values

x	0	2	4
$3x$	0	6	12
-1	-1	-1	-1
$y = 3x - 1$	-1	5	11

Above can be seen the *table of values*, for the *equation* $y = 3x - 1$, for the *domain* $0 \leqslant x \leqslant 4$.

Using the *table of values*, the *graph* representing the *line* $y = 3x - 1$, was then *drawn* on graph paper.

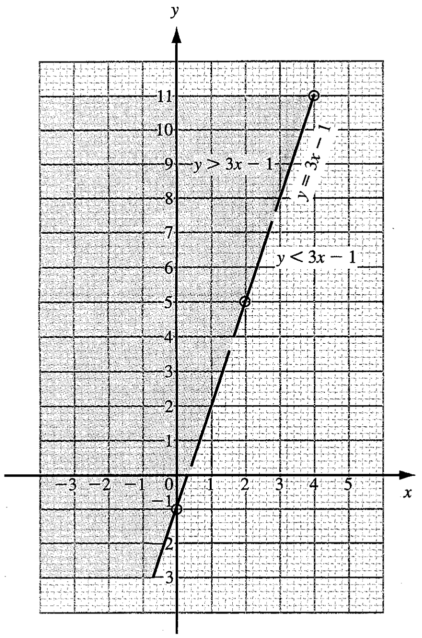


Fig. 7.63 Linear inequality

The *boundary line* $y = 3x - 1$ was drawn *broken* to indicate that it is *not part* of the region representing the *linear inequality* $y > 3x - 1$. The *region* representing the *linear inequality* $y > 3x - 1$ can be seen *shaded* in the graph shown previously. This *region* is *above* the *boundary line* $y = 3x - 1$.

Hence the *solution set* is $\{(x, y): y > 3x - 1\}$.

Where the symbol $\{x, y: \ldots\}$ means '*the set of all points* (x, y) *such that*'.

(b) We first need to set up a *table of values* for the *equation* $y = -4x + 3$.

Table 7.21 Table of values

x	-2	0	2
$-4x$	8	0	-8
$+3$	$+3$	$+3$	$+3$
$y = -4x + 3$	11	3	-5

Above can be seen the *table of values*, for the *equation* $y = -4x + 3$, for the *domain* $-2 \leqslant x \leqslant 2$.

Using the *table of values*, the *graph* representing the *line* $y = -4x + 3$, was then *drawn* on graph paper.

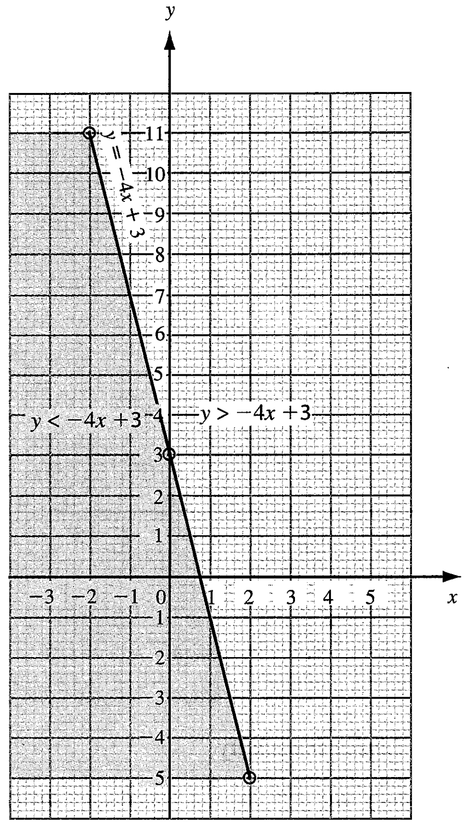


Fig. 7.64 Linear inequality

The *boundary line* $y = -4x + 3$ was drawn *unbroken* to indicate that it is *part* of the region representing the *linear inequality* $y \leqslant -4x + 3$. The *region* representing the *linear inequality* $y < -4x + 3$ can be seen *shaded* in the graph shown previously. This *region* is *below* the *line* $y = -4x + 3$. Thus the *linear inequality* $y \leqslant -4x + 3$ is represented by the *boundary line* $y = -4x + 3$ and the *shaded region* $y < -4x + 3$. Hence the *solution set* is $\{(x, y): y \leqslant -4x + 3\}$.

Alternative Method

An *alternative method* of representing an *inequality* on a graph is by shading the *complement* of the 'less than' or 'greater than' region.

The *boundary line* is drawn broken or unbroken, depending on whether it is part of the solution or not (as usual).

Hence the region representing the *inequality* is left *unshaded*.

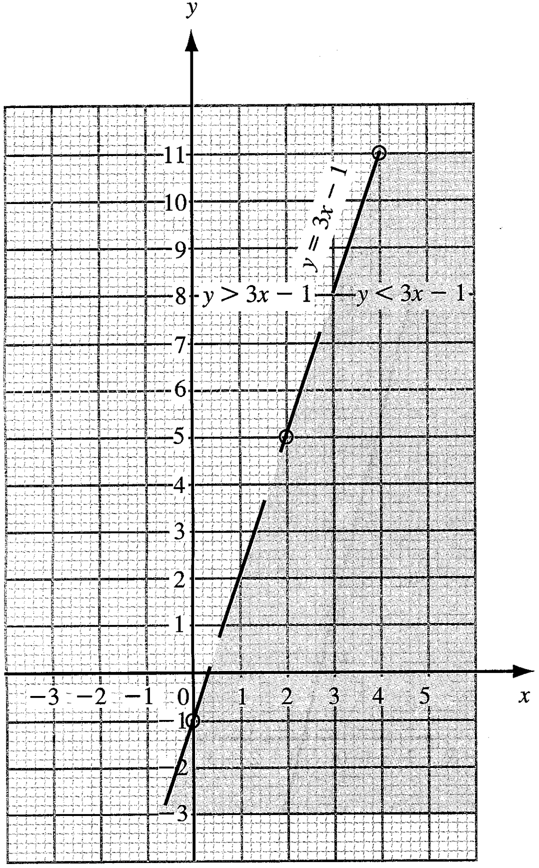


Fig. 7.65 Linear inequality

The *boundary line* $y = 3x - 1$ was drawn *broken* to indicate that it is *not part* of the region representing the *linear inequality* $y > 3x - 1$. The *region* representing the *linear inequality* $y < 3x - 1$ can be seen *shaded* in the graph shown above. This *region* is below the *boundary line* $y = 3x - 1$.

Thus the *region* representing the *linear inequality* $y > 3x - 1$ can be seen *unshaded* in the graph shown above. This *region* is *above* the *boundary line* $y = 3x - 1$.

Hence the *solution set* is $\{(x, y): y > 3x - 1\}$.

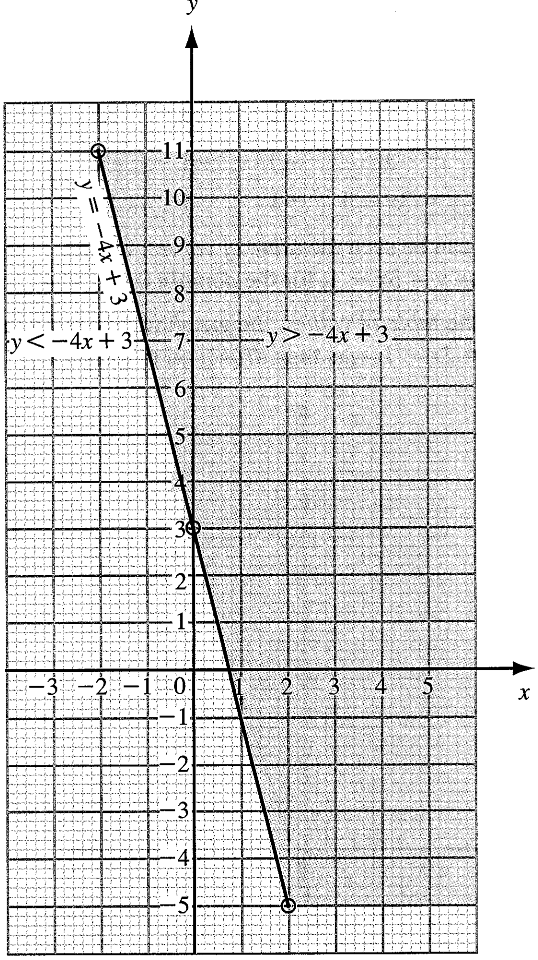


Fig. 7.66 Linear inequality

The *boundary line* $y = -4x + 3$ was drawn *unbroken* to indicate that it is *part* of the *linear inequality* $y \leqslant -4x + 3$. The *region* representing the *linear inequality* $y > -4x + 3$ can be seen *shaded* in the graph shown above. This *region* is *above* the *boundary line* $y = -4x + 3$. Thus the *linear inequality* $y \leqslant -4x + 3$ is represented by the *boundary line* $y = -4x + 3$ and the *unshaded region* $y < -4x + 3$.

Hence the *solution set* is $\{(x, y): y \leqslant -4x + 3\}$.

Exercise 7s

Illustrate on a graph the region that represents each of the following linear inequalities and state its solution set:

1. $y \geqslant 2x + 3$

2. $y \geqslant 3x + 1$

3. $y \geqslant 5x + 3$

4. $y \geqslant -4x + 1$

5. $y \geqslant -2x + 3$

6. $y \geqslant -5x + 4$

7. $y \geqslant 2x - 5$

8. $y \geqslant 3x - 2$

9. $y \geqslant 4x - 1$

10. $y \geqslant -3x - 2$

11. $y \geqslant -4x - 1$

12. $y \geqslant -5x - 3$

13. $y > x + 3$

14. $y > 2x + 1$

15. $y > 3x + 2$

16. $y > -x + 2$

17. $y > -2x + 3$

18. $y > -3x + 1$

19. $y > 4x - 3$

20. $y > 3x - 2$

21. $y > 5x - 4$

22. $y > -x - 1$

23. $y > -2x - 3$

24. $y > -3x - 4$

25. $y \leqslant 4x + 3$

26. $y \leqslant 5x + 4$

27. $y \leqslant 6x + 1$

28. $y \leqslant 7x - 1$

29. $y \leqslant 8x - 3$

30. $y \leqslant 6x - 5$

31. $y \leqslant -5x + 4$

32. $y \leqslant -7x + 3$

33. $y \leqslant -8x + 5$

34. $y \leqslant -9x - 5$

35. $y \leqslant -8x - 7$

36. $y \leqslant -7x - 6$

37. $y < 4x + 3$

38. $y < 5x + 7$

39. $y < 8x + 5$

40. $y < 7x - 3$

41. $y < 8x - 7$

42. $y < 9x - 8$

43. $y < -8x + 5$

44. $y < -9x + 7$

45. $y < -10x + 9$

46. $y < -11x - 7$

47. $y < -12x - 1$

48. $y < -13x - 9$

Solution of Simultaneous Linear Inequations

A *linear inequation* can be written in one of the *forms*:

(i) $y < mx + c$ (ii) $y \leqslant mx + c$

(iii) $y > mx + c$ (iv) $y \geqslant mx + c$.

The *linear inequation* $y < mx + c$ is satisfied by *all points below* the *line* $y = mx + c$.

And the *linear inequation* $y > mx + c$ is satisfied by *all points above* the *line* $y = mx + c$.

When *two or more linear inequations* are drawn *simultaneously* on graph paper, using the same scales and axes, then their *common solution* lies in the area where the *region* representing the inequations *intersect* (that is, *overlap*) at the same time. This area, called the *common region* is usually in the shape of a *polygon*. And the *maximum* and *minimum values* for an expression containing x and y always occur at the *vertices* of the *common region*. If *no vertex* can be a *solution*, then the *maximum* and *minimum values* will be satisfied by *all points* along one of the *sides* of the polygon. However we are not interested in determining a *maximum or* a *minimum value* as yet.

Example 32

Determine graphically the common region representing the inequations $y \geqslant x$, $y \leqslant 5$ and $y \geqslant -3x + 5$. Hence state the vertices in the common region.

Solution

The *equation* $y = x$ is a *straight line* passing through the *origin* and making and *angle of* 45° with the *positive x-axis*.

We need to draw up a *table of values* for the *equation* $y = -3x + 5$.

Table 7.22 Table of values

x	0	1	3
$-3x$	0	-3	-9
$+5$	$+5$	$+5$	$+5$
$y = -3x + 5$	5	2	-4

Above can be seen the *table of values*, for the *equation* $y = -3x + 5$, for the *domain* $0 \leqslant x \leqslant 3$.

The *lines* $y = x$, $y = 5$ and $y = -3x + 5$ were *drawn* on the graph paper, using the *same scales and axes* as seen in the two graphs on the next page.

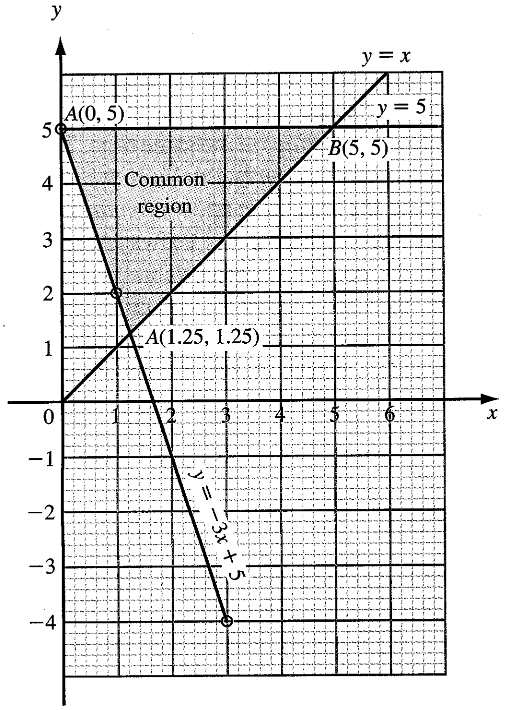


Fig. 7.67 Inequations

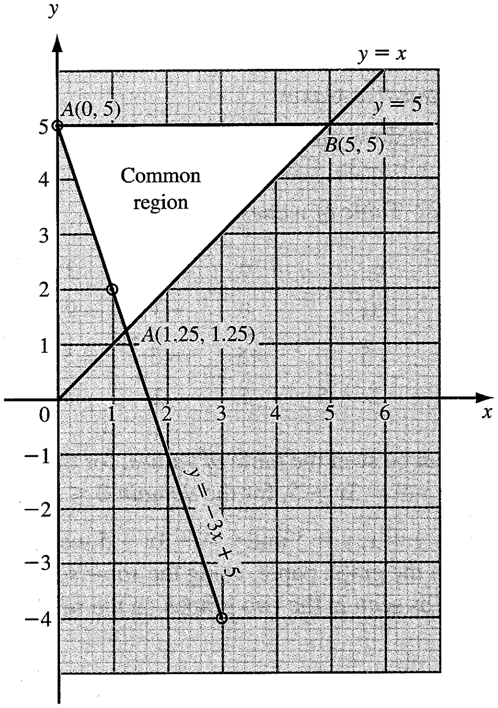


Fig. 7.68 Inequations

The *region* which contains the *common solution* is $\triangle ABC$. The *common region* can be seen *shaded* in Fig. 7.67 and *unshaded* in Fig. 7.68.

The *vertices* in the *common region* are $A(0, 5)$, $B(5, 5)$ and $C(1.25, 1.25)$.

Exercise 7t

Determine graphically the common region representing each of the following sets of inequations. Hence state the vertices in the common region.

1. $x \geqslant 0, y \geqslant 0$ and $y \leqslant -4x + 3$

2. $x \geqslant 0, y \geqslant 0$ and $y \leqslant -5x + 2$

3. $x \geqslant 0, y \geqslant 0$ and $y \leqslant -6x + 3$

4. $x \geqslant 0, y \geqslant 0$ and $y \leqslant -8x + 5$

5. $x \geqslant 1, y \leqslant 4$ and $y \geqslant 3x - 2$

6. $x \geqslant 2, y \leqslant 5$ and $y \geqslant 4x - 3$

7. $x \geqslant 1, y \leqslant 4$ and $y \geqslant 5x - 4$

8. $x \leqslant 2, x \geqslant 0, y \leqslant 5$ and $y \geqslant 7x - 5$

9. $x \geqslant 0, y \geqslant 0, y \leqslant 5$ and $y \geqslant -4x + 5$

10. $x \geqslant 0, y \geqslant 0, y \leqslant x$ and $y \leqslant -5x + 8$

11. $x \geqslant 0, y \geqslant 0, y \leqslant 3x$ and $y \leqslant -3x + 4$

12. $x \geqslant 0, y \geqslant 0, y \leqslant 2x + 1$ and $y \leqslant -2x + 5$

13. $y \leqslant 3, y \geqslant -2x + 4$ and $y \geqslant x - 2$

14. $y \leqslant 4, y \geqslant -\frac{3}{5}x + 3$ and $y \geqslant -\frac{4}{5}x + 4$

15. $y \leqslant 4, y \geqslant -\frac{5}{7}x + 5$ and $y \geqslant \frac{7}{15}x - 1$

General Form of the Quadratic Function

The *general form* of the *quadratic function* is:

$$f: x \to ax^2 + bx + c,\ a \neq 0$$

or $$f(x) = ax^2 + bx + c$$

or $$y = ax^2 + bx + c$$

or $$\{(x, y): y = ax^2 + bx + c\},$$

where a = the *coefficient of* x^2,
b = the *coefficient of* x,
c = the *constant term*, which is the *intercept of the curve on the y-axis*,
x = the *independent variable*
and y = the *dependent variable*.

Further, a, b and c are real numbers, that is, $a, b, c \in R$.

Graph of the Quadratic Function

One *method* of drawing the *graph* of the *quadratic function* $f: x \to ax^2 + bx + c$, is to use a *table of values* to calculate a set of *ordered pairs* (x, y), from which a graph of *y against x* (or *y versus x*) can be drawn, using *graph paper* and suitable *scales*. The graph representing a *quadratic function* is a *smooth curve* called a *parabola*. This *method* is illustrated in the example below.

Example 33

Draw the graphs of the quadratic functions:

(a) $f: x \to x^2 + 2x - 3$

(b) $f: x \to -x^2 - 2x + 3$

for the domain $-5 \leqslant x \leqslant 3$, *using two different sheets of graph paper.*

Solution

(a) The *table of values* representing the *quadratic function* $f: x \to x^2 + 2x - 3$, for the *domain* $-5 \leqslant x \leqslant 3$, can be seen *constructed* below.

Table 7.23 Table of values

x	−5	−4	−3	−2	−1	0	1	2	3
x^2	25	16	9	4	1	0	1	4	9
$+2x$	−10	−8	−6	−4	−2	0	+2	+4	+6
−3	−3	−3	−3	−3	−3	−3	−3	−3	−3
$f(x) = x^2 + 2x - 3$	12	5	0	−3	−4	−3	0	5	12

Using the *table of values* above, the *graph* of the *quadratic function*, for the given *domain*, was then *drawn* on graph paper.

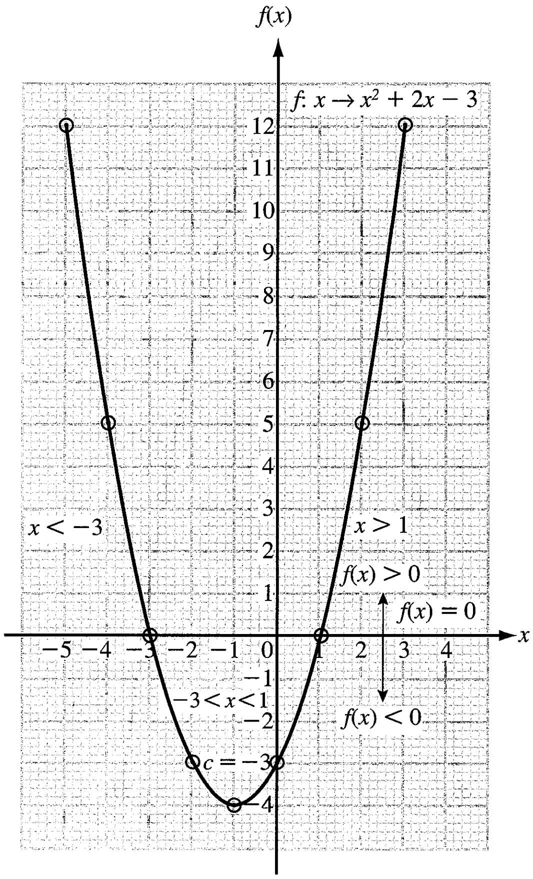


Fig. 7.69 Parabola

From the *graph*:

Note that the *range is negative*, that is, $f(x) < 0$, for the *domain interval* $\{x: -3 < x < 1\}$. And the *range is positive*, that is, $f(x) > 0$, for the *domain interval* $\{x: x < -3 \text{ and } x > 1\} = \{x: -3 \leqslant x \leqslant 1\}'$.

(b) The *table of values* representing the *quadratic function* $f: x \to -x^2 - 2x + 3$, for the *domain* $-5 \leqslant x \leqslant 3$, can be seen *constructed* below.

Table 7.24 Table of values

x	−5	−4	−3	−2	−1	0	1	2	3
$-x^2$	−25	−16	−9	−4	−1	0	−1	−4	−9
$-2x$	+10	+8	+6	+4	+2	+0	−2	−4	−6
+3	+3	+3	+3	+3	+3	+3	+3	+3	+3
$f(x) = -x^2 - 2x + 3$	−12	−5	0	3	4	3	0	−5	−12

Using the *table of values* above, the *graph* of the *quadratic function*, for the given *domain*, was then *drawn* on graph paper.

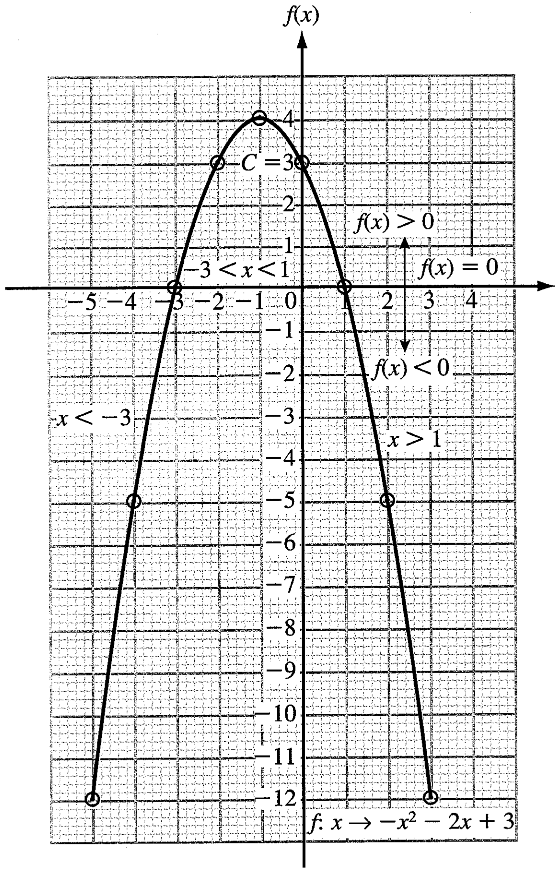


Fig. 7.70 Parabola

From the *graph*:

Note that the *range is positive*, that is, $f(x) > 0$, for the *domain interval* $\{x: -3 < x < 1\}$. And the *range is negative*, that is, $f(x) < 0$, for the *domain interval* $\{x: x < -3 \text{ and } x > 1\} = \{x: -3 \leqslant x \leqslant 1\}'$.

Alternative Method

A *second method* of drawing the *graph* of the *quadratic function* $f: x \to ax^2 + bx + c$, is to *substitute* values of x from the given *domain* in the *equation* $f(x) = ax^2 + bx + c$, and then *calculate* the *particular value* of $f(x)$. The required *set of ordered pairs* will then be obtained.

This *method* can be seen illustrated below

(a) Given the *quadratic function* $f(x) = x^2 + 2x - 3$.

Then $f(-5) = (-5)^2 + 2(-5) - 3$
$= 25 - 10 - 3 = 25 - 13 = 12.$

$f(-4) = (-4)^2 + 2(-4) - 3$
$= 16 - 8 - 3 = 16 - 11 = 5.$

$f(-3) = (-3)^2 + 2(-3) - 3$
$= 9 - 6 - 3 = 9 - 9 = 0.$

$f(-2) = (-2)^2 + 2(-2) - 3$
$= 4 - 4 - 3 = 4 - 7 = -3.$

$f(-1) = (-1)^2 + 2(-1) - 3$
$= 1 - 2 - 3 = 1 - 5 = -4.$

$f(0) = (0)^2 + 2(0) - 3$
$= 0 + 0 - 3 = -3.$

$f(1) = (1)^2 + 2(1) - 3 = 1 + 2 - 3$
$= 3 - 3 = 0.$

$f(2) = (2)^2 + 2(2) - 3 = 4 + 4 - 3$
$= 8 - 3 = 5.$

And $f(3) = (3)^2 + 2(3) - 3 = 9 + 6 - 3$
$= 15 - 3 = 12.$

So the *set of ordered pairs* representing the *quadratic function* $f: x \to x^2 + 2x - 3$, for the *domain* $-5 \leqslant x \leqslant 3$ is $\{(-5, 12), (-4, 5), (-3, 0), (-2, -3), (-1, -4), (0, -3), (1, 0), (2, 5), (3, 12)\}$.

The *graph* of the *quadratic function* for the given *domain* can then be *drawn* on graph paper.

(b) Given the *quadratic function* $f(x) = -x^2 - 2x + 3$.

Then $f(-5) = -(-5)^2 - 2(-5) + 3$
$= -25 + 10 + 3 = -25 + 13$
$= -12.$

$f(-4) = -(-4)^2 - 2(-4) + 3$
$= -16 + 8 + 3 = -16 + 11$
$= -5.$

$f(-3) = -(-3)^2 - 2(-3) + 3$
$= -9 + 6 + 3 = -9 + 9 = 0.$

$f(-2) = -(-2)^2 - 2(-2) + 3$
$= -4 + 4 + 3 = -4 + 7 = 3.$

$f(-1) = -(-1)^2 - 2(-1) + 3$
$= -1 + 2 + 3 = -1 + 5 = 4.$

$f(0) = -(0)^2 - 2(0) + 3$
$= 0 + 0 + 3 = 3.$

$f(1) = -(1)^2 - 2(1) + 3 = -1 - 2 + 3$
$= -3 + 3 = 0.$

$f(2) = -(2)^2 - 2(2) + 3 = -4 - 4 + 3$
$= -8 + 3 = -5.$

And $f(3) = -(3)^2 - 2(3) + 3 = -9 - 6 + 3$
$= -15 + 3 = -12.$

So the *set of ordered pairs* representing the *quadratic function* $f: x \to -x^2 - 2x + 3$, for the *domain* $-5 \leqslant x \leqslant 3$ is $\{(-5, -12), (-4, -5), (-3, 0), (-2, 3), (-1, 4), (0, 3), (1, 0), (2, -5), (3, -12)\}$.

The *graph* of the *quadratic function* for the given *domain* can then be *drawn* on graph paper.

Example 34

Determine the set of ordered pairs (x, y) for the quadratic function $f: x \rightarrow (x + 3)(x - 2)$ *when* $-3 \leqslant x \leqslant 5$*, in order to plot a graph.*

Solution

The *set of ordered pairs* (*x*, *y*) for the *quadratic function* with the given *domain* can then be obtained from the *table of values* below.

Table 7.25 Table of values

x	−3	−2	−1	0	1	2	3	4	5
$x + 3$	0	1	2	3	4	5	6	7	8
$x - 2$	−5	−4	−3	−2	−1	0	1	2	3
$f(x) = (x + 3)(x - 2)$	0	−4	−6	−6	−4	0	6	14	24

So the *set of ordered pairs* is $\{(-3, 0), (-2, -4), (-1, -6), (0, -6), (1, -4), (2, 0), (3, 6), (4, 14), (5, 24)\}$.

or

Given that $f: x \rightarrow (x + 3)(x - 2)$.

Then $f(-3) = (-3 + 3)(-3 - 2) = (0)(-5) = 0.$

$f(-2) = (-2 + 3)(-2 - 2) = (1)(-4) = -4.$

$f(-1) = (-1 + 3)(-1 - 2) = (2)(-3) = -6.$

$f(0) = (0 + 3)(0-2) = (3)(-2) = -6.$

$f(1) = (1 + 3)(1-2) = (4)(-1) = -4.$

$f(2) = (2 + 3)(2-2) = (5)(0) = 0.$

$f(3) = (3 + 3)(3-2) = (6)(1) = 6.$

$f(4) = (4 + 3)(4-2) = (7)(2) = 14.$

And $f(5) = (5 + 3)(5-2) = (8)(3) = 24.$

The *set of ordered pairs* follows from what was done above.

Exercise 7b

1. (a) Draw the graph of the quadratic function

$$f: x \rightarrow x^2 + 2x - 8$$

for the domain $-5 \leqslant x \leqslant 3$.

(b) State the domain interval for which the range is negative.

2. (a) Draw the graph of the quadratic function

$$f(x) = -x^2 - 2x + 8$$

for the domain $-5 \leqslant x \leqslant 3$.

(b) State the domain interval for which the range is negative.

3. (a) Draw the graph of the quadratic function

$$\{(x, y): y = x^2 - 2x - 3\}$$

for the domain $-2 \leqslant x \leqslant 4$.

(b) State the domain interval for which the range is positive.

4. (a) Draw the graph of the quadratic function

$$\{(x, y): y = -x^2 + 2x + 3\}$$

for the domain $-2 \leqslant x \leqslant 4$.

(b) State the domain interval for which the range is negative.

5. (a) Draw the graph of the quadratic equation

$$y = 2x^2 + 7x + 3$$

for the domain $-2 \leqslant x \leqslant 5$.

(b) State the domain interval for which the range is

(i) positive (ii) negative

6. (a) Draw the graph of the quadratic equation

$$y = -2x^2 - 5x + 3$$

for the domain $-4 \leqslant x \leqslant 2$.

(b) State the domain interval for which the range is

(i) positive (ii) negative

7. (a) Draw the graph of the quadratic function

$$f: x \rightarrow (x + 3)(x - 5)$$

for the domain $-5 \leqslant x \leqslant 7$.

(b) State the domain interval for which the elements of the range is

(i) less than 9 (ii) greater than 9

8. (a) Draw the graph of the quadratic equation

$$\{(x, y): y = (3 + x)(2 - x)\}$$

for the domain $-5 \leqslant x \leqslant 4$.

(b) State the domain interval for which the elements of the range is

(i) less than −6 (ii) greater than −6

Solutions of a Quadratic Equation by the Method of Intersecting Graphs

In this *method*, we first have to draw the *graphs* representing a *quadratic function* $f: x \rightarrow ax^2 + bx + c$ and a *constant function* $f(x) = q$ on the same graph paper, using the same scales and axes. The *solutions* are then given by the *points of intersection* of the *parabola* and the *straight line.*

The *roots* of a *quadratic equation* $y = ax^2 + bx + c$ are the values of x when $y = 0$.

Example 35

(a) Using a graphical method, solve each of the following quadratic equations:

- *(i)* $2x^2 + 5x - 3 = 0$
- *(ii)* $2x^2 + 5x - 3 = 4$
- *(iii)* $2x^2 + 5x - 52 = 0$

(b) Determine and state the roots of the quadratic equation $y = 2x^2 + 5x - 3$.

Solution

Table 7.26 Table of values

(a)

x	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
x^2	49	36	25	16	9	4	1	0	1	4	9	16	25
$2x^2$	98	72	50	32	18	8	2	0	2	8	18	32	50
$+5x$	−35	−30	−25	−20	−15	−10	−5	0	+5	+10	+15	+20	+25
-3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3
y	60	39	22	9	0	−5	−6	−3	4	15	30	49	72

Above is the *table of values* for the *quadratic equation* $y = 2x^2 + 5x - 3$, for the *domain* $-7 \leqslant x \leqslant 5$.

Using this *table*, a *graph* of $y = 2x^2 + 5x - 3$ was *drawn.*

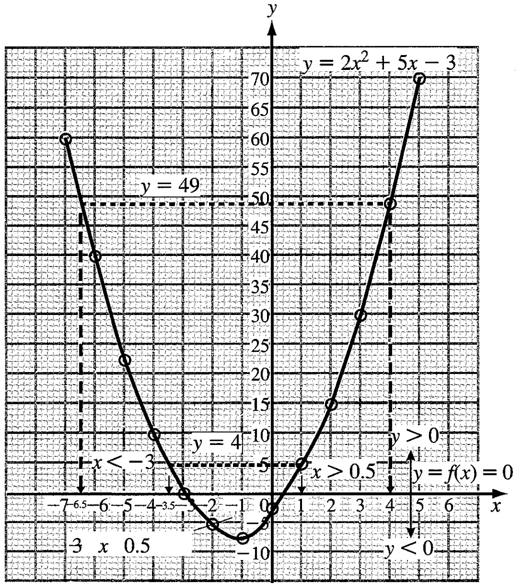


Fig. 7.71 Parabola

(i) Now $2x^2 + 5x - 3 = 0 \quad \Rightarrow y = 0$

When $y = 0$

then $x = -3$ and $x = 0.5$

Hence the *solutions* of the *quadratic equation* $2x^2 + 5x - 3 = 0$ are:

$x = -3$ and $x = 0.5$

(ii) Now $2x^2 + 5x - 3 = 4 \quad \Rightarrow y = 4$

When $y = 4$

then $x = -3.5$ and $x = 1$

Hence the *solutions* of the *quadratic equation* $2x^2 + 5x - 3 = 4$ are:

$x = -3.5$ and $x = 1$

(iii) Now $2x^2 + 5x - 52 = 0$

i.e. $2x^2 + 5x - 52 + 49 = 49$

$\therefore$ $2x^2 + 5x - 3 = 49 \Rightarrow y = 49$

When $y = 49$

then $x = -6.5$ and $x = 4$

Hence the *solutions* of the *quadratic equation* $2x^2 + 5x - 52 = 0$ are:

$$x = -6.5 \text{ and } x = 4$$

(b) The *roots* of the *quadratic equation* $y = 2x^2 + 5x - 3$

i.e. $0 = 2x^2 + 5x - 3$ are:

$x = -3$ and $x = 0.5$

From the *graph*:

> Note that the *range is negative*, that is $y < 0$, for the *domain interval* $\{x : -3 < x < 0.5\}$.
> And the *range is positive*, that is $y > 0$, for the *domain interval*
> $\{x: x < -3 \text{ and } x > 0.5\} = \{x: -3 \leq x \leq 0.5\}'$.

Exercise 7v

1. Using a graphical method, solve the quadratic equation:

$$3x^2 - 14x - 5 = 0.$$

2. A bird taking a dive follows the path given by the quadratic equation $y = 2x^2 - 6x$.
Using a graphical method, solve to find the values of x to 2 significant figures, when $y = -1$.

3. A bird taking a dive follows the path given by the quadratic equation $y = 2x^2 - 5x$.
Solve to determine the possible values for x, when $y = -3$.

4. Solve the quadratic equation $2x^2 - 9x - 5 = 0$ using graphs.

5. Solve the quadratic equation $3x^2 - 14x = 5$ using a graphical method.

6. Solve the quadratic equation $-5x^2 + 3x = -4$ using a graphical method.

7. Solve the quadratic equation $5x^2 - 2 = -10x$ using a graphical method.

8. Draw a graph in order to solve the equation $4x^2 - 7x + 3 = 0$.
State the solutions.

9. Draw a suitable graph in order to solve the quadratic equation $-4x^2 + 3x = -2$.
State the values of the solution.

10. Draw a suitable graph and solve the quadratic equation $5x^2 + 9x = 2$.

11. Using a graphical method, solve the quadratic equation:

$$5x^2 - 19x - 4 = 0.$$

12. Using a graphical method, solve the quadratic equation:

$$4x^2 - 19x - 5 = 0.$$

13. Solve the quadratic equation $-5x^2 + 4x = -7$ using a graphical method.

14. Using a graphical method, solve the quadratic equation:

$$2x^2 + 5x = 9.$$

15. Solve the quadratic equation $35x^2 - 31x + 6 = 0$.

16. The radius of a circular pool is given by the quadratic equation $r^2 - 16r - 16 = 0$.
Use a graphical method to find the radius of the pool.

17. The length of a parallelogram is given by the quadratic equation:

$$2l^2 - 13l - 70 = 0.$$

Use a graphical method to find the length of the parallelogram.

18. The numerator of a fraction is given by the quadratic equation $n^2 + 9n - 22 = 0$.
Determine the numerator of the fraction if n is a natural number.

19. The length of a rectangle is given by the quadratic equation $x^2 + 8x - 20 = 0$.
Determine the length of the rectangle using graphs.

20. The quadratic equation $y = 8x^2 + 10x$ represents the path taken by an aeroplane. Solve the quadratic equation in order to determine two values of x when the aeroplane is at the horizontal level given by $y = 3$.

21. The quadratic equation $y = -10x^2 + 11x$ represents the track followed by a missile. State two values of x when the missile is in the horizontal plane given by $y = -8$.

22. The width of a path is given by the quadratic equation:

$$x^2 + 80x - 164 = 0.$$

Draw a graph and determine the width of the path.

23. The width of a rectangle is given by the quadratic equation $x(x + 5) = 14$. Solve to determine the width of the rectangle, using a graphical method.

24. (a) Draw a graph of the relation $y = x^2 - x - 9$ for $-4 \leqslant x \leqslant 4$.

(b) Use your graph to find the solution to:

(i) $x^2 - x - 9 = 0$ (ii) $x^2 - x - 9 = 3$

25. Plot the graph of the function $y = 2x^2 - 5x - 12$ on graph paper. Hence solve each of the following equations:

(a) $2x^2 - 5x - 12 = 0$

(b) $2x^2 - 5x - 8 = 0$

(c) $2x^2 - 5x + 1 = 0$

(d) $4x^2 - 10x - 26 = 0$

26. Plot the graph of $y = -3x^2 + 2x - 1$ for $-3 \leqslant x \leqslant 4$. Hence determine the solutions of each of the following quadratic equations:

(a) $-3x^2 + 2x + 1 = 0$

(b) $-3x^2 + 2x + 8 = 0$

27. Plot the graph of $y = -3x^2 + 2x - 1$ for $-3 \leqslant x \leqslant 4$. Hence determine the solutions of each of the following quadratic equations:

(a) $-3x^2 + 2x + 1 = 0$

(b) $-3x^2 + 2x + 8 = 0$

28. Write each of the following quadratic equations in the form $ax^2 + bx + c = 0$. Hence determine the values of x for:

(a) $x^2 + 3x = -1$

(b) $4x^2 + 3 = 8x$

(c) $-5x^2 - 2x = -3$

29. Use a graphical method to find the values of x satisfying each of the following equations:

(a) $x(x + 3) = 0$

(b) $(x - 9)(x - 7) = 0$

(c) $(5x + 2)(4x + 3) = 0$

(d) $(4x + 7)(5x - 3) = 0$

30. Using a graphical method, solve each of the following quadratic equations:

(a) $x^2 + x - 12 = 0$ (b) $x^2 + 7x + 10 = 0$

(c) $25x^2 - 64 = 0$ (d) $8x^2 - 3x = 0$

31. Use a graphical method to solve each of the following quadratic equations:

(a) $15x^2 + 31x = -10$

(b) $20x^2 + 15 = 37x$

(c) $(x + 4)(x + 3) = 2$

32. Draw suitable graphs and solve each of the following quadratic equations:

(a) $(x - 8)(2x + 5) = 0$

(b) $(3x + 4)(x - 1) = 0$

(c) $2x^2 + x - 21 = 0$

(d) $10x^2 + 37x + 7 = 0$

33. (a) Copy and complete the table below for the function $y = x^2 + 2x - 1$.

Table 7.27 Table of values

x	-4	-3	-2	-1	0	1	2
x^2	16		4			1	
$+2x$	-8		-4			2	
-1	-1		-1			-1	
y	7		-1			2	

(b) Using a scale of 2 cm to represent a unit on each axis, draw on graph paper the graph of the function for $-4 \leqslant x \leqslant 2$.

(c) On the same diagram and using the same scale as in part (b), draw the line $y = 2$ and write down the coordinates where $y = 2$ cuts the curve.

(d) Hence, solve the equation $x^2 + 2x - 1 = 2$.

34. (a) Copy and complete the table below for the function $y = x^2 + 2x - 1$.

Table 7.28 Table of values

x	-4	-3	-2	-1	0	1	2
y	7		-1			2	

(b) Using a scale of 2 cm to represent a unit on each axis, draw on graph paper the graph of the function for $-4 \leqslant x \leqslant 2$.

(c) On the same diagram and using the same scale as in part (b), draw the line $y = 7$ and write down the coordinates where $y = 7$ cuts the curve.

(d) Hence, solve the equation $x^2 + 2x - 1 = 7$.

35. (a) Copy and complete the table for the function. $y = x^2 + 3x - 2$.

Table 7.29 Table of values

x	-4	-3	-2	-1	0	1	2
y	2		-4			2	

(b) Using a scale of 2 cm to represent a unit on each axis draw on graph paper the graph of the function for $-4 \leqslant x \leqslant 2$.

(c) On the same graph paper and using the same scale and axes as in part (b), draw the line $y = -2$ and write down the coordinates where $y = -2$ cuts the curve.

(d) Hence, solve the equation $x^2 + 3x - 2 = -2$.

36. (a) Copy and complete the table for the function $y = 2x^2 + 4x - 1$.

Table 7.30 Table of values

x	-4	-3	-2	-1	0	1	2
y	15	5					15

(b) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw on graph paper the graph of the function for $-4 \leqslant x \leqslant 2$.

(c) On the same diagram and using the same scales and axes as in (b), draw the line $y = 5$ and write down the coordinates where $y = 5$ intersects the curve.

(d) Hence or otherwise, solve the equation $2x^2 + 4x - 1 = 5$.

37. (a) Complete the following table for the function $y = 3x^2 + 1$, for $-3 \leqslant x \leqslant 3$.

Table 7.31 Table of values

x	-3	-2	-1	0	1	2	3
$y = 3x^2 + 1$	28				4	13	

(b) Hence, draw the graph of $y = 3x^2 + 1$, for $-3 \leqslant x \leqslant 3$, on graph paper.

(c) State the equation of the axis of symmetry.

(d) Solve the equation $3x^2 + 1 = 7$.

38. (a) Use a graphical method to solve the quadratic equation $2x^2 - 6x - 5 = 0$.

(b) Determine the minimum value of $2x^2 - 6x - 5$.

39. Draw a graph of the function $f(x) = 2x^2 + 5x - 3$. Hence

(i) solve the equation $2x^2 + 5x - 3 = 0$

(ii) determine the minimum value of $2x^2 + 5x - 3$.

40. The path of a missile was tracked by an observer. The missile travelled according to the formula $y = 10 + Bx + Cx^2$ where (x, y) represents its coordinates at any time.

Table 7.32 Table of values

x	-3	-2	-1	0	1	2	3
y	-17	-4	5		11	4	1

The table above shows the observer's record in which he incorrectly wrote one of the y values.

(a) Calculate the value of y when $x = 0$.

(b) Draw a graph of the recorded observations using a scale of 2 cm to 1 unit on the x-axis and 1 cm to 2 units on the y-axis, clearly indicating the incorrect value.

(c) Estimate the correct value of y for the incorrect one given.

(d) Determine the values of B and C in the formula.

Experimental Data

When an *experiment* is carried out, quite often a *graph* of *two variables* is plotted, in order to determine a *relationship* between the *variables*. If the points *approximate* to a *straight line*, then the *best* possible straight line is drawn through the points plotted. This can be seen indicated in the diagram below.

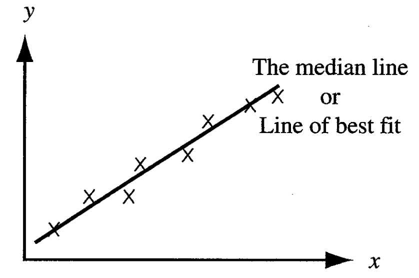
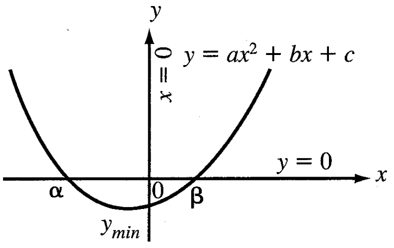


Fig. 7.72 Straight line

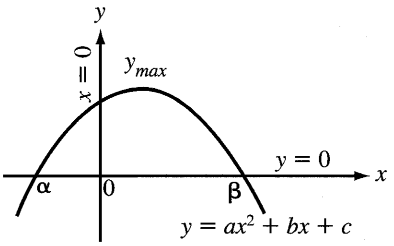
Sometimes a *quadratic curve* (or *parabola*) is obtained. We can then use the *graph* in order to *determine* the *particular equation* or *formula* that represents the *straight line* or *curve*.

In the case of a *straight line*, the *equation* or *formula* is of the *form* $y = mx + c$.

In the case of a *quadratic curve* (or *parabola*), representing a *quadratic equation* or *formula* of the *form* $y = ax^2 + bx + c$; the equation or formula can further be written in the form $y = (x - \alpha)(x - \beta)$, where α and β are the *roots of the equation* or *formula*, corresponding to the values of x when $y = 0$. These facts can be seen indicated in the diagrams below.



When $a > 0$



When $a < 0$

Fig. 7.73 Parabola

Example 36

In an experiment to investigate Ohm's Law for a simple a.c. capacitive circuit, the following table was obtained.

Table 7.33 Table of observations

V (*r.m.s. Volt*)	2	4	6	8	10	12	14	16	18	20
I (*0.33 r.m.s. Amp*)	*1.4*	*2.4*	*4.5*	*6.1*	*7.7*	*9.5*	*11.2*	*12.7*	*14.4*	*16*

It is thought that $I \propto V$.

(*a*) *Draw a graph of I against V to represent the data obtained.*

(*b*) *If the formula is of the form* $I = aV + b$, *determine the values of a and b from the graph.*

(*c*) *Hence, write down the formula representing the data obtained.*

Solution

(a)

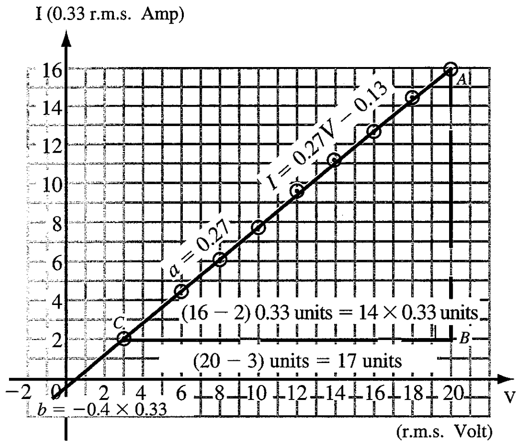


Fig. 7.74 Straight line

The graph of *I against V* was drawn using the *table of observations* given.

(b) From the *graph*:

The *gradient* of the straight line,

$$a = \frac{14 \times 0.33 \cancel{\text{units}}}{17 \cancel{\text{units}}}$$

$$= 0.27 \text{ (correct to 2 d.p.)}$$

The *intercept* of the straight line on the *I-axis*,

$$b = -0.4 \times 0.33$$

$$= -0.13 \text{ (correct to 2 d.p.)}$$

(c) Hence, the *formula* representing the data obtained is:

$$I = 0.27V - 0.13$$

or $$I = \frac{27}{100}V - \frac{13}{100}$$

or $$I = \frac{1}{100}(27V - 13).$$

Example 37

In a projectile experiment, a bullet was shot into the air and its height d metres above a fixed level was measured after t seconds. The observations were recorded in the table below.

Table 7.34 Table of observations

t (s)	0	1	2	3	4	5	6	7	8
d (m)	8	14	18	20	20	18	14	8	0

(a) Draw a distance-time graph to represent the data obtained.

(b) State the type of graph obtained.

(c) From the graph, estimate the values of t when $d = 0$.

(d) Hence, state the formula that represents the data obtained in the form $d = (t - \alpha)(t - \beta)$

Solution

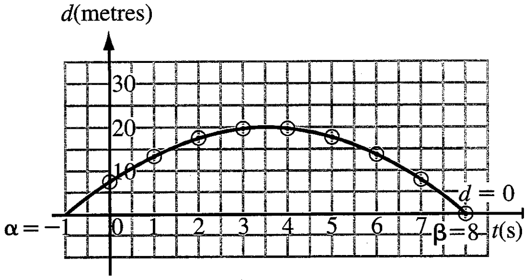


Fig. 7.75 Parabola

(a) The *distance–time graph* was drawn using the *table of observations* given.

(b) A *quadratic curve* (or *parabola*), with a *maximum value* for the height *d* was obtained.

(c) From the *graph*:
The *values of t* when $d = 0$ are $t = -1$ and $t = 8$.

Note that $t = -1$ was obtained by *extrapolating*.

(d) From above:
The *roots* of the *formula* are $\alpha = -1$ and $\beta = 8$.
Hence the *formula* that represents the *data* obtained is:

$$d = (t - \alpha)(t - \beta)$$
$$= (t - [-1])(t - 8)$$

i.e. $$d = (t + 1)(t - 8).$$

Exercise 7w

1. (a) The information given represents a linear relation. Show this on a graph and hence complete the table.

Table 7.35 Table of observations

s, distance travelled in km	*t, time for journey in hours*
17	1
19	2
—	3
23	4
—	5
27	6

(b) Calculate the gradient of the line.

(c) State the intercept on the vertical axis.

(d) Hence, determine the equation of the straight line $s = mt + c$.

2. In determining the coefficient of viscosity for water by capillary flow, the following table of observations was obtained.

Table 7.36 Table of observations

Rate of flow Q (10^{-5} kg m^{-3})	5.69	9.75	15.3	21.0	30.8	37.2
Pressure head h (10^{-1} m)	1.15	1.50	1.90	2.35	2.95	3.80

(a) Plot a graph of rate of flow against pressure head.

(b) From the graph estimate:
  (i) the gradient of the line, m
  (ii) the intercept on the Q-axis, c.

(c) Hence, state the formula that represents the recorded observations in the form:

$$Q = mh + c$$

3. To verify that the frequency f of a sound wave in a stretched wire is inversely proportional to the wavelength l, tension T being constant, the following table was obtained.

Table 7.37 Table of observations

f (cycle/s)	512	480	426	384	320
$\frac{1}{l}$ (10^{-3} cm)	89.3	83.3	76.9	66.6	57.5

(a) Plot a graph of the frequency against the reciprocal of the wavelength.

(b) Does the graph verify that the frequency is inversely proportional to the wavelength?

4. To verify that the refractive index of a solution is proportional to the concentration of dissolved material in the solution (sugar solution in this case), the following table was obtained.

Table 7.38 Table of observations

Refractive index μ	1.38	1.37	1.36	1.35	1.34
Concentration (g ml^{-1})	24.5	19.4	14.8	9.72	3.24

(a) Plot a graph of the refractive index μ against the sugar concentration in g ml^{-1}.

(b) Does the graph verify that the refractive index is proportional to the sugar concentration?

5. In the determination of the capacitance of a capacitor using a flashing neon lamp circuit the following data were collected.

Table 7.39 Table of observations

Capacitance C (μF)	0.1	0.2	0.3	0.4	0.5
Periodic time T (s)	0.2	0.3	0.5	0.7	0.9
Capacitance C (μF)	0.6	0.7	0.8	0.9	1.0
Periodic time T (s)	1.1	1.3	1.5	1.7	1.9

(a) Using the table above, draw a graph of capacitance C against periodic time T, Using a scale of 2 cm to represent 0.1 μF and 1 cm to represent 0.1 s.

(b) From the graph, estimate the capacitance of the capacitor when the periodic time equals:
(i) 0.8 s (ii) 1.2 s (iii) 1.8 s.

(c) From the graph, estimate the periodic time when the capacitance of the capacitor equals:
(i) 0.35 μF (ii) 0.75 μF (iii) 0.85 μF.

6. In the determination of the wavelength of sodium light by Newton's rings the following data were obtained.

Table 7.40 Table of observations

D^2 (mm^2)	10.89	9.61	8.41	7.29	6.25
n	10	9	8	7	6
D^2 (mm^2)	4.84	4.00	2.89	1.69	0.64
n	5	4	3	2	1

(a) Using the table above, draw a graph of D^2 against n, using a scale of 2 cm to represent 1 mm^2.

(b) Given that the wavelength of the sodium light, $\lambda = \dfrac{\text{Slope}}{2} \times 10^{-6}$ m, calculate λ.

7. To investigate Ohm's Law for a simple a.c. inductive circuit the following data were obtained.

Table 7.41 Table of observations

V (r.m.s. Volt)	2	4	6	8	10	12	14	16	18	20
I (r.m.s. Amp.)	16	20	24	28	32	36	40	44	48	52

(a) Draw a graph of I against V using a scale of 1 cm to represent 2 r.m.s. Amps and 1 cm to represent 1 r.m.s. Volt.

(b) Given that the formula representing the linear equation is of the form

$$I = mV + c,$$

determine the experimental formula.

8. To verify the three-halves-power law or Child's Law for a diode, i.e. $I_A = k\,V_A^{3/2}$, the following table of values was calculated from the observations measured.

Table 7.42 Table of observations

LogI_A (10^{-2} μA)	30.11	47.71	60.21	69.90	77.82
Log V_A (10^{-2} V)	39.79	68.12	87.51	102.12	116.14
LogI_A (10^{-2} μA)	84.51	90.31	95.43	100	—
Log V_A (10^{-2} V)	125.50	135.60	143.14	151.59	—

(a) Plot a graph of log I_A against log V_A.

(b) From your graph, can you verify that log I_A is proportional to log V_A?

9. In the determination of the thermal conductivity of ebonite by Lee's Disc method the following table of observations was obtained.

Table 7.43 Table of observations

Temp. (°C)	98.5	97	95.2	93.8	92.2	90.8	89.4	88
Time (min)	1	2	3	4	5	6	7	8
Temp. (°C)	86.9	85.5	84.3	83.2	82	81	80	—
Time (min)	9	10	11	12	13	14	15	—

Draw a cooling curve by plotting temperature in °C against time in minutes.

10. An experiment was carried out to determine the characteristic curve of a diode valve. The observations recorded can be seen in the table below.

Table 7.44 Table of observations

V_A (Volts)	10	20	30	40	50	60	70	80	90
I_A (mA)	24	35	38	40.5	42	43	43.5	44	44

Obtain the characteristic curve for the diode valve by plotting a graph of I_A against V_A.

11. An experiment was performed to determine the characteristic curve of a junction diode (semiconductor diode). The recorded observations for the experiment during the forward bias mode can be seen below.

Table 7.45 Table of observations

V (Volts)	31.05	31.30	31.50	31.65	31.73
I (A)	0.5	1.0	1.5	2.0	2.5
V (Volts)	31.79	31.85	31.90	31.95	—
I (A)	3.0	3.5	4.0	4.5	—

Obtain the characteristic curve for the junction diode (semiconductor diode) by plotting a graph of I against V.

12. To determine the acceleration due to gravity using a compound pendulum, the data seen below was recorded.

Table 7.46 Table of observations

d (cm)	7.8	11.8	15.8	19.8	23.8	31.8	35.8	39.8
T (s)	1.61	1.51	1.52	1.51	1.58	1.86	2.24	3.87

Plot a graph of d in cm against T in seconds.

13. A plane flying follows the path given by the table of values below.

Table 7.47 Table of observations

t (s)	−4	−3	−2	−1	0	1	2
d (m)	5	0	−3	−4	−3	0	5

(a) Draw a distance–time graph to represent the data given.
(b) State the type of graph obtained.
(c) From the graph, estimate the values of t when $d = 0$.
(d) Hence, state the formula that represents the data obtained, in the form:

$$d = (t - \alpha)(t - \beta).$$

14. A bird taking a dive follows a path given by the data recorded below

Table 7.48 Table of observations

t (s)	−3	−2	−1	0	1	2	3	4	5
d (m)	20	12	6	2	0	0	2	6	12

(a) Draw a distance–time graph to represent the data recorded.
(b) State the type of graph obtained.
(c) From the graph, estimate the values of t when $d = 0$.
(d) Hence, state the formula that represents the data obtained in the form:

$$d = (t - \alpha)(t - \beta).$$

15. A missile moves in such a way that its distance d metres after time t seconds is given by the table below.

Table 7.49 Table of observations

t (s)	−3	−2	−1	0	1	2	3	4
d (m)	0	7	12	15	16	15	12	7

(a) Draw a distance–time graph to represent the data recorded.
(b) State the type of curve obtained.
(c) From your graph, estimate the values of t when $d = 0$.
(d) Hence, state the formula that represents the data recorded in the form:

$$d = (t - \alpha)(t - \beta).$$

C.X.C. Past Paper Questions

The following supplementary questions were taken from C.X.C. Past Papers.

Exercise 7x

1. If $y = 10 - x - 2x^2$, complete the table below:

x	$-\frac{5}{2}$	−2	−1	0	1	2
y	0	4			7	

Using a scale of 1 cm to represent 1 unit on the y-axis, and 2 cm to represent 1 unit on the x-axis, draw on the same axes the graphs of

(i) $y = 10 - x - 2x^2$,

(ii) $y = 7 - 2x$.

Hence find the values of x and y which satisfy both $y = 10 - x - 2x^2$ and $y = 7 - 2x$.

Question 9. C.X.C. (Basic). June 1980.

2. Copy and complete the table below for the function $y = 2^x$.

x	-2	-1	0	1	2
y	$\frac{1}{4}$		1		

Using scales of 4 cm to represent 1 unit on each axis, on graph paper plot the graph of $y = 2^x$ for $-2 \leqslant x \leqslant 2$. Using the same axes draw the line $y = x + 2$. From your graphs write down the values of x for which $2^x = x + 2$.

Question 6. C.X.C. (Basic). June 1981.

3. A car is being tested over a course of fixed length. In each trial it is kept as near as possible to a fixed speed and timed. The results are given in the table below where

v = the reported speed in km per hour

t = time in minutes

v	10	20	30	40	50	60	70	80	90
t	14.4	7.4	4.8	3.7	3.0	2.5	2.1	2.0	1.6

(i) Plot the points for a graph of v (on the vertical axis) against t, using 2 cm to represent 10 units on the v-axis and 1 cm to represent 1 unit on the t-axis.

(ii) Draw a smooth graph through the points.

(iii) From your graph determine

(a) the time taken when the speed is 25 kmh^{-1}

(b) the length of the fixed course using your estimate of t when v = 25 kmh^{-1}.

Question 8. C.X.C. (Basic). June 1982.

4. It is known that the cost, C dollars, of producing silver coins of different radii r mm, is given by the formula $C = ar^2 + b$ where b is a fixed overhead cost in dollars.

r^2	4	6.25	9	16	25
C	12.00	13.80	16.00	21.60	28.80

Using the data in the above table and a scale of 2 cm to represent 5 units on BOTH the C-axis and the r^2-axis.

(a) By plotting C against r^2, draw a graph to represent the formula.

(b) From your graph deztermine the fixed overhead cost of producing each coin.

Question 10. C.X.C. (Basic). June 1983.

5. (i) Copy and complete the table for the function

$$y = x^2 + 2x - 2$$

x	-4	-3	-2	-1	0	1	2
y	6	1		-3			

(ii) Using a scale of 2 cm to represent a unit on each axis draw on graph paper the graph of the function for $-4 \leqslant x \leqslant 2$.

(iii) On the same diagram and using the same scale as in part (ii), draw the line $y = 3$ and write down the coordinates where $y = 3$ cuts the curve.

(iv) Hence, solve the equation $x^2 + 2x - 2 = 3$

Question 6. C.X.C. (Basic). June 1984.

6. (a) Using a scale of 1 cm to represent 1 unit on each axis, plot on graph paper the points $P(2, -1)$ and $Q(-2, 5)$.

(b) Calculate the gradient of PQ.

(c) Determine the point where PQ meets the y-axis.

(d) Write down the equation of PQ in the form

$$y = mx + c$$

(e) Hence or otherwise determine the solution of

$$3x + 2y = 4$$
$$x - y = 1$$

(f) Shade the region $y \geqslant x - 1$ for $x \geqslant 0$.

Question 9. C.X.C. (Basic). June 1985.

7. (i) Using a scale of 1 cm to represent one unit on each axis, *draw on graph paper* the graph of $y = 2x + 1$.

(ii) On the same graph, draw the line through the points $P(-4, 3)$ and $Q(0, 1)$ and calculate the gradient of PQ.

(iii) State the relationship between the line $y = 2x + 1$ and the line PQ and state the coordinates of the point of intersection of the lines. Hence or otherwise determine the equation of the line PQ.

Question 9 (b). C.X.C. (Basic). June 1986.

8. (a) Copy and complete the following table for the function $f(x) = 2x^2 - x - 10$.

x	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	3
$f(x)$	11		-7	-9		-10		-7		

(b) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of the function $y = f(x)$ for $-3 \leqslant x \leqslant 3$.

(c) On the same diagram, using the same scale as in (b) above, draw the graph of the line $y = -4$

(d) From your graphs, determine

  (i) the values of x for which $y = 0$,

  (ii) the co-ordinates of the points of intersection of the curve $y = 2x^2 - x - 10$ and of the line $y + 4 = 0$.

Question 10. C.X.C. (Basic). June 1987.

9. (a) (i) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, plot on graph paper the points $P(1, 6)$ and $Q(4, 12)$.

  (ii) Join PQ and calculate the gradient of PQ.

  (iii) Produce QP to cut the y-axis at R. State the co-ordinates or R. Hence, write the equation of PQ.

(b) (i) Given that $y = x^2 + 3$, copy and complete the table below for the range $-3 \leqslant x \leqslant 3$.

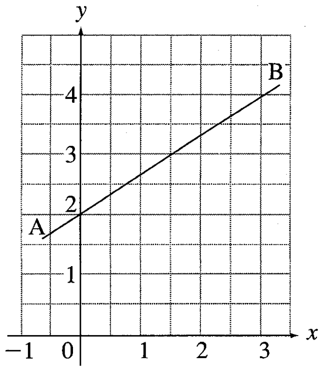
x	-3	-2	-1	0	1	2	3
y	12				4	7	

  (ii) Using the same diagram and the same scale as in part (a) (i) above, draw the graph of $y = x^2 + 3$ in the given range.

  (iii) From your graphs determine the solutions of the equation $x^2 + 3 = 7$.

Question 10. C.X.C. (Basic). June 1988.

10.



(a) Use the graph of the line AB to

  (i) determine the value of y when $x = 0$ and when $x = 3$

  (ii) calculate the gradient of AB

  (iii) determine the value of x when $y = 0$.

(b) (i) Complete the table below for the equation

$$2y + 3x = 4$$

x	-1	2	4
y		-1	

  (ii) Draw on the same axes, the graph of the equation

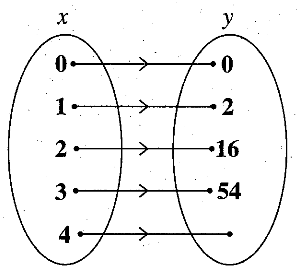
$$2y + 3x = 4.$$

(c) Given the equation of the line AB is $3y - 2x = 6$, hence, or otherwise, solve the equations

$$2y + 3x = 4 \text{ and}$$
$$3y - 2x = 6.$$

Question 10. C.X.C. (Basic). June 1989.

11.

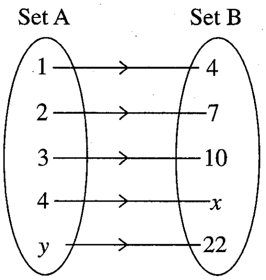


(i) Write an equation in x and y to represent the relation shown by the mapping above.

(ii) Calculate the missing value of y.

Question 4. (a) C.X.C. (Basic). June 1992.

12.



The diagram above shows a mapping from Set A to Set B.

(i) Write an equation to describe the mapping.

(ii) Determine the values of x and y.

Question 9. (b) C.X.C. (Basic). June 1993.

CHAPTER 8

Statistics 1

This chapter will teach you about

- proportionate bar charts, chronological bar charts, pie charts and line graphs.
- variables and its types.
- frequency tables and histograms of grouped and ungrouped data.
- width of a class interval, class intervals, class limits, class boundaries and class mid-points.
- the mean, median and mode.
- frequency curves and its types.
- range, interquartile range and semi-interquartile range.
- probability, relative frequency and theoretical probability.

Introduction

Statistics is the name given to the *science* of *collecting* large quantities of *facts* or *data* and *studying* or *analysing* them. These *facts* or *data* can cover a wide range of subjects and prove very useful in industry, science and everyday planning.

In most Caribbean territories a *census* is conducted a few years before a general election or every ten years. Although the *census* is conducted *primarily* to determine the number of people in the country and those people who should be eligible to vote in the election, quite a lot of *secondary facts* are also obtained at the same time. For example, the average number of children per family, the mean number of people employed per family, and the number of single parent families. These *facts* can prove very useful in *central planning* by the elected government.

The *facts* that are *recorded* initially on paper by the persons conducting the *survey* are called *raw data*. After this *raw data* is collected it will have to be placed into different types of grouping according to the answers given to the various questions asked. *Frequency tables* have, therefore, to be drawn up in order to achieve this.

Using the completed *frequency tables*, various *diagrams* can be drawn in order to carefully *analyse* the *data collected*. After the *data* is *analysed* then various *conclusions* can be reached. From these conclusions *extrapolations* can be made for future planning.

Following can be seen some of the ways in which the *statistician* goes about putting some *order* into the *raw data* and *analysing* the *facts recorded*.

Proportionate Bar Chart (or Composite Bar Chart)

The *proportionate bar chart* (or *composite bar chart*) is a *single rectangular bar* which is drawn *vertically* or *horizontally*. The *rectangular bar* is then *sub-divided* into different *heights* or *lengths* in order to represent various *magnitudes of data*. Thus the *magnitude of the data* represented is *directly proportional* to the *height* or *length* of *each smaller rectangle* or *bar*, since the *width* throughout the *proportionate bar chart* is the *same*.

Each *smaller rectangle* or *bar* is then labelled and coloured brightly. Hence, the *proportionate bar chart* consists of a number of different bright colours. The *colours* enhance the *proportionate bar chart* and make it eye catching and simple to understand and appreciate by the lay person.

The *advantage* of using a *proportionate bar chart* is the fact that it shows how a whole quantity is divided into parts and what size these parts are with respect to each other and to the whole.

Example 1

The table below shows the ways in which the government of a certain country spent its budget for a particular year.

Table 8.1

Facility	*Amount spent in \$ millions*
Wages and salaries	33
Health	24
Education	15
Agriculture	13
Communication	5

(a) Calculate the total amount of money that was budgeted for that particular year.

(b) If the total height of the proportionate bar chart must be 54 mm, calculate the height or length that will represent each of the mentioned facilities.

(c) Hence construct a proportionate bar chart of appropriate width and represent the data recorded above.

Solution

(a) The *total amount of money* that was budgeted for that particular year $= \$(33 + 24 + 15 + 13 + 5)$ million $= \$90$ million

(b) The *total height* of the *proportionate bar chart* $= 54$ mm

∴ the *height* or *length* that will represent the *amount spent* on *wages* and *salaries* $= \dfrac{\$33\ \cancel{\text{million}}}{\$90\ \cancel{\text{million}}} \times 54$ mm $= 19.8$ mm

The *height* or *length* that will represent the *amount spent* on *health* $= \dfrac{\$24\ \cancel{\text{million}}}{\$90\ \cancel{\text{million}}} \times 54$ mm $= 14.4$ mm

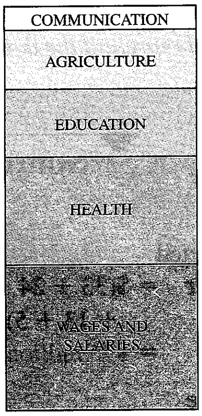
The *height* or *length* that will represent the *amount spent* on *education* $= \dfrac{\$15\ \cancel{\text{million}}}{\$90\ \cancel{\text{million}}} \times 54$ mm $= 9$ mm

The *height* or *length* that will represent the *amount spent* on *agriculture* $= \dfrac{\$13\ \cancel{\text{million}}}{\$90\ \cancel{\text{million}}} \times 54$ mm $= 7.8$ mm

And the *height* or *length* that will represent the *amount spent* on *agriculture* $= \dfrac{\$5\ \cancel{\text{million}}}{\$90\ \cancel{\text{million}}} \times 54$ mm $= 3$ mm

Note that the *total height* of the *proportionate bar chart* $= (19.8 + 14.4 + 9 + 7.8 + 3)$ mm $= 54$ mm

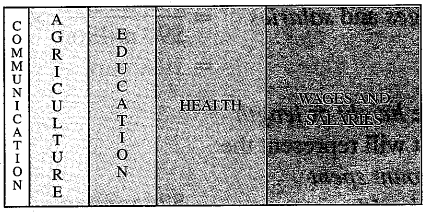
(c)



Total annual budget = $90 million

Fig. 8.1 Proportionate bar chart

Above can be seen the *vertical proportionate bar chart* that represents the data recorded.



Total annual budget = $90 million

Fig. 8.2 Proportionate bar chart

Above can be seen the *horizontal proportionate bar chart* that represents the data recorded.

> Note that it is a good *rule of thumb* to place the *data recorded* in the *proportionate bar chart* either in *ascending order* or *descending order* if possible.

Exercise 8a

1. The table below shows the way in which a certain country spent its national budget for a particular year.

Table 8.2

Facility	*Amount spent in $ millions*
Wages and salaries	37
Health	23
Education	14
Agriculture	19
Communication	3

(a) Calculate the total amount of money that was budgeted for that particular year.

(b) If the total height of the proportionate bar chart must be 96 mm, calculate the height that will represent each of the mentioned facilities.

(c) Hence construct a vertical proportionate bar chart of appropriate width and represent the data recorded above.

2. The table shown below gives the number of graduates by subject from a teacher's training college in 1992.

Table 8.3

Subject	*Mathematics*	*English*	*History*	*Science*	*Modern languages*
No. of teachers	16	35	41	37	11

(a) Calculate the total number of teachers who graduated from the teacher's training college in 1992.

(b) If the total length of the proportionate bar chart must be 14 cm, calculate the length that will represent the number of teachers who graduated in each subject.

(c) Hence construct a horizontal proportionate bar chart of appropriate width and represent the data recorded above.

3. A shopkeeper counted the amount of money that she had in her cash register at the end of the day. She found that she had:

$85 in one-dollar notes
$125 in five-dollar notes
$150 in ten-dollar notes
$420 in twenty-dollar notes
$500 in hundred-dollar notes.

(a) Determine the number of notes for each type of bill.

(b) Hence, construct a composite bar chart of appropriate height and represent the data recorded.

4. Between 1928 and 1975, there were 232 reported visitations by the Virgin Mary in 32 countries. The table below shows the countries where the Virgin Mary was seen most frequently and the number of times seen during that almost half-century period.

Table 8.4

Country	*Number of times seen*
Italy	83
France	30
Germany	20
Belgium	17
Spain	12
United States	9
Canada	6
Switzerland	5

(a) Calculate the total number of times that the Virgin Mary was seen in the countries listed above.

(b) Hence construct a composite bar chart of appropriate height and represent the data recorded.

5. The table below shows the six smallest countries on earth, and their location, size and population.

Table 8.5

Country	*Location*	*Size (sq. km)*	*Population*
Vatican City	Rome, Italy	0.44	750
Monaco	French Riviera on the Mediterranean	1.95	28 000
Nauru	Western Pacific Ocean	21.2	8 000
Tuvalu	South Pacific	26	7 500
San Marino	North-central Italy near Adriatic coast	61	19 000
Liechtenstein	Between Switzerland and Austria	160	27 000

Construct a proportionate bar chart to represent:

(a) each country and its size in square kilometres

(b) each country and its population.

Bar Chart (or Column Graph)

A *bar chart* (*vertical bar chart/column graph* or *horizontal bar chart*) consists of a *number of rectangular bars* of the *same width* which can be drawn *vertically* or *horizontally* and are *evenly spaced* out. The *height* or *length* of each *rectangular bar* is *directly proportional* to the *magnitude of the data* that it is representing.

A *bar chart* is always drawn on graph paper and has a *vertical axis* or *horizontal axis* drawn to *scale* which allows for the *exact magnitude* of *each data* to be *recorded* and *read off* (that is, *interpolated*). *Extrapolations* can also be made in some cases.

Bar charts tend to be used to represent discrete data. Discrete data is data which can only be of certain definite values. Shirt sizes and shoe sizes are examples of discrete data.

Example 2

The table below shows the heights of seven of the principal waterfalls in South America.

Table 8.6

Name of waterfall	*Location of waterfall*	*Height in metres*
Angel	*Venezuela*	*979*
Kukenaäm	*Venezuela*	*610*
King George VI	*Guyana*	*488*
Roraima	*Guyana*	*457*
Glass	*Brazil*	*404*
Catarata de Candelas	*Colombia*	*300*
Kaieteur	*Guyana*	*251*

Draw a bar chart to represent the recorded data shown above.

Solution

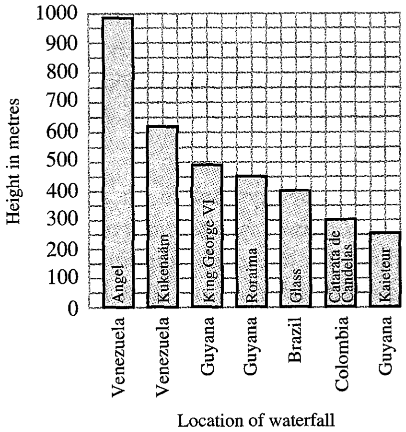


Fig. 8.3 Vertical bar chart (or column graph)

Above can be seen the *vertical bar chart* (or *column graph*) that represents the recorded data.

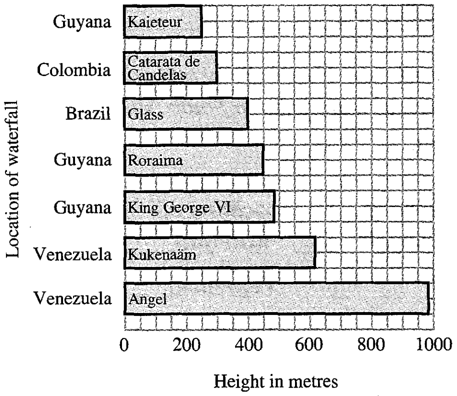


Fig. 8.4 Horizontal bar chart

Above can be seen the *horizontal bar chart* that represents the recorded data.

Exercise 8b

1. The table below shows the heights of some mountains in South America.

Table 8.7

Name	*Location*	*Height in metres*
Mercedario	Argentina	6770
Misti, Volcán	Peru	5821 ≈ 5820
Pular	Chile	6225 ≈ 6230
Sajama	Bolivia	6520
Talima	Colombia	5215 ≈ 5220

Draw a vertical bar chart to represent the recorded data shown above using the approximate heights in metres where necessary.

2. The data below show the lengths of the principal rivers in South America.

Table 8.8

Name	*Length in kilometres*
Amazon	6437 ≈ 6440
Japura	2253 ≈ 2250
Madeira	3315 ≈ 3320
Magdalena	1529 ≈ 1530
Orinoco	2896 ≈ 2900
Paraguay	2076 ≈ 2080
Paraná	3942 ≈ 3940
Purús	3057 ≈ 3060
Sãn Francisco	2896 ≈ 2900
Uruguay	1649 ≈ 1650

Draw a horizontial bar chart to represent the recorded data shown above using the approximate lengths in kilometres.

3. In bionic medicine in the future humans can be expected to be fitted with spare parts. The table below shows the estimated average cost per spare part in United States dollars.

Table 8.9

Spare part	*Average cost (U.S. $)*
Ankle	6600
Ear	1720
Elbow	6600
Finger	3600
Heart	28000
Hip joint	9500

Draw a column graph to represent the information given.

4. In bionic medicine in the future human beings can be expected to be fitted with spare parts. The table below shows the estimated mean cost per spare part in United States currency.

Table 8.10

Spare part	*Mean cost (U.S. $)*
Interocular lens	4 000
Kidney	13 000
Knee	6 600
Lung	10 000
Nose	1 000
Shoulder	6 600
Wrist	3 400

Draw a column graph to represent the given information.

5. The table below shows the speeds of some objects.

Table 8.11

Object	*Speed (km/h)*
Fast elevator	36.3
Human brisk walk	6
Roller coaster	103.7
Cricket ball	159.5
Fast warthog	48
Average wind speed	14.5

Construct a bar chart to represent the data stated in the table above.

6. The table below gives the speeds of some objects.

Table 8.12

Object	*Speed (km/h)*
Fastest recorded pitch	161.6
Fastest bird in level flight	169.6
Head-first free-fall position of a sky diver	296
Nerve pulse along a nerve in your body	328
Sound at sea level at 20 °C	1 212.8

Construct a bar chart to represent the data given in the table above.

7. The table below gives the speeds of some objects.

Table 8.13

Object	*Speed (km/h)*
A fast aircraft	3 504
Bullet from a standard U.S. army M16 rifle	3 600
Space shuttle 9 minutes after takeoff	26 720
Escape velocity from the earth	40 320
Average orbital speed of the earth around sun	106 720

Draw a bar chart to represent the data given.

Chronological Bar Chart

The *chronological bar chart* is similar to the vertical bar chart or column graph, except for the fact that *time* is now indicated along the *horizontal axis.*

Example 3

The table below shows the amount of money spent in dollars on education in a certain country during the period 1989 to 1993

Table 8.14

Year	*Expenditure (in millions of dollars)*
1989	*15*
1990	*17*
1991	*18*
1992	*13*
1993	*10*

Construct a chronological bar chart to represent the information given in the table for the five-year period.

Solution

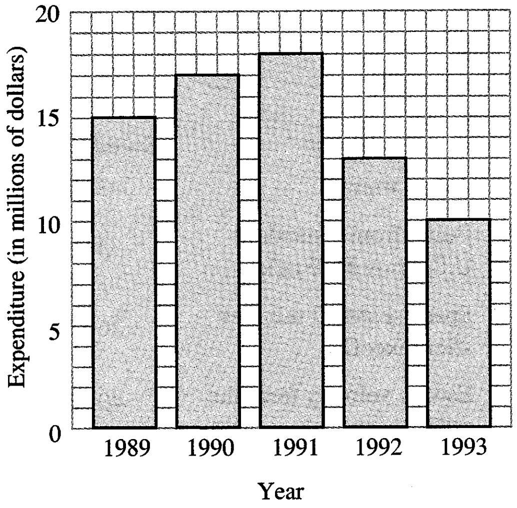


Fig. 8.5 Chronological bar chart

Above can be seen the *chronological bar chart* that represents the information given.

Exercise 8c

1. The table below shows the money spent in dollars on education in a Caribbean country during the period 1980 to 1984.

Table 8.15

Year	*Expenditure (in $m)*
1980	2.0
1981	3.5
1982	1.5
1983	5.0
1984	4.0

Construct a chronological bar chart to represent the information given in the table for the five-year period.

2. The table below shows the amount of money invested in millions of dollars by a manufacturing company over a five-year period.

Table 8.16

Year	*Investment (in $m)*
1989	15
1990	18.5
1991	21.7
1992	23.8
1993	25.9

Draw a chronological bar chart to represent the manufacturing company investment during the five-year period.

3. The distribution of full-time students at a university's Faculty of Natural Sciences was as follows:

Table 8.17

Year	*Number of students*
1975	125
1976	139
1977	157
1978	185
1979	196

Draw a chronological bar chart to represent the distribution of students in the Faculty of Natural Sciences over the five-year period.

4. The annual incidence of flu victims in a city from 1985 to 1989, was as follows:

Table 8.18

Year	*Cases*
1985	3700
1986	3850
1987	3900
1988	4150
1989	4350

Draw a chronological bar chart to represent the incidence of flu victims over the five-year period.

5. The estimated population in a city up to 1983 is shown below.

Table 8.19

Year	*Population (in thousands)*	
	Male	*Female*
1979	2530	2740
1980	2580	2780
1981	2610	2810
1982	2670	2850
1983	2740	2960

Construct a chronological bar chart to represent:

(a) the male population

(b) the female population

(c) the population.

Pie Chart

A *pie chart* is a *circular diagram* (similar to a pizza) which is another way of illustrating statistical information. The *circle* is divided into *sectors* (similar to slices of pizza) of varying *sector angles* or *areas. Each sector angle* or *area* is *directly proportional* to the *magnitude of information* that it is representing. In reality, *each sector* is shaded in a different *bright colour.*

The *advantage* of using a *pie chart* is the fact that it shows how a whole quantity is divided into parts and what size these parts are with respect to each other and to the whole.

Example 4

Table 8.20

Subject	*Mathematics*	*English*	*Physics*	*Chemistry*	*History*
No. of teachers	*17*	*25*	*10*	*15*	*23*

The table above gives the number of graduates by subject from a teacher's training college in 1992.

(a) Calculate the total number of teachers that graduated from the training college in 1992.

(b) Determine the sector angle that will represent the number of teachers graduating in each subject area.

(c) Hence construct a pie chart of radius 2.2 cm to represent the information given in the table above.

Solution

(a) The *total number* of teachers that graduated from the teacher's training college in 1992 = (17 + 25 + 10 + 15 + 23) teachers = 90 teachers

(b) The *sector angle* that represents the number of *Mathematics graduates* $= \frac{17\ \cancel{\text{teachers}}}{\underset{1}{\cancel{90}}\ \cancel{\text{teachers}}} \times \overset{4}{\cancel{360}}°$

$= 68°$

The *sector angle* that represents the number of *English graduates* $= \frac{25\ \cancel{\text{teachers}}}{\underset{1}{\cancel{90}}\ \cancel{\text{teachers}}} \times \overset{4}{\cancel{360}}°$

$= 100°$

The *sector angle* that represents the number of *Physics graduates* $= \frac{10\ \cancel{\text{teachers}}}{\underset{1}{\cancel{90}}\ \cancel{\text{teachers}}} \times \overset{4}{\cancel{360}}°$

$= 40°$

The *sector angle* that represents the number of *Chemistry graduates* $= \frac{15\ \cancel{\text{teachers}}}{\underset{1}{\cancel{90}}\ \cancel{\text{teachers}}} \times \overset{4}{\cancel{360}}°$

$= 60°$

And the *sector angle* that represents the number of *History graduates* $= \frac{23\ \cancel{\text{teachers}}}{\underset{1}{\cancel{90}}\ \cancel{\text{teachers}}} \times \overset{4}{\cancel{360}}°$

$= 92°$

Note that the *sum* of the *sector angles*
= (68° + 100° + 40° + 60° + 92°)
= 360°

(c)

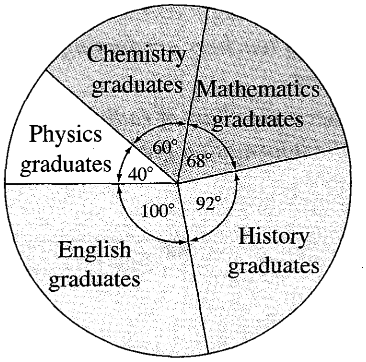


Fig. 8.6 Pie chart

The *pie chart* of *radius* 2.2 cm representing the information given can be seen above.

> Note that it is a good *rule of thumb* to place the *information recorded* in the *pie chart* either in *ascending order* or *descending order* in a *clockwise direction* if possible.

Example 5

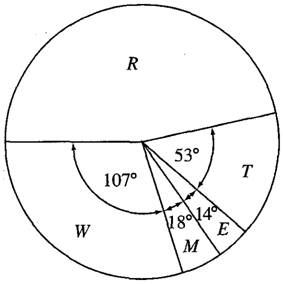


Fig. 8.7 Pie chart

The pie chart above illustrates how a manufacturing company spent its budget for a year on various items as indicated below:

W : Wages and salaries
R : Raw materials
T : Transportation
E : Electricity and telephone bills
M : Miscellaneous

The company spent $875 000 on electricity and telephone bills.
Calculate:

(a) the total budget
(b) the amount spent on raw materials
(c) the percentage of the budget spent on wages.

Solution

(a) The *sector angle* representing the *amount spent* on *electricity and telephone bills* = 14°

And the *amount spent* on *electricity* and *telephone bills* = \$875 000

∴ the *total budget*

$$= \$875\,000 \times \frac{\overset{180}{\cancel{360^\circ}}}{\underset{7}{\cancel{14^\circ}}}$$
$$= \$125\,000 \times 180$$
$$= \$22\,500\,000$$

(b) The *sector* representing the *amount spent* on *raw materials*

$$= 360^\circ - (53^\circ + 14^\circ + 18^\circ + 107^\circ)$$
$$= 360^\circ - 192^\circ$$
$$= 168^\circ$$

∴ the *amount spent* on *raw materials*

$$= \$875\,000 \times \frac{\overset{12}{\cancel{168}}{}^\circ}{\underset{1}{\cancel{14^\circ}}}$$
$$= 875\,000 \times 12$$
$$= \$10\,500\,000$$

Alternatively, the *amount spent* on *raw materials*

$$= \$22\,500\,00\cancel{0} \times \frac{\overset{42}{\cancel{168}}{}^\circ}{\underset{9}{\cancel{360^\circ}}}$$
$$= \$250\,000 \times 42$$
$$= \$10\,500\,000$$

(c) The *percentage* of the *budget spent* on *wages*

$$= \frac{107^\circ \times 10\cancel{0}\%}{36\cancel{0}^\circ}$$
$$= \frac{1\,070}{36}$$
$$= 29.7\%$$

(*correct* to 3 *s.f.*)

Alternatively, the *amount* of the *budget spent* on *wages*

$$= \$875\,000 \times \frac{107^\circ}{14^\circ}$$
$$= \$62\,500 \times 107$$
$$= \$6\,687\,500$$

So the *percentage* of the *budget spent* on *wages*

$$= \frac{\$6\,687\,5\cancel{00}}{\$22\,50\cancel{0}\,\cancel{000}} \times \cancel{100}\%$$
$$= 29.7\%$$

(*correct* to 3 *s.f.*)

Alternative Method (Unitary Method)

(a) The *sector angle* representing the *amount spent* on *electricity* and *telephone bills* = 14°

And the *amount spent* on *electricity* and *telephone bills* = \$875 000

∴ the *amount* that 1° *sector angle* represents $= \dfrac{\$875\,000}{14^\circ}$

= \$62 500 per degree

So the *total budget* = \$62 500 per ~~degree~~ $\times 360\cancel{^\circ}$

= \$22 500 000

(b) The *sector angle* representing the *amount spent* on *raw materials* = 360° − (53° + 14° + 18° + 107°)

= 360° − 192°

= 168°

∴ the *amount spent* on *raw materials* = \$62 500 ~~per degree~~ $\times 168\cancel{^\circ}$

= \$10 500 000

(c) The *amount* of the *budget spent* on *wages* = \$62 500 ~~per degree~~ $\times 107\cancel{^\circ}$

= \$6 687 500

So the *percentage* of the *budget spent* on *wages* $= \dfrac{\$6\,687\,5\cancel{00}}{\$22\,50\cancel{0}\,\cancel{000}} \times 1\cancel{00}\%$

= 29.7%

(*correct* to 3 *s.f.*)

Exercise 8d

1. *Table 8.21*

Subject	*Physics*	*Chemistry*	*Biology*	*Mathematics*	*Geology*
No. of graduates	9	15	19	12	5

The table above gives the number of graduates by subject from a university's Faculty of Natural Sciences in 2002.

(a) Calculate the number of students that graduated from the university's Faculty of Natural Sciences in 2002.

(b) Determine the sector angle that will represent the number of graduates in each subject.

(c) Hence construct a pie chart of radius 4 cm to represent the information given in the table above.

2. An estate valued at \$60 000 is divided among three daughters Annette, Betty and Carol in the ratio 1:2:3 respectively.

(a) Calculate the amount each received.

(b) Draw a pie chart to represent the shares of the three daughters.

3. Sixty students were asked to state their favourite subject chosen from their school timetable. The table below was obtained.

Table 8.22

Subject	*English*	*Mathematics*	*History*	*Geography*	*French*	*spanish*
No. of students	10	16	12	8	8	6

(a) Draw a bar chart to show the information given.

(b) Draw a pie chart to represent the information.

Table 8.23

4.

Type of personnel	*Number employed*
Labourers	54
Operators	29
Supervisors	18
Transportation	19
Total	120

The table above shows the number of people employed on various kinds of work in a factory.

(a) Draw a proportionate bar chart to represent this information.

(b) Draw a simple bar chart to represent this information.

(c) Draw a pie chart to represent this information.

5.

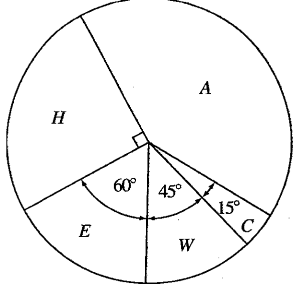


Fig. 8.8 Pie chart

The pie chart above illustrates how a country allocates a budget for 2003 to different ministries as indicated below:

A: Ministry of Agriculture
H: Ministry of Health
E: Ministry of Education
W: Ministry of Works
C: Ministry of Communication

The government allocated \$19.5M for the Ministry of Communication.

Calculate:

(a) the total budget
(b) the amount allocated to the Ministry of Agriculture
(c) the percentage of the budget allocated to the Ministry of Health
(d) the percentage of the budget allocated to the Ministry of Health and Education correct to 3 significant figures.

6.

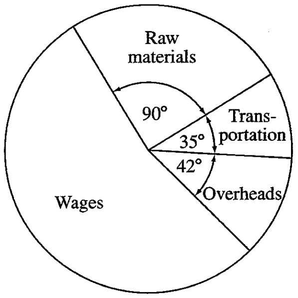


Fig. 8.9 Pie chart

The pie chart above illustrates how a manufacturing company spent its budget for a year on raw materials, transportation, wages and overheads. The company spent \$45 780 on transportation.

Calculate:

(a) the total budget
(b) the amount spent on raw materials
(c) the percentage of the budget spent on wages correct to 3 significant figures.

7.

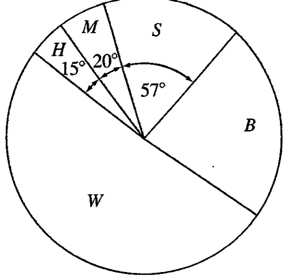


Fig. 8.10 Pie chart

The pie chart above represents the amount of money spent by a firm on various items as indicated below:

W: Wages and Salaries
H: Health
M: Maintenance
S: Sports and games
B: Books and Supplies

The total budget for the year was \$250 200.

(a) Calculate the amounts spent on B and W.
(b) Using a scale of 1 cm to represent \$10 000, draw a bar chart to illustrate the information given in the pie chart above.

8.

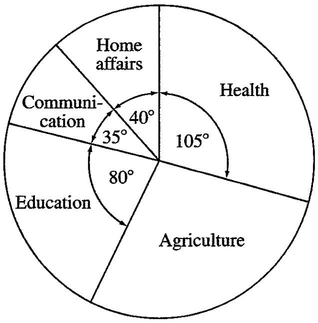


Fig. 8.11 Pie chart

The pie chart above illustrates how a country spent its budget for 2004. It spent \$30.4 million on Education.

Calculate the amount of money spent on

(a) Health
(b) Agriculture.

Line Graph

A *line graph* shows the *data* by means of *drawing a line* as the name suggests. A *line graph* is a graph constructed by joining a *set of points* together in a *consecutive* manner. The *set of points* represents known values of a *given variable*. The *intermediate values* indicated by *elements* of the *line segments* may or may not have a *meaning*. That is, *interpolating* between the *straight line* formed by *consecutive points* may or may not be possible, since *consecutive points* do not necessarily represent *quantities* in *direct proportion*. Normally time is the *variable* that is *plotted* along the *horizontal axis*.

Line graphs are very good for showing *upward* or *downward trends.*

Example 6

The table below shows the quantity of bananas (in tonnes) grown annually on a farm over the period 1986 to 1993.

Table 8.24

Year	*Quantity of bananas grown (tonnes)*
1986	*50*
1987	*200*
1988	*300*
1989	*350*
1990	*450*
1991	*200*
1992	*500*
1993	*600*

(a) Draw a line graph to represent the information given in the table above.

(b) Use the line graph to answer the following questions.

  (i) During which period was there the smallest increase in the quantity of bananas?

  (ii) During which period was there the largest increase in the quantity of bananas?

  (iii) In which year was the quantity of bananas the lowest?

  (iv) During which period was there a decrease in the quantity of bananas?

Solution

(a)

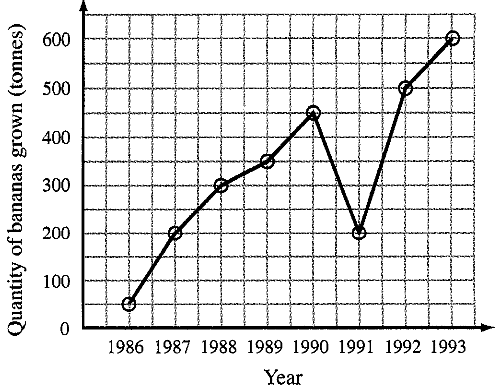


Fig. 8.12 Line graph

Above can be seen the *line graph* which was drawn to represent the information that was given.

(b) From the *line graph*:

  (i) The *period* when there was the *smallest increase in the quantity* of bananas = 1988 to 1989.

  (ii) The *period* when there was the *largest increase in the quantity* of bananas = 1991 to 1992.

  (iii) The *yield* of bananas was the *lowest* in 1986.

  (iv) The *period* when there was a *decrease in the quantity* of bananas = 1990 to 1991.

Exercise 8e

1. A boy's height was measured over several years and the data were recorded below.

Table 8.25

Height (cm)	130	135	150	160	167.5	175	175
Age (years)	9	11	13	15	17	19	21

(a) Draw a line graph to represent the information given in the table above.

(b) Use the line graph to answer the following questions.

  (i) During which period did the boy's height increase the least?

  (ii) During which period did the boy's height increase the most?

  (iii) State the periods during which the boy's height increased by the same amount.

  (iv) Locate and state the period when the boy's height was constant.

2. The 'Ten Year Growth' of a large company is shown in the table below.

Table 8.26

Year	*Profit before tax ($M)*
1984	26
1985	38
1986	70
1987	92
1988	150
1989	164
1990	112
1991	76
1992	94
1993	50

(a) Draw a line graph to represent the data given in the table above.

(b) Use the line graph to answer the following questions:

  (i) Which two periods showed the largest increase in pre-tax profit?

  (ii) State the largest increase in pre-tax profit.

  (iii) Which two periods showed the largest decrease in pre-tax profit?

  (iv) State the largest decrease in pre-tax profit.

3. The infant mortality rates per 1 000 births in a town are shown over a period of years.

Table 8.27

Year	*Mortality rate/1 000 births*
1934	105
1939	93
1944	97
1949	74
1954	61
1959	59
1964	48

(a) Draw a line graph to illustrate this information and comment on it, giving reasons for the trend it shows.

(b) Why do you think that the infant mortality was given as a rate?

(c) Using the line graph answer the following questions:

  (i) During which period was there an increase in the infant mortality rate?

  (ii) During which period was there the largest decrease in the infant mortality rate?

  (iii) State the largest decrease in the infant mortality rate.

4. The employment at the Trinidad and Tobago Electricity Commission (T&TEC), which is a Public Utility is shown in the table below for the period 1970 to 1983.

Table 8.28

Year	*Number of employees*
1970	2 240
1971	$2\,401 \approx 2\,400$
1972	2 120
1973	$2\,445 \approx 2\,450$
1974	$2\,434 \approx 2\,430$
1975	$2\,422 \approx 2\,420$
1976	$2\,426 \approx 2\,430$
1977	$2\,372 \approx 2\,370$
1978	$2\,666 \approx 2\,670$
1979	$2\,781 \approx 2\,780$
1980	$3\,023 \approx 3\,020$
1981	$3\,116 \approx 3\,120$
1982	$3\,128 \approx 3\,130$
1983	$3\,255 \approx 3\,260$

(a) Construct a line graph to represent the information given in the table above using the approximate number of employees where necessary.

(b) Using the line graph, answer the following questions:

  (i) During which period was there the largest increase in employees? State the increase in the number of employees.

  (ii) During which period was there the largest decrease in employees? State the decrease in the number of employees.

(iii) State the three periods when the increase or decrease in the number of employees were the same.

5. The fault rate per 100 stations reported to the Telephone Services of Trinidad and Tobago (TSTT) for the year 1981 are shown below.

Table 8.29

Year	Jan.	Feb.	Mar.	Apr.	May	Jun.
1981	8.5	6.9	8.4	6.3	6.4	8.4
	Jul.	Aug.	Sept.	Oct.	Nov.	Dec.
	9.3	8.6	8.7	8.0	7.1	7.5

(a) Construct a line graph to illustrate the information given.

(b) Why do you think that the faults reported were given as a rate?

(c) Use the line graph to answer the following questions:

(i) During which period was there the greatest increase in the fault rate? State the increase in the rate in the number of faults reported.

(ii) During which periods was there the least increase in the fault rate? State the increase in the rate in the number of faults reported.

(iii) During which period was there the greatest decrease in the fault rate? State the decrease in the rate in the number of faults reported.

Variables

As stated previously, when we collect information which is to be used statistically, we usually want to find out about a *particular characteristic* of a group of people or items. The *particular characteristic* in which we are interested is called the *variable.* This *variable* usually changes from one member of the group to another.

Variables can be divided into two *main types—qualitative* and *quantitative*, defined as follows:

(1) A *qualitative variable* is a defined as a variable which *describes* a characteristic.
For example: The height of a person can be described as short, average or tall.

(2) A *quantitative variable* is defined as a variable which can be given a *numerical value. Quantitative variables* are said to be of two distinct types—*discrete* or *continuous*, defined as follows:

(i) A *discrete variable* is defined as a variable which can only take certain *definite values*, usually *whole numbers.*
For example: The *number of mangoes* on a tree must be a *definite value*, and in this case a *whole number.* Certainly, the number of mangoes on the tree cannot be 476.85.
On the other hand, *shoe sizes* can be bought only in *whole number size*, or a *size and a half*, for example, a *size* 8 or a *size* $8\frac{1}{2}$ and so on, regardless of the actual size of a person's foot. Hence a person with a size 8.25 foot will have to buy a size $8\frac{1}{2}$ as shoes can only be bought in given distinct sizes.

(ii) A *continuous variable* is a variable which can take *any value* within a given *range* and can be obtained by *measurement.*
For example: A person's *height varies continuously* from birth to adulthood and may be found by measurement. That is, if the person is measured 30 cm at birth and 167 cm at adulthood, there was no measurement between 30 cm and 167 cm that the person did not have at some point in time. Hence there were no *gaps* in the *height* of the person within the *interval* (30–167) cm.

Exercise 8f

State whether the following variables are qualitative or quantitative. If the variable is quantitative, state whether it is discrete or continuous:

1. the weight of a baby during the first year of its life on earth.
2. the height of a baby during the first year of its life on earth.
3. the marks of a class in an examination.
4. the colour of a human hair which is black.
5. the colour of a human eye which is brown.
6. the body temperature of a normal person.
7. the temperature of an ill person.

8. the scores of a cricket team in a cricket match.
9. the scores of a football team in a football match.
10. the speed of a ball thrown to the wicket keeper.
11. the speed of a bird in flight.
12. the number of apples on a tree.
13. the length of a child's foot from age 4 to age 9.
14. the shoe sizes of a child from age 4 to age 9.
15. the volume of water used by a household throughout the year.
16. the electricity used by a household throughout the year.
17. the air pressures recorded by a weather balloon released from the ground station.
18. the make of a person's car which is a Mazda 323–1986 model.
19. the description of a straight line (short).
20. the smell of a chaconia flower.

Frequency Table with Ungrouped Data

The *frequency* of an *event* (or *observation* or *score*) is defined as the *number of times* an *event* has *occurred.* For example: If a die is tossed ten times and the results are 3, 1, 4, 6, 5, 2, 1, 3, 1, 5, then we say that the *frequency* of 1 is *three*, since three of the tosses were 1s.

A major way of *organising raw data* into some sort of order is to arrange them in a form of a *frequency distribution*, and a *tally chart* is the best method of doing this. In constructing a *tally chart*, each score in the list of raw data is *crossed out* in the order in which it was *tallied*. *One stroke* is then marked in the *tally column* for each score in the raw data crossed out. And *every fifth stroke* is drawn *diagonally* across the previous four strokes in order to make a *group* or *bundle of* 5. This process is then repeated if necessary. That is, if the *frequency* of that particular score in the raw data is *greater than* 5.

It should be noted that the *total frequency* must always be the *same* as the number of *scores* in the *raw data.*

The *relative frequency* (or *experimental probability*) of an *event* (or *observation* or *score*) occurring is defined as the *frequency of the event* in comparison to the *total frequency*.

The *relative frequency* (or *experimental probability*) is normally stated as a *common fraction* (or *decimal fraction*) or as a *percentage*.

(i) As a *fraction*:

The *relative frequency of an event,*

$$R.F. = \frac{\text{The frequency of the event}}{\text{The total frequency}}$$

(ii) As a *percentage*:

The *relative frequency of an event,*

$$R.F. = \frac{\text{The frequency of the event}}{\text{The total frequency}} \times 100\%$$

Example 7

There are 50 participants in a shooting competition. The score of each participant is listed below. (This is the list of raw data.)

6	3	7	0	2	4	6	7	1	3
1	1	4	1	5	3	3	6	3	0
4	5	6	4	3	1	2	3	4	1
3	0	1	1	1	2	1	1	7	4
0	4	3	5	7	5	0	4	6	1

(a) Construct a frequency table for the scores.

(b) Calculate the relative frequency of the score 5.

(c) Calculate the probability that if a score is chosen at random that it is greater than 4.

Solution

(a)

$\cancel{6}$	$\cancel{3}$	$\cancel{7}$	$\cancel{0}$	$\cancel{2}$	$\cancel{4}$	$\cancel{6}$	$\cancel{7}$	$\cancel{1}$	$\cancel{3}$
$\cancel{1}$	$\cancel{1}$	$\cancel{4}$	$\cancel{1}$	$\cancel{5}$	$\cancel{3}$	$\cancel{3}$	$\cancel{6}$	$\cancel{3}$	$\cancel{0}$
$\cancel{4}$	$\cancel{5}$	$\cancel{6}$	$\cancel{4}$	$\cancel{3}$	$\cancel{1}$	$\cancel{2}$	$\cancel{3}$	$\cancel{4}$	$\cancel{1}$
$\cancel{3}$	$\cancel{0}$	$\cancel{1}$	$\cancel{1}$	$\cancel{1}$	$\cancel{2}$	$\cancel{1}$	$\cancel{1}$	$\cancel{7}$	$\cancel{4}$
$\cancel{0}$	$\cancel{4}$	$\cancel{3}$	$\cancel{5}$	$\cancel{7}$	$\cancel{5}$	$\cancel{0}$	$\cancel{4}$	$\cancel{6}$	$\cancel{1}$

Table 8.30 Frequency table

Score	*Tally*	*Frequency*
0	𝍸	5
1	𝍸 𝍸 ||	12
2	|||	3
3	𝍸 ||||	9
4	𝍸 |||	8
5	||||	4
6	𝍸	5
7	||||	4
Total frequency = 50		

Above can be seen the *frequency table* for the *scores*. The *frequency table* constructed is also called the *frequency distribution table* (or a *simple frequency table* or a *tally chart*).

From the *frequency table*:

(b) The *frequency* of the *score* 5 $= 4$

And the *total frequency* $= 50$

∴ the *relative frequency* of the *score* 5, *R.F.* $= \dfrac{\textit{The frequency of the event}}{\textit{The total frequency}}$

$= \dfrac{4}{50}$

$= 0.08$

Or the *relative frequency* of the *score* 5, *R.F.* $= \dfrac{\textit{The frequency of the event}}{\textit{The total frequency}} \times 100\%$

$= \dfrac{4}{5\cancel{0}} \times 10\cancel{0}\%$

$= 8\%$

(c) The *frequency* of *scores greater than* 4 $= (4 + 5 + 4) = 13$

And the *total frequency* $= 50$

∴ the *probability* that if a *score* is chosen at random that it is *greater than* 4 $= \dfrac{\textit{The frequency of the event}}{\textit{The total frequency}}$

$= \dfrac{13}{50}$

$= 0.26$

Exercise 8g

1. The shoe sizes of pupils in a class are:

4	5	8	7	5	8
7	4	7	9	4	9
4	5	5	8	5	5
6	8	6	5	8	7
5	7	9	6	7	5

(a) Draw a frequency table to represent the information given.

(b) Calculate the relative frequency of a size 6 shoe.

(c) Calculate the probability that if a pupil is chosen at random that he/she wears a size 7 shoe.

2. The number of sessions absent per student of the same form during a certain week are shown below.

0	3	1	0	9	3
1	0	4	10	0	4
4	1	0	0	4	0
0	5	0	2	0	7
5	0	0	4	0	0
0	4	3	7	3	4

(a) Draw a simple frequency table using the following headings:

Table 8.31 Frequency table

Number of sessions absent	0	1
Frequency	15	

(b) Calculate the probability that if a student is selected at random that he/she was absent for:

(i) 6 sessions

(ii) less than 2 sessions

(iii) more than 7 sessions.

3. There are 25 participants in a shooting competition. The score of each participant is listed below.

1	3	5	0	2
2	1	6	5	6
0	3	5	1	1
5	2	1	0	6
1	4	0	3	5

(a) Draw a frequency table to represent the information given.

Calculate the probability that if a participant is chosen at random that he/she scored:

- (i) less than 2
- (ii) exactly 5
- (iii) at least 4.

4. The heights of 50 students correct to the nearest centimetre are given below.

150	151	152	153	153
151	153	154	152	155
153	154	151	153	152
154	155	153	154	154
152	153	155	151	153
153	152	153	152	155
154	153	154	155	152
155	152	156	153	151
153	154	153	156	154
152	153	152	154	153

(a) Construct a tally table to represent the data above.

(b) Calculate the probability that if a student is chosen at random that he/she is:

- (i) not more than 153 cm in height
- (ii) greater than 153 cm in height.

5. The number of tickets bought per person for a calypso show are given below.

3	4	2	3	2	3	3	2	3	3
4	3	4	1	3	4	2	3	2	5
2	6	3	4	4	3	4	6	3	2
3	2	1	3	3	2	3	4	1	1
1	3	2	2	2	3	2	3	4	3
4	1	3	1	3	1	1	2	3	4
2	3	4	3	1	2	3	1	2	2
3	2	2	2	3	3	2	3	5	3
5	5	3	5	2	4	5	4	3	4
4	2	2	3	5	3	2	3	4	2
2	3	1	6	3	2	3	5	2	3
3	2	3	2	2	3	4	2	3	2

(a) Construct a frequency table for the number of tickets bought per person under the headings:

Table 8.32 Frequency table

Number of tickets bought per person for a calypso show	*Tally*	*Frequency*
1 2	𝍸 𝍸 ||	12

(b) Calculate the relative frequency of the number of persons who bought 4 or more tickets for the calypso show as a percentage correct to 3 significant figures.

(c) Calculate the probability that if a person is chosen at random that he/she bought less than 4 tickets correct to 3 decimal places.

6. A biologist takes a sample of 100 grass plants to measure stem length. The following data was obtained correct to the nearest centimetre:

25	29	30	29	28	29	26	29	30	27
30	32	26	30	30	27	28	30	27	30
26	30	29	31	27	30	30	28	32	29
31	26	31	27	32	25	32	27	29	31
29	32	27	26	29	28	29	32	28	32
27	30	32	28	26	31	32	30	30	28
31	27	28	32	32	26	30	29	31	29
30	29	30	29	30	29	29	26	29	31
29	28	31	30	31	30	32	32	26	32
28	31	29	28	29	28	27	31	32	31

(a) Construct a frequency distribution table for the stem lengths.

(b) Calculate the relative frequency for the stems longer than 29 cm.

(c) Calculate the probability that if a stem is selected at random that it is not more than 29 cm in length.

Histogram for Ungrouped Data

A *histogram* consists of a number of *rectangular bars* which can be of *different widths*. These *bars* are always drawn *vertically* and are joined side to side *without leaving any space*, since there is now a regular *scale* along the *horizontal axis*. So the *absence of a bar* implies that the *frequency* of that *observation* (or *variable*) is *zero*.

In the case of a *histogram*, *frequency* is always plotted along the *vertical axis* and the *observation* or *variable* is plotted along the *horizontal axis.*

The *frequency of a variable* is *directly proportional* to the *area* of *each bar*, since the bars can be of *different widths* in general. However, at this level, the *bars* are always of the *same width*, hence the *frequency of an observation* is *directly proportional* to the *height* of *each bar.*

From experience, one of the best ways of representing a *frequency distribution graphically* is by means of a *histogram.*

Example 8

The frequency table below shows the number of points gained by the teams in a series of cricket matches.

Table 8.33 Frequency table

No. of points	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Frequency	*3*	*2*	*0*	*5*	*6*	*7*	*2*	*1*	*0*	*4*	*1*

(a) How many teams took part in the series?

(b) Draw a histogram to represent the data.

(c) If a team is chosen at random, calculate the probability that it gained:

(i) less than 5 points
(ii) exactly 5 points
(iii) greater than 5 points
(iv) at least 5 points.

Solution

(a) The *number of teams* that took part in the series = (3 + 2 + 0 + 5 + 6 + 7 + 2 + 1 + 0 + 4 + 1) teams
= 31 teams

(b)

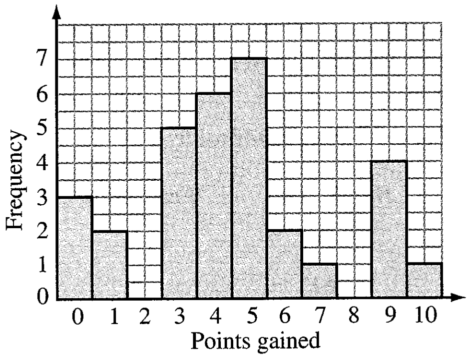


Fig. 8.13 (a) Histogram

Above can be seen the *histogram* that represents the data given.
It should be noted that along the *horizontal axis* the number of *points gained* is written at the *centre* of *each column* (or *rectangular bar*).

NOTE: If the *points gained* can only be *whole numbers*, then the *variable* is *discrete*. This means that a *point gained cannot be*, for example, 1.2, 3.4 or 6.5 Thus, a *discrete variable* can more appropriately be represented by a *vertical bar chart* (or *column graph*) as shown below.

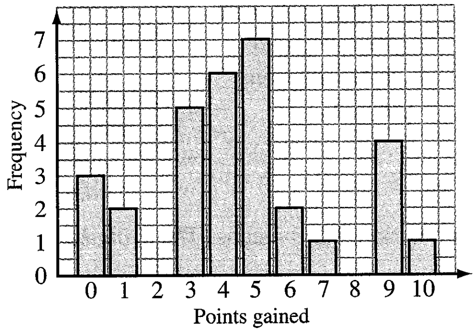


Fig. 8.13 (b) Vertical bar chart (or column graph)

However, representing *discrete data* on a *histogram* as shown still gives the *same total frequency*. What it does is imply that the *distribution* is *continuous*, since adjacent rectangular bars touch each other when the data is continuous.

From the *histogram*:

(c) (i) The *number of teams* that gained *less than 5 points* = (3 + 2 + 0 + 5 + 6) teams
= 16 teams

And the *total number of teams* = 31 teams

∴ the *probability* that a *team* chosen at random gained *less than 5 points*, *P(points gained < 5)*

$$= \frac{\textit{The frequency of the event}}{\textit{The total frequency}}$$

$$= \frac{16\ \cancel{\text{teams}}}{31\ \cancel{\text{teams}}}$$

= 0.52 (*correct* to 2 *d.p.*)

(ii) The *number of teams* that gained *exactly 5 points* = 7 teams

∴ the *probability* that a *team* chosen at random gained *exactly* 5 *points*, *P(points gained* = 5)

$$= \frac{\textit{The frequency of the event}}{\textit{The total frequency}}$$

$$= \frac{7\ \cancel{\text{teams}}}{31\ \cancel{\text{teams}}}$$

= 0.23 (*correct* to 2 *d.p.*)

(iii) The *number of teams* that gained *greater than 5 points* = (2 + 1 + 0 + 4 + 1) teams
= 8 teams

∴ the *probability* that a *team* chosen at random gained *greater than* 5 *points*, *P*(*points gained* > 5)

$$= \frac{\textit{The frequency of the event}}{\textit{The total frequency}}$$

$$= \frac{8\ \cancel{\text{teams}}}{31\ \cancel{\text{teams}}}$$

= 0.26 (*correct* to 2 *d.p.*)

The notation, *P*(...) means *'The probability that'*.

Note that:
P(*points gained* < 5) + *P*(*points gained* = 5) + *P*(*points gained* > 5) $= \frac{16}{31} + \frac{7}{31} + \frac{8}{31} = \frac{31}{31} = 1$

Hence the *probability* that a *team* chosen at random gained between 0 and 10 *points inclusive is* 1. That is, the *probability* of a *certainty is* 1. This is the *maximum value* that the *probability of an event* occurring can take.
And the *probability* of an *impossible event* occurring is *zero*.

For example: *P*(*points gained* = 8) $= \frac{0}{31} = 0.$

(iv) The *number of teams* that gained at *least* 5 *points* = (7 + 2 + 1 + 0 + 4 + 1) teams
= 15 teams

∴ *P*(*points gained* ⩾ 5)

$$= \frac{\textit{The frequency of the event}}{\textit{The total frequency}}$$

$$= \frac{15\ \cancel{\text{teams}}}{31\ \cancel{\text{teams}}}$$

= 0.48 (*correct* to 2 *d.p.*)

Exercise 8h

1. The table below shows the number of children per family in the families of the pupils in a class.

Table 8.34 Frequency table

Number of children per family	1	2	3	4	5	6	7
Frequency	2	4	9	5	7	2	1

(a) Draw a histogram to represent the data given
(b) If a family is chosen at random, calculate the probability that the number of children in the family is:
(i) less than 3
(ii) more than 3
(iii) exactly 3.

2. The frequency table below shows the shoe sizes of pupils in a class.

Table 8.35 Frequency table

Shoe size	4	5	6	7	8	9
Frequency	4	9	3	6	5	3

(a) Construct a histograph to represent the information given.
(b) If a pupil is chosen at random, calculate the probability that his/her shoe size is
(i) less than 5 (ii) greater than 8.

3. The table below shows how many pupils in a form were absent for various numbers of sessions during a certain school week.

Table 8.36 Frequency table

Number of sessions absent	0	1	2	3	4	5	6	7	8	9	10
Frequency	15	3	1	4	7	2	0	2	0	1	1

(a) Draw a histogram to show this information.
(b) If a pupil is selected at random, calculate the probability that he was absent for at least 6 sessions.

4. The heights of 50 students correct to the nearest centimetre are shown in the frequency table below.

Table 8.37 Frequency table

Height (cm)	*Frequency*
150	1
151	5
152	10
153	16
154	10
155	6
156	2

(a) Construct a histogram to represent this information.
(b) Calculate the probability that if a student is selected at random he/she is
(i) shorter than 152 cm
(ii) taller than 154 cm.

5. A biologist takes a sample of 100 grass plants to measure stem length. The following data was obtained:

Table 8.38 Frequency table

Length (cm)	*Frequency*
25	2
26	9
27	10
28	12
29	20
30	19
31	13
32	15

(a) Draw a histogram to represent this data.

(b) Calculate the probability that a stem selected at random was less than 28 cm in length.

6. The number of tickets purchased per person for a calypso show can be seen in the frequency table below.

Table 8.39 Frequency table

No. of tickets purchased per person for a calypso show	*Frequency*
1	12
2	35
3	44
4	18
5	8
6	3

(a) Construct a histogram to represent this data.

(b) Calculate the probability that a person chosen at random purchased exactly 4 tickets.

Frequency Polygon for Ungrouped Data

Another way of representing a *frequency distribution graphically* is by drawing a *line graph* called a *frequency polygon*. A *frequency polygon* is a statistical diagram that indicates the *spread* of a given *distribution*. This concept of *spread* we will deal with in detail later on.

The *frequency polygon* for *ungrouped data* is obtained by plotting the *observation* (or *variable*) against the *corresponding frequency* and then *drawing straight lines* in order to join *consecutive points*. These *observations* are equivalent to the *mid-points* of the *tops* of the *columns* of the *histogram*.

The *area* under a *frequency polygon* is *directly proportional* to the *total frequency* of the *distribution*.

Frequency polygons are most useful in *comparing distributions*, since it is very easy to draw two or more polygons on the same graph paper with clarity. However, the *total frequency* of *each distribution* must be the *same* in order to perform a *fair comparison*.

Example

The marks obtained by 50 students in a test in which the maximum mark was 10 were as follows:

Table 8.40 Frequency table

Mark	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Frequency	*1*	*2*	*4*	*7*	*9*	*10*	*8*	*5*	*3*	*1*

(a) Draw a frequency polygon representing the data on graph paper.

(b) Calculate the area enclosed by the frequency polygon and the horizontal axis.

(c) If a student is selected at random, calculate the probability that he scored

  (i) no more than 5 marks

  (ii) at least 6 marks.

Solution

(a)

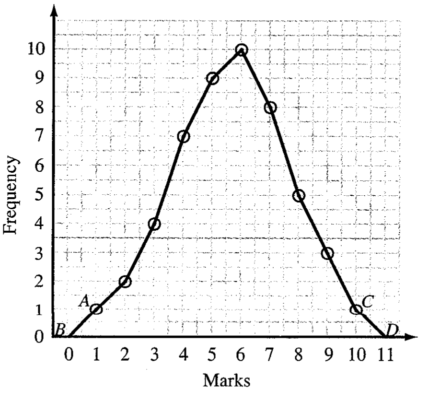


Fig. 8.14 (a) Frequency polygon

Above can be seen the *frequency polygon* that represents the data given. It can also be seen that in order to *complete* the *frequency polygon*, the *line* at *each end* was continued to the horizontal axis to where the *next mark* would have been *found* if it was present. The *continuation* of the *line* is denoted by *AB* and *CD* in the diagram.
The reason for this *procedure* will be explained shortly.

(b)

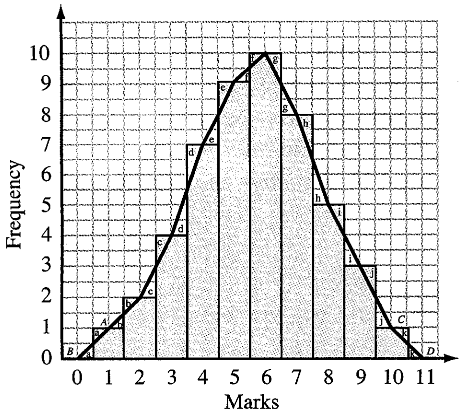


Fig. 8.14 (b) Histogram and frequency polygon

The diagram above shows the *frequency polygon superimposed* on the *histogram*. It can be seen that the *frequency polygon* can be drawn by *joining consecutive mid-points* of the *tops* of the *columns* of the *histogram* by *straight lines*.

Also note that *each pair of regions* marked with the *same letter*, for example, *a*, are *equal in area*. Hence in order for the *frequency polygon* to have the *same area* as the *histogram*, we have to finish off the *polygon* by drawing the *lines AB* and *CD*.

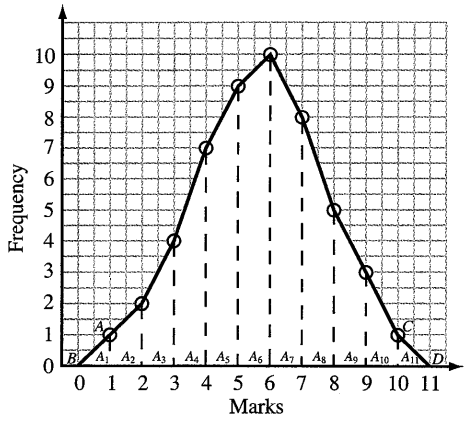


Fig. 8.14 (c) Frequency polygon

The *area enclosed* by the *frequency polygon* and the *horizontal axis*

$$= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 + A_9 + A_{10} + A_{11}$$

$$= \frac{1}{2} \times 1 \times 1 + \frac{1}{2}(1 + 2) \times 1 + \frac{1}{2}(2 + 4) \times 1 + \frac{1}{2}(4 + 7) \times 1 + \frac{1}{2}(7 + 9) \times 1 + \frac{1}{2}(9 + 10) \times 1 + \frac{1}{2}(10 + 8) \times 1 + \frac{1}{2}(8 + 5) \times 1 + \frac{1}{2}(5 + 3) \times 1 + \frac{1}{2}(3 + 1) \times 1 + \frac{1}{2} \times 1 \times 1$$

$$= \frac{1}{2}(1 + 3 + 6 + 11 + 16 + 19 + 18 + 13 + 8 + 4 + 1)$$

$$= \frac{1}{2} \times 100$$

= 50 students

Note that the *regions* A_1 and A_{11} are *triangles*. And the *regions* A_2 to A_{10} are all *trapeziums*.

Note that the *area* of a *triangle*, $A = \frac{1}{2}bh$.

And the *area* of a *trapezium*, $A = \frac{1}{2}(a + b)h$.

In both cases, the *altitude h* is *equal* to the *width* of a *bar* of the *histogram* or the *distance* between *consecutive marks* in the *frequency polygon*.

From the *frequency table*:

(c) (i) The *number of students* who scored *no more than* 5 *marks* (i.e. ⩽ 5 *marks*) = (1 + 2 + 4 + 7 + 9) students = 23 students

And the *total number of students* = 50 students

$$\therefore P(\textit{student's mark} \leqslant 5) = \frac{\textit{The frequency of the event}}{\textit{The total frequency}} = \frac{23 \cancel{\text{ students}}}{50 \cancel{\text{ students}}} = 0.46$$

(ii) The *number of students* who scored *at least* 6 *marks* (i.e. ⩾ 6) = (10 + 8 + 5 + 3 + 1) students = 27 students

$$\therefore P(\textit{student's mark} \geqslant 6) \frac{\textit{The frequency of the event}}{\textit{The total frequency}} = \frac{27 \cancel{\text{ students}}}{50 \cancel{\text{ students}}} = 0.54$$

Exercise 8i

1. The shoe sizes of pupils in a class are given by the frequency table shown below.

Table 8.41 Frequency table

Shoe size	4	5	6	7	8	9
Frequency	4	9	3	6	5	3

(a) Draw a frequency polygon representing the data on graph paper.

(b) Calculate the area enclosed by the frequency polygon and the horizontal axis.

(c) If a pupil is selected at random, calculate the probability that he/she wears a size 5 or 6.

2. The table below shows the number of children per family in the families of the pupils in a class.

Table 8.42 Frequency table

No. of children per family	1	2	3	4	5	6	7
Frequency	2	4	9	5	7	2	1

(a) Draw a frequency polygon to represent the data given on graph paper.

(b) Calculate the area enclosed by the frequency polygon and the horizontal axis.

(c) If a pupil is chosen at random, calculate the probability that the number of children in his/her family is 4 or 5.

3. In a shooting contest in which 50 people participated, the following frequency table was obtained.

Table 8.43 Frequency table

Score	*Frequency*
1	3
2	1
3	4
4	10
5	15
6	9
7	3
8	5

(a) Construct a frequency polygon to represent the frequency distribution on graph paper using suitable scales.

(b) Determine the area enclosed by the frequency polygon and the horizontal axis.

(c) Calculate the probability that if a participant is selected at random he scored 7 to 8 points.

4. The frequency distribution of the heights of 50 students correct to the nearest centimetre is given below.

Table 8.44 Frequency table

Height (cm)	*Frequency*
150	1
151	5
152	10
153	16
154	10
155	6
156	2

(a) Construct a frequency polygon to represent the distribution, on graph paper, using suitable scales.

(b) Determine the area enclosed by the frequency polygon and the horizontal axis.

(c) Calculate the probability that a student chosen at random is 153 cm or 154 cm in height.

5. The number of tickets bought per person for a calypso show can be seen in the frequency distribution below.

Table 8.45 Frequency table

Number of tickets bought per person for a calypso show	*Frequency*
1	12
2	35
3	44
4	18
5	8
6	3

(a) Construct a frequency polygon to represent the distribution, on graph paper, using suitable scales.

(b) Calculate the area enclosed by the frequency polygon and the horizontal axis.

(c) Calculate the probability that a person chosen at random bought 2 or 3 tickets.

6. The marks obtained by 30 students in a test in which the maximum mark was 10 were as follows:

7	8	5	3	6	2	1	1	7	6
5	5	4	4	4	1	4	6	8	6
6	9	6	5	5	3	2	4	7	7

(a) Construct a frequency distribution from the data given.

(b) Draw a frequency polygon representing the data on graph paper.

(c) If a student is chosen at random, calculate the probability that he received
  (i) less than 5 marks
  (ii) at least 5 marks.

7. There are 30 participants in a shooting competition. The score of each participant is listed below.

5	1	3	1	0	6
1	5	1	6	3	1
3	1	4	5	5	2
0	0	2	0	1	6
2	2	3	5	0	5

(a) Set up a frequency table for the scores.

(b) Draw the frequency polygon representing the data on graph paper.

(c) Calculate the probability that a competitor selected at random has a score less than 4.

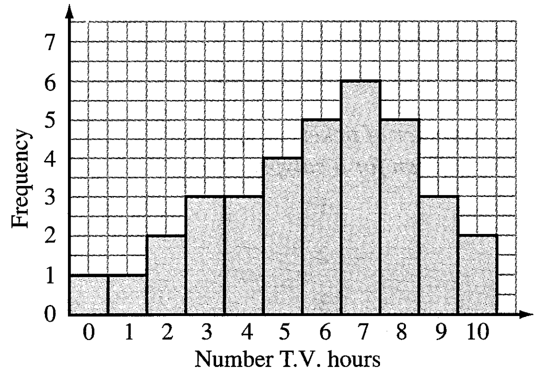


Fig. 8.15 Histogram

The histogram above shows the number of hours a group of children watched television on a Sunday.

(a) Construct a frequency table to represent the data shown on the histogram under the headings, class mark and frequency.

(b) Calculate the probability that a child chosen random from this group watched television for 7 hours or more.

Frequency Table with Grouped Data

Sometimes the data under consideration has such a *large range of values* that it is most useful to collect these values into *groups* (or *classes*). And the *class interval* is defined as the *size* of the *group* (or *class*) chosen.

Example 10

The masses of 100 students correct to the nearest kilogram are as follows:

35	*44*	*81*	*82*	*37*	*44*	*38*	*84*	*55*	*61*
41	*59*	*37*	*68*	*38*	*49*	*35*	*83*	*58*	*83*
47	*52*	*42*	*67*	*47*	*51*	*37*	*64*	*59*	*54*
53	*63*	*38*	*75*	*45*	*58*	*41*	*79*	*45*	*39*
64	*72*	*39*	*78*	*63*	*67*	*48*	*68*	*47*	*75*
71	*81*	*45*	*69*	*64*	*68*	*54*	*61*	*49*	*42*
84	*54*	*63*	*64*	*69*	*42*	*58*	*64*	*37*	*57*
75	*63*	*65*	*58*	*68*	*58*	*67*	*74*	*39*	*81*
63	*65*	*71*	*57*	*73*	*59*	*68*	*71*	*43*	*62*
52	*70*	*81*	*49*	*75*	*60*	*75*	*64*	*58*	*70*

(a) Draw up a tally chart for the classes 35–39, 40–44, 45–49, 50–54, 55–59, 60–64, 65–69, 70–74, 75–79 and 80–84.

(b) Calculate the probability that the mass of a student selected at random is:
  (i) less than 60 kg
  (ii) at least 60 kg.

Solution

(a)

~~35~~	~~44~~	~~81~~	~~82~~	~~37~~	~~44~~	~~38~~	~~84~~	~~55~~	~~61~~
~~41~~	~~59~~	~~37~~	~~68~~	~~38~~	~~49~~	~~35~~	~~83~~	~~58~~	~~83~~
~~47~~	~~52~~	~~42~~	~~67~~	~~47~~	~~51~~	~~37~~	~~64~~	~~59~~	~~54~~
~~53~~	~~63~~	~~38~~	~~75~~	~~45~~	~~58~~	~~41~~	~~79~~	~~45~~	~~39~~
~~64~~	~~72~~	~~39~~	~~78~~	~~63~~	~~67~~	~~48~~	~~68~~	~~47~~	~~75~~
~~71~~	~~81~~	~~45~~	~~69~~	~~64~~	~~68~~	~~54~~	~~61~~	~~49~~	~~42~~
~~84~~	~~54~~	~~63~~	~~64~~	~~69~~	~~42~~	~~58~~	~~64~~	~~37~~	~~57~~
~~75~~	~~63~~	~~65~~	~~58~~	~~68~~	~~58~~	~~67~~	~~74~~	~~39~~	~~81~~
~~63~~	~~65~~	~~71~~	~~57~~	~~73~~	~~59~~	~~68~~	~~71~~	~~43~~	~~62~~
~~52~~	~~70~~	~~81~~	~~59~~	~~75~~	~~60~~	~~75~~	~~64~~	~~58~~	~~60~~

Table 8.46 Frequency table

Mass (kg)	*Tally*	*Frequency*
35–39	𝍸 𝍸 ||	12
40–44	𝍸 |||	8
45–49	𝍸 𝍸	10
50–54	𝍸 ||	7
55–59	𝍸 𝍸 ||	12
60–64	𝍸 𝍸 𝍸	15
65–69	𝍸 𝍸 ||	12
70–74	𝍸 |||	8
75–79	𝍸 ||	7
80–84	𝍸 ||||	9
	Total frequency = 100	

Above can be seen the *tally chart* which was constructed using the given *classes* for the data.

The *tally chart* constructed is also called a *grouped frequency table* (or a *grouped frequency distribution table*).

(b) From the *frequency table*:

(i) The *number of students* with a *mass less than* 60 kg (i.e. < 60 kg) $= (12 + 8 + 10 + 7 + 12)$ students $= 49$ students

The *total number of students* $= 100$ students

$$\therefore P(\textit{student's mass} < 60\text{ kg}) = \frac{\textit{The frequency of the event}}{\textit{The total frequency}} = \frac{49 \cancel{\text{ students}}}{31 \cancel{\text{ students}}} = 0.49$$

(ii) The *number of students* with a *mass of at least* 60 kg (i.e. $\geqslant 60$ kg) $= (15 + 12 + 8 + 7 + 9)$ students $= 51$ *students*

$$\therefore P(\textit{student's mass} \geqslant 60\text{ kg}) = \frac{\textit{The frequency of the event}}{\textit{The total frequency}} = \frac{51 \cancel{\text{ students}}}{100 \cancel{\text{ students}}} = 0.51$$

Width of a Class Interval (or Class Size)

The masses of 100 students correct to the nearest kilogram are shown in the frequency table below.

Table 8.47 Frequency table

Class (kg)	*Frequency*
35–39	12
40–44	8
45–49	10
50–54	7
55–59	12
60–64	15
65–69	12
70–74	8
75–79	7
80–84	9

Class Interval

A *class interval* is defined as a *grouping* of *statistical data. Class intervals* enable the *data* to be *represented* and *interpreted* in a much *simpler way.*

From the *frequency table* above:

(i) The *first class* is the *class interval* (35–39) kg.

(ii) The *second class* is the *class interval* (40–44) kg.

(iii) The *third class* is the *class interval* (45–49) kg.
And so on.

Class Limits

The *class limits* are the *end values* of a *class interval.* Each *class interval* has two class limits—a *lower class limit* to the *left* and an *upper class limit* to the *right.*

From the *frequency table* above:

(i) The *lower class limit* for the *first class interval* is 35 kg.

(ii) The *lower class limit* for the *second class interval* is 40 kg.

(iii) The *lower class limit* for the *third class interval* is 45 kg.
And so on.

(i) The *upper class limit* for the *first class interval* is 39 kg.

(ii) The *upper class limit* for the *second class interval* is 44 kg.

(iii) The *upper class limit* for the *third class interval* is 49 kg.
And so on.

Class Boundaries

Now $34.1 \approx 34$ And $39.1 \approx 39$ Also $44.1 \approx 44$
$34.2 \approx 34$ $39.2 \approx 39$ $44.2 \approx 44$
$34.3 \approx 34$ $39.3 \approx 39$ $44.3 \approx 44$
$34.4 \approx 34$ $39.4 \approx 39$ $44.4 \approx 44$
$34.5 \approx 35$ $39.5 \approx 40$ $44.5 \approx 45$
$34.6 \approx 35$ $39.6 \approx 40$ $44.6 \approx 45$
$34.7 \approx 35$ $39.7 \approx 40$ $44.7 \approx 45$
$34.8 \approx 35$ $39.8 \approx 40$ $44.8 \approx 45$
$34.9 \approx 35$ $39.9 \approx 40$ $44.9 \approx 45$
$35.0 = 35$ $40.0 = 40$ $45.0 = 45$

The *examples* above define what we call *class boundaries. Theoretically*, the *first class interval* (35–39) kg includes all the students with masses between 34.5 kg and 39.4 kg, such that $34.5 \leqslant x < 39.5$. We say that the *lower class boundary* for the *first class interval* is 34.5 kg and the *upper class boundary* is 39.5 kg.

Theoretically, the *second class interval* (40–44) kg includes all the students with masses between 39.5 kg and 44.4 kg, such that $39.5 \leqslant x < 44.5$. We say that the *lower class boundary* for the *second class interval* is 39.5 kg and the *upper class boundary* is 44.5 kg.

Theoretically, the *third class interval* (45–49) kg includes all the students with masses between 44.5 kg and 49.4 kg, such that $44.5 \leqslant x < 49.5$. We say that the *lower class boundary* for the *third class interval* is 44.5 kg and the *upper class boundary* is 49.5 kg.
And so on.

If the *first three classes* are written in a *row* it helps to make the *results clearer.*

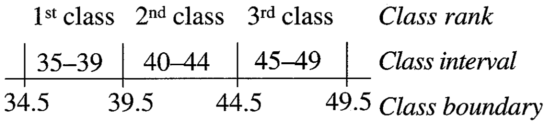


Fig. 8.16 Class boundaries

Thus, 39.5 kg is the *boundary separating* the *first* and *second classes.* That is, 39.5 kg is the *upper class boundary* of the *first class interval* and the *lower class boundary* of the *second class interval.*

And 44.5 kg is the *boundary separating* the *second* and *third classes.* That is, 44.5 kg is the *upper class boundary* of the *second class interval* and the *lower class boundary* of the *third class interval.*

The complete table of *theoretical class intervals* can be seen below.

Table 8.48 Theoretical class intervals

Class interval (kg)	*Theoretical class interval (kg)*
35–39	$34.5 \leqslant x < 39.5$
40–44	$39.5 \leqslant x < 44.5$
45–49	$44.5 \leqslant x < 49.5$
50–54	$49.5 \leqslant x < 54.5$
55–59	$54.5 \leqslant x < 59.5$
60–64	$59.5 \leqslant x < 64.5$
65–69	$64.5 \leqslant x < 69.5$
70–74	$69.5 \leqslant x < 74.5$
75–79	$74.5 \leqslant x < 79.5$
80–84	$79.5 \leqslant x < 84.5$

Thus:

$$\textit{The boundary between two classes} = \frac{\textit{The upper class limit of the lower rank class} + \textit{The lower class limit of the higher rank class}}{2}$$

$$\text{So the } \textit{lower class boundary} \text{ of the } \textit{second class interval} = \frac{\textit{The upper class limit of the first class} + \textit{The lower class limit of the second class}}{2}$$

$$= \frac{(39 + 40)\text{ kg}}{2} = \frac{79\text{ kg}}{2} = 39.5\text{ kg}$$

$$\text{And the } \textit{upper class boundary} \text{ of the } \textit{second class interval} = \frac{\textit{The upper class limit of the second class} + \textit{The lower class limit of the third class}}{2}$$

$$= \frac{(44 + 45)\text{ kg}}{2} = \frac{89\text{ kg}}{2} = 44.5\text{ kg}$$

Hence the *class boundary* is the *average* of the *class limits* involved.

Class Mid-point

Sometimes it is necessary to *determine* the *mid-point of a class interval*. The *mid-point of a class interval* is defined as the *average* of the *lower and upper boundaries* of the *class*. The *mid-point of a class interval* is very important as it is sometimes used to stand for the *whole group*. The *class mid-point* is also called the *class mid-mark* (or the *class value*).

Thus:

$$\textit{The class mid-point} = \frac{\textit{The lower class boundary} + \textit{The upper class boundary}}{2}$$

$$\text{So the } \textit{mid-point} \text{ of the } \textit{first class interval} = \frac{\textit{The lower class boundary} + \textit{The upper class boundary}}{2}$$

$$= \frac{(34.5 + 39.5)\text{ kg}}{2} = \frac{74\text{ kg}}{2} = 37\text{ kg}$$

$$\text{And the } \textit{mid-point} \text{ of the } \textit{second class interval} = \frac{\textit{The lower class boundary} + \textit{The upper class boundary}}{2}$$

$$= \frac{(39.5 + 44.5)\text{ kg}}{2} = \frac{84\text{ kg}}{2} = 42\text{ kg}$$

Once the *pattern* of obtaining *one class mid-point* is *known*, it is quite easy to *determine* the *other class mid-points* without much unnecessary calculation.

It can be *seen* that:

$$\text{The } \textit{mid-point} \text{ of the } \textit{first class interval} = \textit{The lower class limit} + 2$$

$$= (35 + 2)\text{ kg} = 37\text{ kg}$$

$$\text{Thus the } \textit{mid-point second class interval} = \textit{The lower class limit} + 2$$

$$= (40 + 2)\text{ kg} = 42\text{ kg}$$

And so on.

If the *first three classes* are written in a *row* it helps to make the *results clearer.*

1st class	2nd class	3rd class	*Class rank*
37	42	47	*Class mid-point*
35 39	40 44	45 49	*Class interval*
34.5 39.5	44.5	49.5	*Class boundary*

It should also be noted that the *mid-point* of a *class interval* can also be defined as the *average* of the *lower and upper limits* of the *class*.

Thus:

$$\textit{The class mid-point} = \frac{\textit{The lower class limit} + \textit{The upper class limit}}{2}$$

$$\text{So the } \textit{mid-point} \text{ of the } \textit{first class interval} = \frac{\textit{The lower class limit} + \textit{The upper class limit}}{2}$$

$$= \frac{(35 + 39)\text{ kg}}{2} = \frac{74\text{ kg}}{2} = 37\text{ kg}$$

The complete table of *class mid-points* can be seen below.

Table 8.49 Class mid-points

Class interval (kg)	*Class mid-point (kg)*
35–39	37
40–44	42
45–49	47
50–54	52
55–59	57
60–64	62
65–69	67
70–74	72
75–79	77
80–84	82

The Width of a Class Interval (or Class Size)

Sometimes it is necessary to determine the *width of a class interval* (or *class size*). The *width of a class interval* also known as the *class size* is defined as the *difference* between the *upper and lower class boundaries.*

Thus:

$$\textit{The width of a class interval} = \textit{The upper class boundary} - \textit{The lower class boundary}$$

$$\text{So the } \textit{width} \text{ of the } \textit{first class interval} = \textit{The upper class boundary} - \textit{The lower class boundary}$$

$$= (39.5 - 34.5)\text{ kg} = 5\text{ kg}$$

The *width* of the *second class interval* = *The upper class boundary* − *The lower class boundary*

= (44.5 − 39.5) kg = 5 kg

And the *width* of the *third class interval* = *The upper class boundary* − *The lower class boundary*

= (49.5 − 44.5) kg = 5 kg

It should be noted that at this level, the *widths of the class intervals* for a particular *grouped frequency distribution* are always *equal* to a *single value.* Hence, once we have *calculated* the *width of a class interval* (or *class size* or *unit size*) for *one class interval* it is *not necessary* to repeat the *process* again.

If the *first three classes* are written in a *row* it helps to make the *results clearer.*

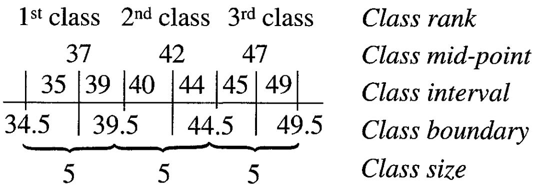


Fig. 8.18 Width of a class interval

It should be noted that the *width of a class interval* (or *class size* or *unit size*) is *not equal* to the *difference* between the *upper* and the *lower class limits.*

Since for the *first class interval*:
The upper class limit − *The lower class limit*
= (39 − 35) kg = 4 kg
4 kg is obviously *not* the *width* of the *first class interval.*

[illegible]

1. The result of a survey of the income earned per hour in dollars for a sample of 60 families is given below.

1	6	17	7	19	10
16	16	22	1	27	2
21	22	18	14	15	15
5	2	23	18	27	20
22	21	26	4	24	21
17	18	19	19	29	17
20	23	24	24	2	28
18	25	26	20	28	3
3	19	4	15	16	29
15	17	3	23	20	16

(a) Draw up a tally chart for the classes 0–4, 5–9, 10–14, 15–19, 20–24 and 25–29.

(b) Calculate the probability that a family selected at random earned:
  (i) less than 15 dollars per hour
  (ii) at least 15 dollars per hour.

2. The heights of 50 children in a survey can be seen recorded below correct to the nearest centimetre.

130	140	137	143	147
143	145	150	148	144
135	136	142	145	142
141	149	146	138	140
154	146	147	141	149
138	142	143	151	153
144	157	155	142	139
145	141	164	149	148
163	147	144	143	140
140	134	141	146	159

(a) Construct a grouped frequency table using the classes 130–134, 135–139, 140–144, 145–149, 150–154, 155–159 and 160–164.

(b) Calculate the probability that a child chosen at random is between 145 cm and 149 cm in height.

3. In a survey the masses of 100 adults were measured correct to the nearest kilogram. The list of the raw data obtained can be seen below.

50	66	73	83	68	60	86	87	88	70
83	51	81	92	84	85	104	76	105	106
65	82	72	86	73	74	61	92	77	89
99	88	52	80	84	75	93	87	91	90
80	71	87	100	74	94	75	62	88	78
79	98	67	53	95	85	95	82	81	80
89	81	97	96	54	103	73	96	63	79
70	78	77	79	93	74	86	72	97	84
89	91	101	102	75	94	76	86	85	64
71	72	82	76	85	84	83	77	71	107

(a) Construct a grouped frequency distribution table for the classes 50–59, 60–69, 70–79, 80–89, 90–99 and 100–109.

(b) Calculate the probability that a person selected at random has a mass between 80 kg and 99 kg inclusive.

4. The marks awarded to 120 candidates in an examination are as follows:

1	22	8	19	25	26	12	23	34	14
16	2	18	44	20	6	17	47	22	49
26	7	37	24	39	31	29	33	48	21

21	27	3	46	14	16	22	28	24	35
6	12	28	28	30	40	32	12	19	25
31	29	29	4	24	21	9	24	39	11
11	17	23	27	10	15	44	5	27	26
30	32	13	38	45	35	23	43	13	40
10	23	24	9	5	25	34	18	42	20
25	36	19	34	26	45	11	36	30	41
17	50	33	20	35	1	37	33	7	8
15	18	13	22	21	30	29	25	32	31

(a) Draw a tally chart for the groups 1–5, 6–10, 11–15, 16–20, etc.

(b) Calculate the probability that a candidate chosen at random was awarded more than 35 marks.

5. Construct a table showing the class intervals and the theoretical class intervals representing the income earned per hour in dollars for the sample of 60 families given in Question 1.

6. Draw up a table showing the class intervals and the theoretical class intervals representing the heights of the children in centimetres recorded in Question 2.

7. Draw a table stating the class intervals and the theoretical class intervals representing the masses of the adults in kilograms shown in Question 3.

8. Construct a table indicating the class intervals and the theoretical class intervals representing the marks awarded to the candidates in Question 4.

9. Construct a table showing the class intervals and the class mid-points for the sample given in Question 1.

10. Draw up a table showing the class intervals and the class mid-points for the survey of heights recorded in Question 2.

11. Draw a table stating the class intervals and the class mid-marks for the survey of masses indicated in Question 3.

12. Construct a table indicating the class intervals and the class values for the marks awarded to the candidates in Question 4.

13. Calculate the width of the class intervals in Question 1.

14. Determine the unit size of the class intervals in Question 2.

15. Calculate the class size of the class intervals in Question 3.

16. Determine the width of the class intervals in Question 4.

Histogram for Grouped Data

A *histogram* can also be used to represent *graphically*, a *frequency distribution* with *grouped data.* In the case of *grouped data*, we plot either the *class boundaries* or *class mid-points* along the *horizontal axis* against corresponding *frequencies.*

Example 11

The masses of 100 students correct to the nearest kilogram are given in the frequency table below.

Table 8.50 Frequency table

Class (kg)	*Frequency*
35–39	*5*
40–44	*7*
45–49	*8*
50–54	*10*
55–59	*13*
60–64	*17*
65–69	*15*
70–74	*12*
75–79	*9*
80–84	*4*

(a) Draw a histogram to represent the information given above.

(b) What is the relative frequency of the class (60–64) kg?

(c) What percentage of students had a mass between 55 kg and 64 kg?

Solution

(a)

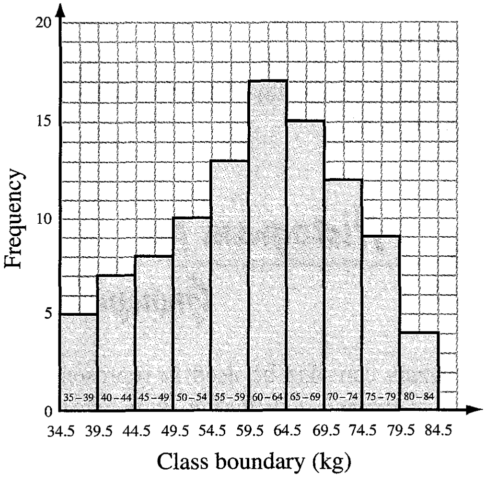


Fig. 8.19 (a) Histogram

or

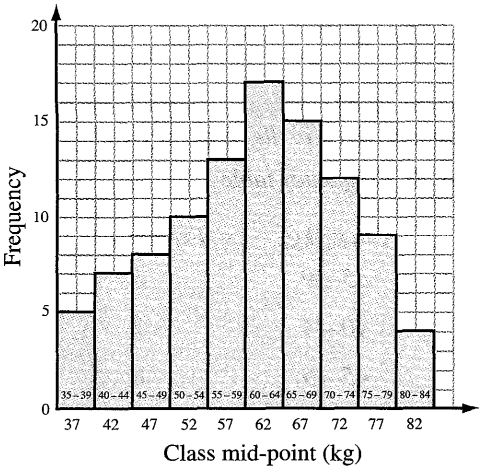


Fig. 8.19 (b) Histogram

Above can be seen the *histogram* that represents the information given.

(b) The *frequency* of *class* (60–64) kg = 17 students

And the *total frequency* = 100 students

∴ the *relative frequency* of the *class* (60–64) kg, *R.F.*

$$= \frac{\textit{The frequency of the event}}{\textit{The total frequency}}$$

$$= \frac{17\ \cancel{\text{students}}}{100\ \cancel{\text{students}}}$$

$$= 0.17$$

(c) The *number of students* with a *mass between* 55 kg *and* 64 kg = (13 + 17) students = 30 students

And the *total number of students* = 100 students

∴ the *percentage* of *students* with a *mass between* 55 kg *and* 64 kg (i.e. $55 \leqslant x \leqslant 64$)

$$= \frac{30\ \cancel{\text{students}}}{\cancel{100}\ \cancel{\text{students}}} \times 1\cancel{00}\%$$

$$= 30\%$$

Exercise 8k

1. The table below shows the distribution of masses of 100 adults measured to the nearest kilogram.

Table 8.51 Frequency table

Mass (kg)	*Frequency*
50– 59	5
60– 69	9
70– 79	28
80– 89	33
90– 99	17
100–109	8

(a) Draw a histogram to represent the information given above.

(b) What is the relative frequency of the class (80–89) kg?

2. The frequency distribution of the marks awarded to 100 candidates in an examination is as follows:

Table 8.52 Frequency table

Marks	*No. of candidates*
1– 5	5
6–10	8
11–15	11
16–20	12
21–25	20
26–30	16
31–35	13
36–40	7
41–45	5
46–50	3

(a) Construct a histogram to represent the frequency distribution given above.

(b) A candidate is selected at random. Calculate the probability that his mark is less than 25.5.

3. An industrial organisation gives an aptitude test to all applicants for employment. The results of 150 people taking the test were:

Table 8.53 Frequency table

Score	*Frequency*
1– 10	6
11– 20	12
21– 30	15
31– 40	21
41– 50	35
51– 60	24
61– 70	20
71– 80	10
81– 90	6
91–100	1

(a) Draw a histogram to represent this information.

(b) What percentage of the applicants scored between 60.5 and 90.5?

4. A frequency table recording the heights of 50 children is shown below.

Table 8.54 Frequency table

Height (cm)	*Frequency*
130–134	2
135–139	6
140–144	19
145–149	14
150–154	4
155–159	3
160–164	2

(a) Construct a histogram to represent the data recorded.

(b) What percentage of the children were less than 149.5 cm in height?

5. The frequency distribution of the lengths of 100 steel rods measured in mm is given in the table following.

Table 8.55 Frequency table

Length (mm)	*Frequency*
100–104	4
105–109	9
110–114	10
115–119	17
120–124	25
125–129	21
130–134	9
135–139	5

(a) Draw a histogram to represent the frequency distribution.

(b) If a steel rod is chosen at random, calculate the probability that it is greater than 124.5 mm in length.

Frequency Polygon for Grouped Data

A *frequency polygon* can also be used to represent a *frequency distribution* with *grouped data.*

The *frequency polygon* for *grouped data* is obtained by plotting the frequency against the corresponding *mid-point* of the *class interval* and then *drawing* a *straight line* in order to join *consecutive points.*

We can also draw a *frequency polygon* by *joining consecutive mid-points* of the *tops* of the *columns* of the *histogram* by *straight lines.*

Example 12

(a) Draw the frequency polygon for the following distribution of examination marks obtained by 115 students.

Table 8.56 Frequency table

Mark	*1–10*	*11–20*	*21–30*	*31–40*	*41–50*
Frequency	*3*	*3*	*8*	*15*	*19*
Mark	*51–60*	*61–70*	*71–80*	*81–90*	*91–100*
Frequency	*24*	*20*	*13*	*7*	*3*

(b) Calculate the area enclosed by the frequency polygon and the horizontal axis and hence determine the number of students who wrote the examinations.

Solution

(a) We first need to *find* the *mid-points* of the *class intervals* given.

Table 8.57 Class mid-points

Mid-point	5.5	15.5	25.5	35.5	45.5
Frequency	3	3	8	15	19
Mid-point	55.5	65.5	75.5	85.5	95.5
Frequency	24	20	13	7	3

(a)

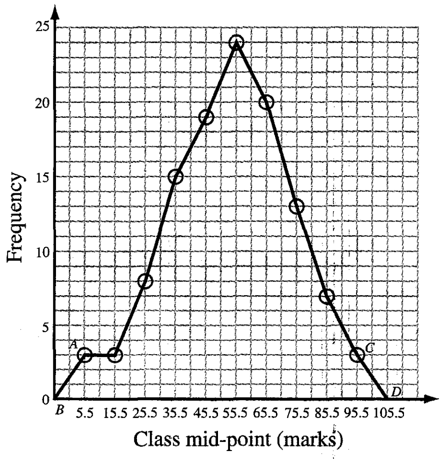


Fig. 8.20 (a) Frequency polygon

Above can be seen the *frequency polygon* that represents the distribution of examination marks given.

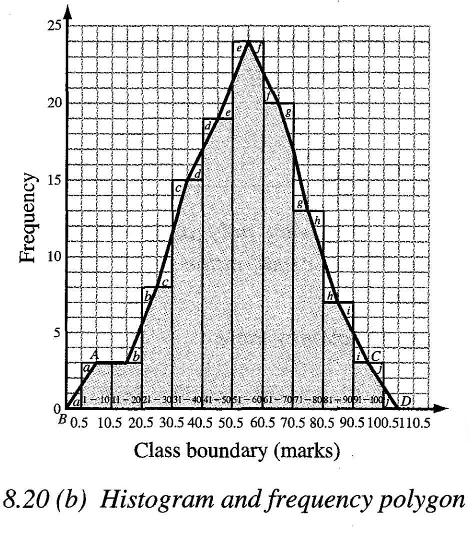


Fig. 8.20 (b) Histogram and frequency polygon

The diagram above shows the *frequency polygon superimposed* on the *histogram.* It can be seen that the *frequency polygon* can be drawn by *joining consecutive mid-points* of the *tops* of the *columns* of the *histogram* by *straight lines.*

Also note that *each pair of regions* marked with the *same letter*, for example *a*, are *equal in area.* Hence in order for the *frequency polygon* to have the *same area* as the *histogram*, we have to finish off the *polygon* by drawing *lines AB* and *CD.*

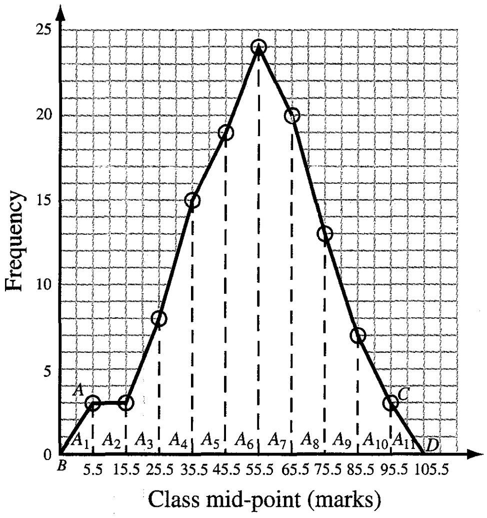


Fig. 8.20 (c) Frequency polygon

(b) The *area enclosed* by the *frequency polygon* and the *horizontal axis*

$$
\begin{aligned}
&= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 \\
&\quad + A_9 + A_{10} + A_{11} \\
&= \frac{1}{2} \times 3 \times 10 + 3 \times 10 + \frac{1}{2}(3 + 8) \times 10 \\
&\quad + \frac{1}{2}(8 + 15) \times 10 + \frac{1}{2}(15 + 19) \times 10 \\
&\quad + \frac{1}{2}(19 + 24) \times 10 + \frac{1}{2}(24 + 20) \times 10 \\
&\quad + \frac{1}{2}(20 + 13) \times 10 + \frac{1}{2}(13 + 7) \times 10 \\
&\quad + \frac{1}{2}(7 + 3) \times 10 + \frac{1}{2} \times 3 \times 10 \\
&= \frac{1}{2} \times 10(3 + 6 + 11 + 23 + 34 + 43 + 44 \\
&\quad + 33 + 20 + 10 + 3) \\
&= 5 \times 230 \\
&= 1\,150
\end{aligned}
$$

And the *unit size* $= \begin{matrix}\textit{The upper} \\ \textit{class boundary}\end{matrix} - \begin{matrix}\textit{The lower} \\ \textit{class boundary}\end{matrix}$

$= (10.5 - 0.5)$ marks $= 10$ marks

∴ the *number of students* who *wrote the examinations*

$$= \frac{\textit{The area enclosed by the frequency polygon and the horizontal axis}}{\textit{The unit size}}$$

$$= \frac{1150}{10} = 115 \text{ students}$$

Note that the *regions* A_1 and A_{11} are *triangles.*
The *region* A_2 is a *rectangle.*
And the *regions* A_3 to A_{10} are *trapeziums.*

The *altitude h*, is *equal* to the *width* of a *bar* of the *histogram* or the *distance* between *consecutive mid-points* (*marks*) in the *frequency polygon.*

Exercise 8I

1. (a) Draw a frequency polygon for the following distribution of the marks obtained by 100 candidates on a Mathematics paper.

Table 8.58 Frequency table

Number of marks	*No. of candidates*
1– 10	5
11– 20	7
21– 30	8
31– 40	11
41– 50	19
51– 60	13
61– 70	12
71– 80	11
81– 90	8
91–100	6

(b) Calculate the area enclosed by the frequency polygon and the horizontal axis and hence, determine the number of candidates who wrote the paper.

2. (a) Draw the frequency polygon for the following distribution of heights in centimetres.

Table 8.59 Frequency table

Height (cm)	*Frequency*
131–135	7
136–140	12
141–145	13
146–150	18
151–155	35
156–160	11
161–165	4

(b) Calculate the area enclosed by the frequency polygon and the horizontal axis and hence, determine the number of persons whose heights were measured in the survey.

3. The frequency distribution of the marks awarded to the candidates in an examination is as follows:

Table 8.60 Frequency table

Marks	*Frequency*
1– 5	5
6–10	8
11–15	11
16–20	12
21–25	20
26–30	16
31–35	13
36–40	7
41–45	5
46–50	3

(a) Construct a frequency polygon to represent the information given.

(b) Calculate the area enclosed by the frequency polygon and the horizontal axis and determine the total number of candidates.

4. The masses of some pupils in the same school are shown in the table below:

Table 8.61 Frequency table

Mass (kg)	*Number of pupils*
15–23	3
24–32	10
33–41	17
42–50	12
51–59	5
60–68	3

(a) Construct a frequency polygon to represent the data given above.

(b) Calculate the area enclosed by the frequency polygon and the horizontal axis and hence determine the total number of pupils weighed in the survey.

5. In an intelligence test the following frequency table was obtained:

Table 8.62 Frequency table

Mark	*Frequency*
1– 100	7
101– 200	10
201– 300	13
301– 400	14
401– 500	22
501– 600	18
601– 700	15
701– 800	9
801– 900	7
901–1000	5

(a) Draw a frequency polygon to represent the frequency distribution.

(b) Calculate the area enclosed by the frequency polygon and the horizontal axis and hence, determine the number of people who wrote the intelligence test.

Measures of Central Tendency

As was shown previously in this chapter, *raw data* can be more easily understood, when it is *tabulated* in an orderly fashion in a *frequency distribution* and then shown *diagrammatically* in proportionate bar charts, bar charts (or column graphs), chronological bar charts and pie charts, or *graphically* in line graphs, histograms and frequency polygons.

Sometimes there is a need to find or use a *single value* which *represents* or *characterises* the *group* (or *set of data*) as a *whole*. This *single value* is called a *statistical average* (or a *measure of central tendency*).

The *three statistical averages* (or *measures of central tendency*) that we need to know are:

1. The *arithmetic mean*, simply called the *mean* for short.
2. The *median.*
3. The *mode.*

Later on it will be seen that *one average* is *more appropriate* to use than another average under a given *set of circumstances* or *conditions.*

Mean

The *mean* for a given *set of data* is the *average* we previously came across in our *arithmetic calculations.* However, in Statistics we always call it the *mean.*

Calculating the Mean from Raw Data

The *formula* that we use to calculate the *mean* from *raw data* is:

$$\overline{x} = \frac{\Sigma x}{\Sigma f} = \frac{\Sigma x}{n},$$

where the symbol Σ means *'the sum of'*,

$\overline{x}$ = the *mean,*

x = the *value of an observation* (or *variable*),

f = the *frequency* (or the *number of observations*),

Σx = the *sum of the values of the observations*,

and $n = \Sigma f$ = the *sum of the frequencies* (or the *total frequency* or the *total number of observations*).

Example 13

Evaluate the mean of the following numbers:
1, 3, 5, 7, 11, 12, 13, 15, 16, 17.

Solution

The *mean number*, $\overline{x} = \dfrac{\Sigma x}{n}$

$$= \frac{1 + 3 + 5 + 7 + 11 + 12 + 13 + 15 + 16 + 17}{10}$$

$$= \frac{10\cancel{0}}{1\cancel{0}}$$

$$= 10$$

The Mean from a Frequency Distribution with Ungrouped Data

The *mean* of a *frequency distribution* with *ungrouped data* can be *calculated* by using the *formula*:

$$\overline{x} = \frac{\Sigma fx}{\Sigma f} = \frac{\Sigma fx}{n},$$

where fx = the *product of the frequency and the value of the corresponding observation*

and Σfx = the *sum of the products fx.*

Example 14

The marks obtained by 100 students in a test in which the maximum possible mark was 10 are shown in the table below.

Table 8.63 Frequency table

Mark	*Frequency*
0	*2*
1	*5*
2	*8*
3	*17*
4	*23*
5	*0*
6	*15*
7	*12*
8	*9*
9	*6*
10	*3*

Calculate the mean mark for the frequency distribution.

Table 8.64 Frequency table

Mark x	*Frequency* f	*Frequency × mark* fx
0	*2*	*0*
1	*5*	*5*
2	*8*	*16*
3	*17*	*51*
4	*23*	*92*
5	*0*	*0*
6	*15*	*90*
7	*12*	*84*
8	*9*	*72*
9	*6*	*54*
10	*3*	*30*
$n = \Sigma f = 100$		$\Sigma fx = 494$

Solution

In the *last column*, we *multiply* the *frequency* by the value of the *corresponding observation* (i.e. *mark*) to get the *product fx.* We then *add* the *column* containing the *values* of the *product fx,* in order to obtain the *sum* of the *products fx*, that is Σfx.

The *mean mark* for the *frequency distribution,*
$$\bar{x} = \frac{\Sigma fx}{n} = \frac{494 \text{ marks}}{100} = 4.94 \text{ marks}$$

It should be noted that the symbol Σ means *'the sum of'* or *'total'* and in *statistics* it implies that we *add a whole column* in the *frequency table.*

Example 15

The mean height of nine choir members is 157 cm. Calculate the mean height if:

(a) a man of height 169 cm leaves the choir

(b) a woman of height 165 cm joins the original choir.

Solution

(a) The *mean height,* $\bar{x} = \dfrac{\Sigma fx}{n}$

So the *total height* of the 9 *choir members,*
$$\Sigma fx = n\bar{x} = 9 \times 157 \text{ cm} = 1\,413 \text{ cm}$$

∴ the *total height* of the remaining 8 *choir members,*
$$\Sigma fx = (1\,413 - 169) \text{ cm} = 1\,244 \text{ cm}$$

Hence the *mean height* of the 8 *choir members,*
$$\bar{x} = \frac{\Sigma fx}{n} = \frac{1\,244 \text{ cm}}{8} = 155.5 \text{ cm}$$

(b) The *total height* of the 10 *choir members,*
$$\Sigma fx = (1\,413 + 165) \text{ cm} = 1\,578 \text{ cm}$$

∴ the *mean height* of the 10 *choir members,*
$$\bar{x} = \frac{\Sigma fx}{n} = \frac{1\,578 \text{ cm}}{10} = 157.8 \text{ cm}$$

Exercise 8m

1. Determine the mean of the following numbers:
2, 3, 4, 4, 5.

2. Calculate the mean of the following numbers:
7, 4, 3, 5, 6, 5.

3. Erica's marks in eight consecutive Mathematics examinations were:
94, 83, 75, 52, 71, 68, 75, 49.

(a) Determine the total marks that she scored

(b) What was her mean mark?

4. The height of 13 men in centimetres are given below:
162, 160, 163, 160, 165, 167, 170, 167, 174, 176, 178, 179, 178.
Determine the mean of the heights correct to two decimal places.

5. The table below shows the revenues of two public utilities for the period 1970−79 in millions of Trinidad and Tobago dollars ($TT M).

Table 8.65

YEAR	*T&TEC*	*TSTT*
1970	30.5	12.0
1971	33.6	13.1
1972	37.3	15.2
1973	39.4	16.0
1974	45.8	23.4
1975	45.4	25.4
1976	52.6	27.2
1977	57.9	28.0
1978	64.5	28.8
1979	69.6	29.3

Calculate the mean revenue collected for the ten-year period by:

(a) *T&TEC* (b) *TSTT.*

6. The heights of 10 girls in centimetres are:
154, 149, 152, 154, 155, 148, 161, 154, 156, 153.
Evaluate the mean height.

7. The heights of a group of children in centimetres are:
158, 154, 152, 153, 156, 161, 151, 159, 160, 156.
Calculate their mean height.

8. The table shows the number of children per family in the families of the pupils in a class.

Table 8.66 Frequency table

No. of children per family	1	2	3	4	5	6	7
Frequency	2	3	9	5	6	4	1

Evaluate the mean number of children per family for the frequency distribution.

9. The frequency table below shows the number of tickets bought per person for a calypso show.

Table 8.67 Frequency table

No. of tickets bought per person for a calypso show	*Frequency*
1	12
2	35
3	44
4	18
5	8
6	3

Calculate the mean number of tickets bought per person for the calypso show.

10. A biologist takes a sample of 100 grass plants to measure stem length. The following data were obtained:

Table 8.68 Frequency table

Length (cm)	*Frequency*
25	2
26	9
27	10
28	12
29	20
30	19
31	13
32	15

Calculate the mean length per stem.

11. The frequency distribution below shows the marks obtained by 40 students in a test.

Table 8.69 Frequency table

Mark	*Frequency*
1	3
2	5
3	6
4	9
5	5
6	2
7	6
8	4

Evaluate the mean mark for the distribution.

12. In a shooting contest in which 50 people participated, the following frequency table was obtained.

Table 8.70 Frequency table

Score	*Frequency*
1	3
2	1
3	4
4	10
5	15
6	9
7	3
8	5

Calculate the mean score, correct to the nearest whole number.

13. The mean height of 12 cricketers is 165 cm. Calculate the mean height if:

(a) a cricketer of height of 176 cm leaves the team

(b) a cricketer of height of 152 cm joins the original team.

14. The mean mass of 15 women is 53 kg. Calculate the mean mass if:

(a) a woman of mass 60 kg leaves the group

(b) a woman of mass 69 kg joins the original group.

15. The mean mark of 25 students in a test is 73. Determine the mean mark if:

(a) a student whose mark was 85 was absent from the test.

(b) a student whose mark was 25 was absent from the test.

Median

The *median* is defined as the *'middle'* or *central value* in a *set of ascending* or *descending observations* and it is represented by the symbol Q_2. The *median* always has the *same number of values above* it as there are values *below*. When there is an *odd number* of *observations* then the *'middle' value* or *median* is easily ascertained. However, when there is an *even number* of *observations*, then the *median* is the *average* of the *two central observations*, since there is no *single 'middle' value*.

Finding the Median from Raw Data

Example 16

Find the median of the following heights which are stated in centimetres:

(a) 163, 158, 154, 161, 156, 159, 155.

(b) 158, 163, 154, 161, 157, 156, 159, 155.

Solution

(a) The *heights* in *ascending order* are:

$$\underbrace{154,\ 155,\ 156,}\quad \boxed{158,}\quad \underbrace{159,\ 161,\ 163}$$

Central height

3 heights below the median $\downarrow$ $Q_2 = 158$ 3 heights above the median

∴ the *median height*, $Q_2 = 158$ cm.

(b) The *heights* in *ascending order* are:

$$\underbrace{154,\ 155,\ 156,}\quad \boxed{157,\ 158,}\quad \underbrace{159,\ 161,\ 163}$$

Central heights

3 heights below the median $\downarrow$ $Q_2 = 157.5$ cm 3 heights above the median

$$\therefore \text{ the median height, } Q_2 = \frac{(157 + 158)\text{ cm}}{2}$$
$$= 157.5\text{ cm}$$

The Median from a Frequency Distribution with Ungrouped Data

When the *observations* in a *set of data* are given as a *frequency distribution* with *ungrouped data*, then the *position* of the *median* is given by the $\frac{1}{2}(n + 1)^{th}$ *rank*, and the *median* which is represented by the symbol Q_2 is the *value corresponding* to the $\frac{1}{2}(n + 1)^{th}$ *rank.*

A *summation* of the *frequencies* called the *'running total'* or the *cumulative frequency* is used to help us find the *rank* easily, since the *rank* is a *frequency value*. So we have to *construct* what is called a *cumulative frequency table. A cumulative frequency table* is very helpful in determining how many *observations* were *less than a given value* or *greater than a given value.*

Example 17

The masses of 100 pupils in a school are shown in the table below.

Table 8.71 Frequency table

Mass (kg)	*Number of pupils*
51	*7*
52	*8*
53	*10*
54	*12*
55	*13*
56	*15*
57	*12*
58	*9*
59	*8*
60	*6*

(a) Find the median of the masses shown in the frequency distribution given above.

(b) Determine the probability that if a pupil is chosen at random:

- *(i) the mass of the pupil is 56 kg or less*
- *(ii) the mass of the pupil is less than 53 kg*
- *(iii) the mass of the pupil is more than 58 kg.*

Solution

(a) We first construct the *cumulative frequency table* from the *frequency table* as shown below.

Table 8.72 Cumulative frequency table

Mass interval (kg)	*Cumulative frequency*
≤51	7
≤52	7 + 8 = 15
≤53	15 + 10 = 25
≤54	25 + 12 = 37
≤55 ←	37 + 13 = 50 ← 50th rank
$Q_2 = 55.5$ ←	
≤56 ←	50 + 15 = 65 ← 51st rank
≤57	65 + 12 = 77
≤58	77 + 9 = 86
≤59	86 + 8 = 94
≤60	94 + 6 = 100

It can be *seen* that:

- (i) The *cumulative frequency* is obtained by *adding each frequency* to the *total frequency* of its *predecessors.*
- (ii) The *final cumulative frequency* is *equal* to the *total frequency.*
- (iii) *Each observation* is stated as *less than or equal to* its *original value.*

The *position* of the *median*
$$= \frac{1}{2}(n + 1)^{\text{th}} \textit{ rank}$$
$$= \frac{1}{2}(100 + 1)^{\text{th}} \text{ rank}$$
$$= \frac{1}{2}(101)^{\text{th}} \text{ rank}$$
$$= 50.5^{\text{th}} \text{ rank}$$

This implies that the *median* is the *average* of the 50th and the 51st *observations.*

From the *cumulative frequency table* it can be *seen* that:

The 50th *rank* (i.e. the 50th *pupil) has a mass of* 55 kg.

And the 51st *rank* (i.e. the 51st *pupil) has a mass of* 56 kg.

∴ The *median of the masses,* Q_2
$$= \frac{(50^{\text{th}} + 51^{\text{st}}) \textit{ observations}}{2}$$
$$= \frac{(55 + 56) \text{ kg}}{2}$$
$$= 55.5 \text{ kg}$$

(b) From the *cumulative frequency table*:

(i) The *number* of pupils with a *mass* 56 kg *or less* = 65 pupils

And the *total number of pupils* = 100 pupils

$$\therefore P(\textit{pupil's mass} \leqslant 56 \text{ kg}) = \frac{\textit{The frequency of the observation}}{\textit{The total frequency}} = \frac{65 \cancel{\text{ pupils}}}{100 \cancel{\text{ pupils}}} = 0.65$$

(ii) The *number* of pupils with a *mass less than* 53 kg = 15 pupils

$$\therefore P(\textit{pupil's mass} < 53 \text{ kg}) = \frac{\textit{The frequency of the observation}}{\textit{The total frequency}} = \frac{15 \cancel{\text{ pupils}}}{100 \cancel{\text{ pupils}}} = 0.15$$

(iii) The *number* of pupils with a *mass more than* 58 kg = (100 − 86) pupils = 14 pupils

$$\therefore P(\textit{pupil's mass} > 58 \text{ kg}) = \frac{\textit{The frequency of the observation}}{\textit{The total frequency}} = \frac{14 \cancel{\text{ pupils}}}{100 \cancel{\text{ pupils}}} = 0.14$$

Exercise 8n

1. Find the median of the following numbers:
2, 4, 3, 5, 4.

2. Determine the median of the following numbers:
7, 4, 3, 5, 6, 5.

3. The heights of 10 girls stated in centimetres are:
152, 154, 149, 155, 148, 154, 161, 156, 153, 154.
Find the median height.

4. The heights of a group of children stated in centimetres are:
152, 158, 154, 156, 161, 153, 159, 151, 160, 156.
Determine their median height.

5. The masses of 13 children in kg are:
69, 71, 65, 66, 68, 72, 66, 67, 73, 67, 71, 70, 68.
State their median mass.

6. The marks obtained by 40 students in a test are shown in the table below.

Table 8.73 Frequency table

Marks	*Frequency*
1	3
2	5
3	6
4	9
5	5
6	2
7	6
8	4

(a) Find the median of the marks shown in the frequency distribution given above.

(b) Determine the probability that if a student is chosen at random his marks are 5 or less.

7. In a shooting contest in which 50 people participated, the following frequency table was obtained.

Table 8.74 Frequency table

Score	*Frequency*
1	3
2	1
3	4
4	10
5	15
6	9
7	3
8	5

(a) Find the median score.

(b) Determine the probability that if a participant is chosen at random he scored less than 6.

8. The table below shows the number of children per family in the families of the pupils in a class.

Table 8.75 Frequency table

No. of children per family	1	2	3	4	5	6	7
Frequency	2	3	9	5	6	4	1

(a) Calculate the median.

(b) Determine the probability that if a family is chosen at random it has more than 5 children.

9. A biologist takes a sample of 100 grass plants to measure stem length. The following data were obtained:

Table 8.76 Frequency table

Length (cm)	*Frequency*
25	2
26	9
27	10
28	12
29	20
30	19
31	13
32	15

Determine the median stem length.

10. The shoe sizes of pupils in a class are:
4, 7, 4, 6, 5, 5, 5, 4, 8, 7, 8, 8, 7, 5, 7, 6, 8, 5, 8, 9, 9, 6, 5, 4, 5, 7, 7, 5, 9, 5.

(a) Draw a frequency table to represent the information given.

(b) What is the median shoe size?

Mode

The mode of a distribution is defined as the observation with the highest frequency—it occurs most frequently. That is, it is the most common observation occurring.

In everyday life, we say that the *mode* of a *distribution* is the '*most popular*' or the '*most fashionable*' item. If a *distribution* has a *single mode*, or *two modes* or *three modes*, then it is said to be *unimodal*, *bimodal* or *trimodal*, respectively.

Determining the Mode from Raw Data

Example 18

Determine the mode(s) of the basic wages in the following distributions:

(a) \$125, \$175, \$195, \$175, \$205, \$125, \$175, \$210.

(b) \$155, \$209, \$155, \$200, \$160, \$185, \$160, \$195.

(c) \$160, \$125, \$140, \$159, \$175, \$140, \$125, \$180, \$159.

Solution

(a) Given the *distribution* of the *basic wages*:
\$125, \$175, \$195, \$175, \$205, \$125, \$175, \$210.

There are *three* \$175s.
So the *modal basic wage* is \$175.
This *distribution* is said to be *unimodal.*

(b) Given the *distribution* of *basic wages*:
\$155, \$209, \$155, \$200, \$160, \$185, \$160, \$195.

There are *two* \$155s and *two* \$160s.
So the *modal basic wages* are \$155 and \$160.
This *distribution* is said to be *bimodal.*

(c) Given the *distribution* of *basic wages*:
\$160, \$125, \$140, \$159, \$175, \$140, \$125, \$180, \$159.

There are *two* \$125s, *two* \$140s and *two* \$159s.
So the *modal basic wages* are \$125, \$140 and \$159. This *distribution* is said to be *trimodal.*

The Mode from a Frequency Distribution with Ungrouped Data

Example 19

The following table shows the number of children per family in the families of the students in a Form Five.

Table 8.77 Frequency table

No. of children per family	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Frequency	*3*	*5*	*6*	*7*	*10*	*5*	*3*	*2*	*1*	*0*

Determine the modal number of children per family.

Solution

Table 8.78 Frequency table

No. of children per family	1	2	3	4	5	6	7	8	9	10
Frequency	3	5	6	7	↑ ⑩	5	3	2	1	0

The *highest frequency*, $f_{max} = 10$

∴ the *modal number of children per family*, $x = 5$ children.

Exercise 8o

1. Find the mode of the following numbers:

2, 3, 4, 5, 4.

2. Find the mode of the following numbers:

3, 7, 4, 5, 6, 5.

3. The heights of 10 girls stated in centimetres are:

153, 156, 154, 161, 148, 155, 154, 152, 149, 154.

Determine their modal height.

4. The heights of a group of children stated in centimetres are:

156, 160, 159, 151, 161, 156, 153, 152, 154, 158.

Find their modal height.

5. The table shows the number of children per family in the families of the pupils in a class.

Table 8.79 Frequency table

No. of children per family	1	2	3	4	5	6	7
Frequency	2	3	9	5	6	4	1

Determine the mode.

6. The distribution of the marks in a test is given below.

Table 8.80 Frequency table

Marks	*Frequency*
1	3
2	5
3	6
4	9
5	5
6	2
7	6
8	4

State the modal mark.

7. The shoe sizes of pupils in a class are given by the frequency table below.

Table 8.81 Frequency table

Shoe size	4	5	6	7	8	9
Frequency	4	9	3	6	5	3

What is the modal shoe size?

8. The table below shows the number of children per family in the families of the pupils in a class.

Table 8.82 Frequency table

No. of children per family	1	2	3	4	5	6	7
Frequency	2	4	9	5	7	2	1

(a) Draw a bar chart to show these results.

(b) Determine:

- (i) the mode
- (ii) the median
- (iii) the mean.

9. The table shows how many pupils in a form were absent for various numbers of sessions during a certain school week.

Table 8.83 Frequency table

No. of sessions absent	0	1	2	3	4	5	6	7	8	9	10
Frequency	15	3	1	4	7	2	0	2	0	1	1

(a) Draw a bar chart to show this information.

(b) Determine:

- (i) the mean
- (ii) the median
- (iii) the mode.

10. The frequency distribution of the marks of 125 candidates in an examination is shown below:

Table 8.84 Frequency table

Marks	*No. of candidates*
5	9
15	11
25	14
35	15
45	23
55	19
65	13
75	10
85	7
95	4

(a) Determine the median mark for the distribution.

(b) Calculate the mean to the nearest whole number.

(c) State the mode of the distribution.

(d) A candidate is selected at random. Calculate the probability that his marks are:

  (i) less than or equal to 45

  (ii) at least 75.

11. A frequency distribution indicating the heights of a sample of people is shown below.

Table 8.85 Frequency table

Height (cm)	*Frequency*
150	1
151	5
152	10
153	16
154	10
155	6
156	2

(a) Calculate the mean height of the frequency distribution.

(b) Determine the median height.

(c) State the modal height.

12.

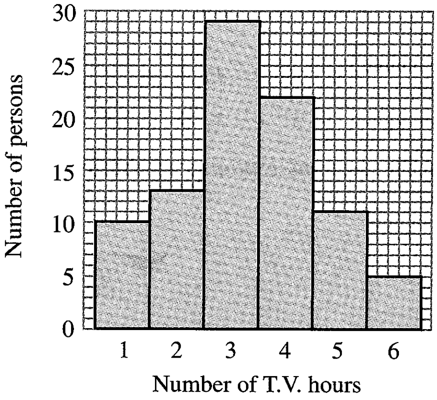


Fig. 8.21 Histogram

The histogram above shows the number of hours for which a group of ladies watched television during a particular evening.

(a) Construct a frequency table to represent the data shown in the histogram under the headings Class mark and Frequency.

(b) Calculate the mean number of T.V. hours.

(c) Calculate the probability that a person chosen at random from this group watched television for 4 hours or more.

13.

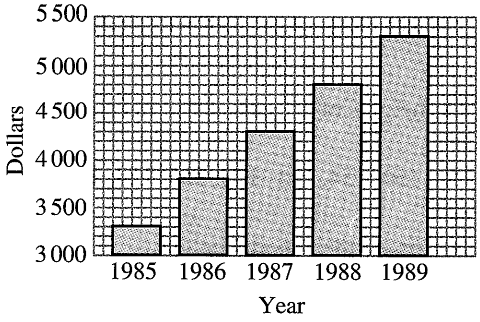


Fig. 8.22 Bar chart

(a) The bar chart above shows the amount of money invested by a firm over a five-year period.

  (i) Write down the amounts invested in 1985 and 1988.

  (ii) Calculate the mean amount invested per year over the 5-year period.

  (iii) Estimate the amount invested in 1990.

  (iv) Calculate the sector angle which would represent the amount invested in 1988 if the information illustrated in the bar chart above is to be represented on a pie chart. State your answer correct to 3 significant figures.

(b) A box contains 10 similar balls, 4 of which are green. The first ball taken out at random was green. It was not replaced. Calculate the probability that a second ball taken out at random is also green.

Table 8.86

14.

Subjects	*Mathematics*	*English*	*History*	*Science*	*Modern languages*
No. of teachers	18	39	43	38	12

The table above gives the number of graduates by subject from a teacher's training college in 1990.

(a) Using graph paper draw a bar chart to represent the data.

(b) Calculate the probability that a teacher chosen at random is an English teacher.

(c) A pie chart is drawn to represent the data in the table. Calculate the sector angle representing the number of Science teachers.

(d) The mean number of Mathematics teachers who graduated in the three-year period 1991–1993 is 32.

- (i) Calculate the total number of Mathematics teachers who graduated over the period 1991–1993.
- (ii) Hence, calculate the mean number of Mathematics teachers who graduated over the period 1991–1993.

15. A shopkeeper counted the amount of money that she had in her cash register at closing time. She found that she had

$79 in one-dollar notes
$80 in five-dollar notes
$350 in ten-dollar notes
$400 in twenty-dollar notes
$500 in fifty-dollar notes
$700 in hundred-dollar notes

Table 8.87 Frequency table

(a)

Value	*Type of note*	*Number of notes*
$79	$1.00	79
$80	$5.00	
$350	$10.00	
$400	$20.00	
$500	$50.00	
$700	$100.00	

Complete the frequency table to show the number of notes for each type.

(b) Represent the information in the completed frequency table by drawing a bar graph, using a scale of 1 cm to represent 5 notes of each type.3

(c) Estimate the median value of this distribution

(d) If a note is selected at random, calculate the probability that

- (i) it is a five-dollar note
- (ii) it is not a hundred-dollar note.

16. A teacher kept a record of the length of time by which 100 students were late for class. The results are shown in the following frequency distribution table:

Table 8.88 Frequency table

Minutes late	0	1	2	3	4	5	6	7
No. of students	5	9	12	8	17	23	16	10

(a) State the mode of the distribution.

(b) Calculate the median of the distribution.

(c) Draw a histogram to represent the data.

(d) Calculate the total time lost by the students.

(e) Calculate the mean time lost per student, correct to the nearest minute.

(f) Calculate the probability that a student of the class chosen at random was late

- (i) by exactly 6 minutes
- (ii) by at least 6 minutes
- (iii) less than 6 minutes.

17. A survey was taken to determine the approximate times, to the nearest 10 minutes that school children wait for their maxi taxis. The results are given below.

Table 8.89

Time (minutes)	10	20	30	40	50	60
No. of children	40	50	10	70	0	30

(a) Calculate the mean waiting time per child.

(b) Calculate the median of the distribution.

(c) State which statistical average (mean, median or mode) you would focus on if you wanted to highlight the need to improve the punctuality of the maxi taxis. Give a reason for your choice.

(d) Calculate the probability that a child chosen at random waits for the bus for at least half an hour.

18. The graph below shows the money spent in dollars on education in a Caribbean country during the period 1980 − 1984.

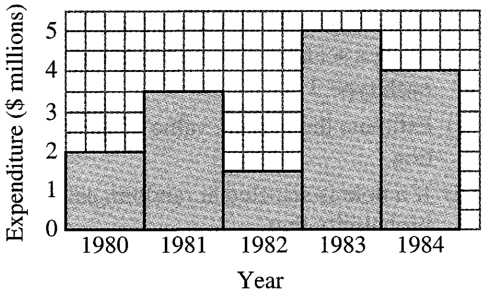


Fig. 8.23 Graph

(a) Using the graph, determine the increase of the 1981 expenditure over the 1980 expenditure.

(b) For the period 1980 to 1984
  (i) calculate the mean annual expenditure
  (ii) state the median annual expenditure.

(c) Calculate the probability that a year chosen at random during this 5-year period had an expenditure greater than the median.

(d) If the above data is represented on a pie chart, calculate the sector angle needed to represent the expenditure for each year.

(e) Draw a pie chart to represent the data.

Table 8.90

19.

Age (years)	12	13	14	15	16	17	18
No. of children	4	7	11	8	5	3	2

The table above shows the distribution of the ages of 40 children in a school choir.

(a) Calculate both the mean age and the median age of this distribution.

(b) Calculate the probability that a child chosen at random is:
  (i) under 15 years of age
  (ii) at least 15 years of age.

Frequency Curves

If a *large sample* is taken from a very *large population* and a *frequency polygon* is drawn for midpoints which are relatively close to each other consecutively, we can literally draw a *curve* through the points instead of straight lines. We say that the frequency polygon tend towards a *smooth continuous curve* called a *frequency curve*. This *fact* can be seen illustrated in the diagram shown below.

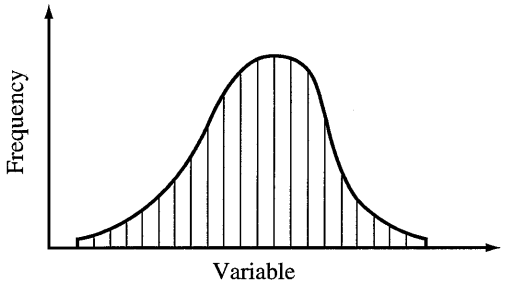


Fig. 8.24 Frequency curve

Types of Frequency Curves

The *three types of frequency curves* that we need to understand are:

1. The *normal curve.*
2. The *negatively skewed curve.*
3. The *positively skewed curve.*

The Normal Curve

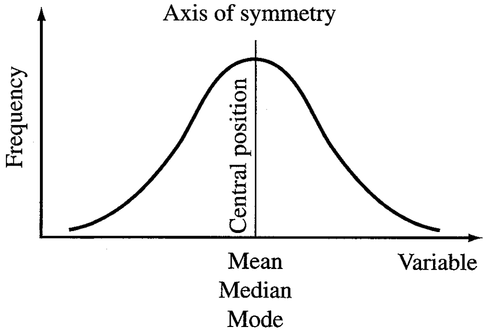


Fig. 8.25 Normal distribution

In *experiments* dealing with large samples from a very large population, a *symmetrical bell-shaped curve* is obtained when the *variable* is plotted against the *corresponding frequency.* The *bell-shaped curve* is known as the *normal probability curve* and it is said to represent a *normal distribution.*

In a *normal distribution*, the measures of central tendency the *mean*, the *median* and the *mode*, all *coincide*. That is, they all have the *same value*, as shown in the diagram above.

Some examples of *data* that will give a *normal distribution* are: height, weight, mass, intelligence quotients, and achievement test scores taken from a human population.

The Negatively Skewed Curve

When a *frequency curve* is drawn and the graph obtained is *non-symmetrical*, then the *curve* is said to be '*lopsided*' or *skewed*. And the *data* is said to represent a *skewed distribution*. A *skewed distribution* can be obtained if a *small sample* is taken from a population which would *otherwise* have given a *normal distribution.*

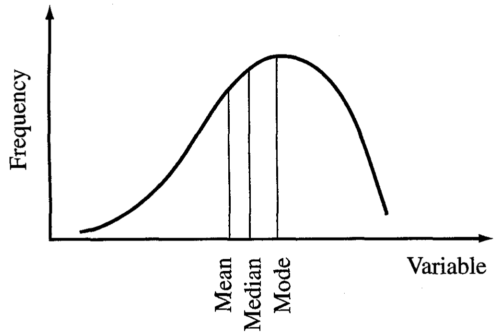


Fig. 8.26 Negatively-skewed distribution

The diagram above shows a *negatively-skewed distribution*, that is, it is *skewed to the left. A negatively skewed distribution* can be obtained from the results of a test in which most of the students performed well and only a few students performed unsatisfactorily.

Negatively-skewed distributions occur very rarely in their own right.

In a *negatively-skewed distribution*, the *measures of central tendency* are all *different* in such a way that, the mean is less than the median, and the median is less than the mode. That is, the *mean* < *median* < *mode* (in general). So the *mean* tends to be *'pulled'* away from the *mode* in the direction of *extreme values.*

The Positively Skewed Curve

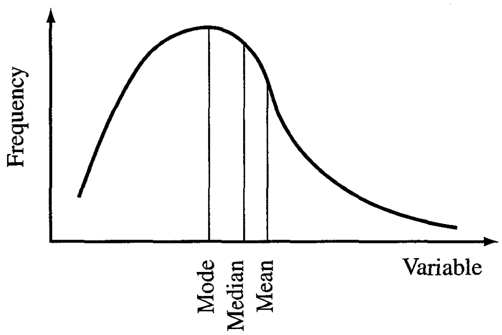


Fig. 8.27 Positively-skewed distribution

The diagram above shows a *positively-skewed distribution*, that is, it is *skewed to the right*. A *positively-skewed distribution* can be obtained from the results of a very difficult test in which only a few students performed well and most of the student performed poorly.

A number of *positively-skewed distributions* occur in their own right. For example: The number of children per family.

In a *positively-skewed distribution*, the *measures of central tendency* are all *different* in such a way that, the mean is greater than the median and the median is greater than the mode. That is, the *mean* > *median* > *mode* (in general). So the *mean* tends to be '*pulled*' away from the *mode* in the direction of *extreme values.*

Comparing the Three Measures of Central Tendency

Below can be seen the *advantages* and *disadvantages* of the *three measures of central tendency.*

The Mean

Table 8.91

Advantages	*Disadvantages*
1. It is the *most commonly* used measure.	It can be *greatly affected* by a single extremely high or low value.
2. It can be *exactly* calculated.	It can sometimes give an *impossible value* when the data is discrete. Especially when the expected value is a whole number.
3. *All the information in the set of data* is used in its calculation.	It cannot be obtained *graphically*.
4. It can *interface* with further statistical calculations.	

The Median

Table 8.92

Advantages	*Disadvantages*
1. It is *very simple* to understand.	It cannot *interface* with further statistical calculations.
2. It is *not affected* by extremely high or low values.	In a limited set of data it *may not be characteristic* of the group.
3. It can be *characteristic* of the set of data and sometime represents an actual member.	In grouped frequency distributions it is mostly estimated from a *cumulative frequency curve*.

The Mode

Table 8.93

Advantages	*Disadvantages*
1. It is *very simple* to understand.	It *cannot interface* with further statistical calculations.
2. It is *not affected* by extremely high or low values.	It *cannot be determined exactly* from a grouped frequency distribution.
3. It is *easily obtained* from a histogram	A set of data can have *more than one mode.*
4. It is *user friendly.*	

Choosing a Measure of Central Tendency

Sometimes it is a problem to decide which of the *three measures of central tendency* to use, as one may be more appropriate to a *particular problem* than another.

Use the Mean

1. When the observations in a distribution are more or less *symmetrically grouped about a central point.*
2. When the *measure of central tendency* will also form the *basis of other statistics.*
3. When the problem requires the *combination of the mean* with the *means of other sets of data* measured on the *same variable.*

Use the Median

1. When the problem calls for *knowledge* of the *exact mid-point of a distribution.*
2. When *extreme values are included* in the set of data.
3. When a distribution has a *high proportion of extremely high values* as well as a *low proportion of extremely low ones.*

Use the Mode

1. When a *quick and approximate way* of determining central tendency is needed.
2. When the measure of central tendency is referred to as *'typical'* or the *'most usual'* or the *'most fashionable'* or the *'most popular'*.

Measures of Dispersion

The *measures of location* (or *central tendency*) are very important because they give us a picture of the *location* of the *set of data* which they represent. However, they only give us a limited view of the whole picture taken by themselves. We also need to know how a *set of data* is *grouped* around the *central position*—if the set of data is *relatively close* to the *central position* or if it is *widely spread.* That is, how *homogeneous* is the *distribution*? We, therefore, need to define a *measure of spread* (or *dispersion* or *scattering*) for a *set of data* or distribution.

The *four measures of spread* (or *measures of dispersion*) that we need to know are:

1. The *range.*
2. The *interquartile range.*
3. The *semi-interquartile range* (or *quartile deviation*).
4. The *standard deviation.*

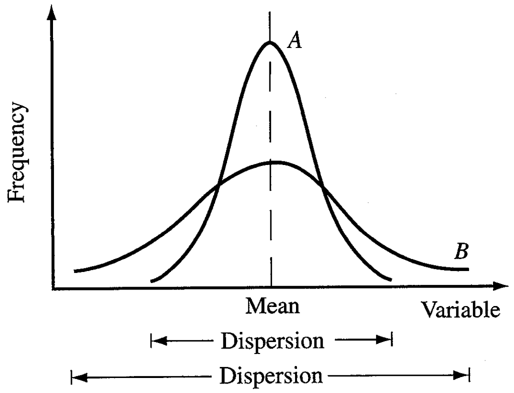


Fig. 8.28 Frequency curve

Dispersion measures the extent to which a *random variable* (or set of observations) is *spread* about its *mean. Two different distributions* may have the same *mean*, but *different dispersions*. In the diagram shown above, *both distributions* have the *same mean*, however *distribution A* is *more homogeneous* than distribution *B*. This is so because *distribution B* has a *greater dispersion* than distribution *A*, although they share the *same mean.*

If the *two distributions* represent the marks obtained by the *same class* of students on *two different tests*—then they *performed better* in *test A* than test *B*. This is so because the *marks* are *more equitably distributed* in *test A* than *test B* and the *mean mark* for both tests were the *same.*

If the *two distributions* represent the marks obtained by *two different classes* of the *same size* on the *same test*—then *class A performed better* than class B. This is so because *both classes* obtained the *same mean mark*, but the *dispersion* of marks was *smaller* for *class A* than *class B*. That is, the *marks* obtained by the students in *class A* were *more equitably distributed.*

Range

The *range* of a *set of a data* is defined as the *difference* between the *largest and the smallest observations*. This fact can be seen illustrated in the *histogram* shown below.

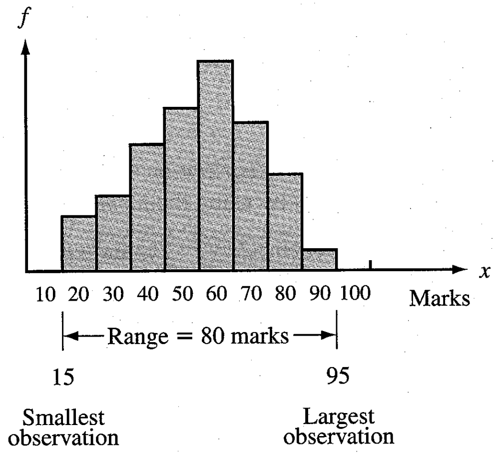


Fig. 8.29 Histogram

The *range* = (95 − 15) marks = 80 marks.

Calculating the Range from Raw Data

The *formula* that we use to *calculate* the *range* from *raw data* is:

The range = *The largest observation* − *The smallest observation*

Example 20

The basic wages of workers in a factory are:

$175, $160, $195, $149, $185, $167, $148.

Calculate the range of the basic wages.

Solution

The *basic wages* of the workers in the factory are:

$148, $149, $160, $167, $175, $185, $195

|← Range = $47 →|

$148 Smallest observation

$195 Largest observation

So the *range* of the *basic wages* = *The largest observation* − *The smallest observation*

= $(195 − 148)

= $47

Calculating the Range from a Frequency Distribution with Ungrouped Data

The *range* of a *frequency distribution* with *ungrouped data* can be *calculated* by using the *formula*:

The range = *The upper boundary limit of the largest observation* − *The lower boundary limit of the smallest observation*

Example 21

The masses of 50 lambs were estimated to the nearest kilogram. The results can be seen tabulated below.

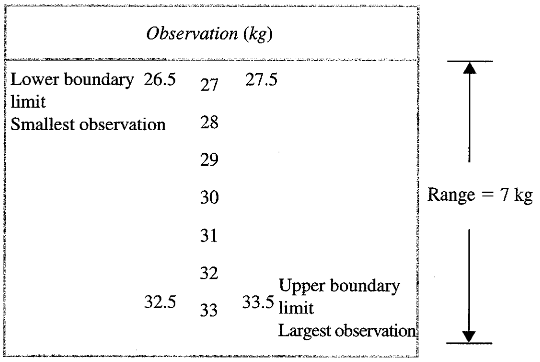
Table 8.94 Frequency table

Mass (kg)	*Frequency*
27	*4*
28	*9*
29	*16*
30	*13*
31	*5*
32	*2*
33	*1*

What value is the range of these estimates?

Solution

Table 8.95 Observation



The *range* of the *estimated masses* of the *lambs*

= *The upper boundary limit of the largest observation* − *The lower boundary limit of the smallest observation*

= (33.5 − 26.5) kg

= 7 kg

The *range* is the *easiest* measure of dispersion to *determine*. However, it is *influenced* too much by *extreme values* in the *set of data*. So it is used mainly as a *measure of dispersion* for *small samples* when it is most effective.

Interquartile Range and Semi-interquartile Range

A *quartile* by definition is one of three values that divide an *ordered set of data* into *four equal parts*. The *first* (or *lower*) *quartile* Q_1 is the *value below* which *one-quarter* of the data lies.

The *second* (or *middle*) *quartile* Q_2 is the *value below* which *one-half* of the data lies. This *quartile* we know as the *median*.

And the *third* (or *upper*) *quartile* Q_3 is the *value below* which *three-quarters* of the data lies.

The *quartiles* and their *positions* are illustrated in the diagram below.

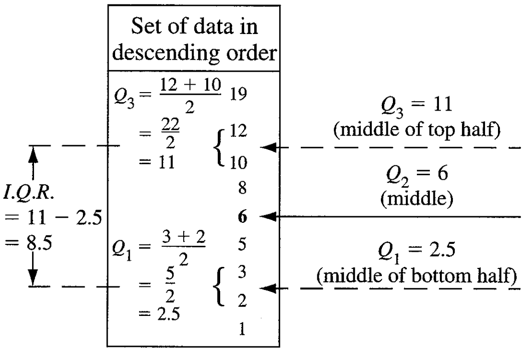


Fig. 8.30 Quartiles

From the *diagram* shown above it can be *seen* that:

- (i) The *median* Q_2 is the *middle value* of the *whole* set of data.
- (ii) The *lower quartile* Q_1 is the *middle value* of the *bottom half* of the data.
- (iii) The *upper quartile* Q_3 is the *middle value* of the *top half* of the data.

The *interquartile range* of a *distribution* is defined as the *difference* between its *upper* and *lower quartiles*.

Thus:

The *interquartile range, I.Q.R.* = $Q_3 - Q_1$

And the *semi-interquartile range* (or *quartile deviation*) of a *distribution* is defined as *half* the *difference* between its *upper* and *lower quartiles*. Hence it is *half* of the *interquartile range*.

Thus:

The *semi-interquartile range, S.I.Q.R.* $= \dfrac{Q_3 - Q_1}{2}$

From the *diagram* above:

The *interquartile range, I.Q.R.*

$$\begin{aligned} &= Q_3 - Q_1 \\ &= 11 - 2.5 \\ &= 8.5 \end{aligned}$$

And the *semi-interquartile range,*

$$\begin{aligned} S.I.Q.R. &= \frac{Q_3 - Q_1}{2} \\ &= \frac{11 - 2.5}{2} \\ &= \frac{8.5}{2} \\ &= 4.25 \end{aligned}$$

Interquartile Range and Semi-interquartile Range from Raw Data

Example 22

Calculate the interquartile range and semi-interquartile range of the following heights stated in centimetres:

(a) 163, 158, 154, 161, 156, 159, 155

(b) 158, 163, 154, 161, 157, 156, 159, 155.

Solution

(a) The *heights* in *ascending order* are:

1st	2nd	3rd	4th	5th	6th	7th	*Rank*
154,	$\boxed{155,}$	156,	$\textcircled{158,}$	159,	$\boxed{161,}$	163	*Height*
	↑				↑		
	Q_1		Q_2		Q_3		*Quartile*

|← I.Q.R. = 6 cm →|

We first *fix* the *position* of the *median* Q_2. Since there are now three heights below the median and three heights above the median, then the *lower quartile* Q_1 is the *second height* and the *upper quartile* Q_3 is the *sixth height*.

Thus the *lower quartile*, Q_1 = 155 cm.

And the *upper quartile*, Q_3 = 161 cm.

So the *interquartile range, I.Q.R.*

$$\begin{aligned} &= Q_3 - Q_1 \\ &= (161 - 155)\text{ cm} \\ &= 6\text{ cm} \end{aligned}$$

And the *semi-interquartile range, S.I.Q.R.*

$$\begin{aligned} &= \frac{Q_3 - Q_1}{2} \\ &= \frac{(161 - 155)\text{ cm}}{2} \\ &= \frac{6\text{ cm}}{2} \\ &= 3\text{ cm} \end{aligned}$$

(b) The *heights* in *ascending order* are:

1st 2nd 3rd 4th 5th 6th 7th 8th *Rank*

$$154,\ \boxed{\begin{array}{c}155, 156,\\ 155.5\end{array}}\ 157, 158,\ \boxed{\begin{array}{c}159, 161,\\ 160\end{array}}\ 163\ \textit{Height}$$

↑ ↑ ↑

Q_1 Q_2 Q_3 *Quartile*

|← I.Q.R. = 4.5 cm →|

We first *locate* the *position* of the *median* Q_2. Since there are four heights below and above the *median*, then the *lower quartile* Q_1 is the *average* of the *second and third heights*, and the *upper quartile* Q_3 is the *average* of the *sixth and seventh heights.*

Thus the *lower quartile*, Q_1

$$\begin{aligned} &= \frac{(155 + 156)\text{ cm}}{2} \\ &= \frac{311\text{ cm}}{2} \\ &= 155.5\text{ cm} \end{aligned}$$

And the *upper quartile*, Q_3

$$\begin{aligned} &= \frac{(159 + 161)\text{ cm}}{2} \\ &= \frac{320\text{cm}}{2} \\ &= 160\text{ cm} \end{aligned}$$

So the *interquartile range, I.Q.R.*

$$\begin{aligned} &= Q_3 - Q_1 \\ &= (160 - 155.5)\text{ cm} \\ &= 4.5\text{ cm} \end{aligned}$$

And the *semi-interquartile range, S.I.Q.R.*

$$\begin{aligned} &= \frac{Q_3 - Q_1}{2} \\ &= \frac{(160 - 155.5)\text{ cm}}{2} \\ &= \frac{4.5\text{ cm}}{2} \\ &= 2.25\text{ cm} \end{aligned}$$

Because the *range* is *influenced* too much by *extreme values* in the *set of data* it was necessary to define *another measure of dispersion* called the *interquartile range*.

The *interquartile range* is *not affected* by *extreme values* as it is *centred* around the *middle half of the data* containing the *median*. Thus it *does not* show the *dispersion* of the *set of data* as *a whole*.

However the *range* and the *interquartile range* can be combined in order to give us a *more accurate* picture of the *distribution*.

Interquartile Range and Semi-interquartile Range from a Frequency Distribution with Ungrouped Data

When the *observations* in a *set of data* are given as a *frequency distribution* with *ungrouped data*, then the *position* of the *median* Q_2 is given by the $\frac{1}{2}(n + 1)^{\text{th}}$ *rank* and the *median* Q_2 is the *value corresponding* to this *rank*. Similarly, the *positions* of the *lower quartile* Q_1 and the *upper quartile* Q_3 are given by the $\frac{1}{4}(n + 1)^{\text{th}}$ *rank* and the $\frac{3}{4}(n + 1)^{\text{th}}$ *rank*, respectively. So the *lower quartile* Q_1 and the *upper quartile* Q_3 are the *values corresponding* to the $\frac{1}{4}(n + 1)^{\text{th}}$ *rank* and the $\frac{3}{4}(n + 1)^{\text{th}}$ *rank*, respectively.

Example 23

The masses of 100 pupils in a school are shown in the table below.

Table 8.96 Frequency table

Weight (kg)	*Number of pupils*
51	*7*
52	*8*
53	*10*
54	*12*
55	*13*
56	*15*
57	*12*
58	*9*
59	*8*
60	*6*

(a) Determine for the distribution given:

(i) its lower quartile

(ii) its upper quartile.

(b) Hence find the value of:

(i) the interquartile range of the masses

(ii) the semi-quartile range of the masses.

Solution

(a) We first *construct* the *cumulative frequency table* from the frequency table as shown below:

Table 8.97 Cumulative frequency table

	Mass interval (kg)	*Cumulative frequency*	
	≤ 51	7	
	≤ 52	7 + 8 = 15	
$Q_1 = 53.5$	≤ 53 ←	15 + 10 = 25 ←	25th rank
I.Q.R. = 3.5 kg	≤ 54 ←	25 + 12 = 37 ←	26th rank
	≤ 55	37 + 13 = 50	
	≤ 56	50 + 15 = 65	
$Q_3 = 57$	≤ 57 ←	65 + 12 = 77 ←	75th rank 76th rank
	≤ 58	77 + 9 = 86	
	≤ 59	86 + 8 = 94	
	≤ 60	94 + 6 = 100	

(i) The *position* of the *lower quartile* $= \frac{1}{4}(n + 1)^{\text{th}}$ *rank*

$$= \frac{1}{4}(100 + 1)^{\text{th}} \text{ rank}$$

$$= \frac{1}{4}(101)^{\text{th}} \text{ rank}$$

$$= 25.25^{\text{th}} \text{ rank}$$

This implies that the *lower quartile* is the *average* of the 25th and 26th *observations*.

From the *cumulative frequency table* it can be seen that:

The 25th *rank* (i.e. the 25th *pupil) has a mass of* 53 kg.

And the 26th *rank* (i.e. the 26th *pupil) has a mass of* 54 kg.

∴ the *lower quartile* of the distribution, $Q_1 = \frac{(25^{\text{th}} + 26^{\text{th}}) \textit{ observations}}{2}$

$$= \frac{(53 + 54) \text{ kg}}{2}$$

$$= 53.5 \text{ kg}$$

(ii) The *position* of the

$$\textit{upper quartile} = \frac{3}{4}(n+1)^{\text{th}}\ \textit{rank}$$
$$= \frac{3}{4}(100+1)^{\text{th}}\ \text{rank}$$
$$= \frac{3}{4}(101)^{\text{th}}\ \text{rank}$$
$$= 75.75^{\text{th}}\ \text{rank}$$

This implies that the *upper quartile* is the *average* of the 75th and 76th *observations.* From the *cumulative frequency table* it can be seen that:
Both the 75th and 76th *ranks* (i.e. the 75th and 76th *pupils*) *have a mass of* 57 *kg each.*

∴ the *upper quartile* of the distribution, Q_3 = 57 kg

(b) (i) The *interquartile range* of the *masses, I.Q.R.*

$$= Q_3 - Q_1$$
$$= (57 - 53.5)\ \text{kg}$$
$$= 3.5\ \text{kg}$$

(ii) The *semi-interquartile range* of the *masses, S.I.Q.R.*

$$= \frac{Q_3 - Q_1}{2}$$
$$= \frac{(57 - 53.5)\ \text{kg}}{2}$$
$$= \frac{3.5\ \text{kg}}{2}$$
$$= 1.75\ \text{kg}$$

Exercise 8p

1. Given the raw data of numbers:

7, 3, 2, 4, 5, 4, 6.

Calculate:
(a) the range
(b) the interquartile range
(c) the semi-interquartile range.

2. Given the raw data of numbers:

9, 6, 4, 3, 5, 7, 5, 8,

determine:
(a) the range
(b) the interquartile range
(c) the semi-interquartile range.

3. The heights of 13 men in centimetres are given below:

162, 160, 163, 160, 165, 167, 170, 167, 174, 176, 178, 179, 178.

Determine:
(a) the range
(b) the interquartile range
(c) the semi-interquartile range.

4. The masses of 12 men in kilograms are:

69, 70, 65, 68, 66, 72, 66, 67, 73, 67, 71, 70.

Calculate:
(a) the range
(b) the interquartile range
(c) the semi-interquartile range.

5. Twenty-five students wrote a mathematics test in which the maximum mark that could be obtained was 10. The mark of each participant is listed below.

0	1	5	2	8
3	2	4	8	6
5	9	3	9	4
7	6	7	5	0
8	10	8	4	3

(a) State the range of the marks.
(b) Determine the median mark and the semi-interquartile range.
(c) Calculate the mean mark for the distribution.
(d) What is the probability that a student chosen at random has a mark greater than 7?

6. 100 students wrote a test in which the maximum mark that could be obtained was 5. The mark of each student is listed in the frequency table below.

Table 8.98 Frequency table

Mark	*Frequency*
1	30
2	26
3	20
4	14
5	10

(a) Calculate the range of the marks.
(b) Determine the median mark and the semi-interquartile range.
(c) Calculate the mean mark.

7. In a shooting contest in which 50 people participated, the following frequency table was obtained.

Table 8.99 Frequency table

Score	*Frequency*
1	3
2	1
3	4
4	10
5	15
6	9
7	3
8	5

(a) Calculate the range of the scores.

(b) Determine the semi-interquartile range or quartile deviation for the distribution.

8. The masses of 120 pupils in a school are shown in the table below.

Table 8.100 Frequency table

Mass (kg)	*Number of pupils*
51	7
52	10
53	13
54	14
55	22
56	18
57	15
58	9
59	7
60	5

(a) Determine for the distribution given:
  (i) its lower quartile
  (ii) its upper quartile.

(b) Hence calculate the value of:
  (i) the interquartile range of the masses
  (ii) the semi-interquartile range of the masses.

(c) State the range of the masses.

9. The frequency distribution of the heights of 124 people is shown below.

Table 8.101 Frequency table

Height (cm)	*Frequency*
151	9
152	11
153	14
154	15
155	23
156	19
157	13
158	9
159	7
160	4

(a) State the range of the heights.

(b) Calculate the value of:
  (i) the interquartile range
  (ii) the semi-interquartile range.

10. The frequency distribution of the length of 150 steel rods measured in millimetre is given in the table below.

Table 8.102 Frequency table

Length (mm)	*Frequency*
201	9
202	11
203	19
204	20
205	28
206	24
207	18
208	10
209	7
210	4

(a) State the range for the distribution.

(b) Calculate:
  (i) the interquartile range
  (ii) the quartile deviation.

Probability

Probability is defined as the *measure* of *how likely* an *event* is to *occur*. The *probability* of an *event* is a number between 0 (the *impossible event*) and 1 (the *certain event*). For example:

The probability of a person having three heads is 0.
The probability of a person walking on the sun is 0.
The probability that a person will die is 1.
The probability of the sun setting in the west is 1.

Sample Space, Outcomes and Events

The *sample space U* is the *set* of *all possible outcomes* of a given *experiment*. *Each element* of the *sample space U* is called a *sample point* or *outcome a*. That is $a \in U$. An *event A* is a *set of outcomes*. That is, A is a *subset* of $U, A \subset U$.

Equally Likely Events

Probability problems are usually based on *mathematical ideas* like '*a fair coin*'. In *practice*, many *coins* are *slightly unfair* and therefore tend to give *slightly uneven results*. However a '*fair*' *coin* will tend to give an *equal number* of '*heads*' and '*tails*' over a *large number of throws*. The events '*a head will come up*' and '*a tail will come up*' are then said to be *equally likely*.

In situations where *several equally likely outcomes* are possible, the *probability* of a *particular event* is *measured* by:

The *probability of an event occurring*

$$= \frac{\textit{The number of favourable outcomes}}{\textit{The total number of possible outcomes}}$$

That is $P(A) = \dfrac{n(A)}{n(U)}$.

The *probability* of an *event* occurring *ranges* from a *minimum* of 0 to a *maximum* of 1. That is, for any *event* A, $0 \leqslant P(A) \leqslant 1$.

The Impossible Event

If $P(A) = 0$, then the *event* is an *absolute impossibility*, i.e. it will *never occur*. For example, the probability of a car travelling with the speed of light.

That is, if $A = \emptyset$, the *empty set*,

then $P(A) = \dfrac{n(\text{A})}{n(U)} = \dfrac{n(\emptyset)}{n(U)} = \dfrac{0}{n(U)} = 0 = P(\emptyset)$

Hence $P(\emptyset) = 0$.

The Certain Event

If $P(A) = 1$, then the *event* is an *absolute certainty*, i.e. it *will occur*. For example, the probability that a person will eventually die.

That is, if $A = U$, the *universal set*,

then $P(A) = \dfrac{n(A)}{n(U)} = \dfrac{n(U)}{n(U)} = 1 = P(U)$

Hence $P(U) = 1$.

Probability Dealing with One Event and its Complement

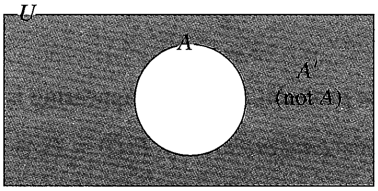


Fig. 8.31 Venn diagram

Given any *event A*,

$$\begin{aligned}
&\text{then} && P(U) = P(A) + P(A')\\
&\text{So} && 1 = P(A) + P(A')\\
&\text{i.e.} && P(A) = 1 - P(A')\\
&\text{Since} && P(U) = 1\\
&\text{Here} && P(A) = \frac{n(A)}{n(U)}\\
&\text{and} && P(A') = \frac{n(A')}{n(U)}.
\end{aligned}$$

Example 24

A fair coin is tossed once and the symbol that appears on top is observed.
Calculate the probability that:
(i) a head appears
(ii) a tail appears.
Assume that either a head or a tail appears.

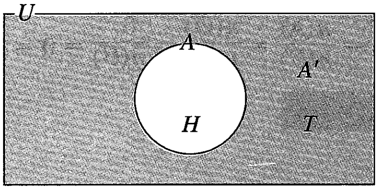


Fig. 8.32 Venn diagram

Solution

The *sample space* $U = \{H, T\}$
That is $n(U) = 2$
Let $A = \{\text{head appears}\} = \{H\}$
Then $n(A) = 1$
And $A' = \{\textit{tail appears}\} = \{T\}$
Then $n(A') = 1$

(i) $P(A) = \dfrac{n(A)}{n(U)} = \dfrac{1}{2} = 0.5$

That is, the *probability* of a *head* appearing is 0.5.

(ii) Now $P(A') = \dfrac{n(A')}{n(U)} = \dfrac{1}{2} = 0.5$

or
$$\begin{aligned} P(A') &= 1 - P(A) \\ &= 1 - \frac{1}{2} \\ &= \frac{1}{2} \\ &= 0.5 \end{aligned}$$

That is, the *probability* of a *tail* appearing is 0.5.

Note that the *total probability*, $P(U) = P(A) + P(A')$
$$= \frac{1}{2} + \frac{1}{2}$$
$$= 1$$

Theoretical Probability

It is not *always practical* to carry out an *experiment* or *survey* in order to determine the *probability* of an *event* occurring, since it might be time consuming. However, once we can *determine* the *number of favourable outcomes* and the *number of possible outcomes*, we can easily *calculate* the *probability* using the stated *formula*.

Example 25

A bag contains 60 marbles; 45 green ones and 15 red ones.

(a) What is the probability of drawing a green marble?
(b) What is the probability of drawing a red marble?
(c) If 15 green marbles are removed from the bag, what is the chance now of drawing a green marble?
(d) What is the chance of drawing a yellow marble?
(e) What is the probability of drawing either a green marble or a red marble?

Solution

(a) The *number of favourable outcomes* = The number of green marbles
= 45 marbles

And the *total number of possible outcomes* = The total number of marbles
= (45 + 15) marbles
= 60 marbles

$$\therefore P(\textit{marble is green}) = \frac{\textit{The number of favourable outcomes}}{\textit{The total number of possible outcomes}}$$
$$= \frac{45\ \cancel{\text{marbles}}}{60\ \cancel{\text{marbles}}}$$
$$= \frac{3}{4}$$
$$= 0.75$$

(b) The *number of favourable outcomes* = The number of red marbles
= 15 marbles

$$\therefore P(\textit{marble is red}) = \frac{\textit{The number of favourable outcomes}}{\textit{The total number of possible outcomes}}$$

$$= \frac{15\ \cancel{\text{marbles}}}{60\ \cancel{\text{marbles}}}$$

$$= \frac{1}{4}$$

$$= 0.25$$

(c) The *number of favourable outcomes* = The number of green marbles
= (45 − 15) marbles
= 30 marbles

And the *total number of possible outcomes* = The total number of marbles
= (60 − 15) marbles
= 45 marbles

$$\therefore P(\textit{marble is green}) = \frac{\textit{The number of favourable outcomes}}{\textit{The total number of possible outcomes}}$$

$$= \frac{30\ \cancel{\text{marbles}}}{45\ \cancel{\text{marbles}}}$$

$$= \frac{2}{3}$$

= 0.67 (*correct* to 2 *d.p.*)

(d) The *number of favourable outcomes* = The number of yellow marbles
= 0 marbles

$$\therefore P(\textit{marble is yellow}) = \frac{\textit{The number of favourable outcomes}}{\textit{The total number of possible outcomes}}$$

$$= \frac{0\ \cancel{\text{marbles}}}{60\ \cancel{\text{marbles}}}$$

$$= 0$$

That is, there is *no possible chance* of drawing a *yellow marble*, since *none exists* in the bag.

(e) The *number of favourable outcomes* = The total number of green and red marbles
= (45 + 15) marbles
= 60 marbles

$$\therefore P(\textit{marble is green or red}) = \frac{\textit{The number of favourable outcomes}}{\textit{The total number of possible outcomes}}$$

$$= \frac{60\ \cancel{\text{marbles}}}{60\ \cancel{\text{marbles}}}$$

$$= 1$$

That is, it is a *certainty* that *either a green marble or a red marble* is *drawn.*

Example 26

A card is chosen from a standard pack of 52 playing cards. What is the probability that the card is:

(a) an Ace?
(b) a red card?

Solution

(a) The *number of favourable outcomes* = The number of Aces
= 4 cards

The *total number of possible outcomes* = The total number of cards
= 52 cards

$$\therefore P(\textit{card is an Ace}) = \frac{\textit{The number of favourable outcomes}}{\textit{The total number of possible outcomes}}$$

$$= \frac{4\ \cancel{\text{cards}}}{52\ \cancel{\text{cards}}}$$

$$= \frac{1}{13}$$

(b) The *number of favourable outcomes* = The number of red cards
= 26 cards

$$\therefore P(\textit{card is red}) = \frac{\textit{The number of favourable outcomes}}{\textit{The total number of possible outcomes}}$$

$$= \frac{26\ \cancel{\text{cards}}}{52\ \cancel{\text{cards}}}$$

$$= \frac{1}{2}$$

Exercise 8q

1. A fair silver dollar is tossed once and the symbol that appears on top is observed. Calculate the probability that:

(a) a tail appears
(b) a head appears.

2. A fair die is rolled. Calculate the probability that:

(a) a 6 is rolled
(b) a number more than 4 is rolled.

3. A die is thrown. What is the probability that:

(a) a multiple of 2 is thrown?
(b) an odd number is thrown?

4. A die is tossed. What is the probability that:

(a) a prime number appears?
(b) a number less than 5 appears?

5. An urn contains 50 marbles, 40 blue ones and 10 yellow ones.

(a) What is the probability of drawing a blue marble?
(b) If 5 yellow marbles are removed from the urn, what is the chance now of drawing a yellow marble?

6. An urn contains 75 marbles. 50 marbles are blue and the remaining marbles are green.

(a) What is the probability of drawing a green marble?
(b) If 25 blue marbles are removed from the urn, what is the chance of drawing a blue marble?

7. A jar contains 100 one-cent coins. 60 of the one-cent coins are Guyanese coins and the remainder are Jamaican coins.

(a) Calculate the probability of choosing a Guyanese coin.
(b) What is the probability of choosing a Jamaican coin?
(c) Determine the probability of selecting a Guyanese coin or Jamaican coin.

8. A card is chosen at random from a standard pack of 52 playing cards.
Calculate the probability that the card is:

(a) a King
(b) a black card.

9. If we use a standard pack of 52 playing cards, what is the probability of drawing:

(a) the Ace of Hearts?
(b) a Joker?

10. Using a standard pack of 52 playing cards, calculate the probability of selecting

(a) a red Jack
(b) a king or Queen
(c) a King, Queen, or Jack.

11. If a letter is taken at random from the words MATHEMATICS OLYMPIAD, what is the probability that

(a) it is a vowel?
(b) it is a M?
(c) it is an O?

12. If a number is chosen at random from the numbers 1 to 25 inclusive written on pieces of paper and placed in a vase, calculate the probability that

(a) a multiple of 5 is selected
(b) a prime number is selected.

13. A class has 18 boys and 12 girls. A perfect is to be chosen from the class. If each student is equally likely to be chosen as the perfect, calculate the probability that the selected perfect will be:

(a) a boy (b) a girl.

14. A piggy bank contains the following currency notes: thirty $1, fifteen $5 and five $10.
The notes were placed in the piggy bank at random.
Calculate the probability of taking out a

(a) $1 note (b) $5 note (c) $10 note.

15. A piggy bank contains the following currency notes: thirty $1, twenty $5, fifteen $10, thirteen $20 and twelve $100.
The notes were saved in the piggy bank at random.
Calculate the probability of choosing

(a) a $100 note
(b) a $20 note or a $100 note
(c) either a $10 note, a $20 note or a $100 note.

16. A box contains 4 dozen pencils. 18 pencils were sharpened and the remainder were unsharpened. What is the probability of picking out a pencil which was unsharpened?

17. A car park contains twenty-five 2 800 cc cars, thirty 1 500 cc cars and forty-five 1 300 cc cars. If they are all equally likely to leave, what is the probability of

(a) a 2 800 cc car leaving first?

(b) a 1 500 cc car or a 1300 cc car leaving first?

18. The numbers 1 to 100 are written on pieces of paper are placed in an urn. If a number is picked at random from the urn, what is the probability that it is

(a) a square

(b) a cube

(c) exactly divisible by 5.

19. A box contains 18 red pens, 12 blue pens, 6 green pens and 24 black pens. If a teacher selects a pen from the container, what is the probability that it is

(a) blue or green? (b) red or black?

20. A video club has 252 Western, 198 Romance, 154 Mystery and 196 Comedy cassettes. If they are all equally likely to be borrowed, what is the probability that

(a) a Western or Romance cassette is borrowed?

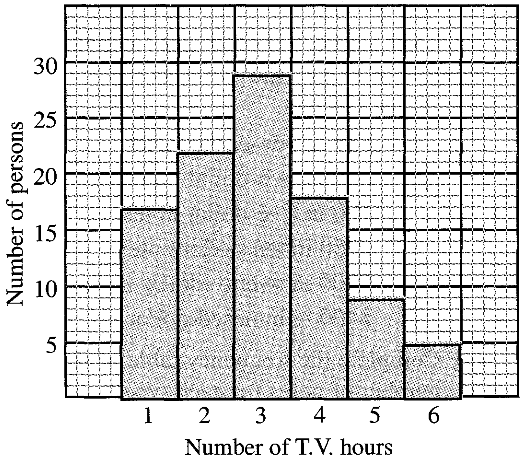
(b) a Mystery or Comedy cassette is borrowed?

C.X.C. Past Paper Questions

The following supplementary questions were taken from C.X.C. Past Papers.

Exercise 8r

1.



The histogram shows the number of hours a group watched television during a particular evening.

(a) Construct the frequency table for the data shown in the above histogram under the headings class mark and frequency.

(b) Calculate the mean number of TV hours.

(c) Calculate the probability that a person selected at random from this group watched television for 5 hours or more.

Question 9. C.X.C. (Basic). June 1979.

2. There are 25 participants in a shooting competition. The score of each participant is listed below.

1	3	6	0	5
0	1	5	1	6
2	3	5	0	1
1	4	0	5	6
5	2	1	3	2

(a) Set up a frequency table for the scores.

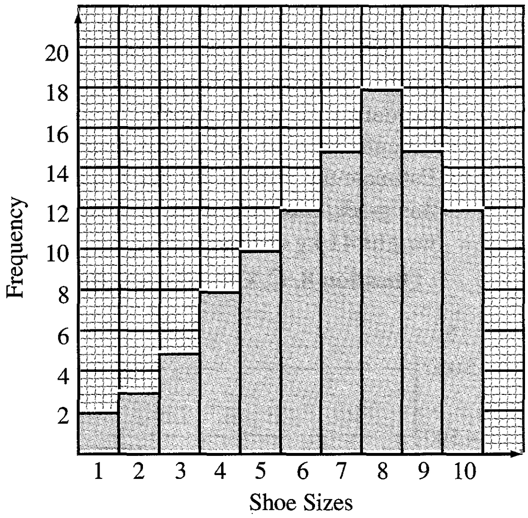
(b) Draw the frequency polygon representing the data. (*Use graph paper*)

(c) Find the median score and the interquartile range.

(d) Find the probability that a competitor chosen at random has a score greater than 4.

Question 6. C.X.C. (Basic). June 1980.

3.



The histogram above shows the frequency of shoe sizes for a random sample of 100 pairs of shoes sold by a large department store at the beginning of the school term.

(a) Draw up a frequency table to represent this information.

(b) Determine the mode size, median and mean size of this sample.

(c) The store manager wishes to replenish his stock. Which of these three measures should he use to determine what size to order in the largest quantity? State a reason for your choice.

(d) Estimate the probability that a pair of shoes chosen at random from this sample of 100 pairs is a size 6.

Question 6. C.X.C. (Basic). June 1982.

4. (a) The heights of 11 women in centimetres are given below:

150, 150, 153, 155, 157, 157, 160, 164, 166, 168, 169.

Determine the interquartile range of these heights.

(b) The weights of 60 pupils in a grade/class are shown in the table below:

Weight (kilograms)	*Number of pupils*
24–32	2
33–41	15
42–50	20
51–59	12
60–68	8
69–77	3

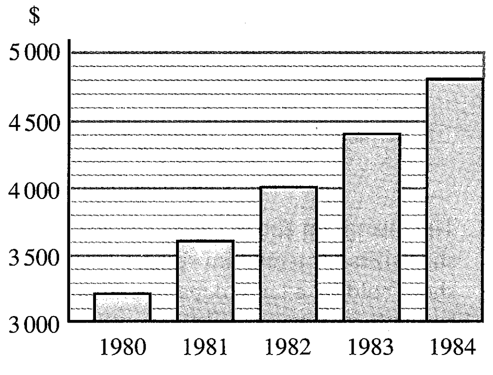
(i) Draw a frequency polygon to represent the data. (Use graph paper)

(ii) Calculate the median of this distribution.

(iii) Estimate the probability that if a boy in this grade/class is chosen at random he weighs 41 kg or less.

Question 8. C.X.C. (Basic). June 1983.

5.



(a) The bar-chart above shows the amount of money invested by a company over a five-year period.

(i) Write down the amounts invested in 1980 and 1983.

(ii) Calculate the mean amount invested per year over the five-year period.

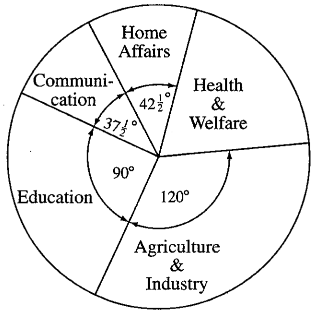
(iii) Estimate the amount invested in 1985, assuming the trend shown in the graph continues. Give a reason for your answer.

(iv) Calculate the angle which would represent the amount invested in 1983 if the information illustrated in the bar-chart above is to be represented on a pie chart.

(b) A box contains 10 similar balls, 4 of which are yellow. The first ball taken out at random is yellow. It is not replaced. Calculate the probability that a second ball taken out at random is yellow.

Question 5. C.X.C. (Basic). June 1985.

6.



The pie chart above illustrates how a country spent its budget for 1985. It spent $22.5 million on Education.

Calculate the amount of money spent on

(a) Agriculture and Industry

(b) Health and Welfare.

Question 9(a). C.X.C. (Basic). June 1986.

7. At closing time, a shopkeeper counted the amount of money she had in her cash box. She found that she had

$38 in one-dollar notes
$20 in two-dollar notes
$90 in five-dollar notes
$250 in ten-dollar notes
$300 in twenty-dollar notes
$400 in hundred-dollar notes

(a) Complete the frequency table to show the number of notes for each type.

Value	*Type of notes*	*Number of notes*
$38	$1.00	38
$20	$2.00	
$90	$5.00	
$250	$10.00	
$300	$20.00	
$400	$100.00	

(b) Represent the information in the completed frequency table by means of a bar graph, using a scale of 1 cm to represent 5 notes of each type.

(c) Estimate the median value of this distribution.

(d) If a note is selected at random, calculate the probability that
  (i) it is a ten-dollar note
  (ii) it is not a hundred-dollar note.

Question 9. C.X.C. (Basic). June 1987.

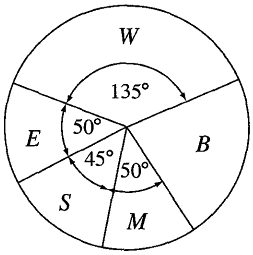
8. (a)

Age	11	12	13	14	15	16	17
No. of children	3	6	6	6	4	3	2

The table above shows a distribution of the ages of 30 children in a school choir.

(i) Calculate the mean age and the median age of this distribution.
(ii) Calculate the probability that a child chosen at random is
— under 15 years old
— at least 15 years old.

(b)



The pie chart above, which is not drawn to scale, represents the amount of money spent by a school on various items as indicated below.

W : Wages and Salaries
B : Book and Supplies
M : Maintenance
S : Sports and Games
E : Other Expenses

The total budget was $72 000.

(i) Calculate the amounts spent on S and on B.
(ii) Using a scale of 1 cm to represent $2 000, draw a bar chart to illustrate the information given in the pie chart above.

Question 9. C.X.C. (Basic). June 1988.

9.

Subjects	*Mathematics*	*English*	*Social Studies*	*Science*	*Modern languages*
No. of teachers	15	26	40	39	10

The table above gives the number of graduates by subject from a teacher's collage in 1984.

(a) Using graph paper draw a bar chart to represent the data.

(b) Calculate the probability that a teacher chosen at random is an English teacher.

(c) A pie chart is drawn to represent the data in the table. Calculate the angle of the sector representing the number of Science teachers.

(d) The mean number of Mathematics teachers who graduated in the three-year period 1985–1987 is 31.
  (i) Calculate the total number Mathematics teachers who graduated over the period 1985–1987.
  (ii) Hence, calculate the mean number of Mathematics teachers who graduated over the period 1984–1987.

Question 8. C.X.C. (Basic). June 1989.

10. (a) The table below shows the amounts and corresponding proportions of salary a manager spends on various items.

Item of expenditure	*Money budgeted*	*Proportion of salary*
Insurance	$450	$\frac{1}{12}$
Income tax	$1 125	$\frac{5}{24}$
Mortgage payment	$1 350	a
Savings	b	$\frac{1}{3}$
Food and Expenses	c	d

(i) Calculate the values of a, b, c and d.
(ii) Calculate the angle of the sector required to represent EACH of the following on a pie chart:
— Savings
— Income tax

(b) The marks obtained by 30 students in a test in which the maximum was 10 marks were as follows:

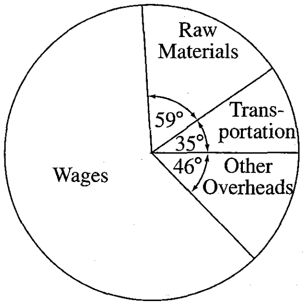
6 3 4 4 3 3 1 1 8 6
5 9 5 6 4 2 4 4 7 7
7 8 6 5 3 1 2 6 7 6

(i) Construct a frequency distribution from the data given.

(ii) If a student is chosen at random, calculate the probability that he got less than 4 marks.

Question 9. C.X.C. (Basic). June 1990.

11.



The pie chart above illustrates how a manufacturing company spends its budget for a year on raw materials, transportation, wages and other overheads. The company spent $35 700 on transportation.

Calculate:

(a) the total budget

(b) the amount spent on raw materials

(c) the fraction of the budget spent on wages.

Question 6(b). C.X.C. (Basic). June 1991.

12. A teacher kept a record of the length of time that 90 students were late for class.
The results are shown in the following frequency distribution table:

Minutes late	0	1	2	3	4	5	6	7
No. of students	4	8	10	8	15	20	15	10

(a) State the mode of the distribution.

(b) Calculate the median of the distribution.

(c) Draw a histogram to represent the data.

(d) Calculate the total time lost by the students.

(e) Calculate, to the nearest minute, the mean time lost per student.

(f) Calculate the probability that a student of the class chosen at random was late

 (i) by exactly 5 minutes

 (ii) by at least 5 minutes.

Question 9. C.X.C. (Basic). June 1991.

CHAPTER 9

Geometry 1

This chapter will teach you about

- defining a point, line, ray, surface and solid
- properties of a triangle, quadrilateral, polygon and circle
- the types of angles, triangles, quadrilaterals and polygons
- constructing an angle, triangle and quadrilateral
- drawing the net of a solid
- drawing the plan and elevations of a solid
- drawing the plan of a house

Introduction

Geometry is a branch of Mathematics that deals with *points*, *lines*, *surfaces* and *solids*. It *examines* their *properties*, *measurement* and *mutual relations* in *space*.

Point

A *point* is a *location* in *space* or on a *surface*. A *point* is so *tiny* that it is said to have a *position* but *no size*. *Points* are often *described* by their *coordinates* as in *graphical* work.

Thus:

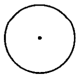


Fig. 9.1 Point

represents a *point*. The *point* is so *tiny* that it is *circled* to indicate that there is a *point* at the *centre*.

Line Segment

A *line segment* is *part* of a *straight line* between *two* given *points*.

If we *mark two points* and then *join* them with a *straight edge*, we have an illustration of a *line segment*.

Thus:

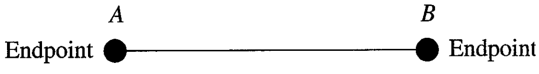


Fig. 9.2 Line segment

represents a *line segment AB*, since the *two endpoints* are *A* and *B*. The *points A* and *B* do not necessarily have to be *part* of the line segment.

In *Geometry*, we think of a *line segment AB* as having a *measurable length* but *no measurable width.*

Line

If the *line segment AB* in Fig. 9.2 *extends indefinitely* in *both directions*, then we get what is called in Geometry, a *line*. Thus:

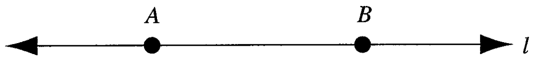


Fig. 9.3 Line

represents a *line AB* or *line l.* A *line* is said to *have length*, but *no breadth* and *no thickness.*

Ray

A *ray* is a *straight line extending* from a *point* called the *origin.*

If we take *two points*, *A* and *B*, *join them* and then *extend segment AB* beyond *B indefinitely*, then we get a *ray AB.*

Thus:

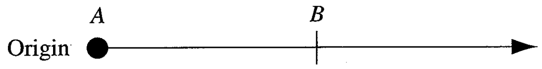


Fig. 9.4 Ray

represents a *ray AB*, since it is a *line extending* from the *point A.*

NOTE: For convenience, '*line AB*', '*line segment AB*', '*ray AB*' and the '*length of AB*' are simply written as *AB*. And instead of drawing them differently, we simply *represent them* as shown below in Fig. 9.5.

Fig. 9.5

Angles

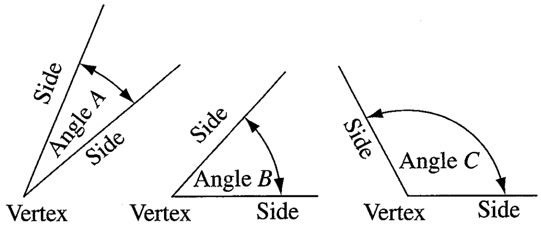


Fig. 9.6 Angles

When *two straight lines meet* at a *point* they *form* an *angle*. The *point* where the *two lines* (or *sides* or *arms*) meet is called a *vertex*, and the *angle* is a *measure* of the *space* or '*opening*' between the *two straight lines* (or *sides* or *arms*) that extend from the *common point* (or *vertex*).

The *magnitude* (or *size*) of the *angle* can also be defined as the *amount of turn* from *one line* (or *side* or *arm*) to *another* about the *vertex.*

In Fig. 9.6, *angle A*, *angle B* and *angle C* can be represented by the *symbols* $\hat{A}$, $\hat{B}$ and $\hat{C}$, respectively; or $\angle A$, $\angle B$ and $\angle C$, respectively. Where the *symbols* ^ and $\angle$ both mean '*angle*'.

From Fig. 9.6, it can be *deduced* that the *magnitude of an angle* is *not proportional* to the *lengths of the sides* (or *arms*) forming the *angle*. That is, the *greater* the *magnitude of the angle does not* mean the *longer* the *lengths of the sides* (or *arms*) forming the *angle*, and vice versa.

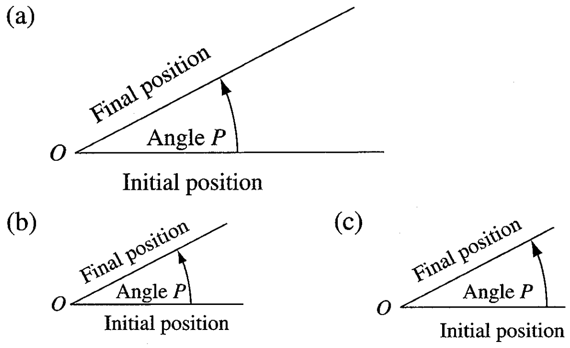


Fig. 9.7 Equal angles

The same *deduction* can be made from Fig. 9.7, which shows *three different size books* opened to the *same angle P.*

Revolution

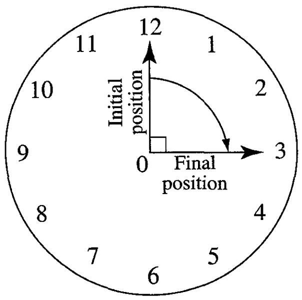


Fig. 9.8 Clock

The *hour hand* of a clock makes *one complete turn* in 12 hours, the *minute hand* of a clock makes *one complete turn* in 1 hour and the *second hand* of a clock makes *one complete turn* in 1 minute. *One complete turn* is called *one revolution*. Hence we can *measure* an *angle* by stating its *magnitude* as a *fraction of a revolution*. *Revolution* is abbreviated to *rev*.

Further, *one quarter of a revolution* (i.e. $\frac{1}{4}$ *turn*) is *equal* to *one right angle*. And the *symbol* for *one right angle* is ∟. The size of one right angle can be seen illustrated in Fig. 9.8, where the *second hand* of a clock *starts* at 12 and *stops* at 3, hence describing *one right angle*. *One right angle* can be abbreviated to 1 rt. ∠.

Example 1

What fraction of a revolution does the second hand of a clock turn through when

(a) it starts at 12 and stops at 7

(b) it starts at 8 and stops at 5?

Solution

(a)

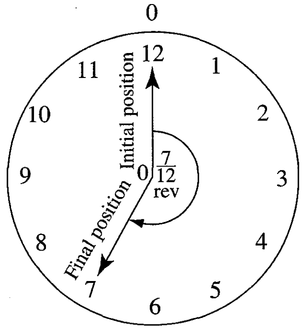


Fig. 9.9 Clock

The *number of hours* between 12 and 7 $= (7 - 0)\text{ h} = 7\text{ h}$

$\therefore$ the *fraction* of a *revolution* $= \frac{7\cancel{h}}{12\cancel{h}}$

$= \frac{7}{12}$ rev.

(b)

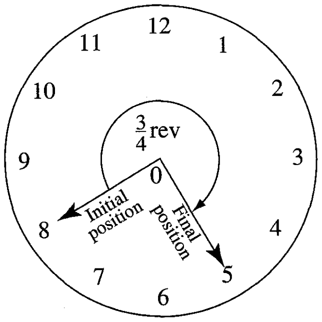


Fig. 9.10 Clock

The *number of hours* between 8 and 5 $= (12 - 8 + 5)\text{ h}$

$= (4 + 5)\text{ h}$

$= 9\text{ h}$

$\therefore$ the *fraction* of a *revolution* $= \frac{9\cancel{h}}{12\cancel{h}}$

$= \frac{3}{4}$ rev.

Example 2

Where does the second hand of a clock stops if:

(a) it starts at 12 and turns through $\frac{1}{6}$ of a revolution

(b) it starts at 4 and turns through $\frac{3}{4}$ of a revolution

Solution

(a)

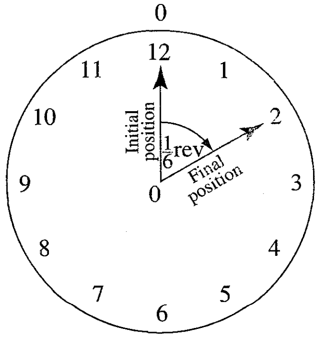


Fig. 9.11 Clock

$\frac{1}{6}$ of a *revolution* $= \frac{1}{6} \times 12\text{ h} = 2\text{ h}$

$\therefore$ the *position* where the second hand stops $= 0 + 2 = 2$

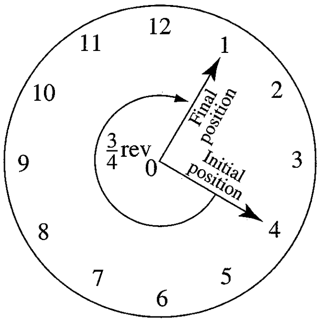


Fig. 9.12 Clock

(b) $\frac{3}{4}$ of a *revolution* $= \frac{3}{4} \times 12\text{ h} = 9\text{ h}$

$\therefore$ the *position* where the second hand stops $= 4 + 9 = 13 = 12 + 1$
$\Rightarrow 1$

Example 3

How many right angles does the second hand of a clock turn through when:

(a) it starts at 12 and stops at 6

(b) it starts at 1 and stops at 4?

Solution

(a)

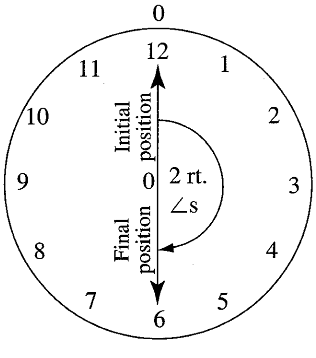


Fig. 9.13 Clock

The *number of hours* between 12 and 6 $= (6 - 0)\text{ h} = 6\text{ h}$

$\therefore$ the *number* of *right angles* $= \frac{6\cancel{h}}{3\cancel{h}} = 2$ rt. ∠s.

(b)

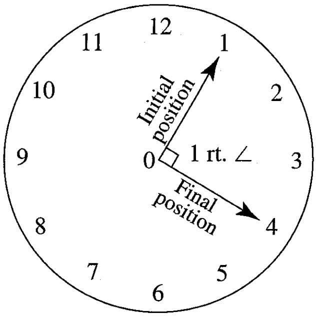


Fig. 9.14 Clock

The *number of hours* between 1 and 4 $= (4 - 1)\text{ h} = 3\text{ h}$

$\therefore$ the *number* of *right angles* $= \frac{3\cancel{h}}{3\cancel{h}} = 1$ rt. ∠.

Clockwise or Anti-clockwise

A *clockwise direction* indicates a movement in the *direction* in which the *hands* of a *conventional clock turn.* Thus:

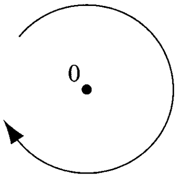


Fig. 9.15 Clockwise

indicates movement in a *clockwise direction.*

An *anti-clockwise* (or *counterclockwise*) *direction* indicates a movement in the *opposite direction* to which the *hands* of a *conventional clock turn.* Thus:

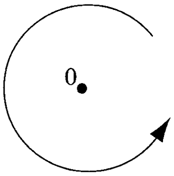


Fig. 9.16 Anti-clockwise

indicates movement in an *anti-clockwise* (or *counter clockwise*) *direction.*

The *four cardinal directions* are *north*, *south*, *east* and *west* as indicated in the diagram below:

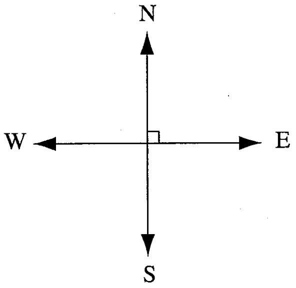


Fig. 9.17 Cardinal directions

Example 4

(a) If you stand facing east and turn clockwise through $\frac{1}{4}$ of a revolution, in which direction will you be facing?

(b) If you stand facing south and turn anti-clockwise through $\frac{3}{4}$ of a revolution, in which direction will you be facing?

Solution

(a)

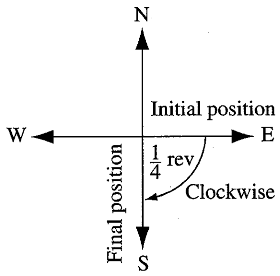


Fig. 9.18 Cardinal directions

The *direction* in which you will *facing* = south.

(b)

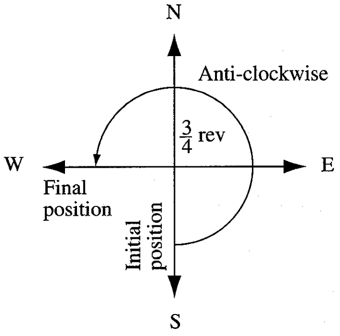


Fig. 9.19 Cardinal directions

The *direction* in which you will *facing* = west.

Example 5

How many right angles do you turn through if you:

(a) face north and turn clockwise to face west

(b) face west and turn anti-clockwise to face east?

Solution

(a)

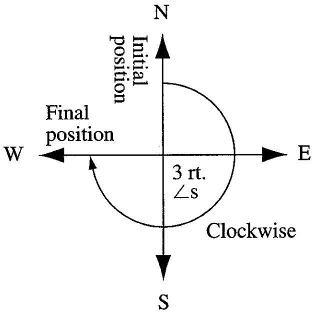


Fig. 9.20 Cardinal directions

The *number of right angles* = 3 rt. ∠s.

(b)

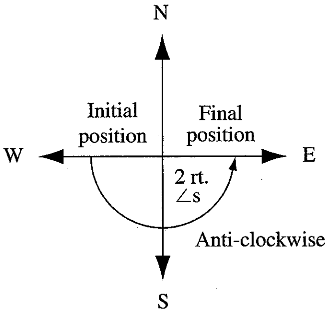


Fig. 9.21 Cardinal directions

The *number of right angles* = 2 rt. ∠s.

Degrees

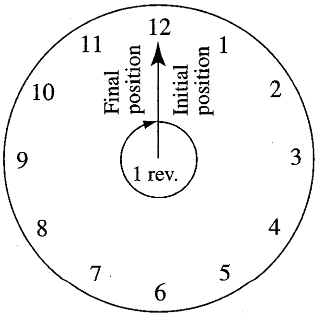


Fig. 9.22 Clock

When one of the hands of a clock makes a *complete turn* it is said to complete 1 *revolution* or describe 360 *degrees*. 360 *degrees* can be abbreviated to 360°, where the symbol ° means '*degrees*'. *Angles* are *measured* most commonly using *degrees*.

Hence $1\ rev. = 360°$

and $360° = 4$ rt. $\angle$s.

Example 6

How many degrees are there in:

(a) $\frac{5}{12}$ *of a revolution* *(b) 0.8 of a revolution*

(c) one right angle?

Solution

(a) Now 1 revolution $= 360°$

So $\frac{5}{12}$ of a *revolution* $= \frac{5}{12} \times 360°$

$= 5 \times 30° = 150°$

Hence $\frac{5}{12}$ of a revolution is 150°.

(b) Now 1 revolution $= 360°$

So 0.8 of a *revolution* $= 0.8 \times 360° = 288°$

Hence 0.8 of a revolution is 288°.

(c) Now 4 right angles $= 360°$

So 1 *right angle* $= \frac{360°}{4} = 90°$

Hence 1 right angle is 90°.

Exercise 9a

1. What fraction of a revolution does the second hand of a clock turn through when:
   (a) it starts at 12 and stops at 8
   (b) it starts at 1 and stops at 10
   (c) it starts at 9 and stops at 4?

2. What fraction of a revolution does the minute hand of a clock turn through when:
   (a) it starts at 12 and stops at 9
   (b) it starts at 5 and stops at 3?

3. What fraction of a revolution does the hour hand of a clock turn through when:
   (a) it starts at 3 and stops at 11
   (b) it starts at 5 and stops at 11?

4. What fraction of a revolution does the second hand of a clock turn through when:
   (a) it starts at 4 and stops at 9
   (b) it starts at 10 and stops at 1?

5. Where does the second hand stop if:
   (a) it starts at 12 and turns through $\frac{1}{3}$ of a revolution
   (b) it starts at 9 and turns through $\frac{3}{4}$ of a revolution?

6. Where does the minute hand stop if:
   (a) it starts at 6 and turns through $\frac{3}{4}$ of a revolution.
   (b) it starts at 12 and turns through $\frac{2}{3}$ of a revolution?

7. Where does the hour hand stop if:
   (a) it starts at 12 and turns through $\frac{3}{4}$ of a revolution.
   (b) it starts at 7 and turns through $\frac{1}{2}$ of a revolution?

8. Where does the second hand stop if:
   (a) it starts at 5 and turns through $\frac{7}{12}$ of a revolution.
   (b) it starts at 9 and turns through $\frac{1}{4}$ of a revolution?

9. How many right angles does the second hand of a clock turn through when:
   (a) it starts at 12 and stops at 9
   (b) it starts at 5 and stops at 2
   (c) it starts at 3 and stops at 3?

10. How many right angles does the minute hand of a clock turn through when:
   (a) it starts at 4 and stops at 1
   (b) it starts at 8 and stops at 2
   (c) it starts at 6 and stops at 3?

11. How many right angles does the hour hand of a clock turn through when:
   (a) it starts at 1 and stops at 7
   (b) it starts at 6 and stops at 9?

12. How many right angles does the second hand of a clock turn through when:
   (a) it starts at 4 and stops at 7
   (b) it starts at 10 and stops at 4?

13. (a) If you stand facing east and turn clockwise through $\frac{3}{4}$ of a revolution, in which direction will you be facing?

(b) If you stand facing south and turn anti-clockwise through $1\frac{1}{4}$ revolutions, in which direction will you be facing?

14. (a) If you stand facing east and turn clockwise through $\frac{1}{2}$ of a revolution, in which direction will you be facing?

(b) If you stand facing east and turn anti-clockwise through $\frac{1}{4}$ of a revolution, in which direction will you be facing?

15. (a) If you stand facing south and turn clockwise through $\frac{1}{2}$ of a revolution, in which direction will you be facing?

(b) If you stand facing north and turn anti-clockwise through $1\frac{1}{2}$ revolutions, in which direction will you be facing?

16. (a) If you stand facing south and turn clockwise through $\frac{3}{4}$ of a revolutions, in which direction will you be facing?

(b) If you stand facing west and turn anti-clockwise through $1\frac{3}{4}$ revolutions, in which direction will you be facing?

17. How many right angles do you turn through if you:
(a) face west and turn clockwise to face south
(b) face east and turn anti-clockwise to face west?

18. How many right angles do you turn through if you:
(a) face north and turn clockwise to face west.
(b) face south and turn anti-clockwise to face west?

19. How many right angles do you turn through if you:
(a) face south and turn clockwise to face west.
(b) face north and turn anti-clockwise to face south?

20. How many right angles do you turn through if you:
(a) face east and turn clockwise to face north
(b) face west and turn anti-clockwise to face south?

21. How many degrees are there in:
(a) three-quarters of a revolution
(b) 0.5 of a revolution
(c) two right angles?

22. How many degrees are there in:
(a) $\frac{5}{9}$ of a revolution
(b) 0.65 of a revolution
(c) three right angles?

23. How many degrees are there in:
(a) $\frac{7}{12}$ of a revolution
(b) 0.85 of a revolution
(c) five right angles?

24. How many degrees are there in:
(a) $\frac{8}{9}$ of a revolution
(b) 0.95 of a revolution
(c) 3.5 right angles?

25. How many degrees has the second hand of a clock turned through when it moves from:
(a) 12 to 7
(b) 3 to 8
(c) 5 to half-way between 6 and 7
(d) 2 to mid-way between 7 and 8?

Types of Angles

Right Angle

An *angle* of 90° is called a *right angle*. That is $\theta = 90^\circ$.

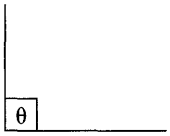


Fig. 9.23 Right angle

Straight Angle

An *angle* of 180° is called a *straight angle*. That is $\theta = 180^\circ$.

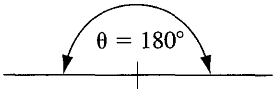


Fig. 9.24 Straight angle

Acute Angle

An *angle* is *acute* if its *magnitude* is *greater than* 0° *but less than* 90°.

That is $0° < \theta < 90°$. For example: 25° and 83° are *two acute angles.*

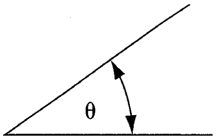


Fig. 9.25 Acute angle

Obtuse Angle

An *angle* is *obtuse* if its *magnitude* is *greater than* 90° *but less than* 180°. That is $90° < \theta < 180°$. For example: 95° and 173° are *two obtuse angles.*

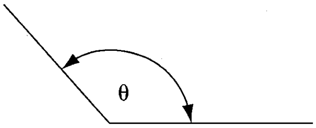


Fig. 9.26 Obtuse angle

Reflex Angle

An *angle* is *reflex* if its *magnitude* is *greater than* 180° *but less than* 360°. That is $180° < \theta < 360°$. For example: 182° and 357° are *two reflex angles.*

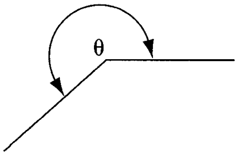


Fig. 9.27 Reflex angle

Complementary Angles

Two angles are said to be *complementary* if their *sum is equal to* 90°. In Fig. 9.28 the *angles A and B* are *complementary angles*, since $\hat{A} + \hat{B} = 90°$.

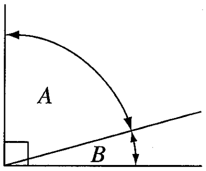


Fig. 9.28 Complementary angles

For example:
In Fig. 9.29 the angles 75° and 15° are *complementary angles*, since 75° + 15° = 90°. *Complementary angles* can be abbreviated to '*comp.* ∠*s*'.

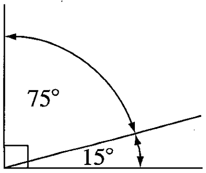


Fig. 9.29 Complementary angles

Supplementary Angles

Two angles are said to be *supplementary* if their *sum* is *equal to* 180°. The *angles A and B* are *supplementary angles*, since $\hat{A} + \hat{B} = 180°$.

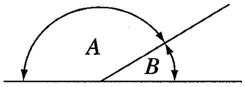


Fig. 9.30 Supplementary angles

For example:
In Fig. 9.31 the *angles* 149° *and* 31° are *supplementary angles*, since 149° + 31° = 180°. *Supplementary angles* can be abbreviated to '*supp.* ∠*s*'.

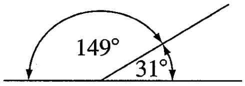


Fig. 9.31 Supplementary angles

Properties of Angles Formed by Intersecting Lines

Adjacent Angles

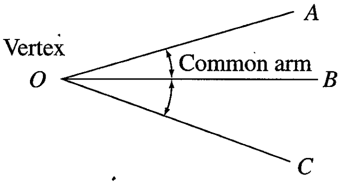


Fig. 9.32 Adjacent angles

Adjacent angles are *two angles* which have a *common vertex* and *lie* on *opposite sides* of a *common arm.*

Thus in Fig. 9.32, $A\hat{O}B$ and $B\hat{O}C$ are *adjacent angles* because:

(i) they have a *common vertex O*
(ii) they have a *common arm OB*
(iii) they *lie* on *opposite sides of OB.*

NOTE: $A\hat{O}B = B\hat{O}A$ and $B\hat{O}C = C\hat{O}B$.
This is another notation used to *denote* an *angle.* In this case *three capital letter*s are being used to *name* an *angle*, where the *middle letter* represents the *position* of the *vertex* of the *angle.*

Class Activity

(a) *Each student* is to take a *ruler* and *pencil* and *draw* a *horizontal straight line* on paper. Now *draw* a *straight line* that is *inclined* to the *left* and *intersects* the *horizontal straight* line at *O* as shown in Fig. 9.33 below.

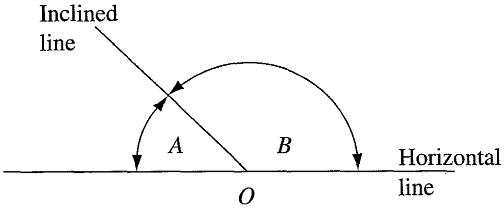


Fig. 9.33 Straight line

Denote the *adjacent angles formed* by the letters *A* and *B*. Use your *protractor* and *measure* the *adjacent angles*. Now *sum* the *two adjacent angles*. *What* do you *observe*?

(b) Draw a *horizontal straight line* on paper. Now *draw* a *straight line* that is *inclined* to the *right* and *intersects* the *horizontal straight line* at *O* as shown in Fig. 9.34 below.

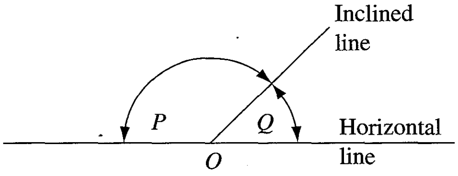


Fig. 9.34 Straight line

Denote the *adjacent angles* formed by the letters *P* and *Q*. Use your *protractor* and *measure* the *adjacent angles*. Now *sum* the two *adjacent angles*. *What* do you *observe*?

(c) *Compare* the *sum* of the *adjacent angles A* and *B*, and the *sum* of the *adjacent angles P and Q*. *What* do you *observe*?

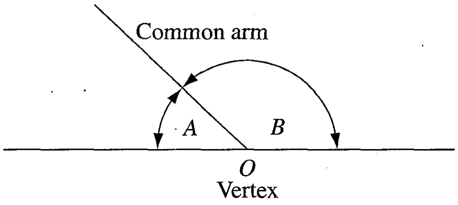


Fig. 9.35 Adjacent angles

THEORY: We say that the *sum of the adjacent angles on a straight line is equal to* 180°, since $\hat{A} + \hat{B} = 180°$.

Adjacent angles can be abbreviated to 'adj. ∠s'.

Vertically Opposite Angles

Class Activity

(a) *Each student* is to take a *ruler* and *pencil* and *draw two lines* to *intersect* on paper. *Denote* the *angles formed* by the *letters*, as shown in Fig. 9.36 below.

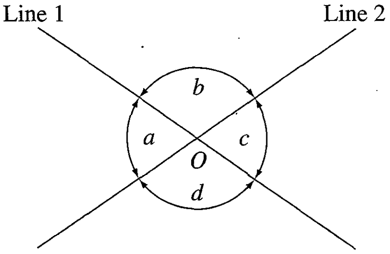


Fig. 9.36 Intersecting lines

Now use your *protractor* to *measure* the *four angles* at the *point of intersection O*.

(b) *Sum angles a* and *b*. *Sum angles b* and *c*. *Sum angles c* and *d*. *Sum angles a* and *d*. *What* do you *observe*?

(c) Now *compare angles a* and *c*. Also *compare angles b* and *d*. *What* do you *observe*?

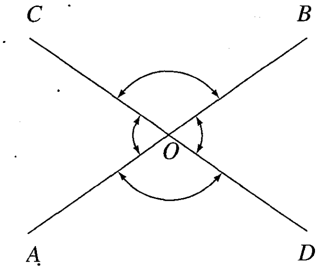


Fig. 9.37 Vertically opposite angles

THEORY: When *two straight lines intersect* at a *point*, *vertically opposite angles* are formed. Thus in Fig. 9.37:

The *straight lines AB and CD intersect at O* and *two pairs* of *vertically opposite angles* are formed. $A\hat{O}C$ and $B\hat{O}D$ are *vertically opposite angles*. $A\hat{O}D$ and $B\hat{O}C$ are *vertically opposite angles*.

Further, *vertically opposite angles* are always *equal*. Thus: $A\hat{O}C = B\hat{O}D$ (*vertically opposite angles*) $A\hat{O}D = B\hat{O}C$ (*vertically opposite angles*).

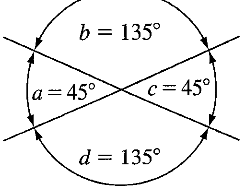
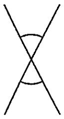
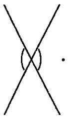


Fig. 9.38 Vertically opposite angles

For example in Fig. 9.38.

$\hat{a} = \hat{c} = 45°$ (*vertically opposite angles*)
and
$\hat{b} = \hat{d} = 135°$ (*vertically opposite angles*).

Vertically opposite angles can be abbreviated to '*vert. opp.* $\angle s$'.

Vertically opposite angles are a pair of angles formed by a  or  .

Corresponding Angles

Class Activity

(a) *Each student* is to take a *ruler* and *pencil* and *draw two parallel lines* on paper (the *lines* on the *page* of an exercise book page are *parallel*). *Draw* the *parallel lines* about 12 *lines apart* on an exercise book page. Now *draw a straight line* that is *inclined* to the *left* and *intersects* the *parallel lines* at *two points* as shown in Fig. 9.39 below. Such a *line* is called a *transversal*.

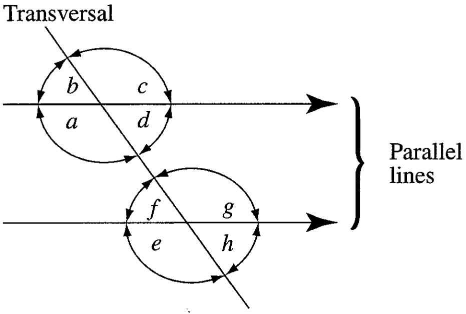


Fig. 9.39 Parallel lines and transversal

Denote the *angles formed* by the *letters* as shown in Fig. 9.39 above.

Now use your *protractor* to *measure* the *eight angles* at the *two points of intersection.*

(b) *Compare angles a* and *c*; *angles b* and *d*; *angles e* and *g*; and *angles f* and *h*. *What* do you *observe*?

(c) *Compare angles a* and *e*; *angles b* and *f*; *angles c* and *g*; and *angles d* and *h*. *What* do you *observe*?

(d) *Draw two parallel lines* with a *transversal inclined* to the *right* as shown in Fig. 9.40. *Denote* the *angles formed* by the *letters* as shown in Fig. 9.40.

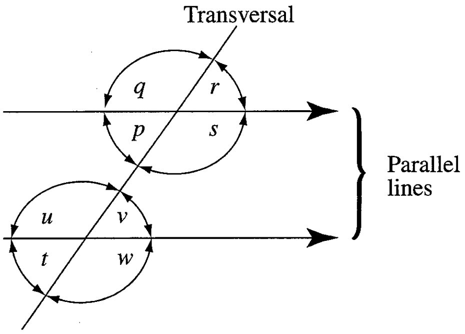


Fig. 9.40 Parallel lines and transversal

(e) *Compare angles p* and *r*; *angles q* and *s*; *angles t* and *v*; and *angles u* and *w*. *What* do you *observe*?

(f) *Compare angles p* and *t*; *angles q* and *u*; *angles r* and *v*; and *angles s* and *w*. *What* do you *observe*?

THEORY: When a *transversal cuts two parallel lines* then the *corresponding angles* formed are always *equal.*

Corresponding angles are *angles* that are in *corresponding positions.*

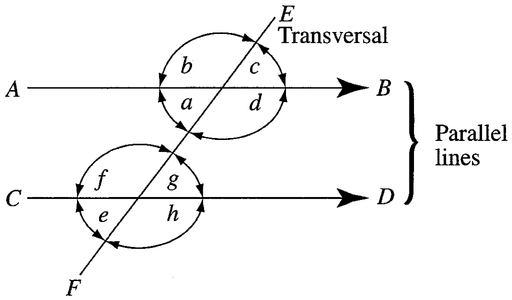


Fig. 9.41 Parallel lines and transversal

Thus in Fig. 9.41:

The *parallel lines AB* and *CD* are *cut* by the *transversal EF*. The *arrows* indicate that the *lines AB* and *CD* are *parallel.*

Hence:

$\hat{a} = \hat{e}$ (*corresponding angles*—two bottom left positions)

$\hat{b} = f$ (*corresponding angles*—two top left positions)

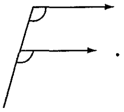
$\hat{c} = \hat{g}$ (*corresponding angles*—two top right positions)

$\hat{d} = \hat{h}$ (*corresponding angles*—two bottom right positions)

Corresponding angles can be abbreviated to '*corres.* ∠*s*'.

AB is parallel to CD can be abbreviated to '*AB // CD*', where the symbol // means '*is parallel to*'.

Corresponding angles are a pair of angles formed by a



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NOTE: $\hat{a} = \hat{c}$, $\hat{b} = \hat{d}$, $\hat{e} = \hat{g}$ and $\hat{f} = \hat{h}$ (*vertically opposite angles*).

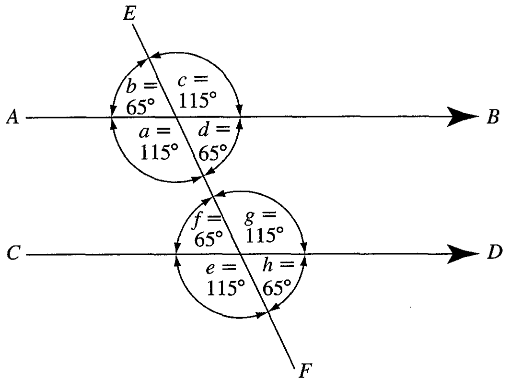


Fig. 9.42 Parallel lines and transversal

For example in Fig. 9.42:

$\hat{a} = \hat{e} = 115°$ (*corresponding angles*)

$\hat{b} = \hat{f} = 65°$ (*corresponding angles*)

$\hat{c} = \hat{g} = 115°$ (*corresponding angles*)

$\hat{d} = \hat{h} = 65°$ (*corresponding angles*).

And $\hat{a} = \hat{c} = 115°$ (*vertically opposite angles*)

$\hat{b} = \hat{d} = 65°$ (*vertically opposite angles*)

$\hat{e} = \hat{g} = 115°$ (*vertically opposite angles*)

$\hat{f} = \hat{h} = 65°$ (*vertically opposite angles*).

Alternate Angles

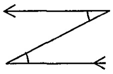
Class Activity

(a) Using your Fig. 9.39, *compare angles a* and *g*; and *angles d* and *f*. *What* do you *observe*?

(b) Using your Fig. 9.40, *compare angles p* and *v*, and *angles s* and *u*. *What* do you *observe*?

THEORY: When a *transversal cuts two parallel lines*, then the *alternate angles* formed are always *equal*.

Alternate angles are a pair of angles *enclosed* by a



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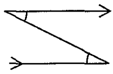

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Thus in Fig. 9.41:

$\hat{a} = \hat{g}$ (*alternate angles*)

$\hat{d} = \hat{f}$ (*alternate angles*).

For example in Fig. 9.42:

$\hat{a} = \hat{g} = 115°$ *(alternate angles)*

$\hat{d} = \hat{f} = 65°$ *(alternate angles)*.

Alternate angles can be abbreviated to '*alt.* ∠*s*'.

Interior Angles

Class Activity

(a) Using your Fig. 9.39, *sum angles a* and *f*; and *angles d* and *g*. *What* do you *observe*?

(b) Using your Fig. 9.40, *sum angles p* and *u*; and *angles s* and *v*. *What* do you *observe*?

THEORY: When a *transversal cuts two parallel lines*, then the *interior angles* on the *same side* of the *transversal* are *supplementary*.

Thus in Fig. 9.41:

$\hat{a} + \hat{f} = 180°$ (*interior angles*)

$\hat{d} + \hat{g} = 180°$ (*interior angles*).

For example in Fig. 9.42:

$\hat{a} + \hat{f} = 115° + 65°$
$= 180°$ (*interior angles*)

$\hat{d} + \hat{g} = 65° + 115°$
$= 180°$ (*interior angles*).

Interior angles can be abbreviated to '*int.* ∠*s*'.

Interior angles (referred to here) are a pair of angles enclosed by a

.

From the above *rules*, we can *conclude* that when a *transversal cuts two parallel lines*:

(i) the *corresponding angles* are *equal*
(ii) the *alternate angles* are *equal*
(iii) the *vertically opposite angles* are *equal*
(iv) the *interior angles* on the *same side* of the *transversal* are *supplementary*.

Two lines in a *plane* are *parallel* if they are *cut* by a *transversal* in such a way that:

(i) the *corresponding angles* are *equal*, or

(ii) the *alternate angles* are *equal*, or

(iii) the *interior angles* on the *same side* of the *transversal* are *supplementary*.

Angles at a Point

Class Activity

(a) Using your Fig. 9.39, *sum* the following *angles*: a, b, c and d; and e, f, g and h. *What* do you *observe*?

(b) Using your Fig. 9.40, *sum* the following *angles*: p, q, r and s; and t, u, v and w. *What* do you *observe*?

THEORY: The *sum* of the *angles* at a *point* is *equal* to 360°.

Thus in Fig. 9.43:

$\hat{a} + \hat{b} + \hat{c} + \hat{d} + \hat{e} = 360°$ (*sum* of the *angles at a point*).

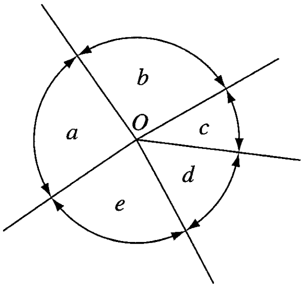


Fig. 9.43 Angles at a point

For example in Fig. 9.44:

$$\hat{a} + \hat{b} + \hat{c} + \hat{d} + \hat{e}$$
$$= 90° + 95° + 37° + 55° + 83°$$
$$= 360°$$ (*sum* of the *angles at a point*)

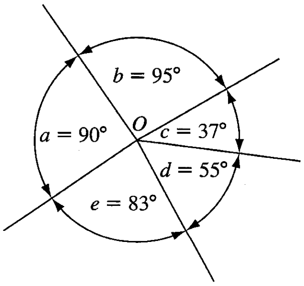


Fig. 9.44 Angles at a point

Example 7

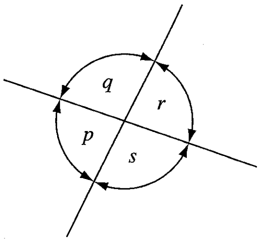


Fig. 9.45 Angles

In Fig. 9.45, angle p is 97°. Calculate the magnitude of angles q, r and s giving a reason for each of your answers.

Solution

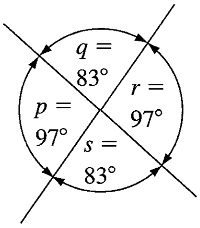


Fig. 9.45 Angles

Given that $\hat{p} = 97°$

Then $\hat{r} = \hat{p} = 97°$ (*vert. opp.* ∠*s*)

Now $\hat{p} + \hat{q} = 180°$ (∠*s on a st. line*)

So $97° + \hat{q} = 180°$

i.e. $\hat{q} = 180° - 97°$

∴ $\hat{q} = 83°$

Now $\hat{s} = \hat{q} = 83°$ (*vert opp.* ∠*s*)

Hence $\hat{q} = 83°$, $\hat{r} = 97°$ and $\hat{s} = 83°$.

Example 8

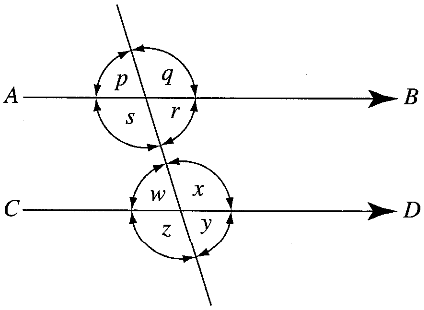


Fig. 9.46 Angles

In Fig. 9.46, AB is parallel to CD and $\hat{w} = 73^\circ$. Determine the size of each angle marked with a letter, giving reasons for your answers.

Solution

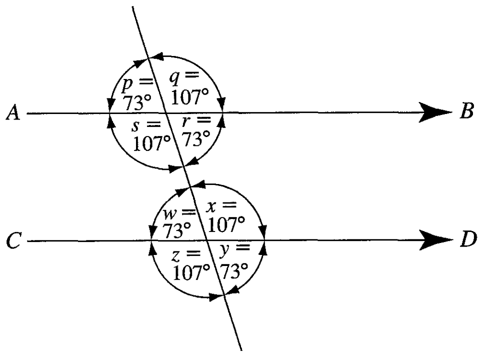


Fig. 9.46 Angles

Given that $\hat{w} = 73^\circ$

Then $\hat{y} = \hat{w} = 73^\circ$ (*vert. opp.* $\angle$*s*)

Now $\hat{x} + \hat{y} = 180^\circ$ ($\angle$*s on a st. line*)

So $\hat{x} + 73^\circ = 180^\circ$

i.e $\hat{x} = 180^\circ - 73^\circ$

$\therefore$ $\hat{x} = 107^\circ$

Now $\hat{z} = \hat{x} = 107^\circ$ (*vert opp.* $\angle$*s*)

Now $\hat{r} = \hat{w} = 73^\circ$ (*alt.* $\angle$*s*)

Now $\hat{s} = \hat{x} = 107^\circ$ (*alt.* $\angle$*s*)

Now $\hat{p} = \hat{w} = 73^\circ$ (*corres.* $\angle$*s*)

Now $\hat{q} = \hat{x} = 107^\circ$ (*corres.* $\angle$*s*)

Hence $\hat{p} = \hat{r} = \hat{y} = 73^\circ$ and
$\hat{q} = \hat{s} = \hat{x} = \hat{z} = 107^\circ$.

Example 9

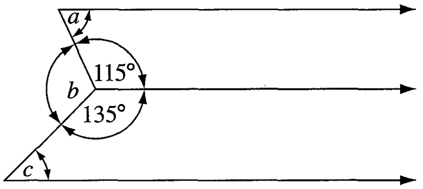


Fig. 9.47 Angles

Calculate the size of each marked angle in Fig. 9.47.

Solution

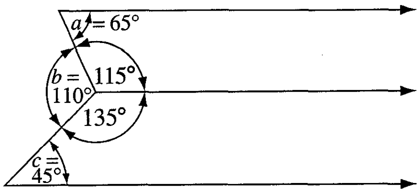


Fig. 9.47 Angles

Now $\hat{a} + 115^\circ = 180^\circ$ (*int.* $\angle$*s*)

So $\hat{a} = 180^\circ - 115^\circ$

i.e. $\hat{a} = 65^\circ$

Now $\hat{c} + 135^\circ = 180^\circ$ (*int.* $\angle$*s*)

So $\hat{c} = 180^\circ - 135^\circ$

i.e. $\hat{c} = 45^\circ$

Now $\hat{b} + 115^\circ + 135^\circ = 360^\circ$ ($\angle$*s at a pt.*)

So $\hat{b} + 250^\circ = 360^\circ$

i.e. $\hat{b} = 360^\circ - 250^\circ$

$\therefore$ $\hat{b} = 110^\circ$

Hence $\hat{a} = 65^\circ$, $\hat{b} = 110^\circ$ and $\hat{c} = 45^\circ$.

NOTE: There are many other methods of solving the Geometry problems given above.

Example 10

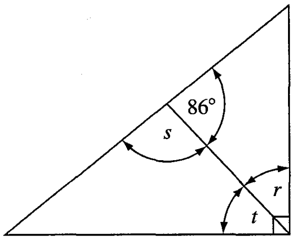


Fig. 9.48 Angles

Angle s is twice angle t. Evaluate angles r, s and t.

Solution

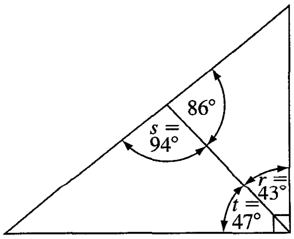


Fig. 9.48 Angles

Now $\hat{s} + 86^\circ = 180^\circ$ ($\angle s$ *on a st. line*)

So $\hat{s} = 180^\circ - 86^\circ$

i.e. $\hat{s} = 94^\circ$

Given that $\hat{s} = 2\hat{t}$

Then $\hat{t} = \frac{\hat{s}}{2} = \frac{94^\circ}{2} = 47^\circ$

Now $\hat{r} + \hat{t} = 90^\circ$ (*comp.* $\angle s$)

So $\hat{r} + 47^\circ = 90^\circ$

i.e. $\hat{r} = 90^\circ - 47^\circ$

$\therefore$ $\hat{r} = 43^\circ$

Hence $\hat{r} = 43^\circ$, $\hat{s} = 94^\circ$ and $\hat{t} = 47^\circ$.

Example 11

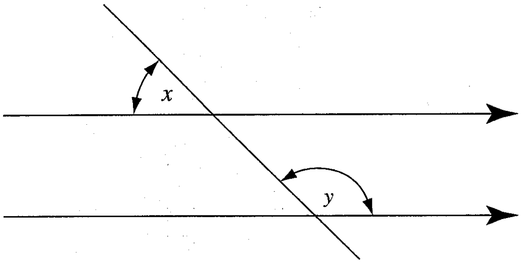


Fig. 9.49 Angles

Angle y is thrice angle x. Form an equation and solve for x.

Solution

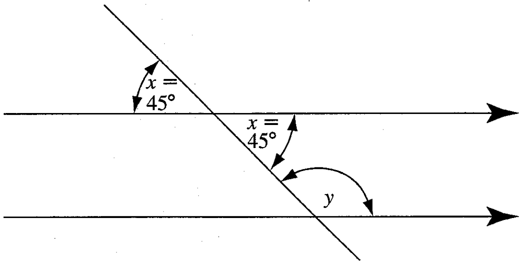


Fig. 9.49 Angles

Given that $\hat{y} = 3\hat{x}$

Then $\hat{x} + \hat{y} = 180^\circ$ (*int.* $\angle s$)

So $\hat{x} + 3\hat{x} = 180^\circ$

i.e. $4\hat{x} = 180^\circ$

$\therefore$ $\hat{x} = \frac{180^\circ}{4}$

$\Rightarrow$ $\hat{x} = 45^\circ$

Hence $\hat{x} = 45^\circ$.

Exercise 9b

1. What type of angle is each of the following:

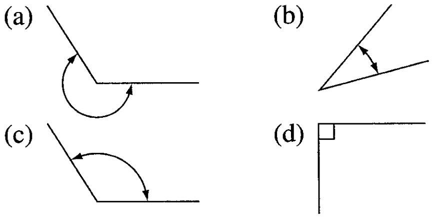


Fig. 9.50 Angles

2. State the name of each of the following angles:

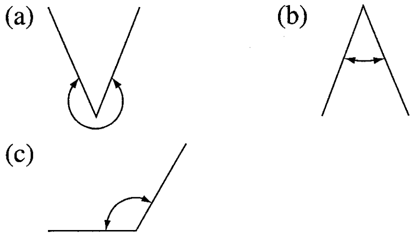


Fig. 9.51 Angles

3. What type of angle is each of the following:

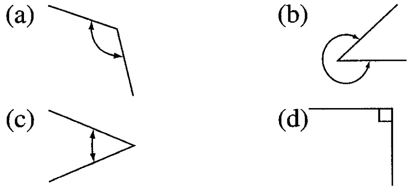


Fig. 9.52 Angles

4. State the name of each of the following angles:

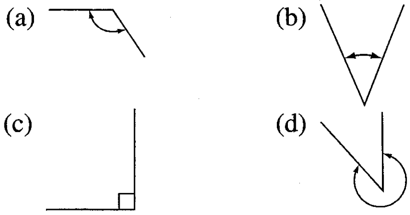


Fig. 9.53 Angles

5. What type of angle is each of the following?

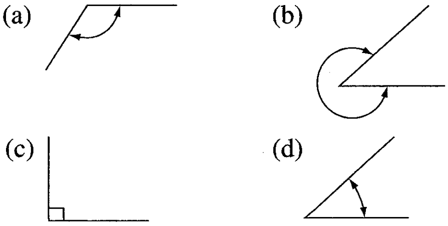


Fig. 9.54 Angles

6. Calculate the magnitude of each of the following marked angles, giving reasons for your answers.

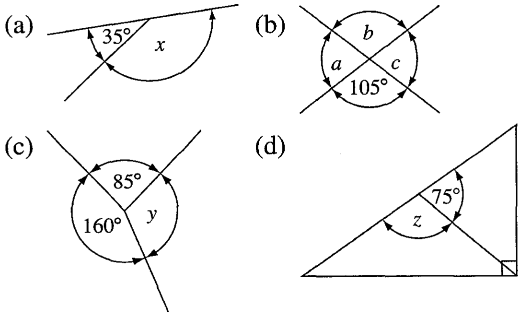


Fig. 9.55 Angles

7. Determine the size of each of the following unknown angles, giving reasons for your answers.

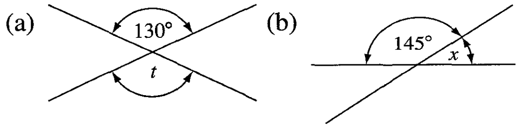


Fig. 9.56 Angles

8. Calculate the size of each of the following unknown angles, giving reasons for your answers.

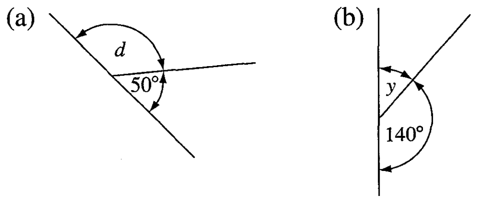


Fig. 9.57 Angles

9. Evaluate each of the following unknown angles, giving reasons for your answers.

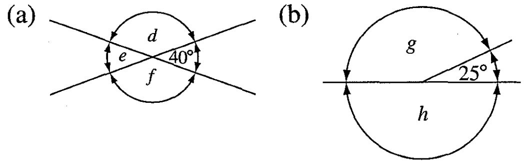


Fig. 9.58 Angles

10. Determine each of the following unknown angles, giving reasons for your answers.

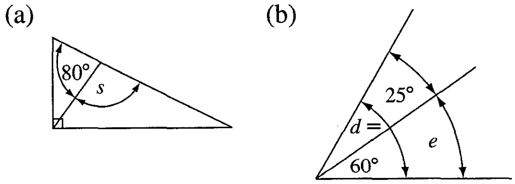


Fig. 9.59 Angles

11. Calculate the magnitude of each of the following marked angles, giving reasons for your answers.

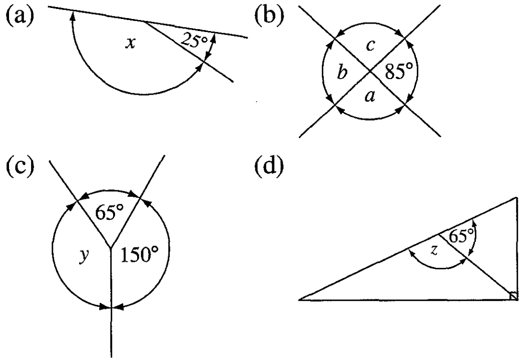


Fig. 9.60 Angles

12.

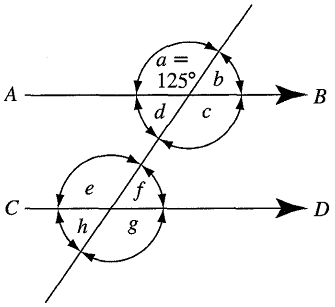


Fig. 9.61 Angles

The diagram above shows two parallel straight lines *AB* and *CD* which are cut by a transversal *EF*. Given that $\hat{a} = 125°$, calculate the angles b, c, d, e, f, g and h. Give a reason for each of your answers.

13. Calculate the size of each of the following marked angles, giving reasons for your answers.

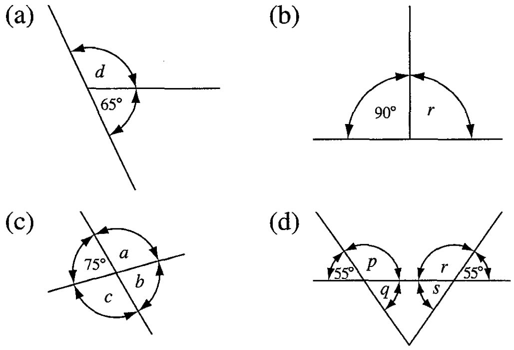


Fig. 9.62 Angles

14. Determine the size of each of the following marked angles, giving reasons for your answers.

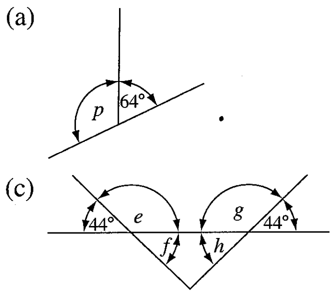


Fig. 9.63 Angles

15. Calculate the size of each marked angle, giving reasons for your answers.

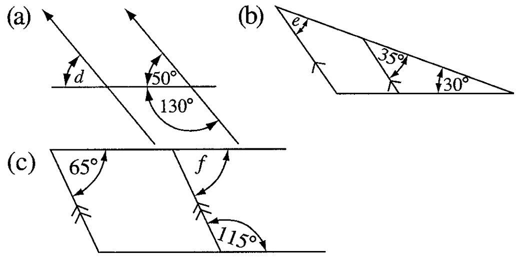


Fig. 9.64 Angles

16. Determine the size of each marked angle, giving reasons for your answers.

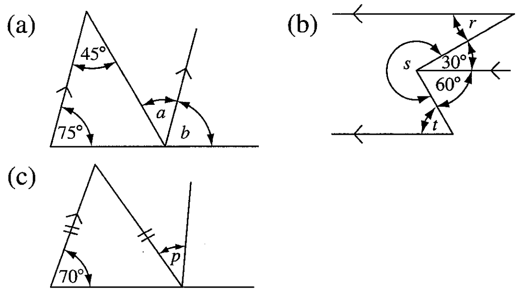


Fig. 9.65 Angles

17. Evaluate the size of each marked angle, giving reasons for your answers.

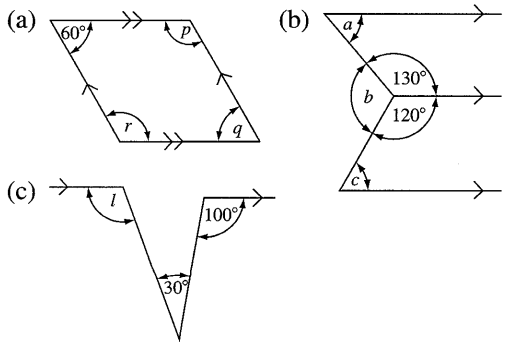


Fig. 9.66 Angles

18. Calculate the size of each of the following unknown angles, giving reasons for your answers.

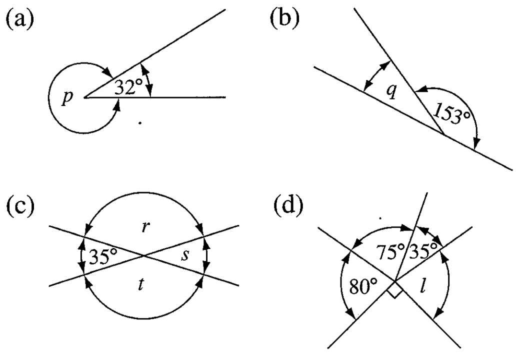
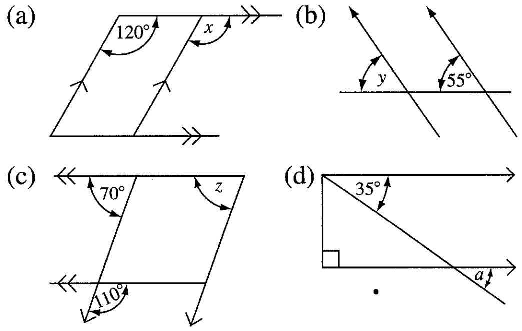


Fig. 9.67 Angles

19. State the magnitude of each of the marked angles in each of the following diagrams:



State a reason for each of your answers.

Fig. 9.68 Angles

20. Calculate the size of each marked angle, giving reasons for your answers.

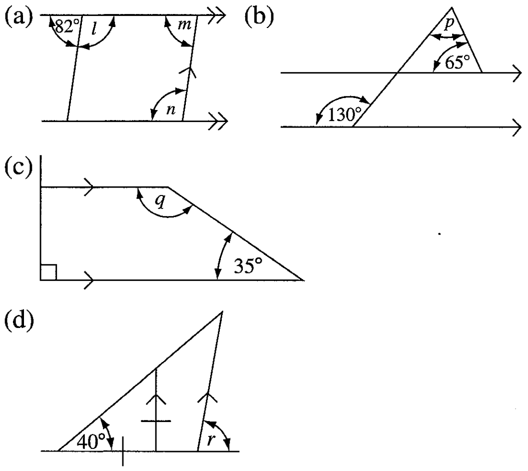


Fig. 9.69 Angles

21. Determine the size of each marked angle, giving a reason for each of your answers.

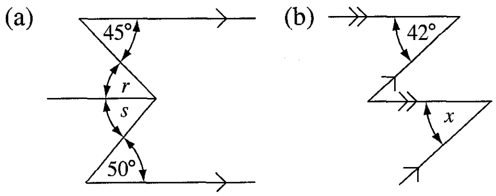


Fig. 9.70 Angles

22. State the size of each marked angle, giving a reason for each of your answers.

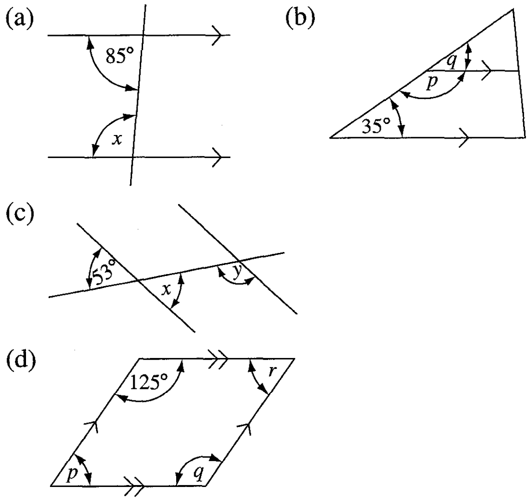


Fig. 9.71 Angles

23. Evaluate each marked angle, giving a reason for each of your answers.

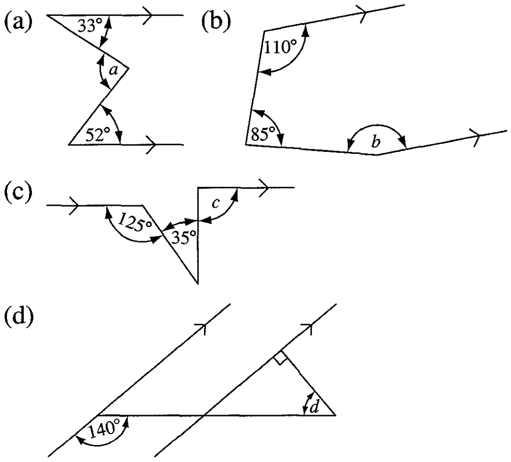


Fig. 9.72 Angles

24. Calculate the size of each marked angle, giving a reason for each answer.

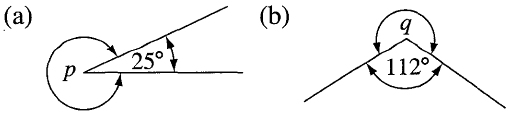
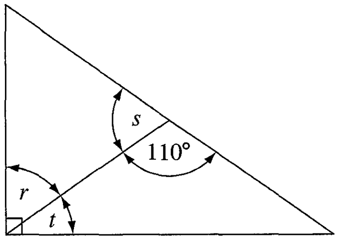


Fig. 9.73 Angles

25. (a) Angle *s* is twice angle *t*. Determine the size of angles *r*, *s* and *t*.



(b) Evaluate the angles marked, *p*, *q*, *r* and *s*.

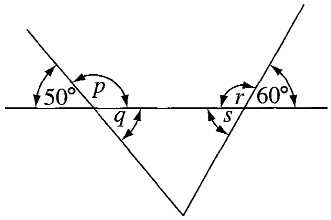
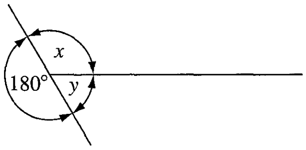


Fig. 9.74 Angles

26. (a)



The angle marked *x* is twice the angle marked *y*. Evaluate angles *x* and *y*.

(b)

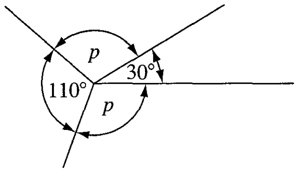


Fig. 9.75 Angles

Determine the magnitude of angle *p*.

27.

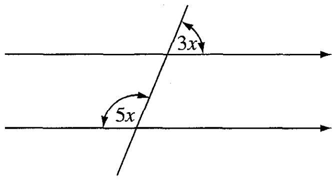


Fig. 9.76 Angles

Form an equation in *x* and solve.

28.

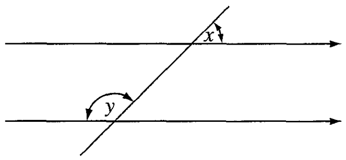


Fig. 9.77 Angles

Angle *y* is thrice angle *x*. Form an equation and determine the value of *x*.

29.

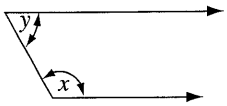


Fig. 9.78 Angles

Angle x is twice angle y. Form an equation and determine the value of y.

30.

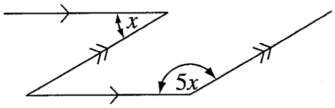


Fig. 9.79 Angles

Form an equation and solve for x.

Measuring Angles

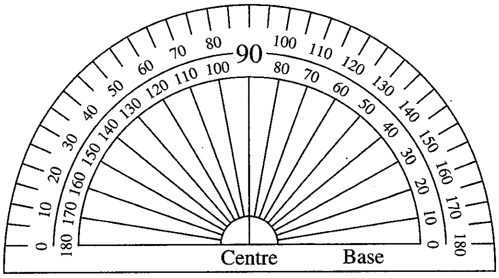
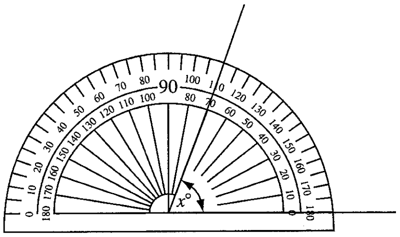


Fig. 9.80 Protractor

A *protractor* is an *instrument* used for *measuring angle* in *degrees*. Fig. 9.80 shows a *protractor* in a diagrammatic form. On your *protractor* there is a *scale* going *clockwise* from 0° *to* 180° (i.e. from *left to right*). And there is another *scale* going *anti-clockwise* from 0° *to* 180° (i.e. from *right to left*). Using *either* of these *scales* we can *measure angles* from 0° *to* 180°. *Angles* between 180° and 360° are obtained by *deduction* as will be explained later on.

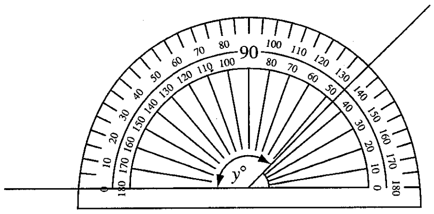
In order to *measure* an *angle*, the *centre of the protractor* is *placed* at the *vertex* of the *angle*. And the *horizontal base is placed* along an *arm of the angle* (this corresponds to the 0° *position*). The *magnitude of the angle* is then *read off* the *protractor* using the *correct scale* (i.e. *the scale* that goes from 0° to $x°$). This method is illustrated in Fig. 9.81 below.

(a)



Measuring an acute angle
The *acute angle* $x = 70^\circ$.

(b)



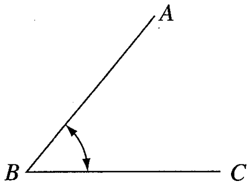
Measuring an obtuse angle
The *obtuse angle* $y = 135^\circ$.

Fig. 9.81

Example 12

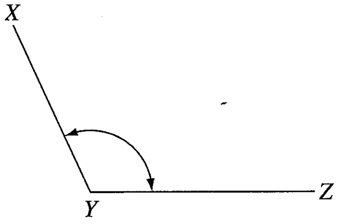
Use your protractor to measure each of the following angles:

(a)



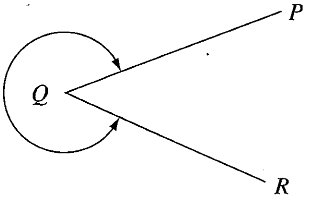
Acute angle

(b)



Obtuse angle

(c)



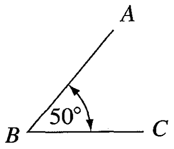
Reflex angle

Fig. 9.82

Solution

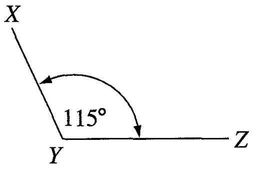
By *measurement*:

(a) *Acute angle ABC* = 50°.



Acute angle

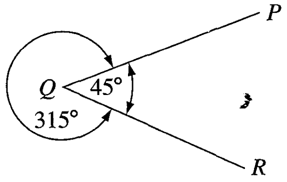
(b) *Obtuse angle XYZ* = 115°.



Obtuse angle

(c) *Acute angle PQR* = 45°.

So *reflex angle PQR* = 360° − 45° = 315°.



Reflex angle

Fig. 9.82

Drawing Angles

The *protractor* can also be used to *draw angles* between 0° and 180° *directly* or *angles* between 180° *and* 360° *indirectly.*

Example 13

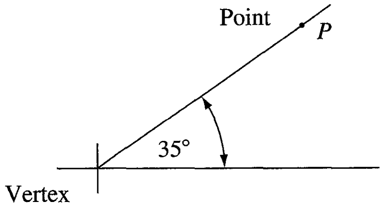
Use your protractor to draw each of the following angles accurately:

(a) 35° *(b) 127°* *(c) 300°*

Solution

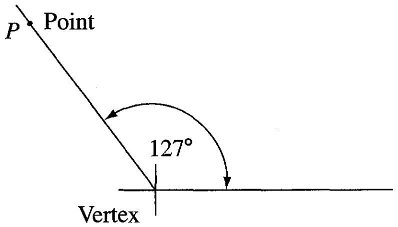
The *procedure* is to *draw a straight line* to *act* as an *arm* of the *angle* and *mark off* a *point* on the *line* to *act* as the *vertex* of the *angle*. The *protractor* is then *set up* as *usual* and the *angle* found by *measurement*. A *point P* is then *placed above* the *protractor* in line with the *measured angle*. A *line* is then *drawn* through the *point* and the *vertex* and the *angle is completed.*

(a)



Acute angle

(b)



Obtuse angle

(c) Now 360° − 300° = 60° (*acute angle*). So we *draw* an *acute angle* of *magnitude* 60° in order to obtain the *reflex angle of* 300°.

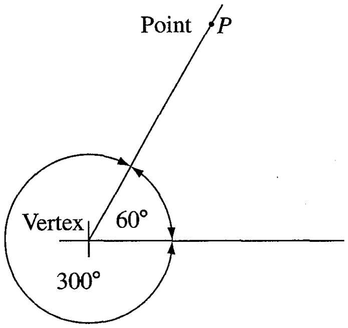


Fig. 9.83 Reflex angle

Parallel Lines

Parallel lines are *lines* that *never meet* no matter how *far* they are *extended*. *Parallel lines* are *always* the *same distance apart* and they will *never intersect.* For example: A pair of railway lines.

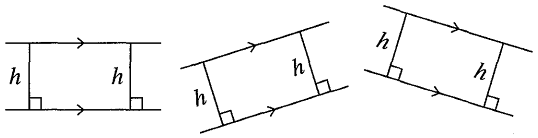


Fig. 9.84 Parallel lines

The diagram above shows *three pairs of parallel lines*. *Each pair* of *parallel lines* is *h units* apart. The two *arrows* indicate that a *pair* of *lines* are *parallel.*

Drawing Parallel Lines

One way of *drawing parallel lines* is by *sliding* a *set-square* along a *ruler* as shown in Fig. 9.85 below.

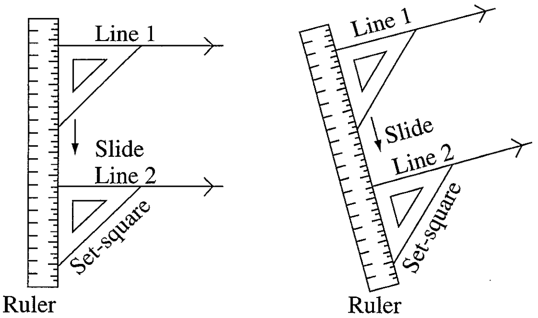


Fig. 9.85 Parallel lines

Two lines are *drawn* using the *upper edge* of the *set square*. These *lines* will be *parallel* to each other.

Perpendicular Lines

Two lines are said to be *perpendicular* if they *meet* (or *cross* or *intersect*) at *right angles* (i.e. 90°).

For example: The *x-axis* and the *y-axis* used to draw graphs are *perpendicular.*

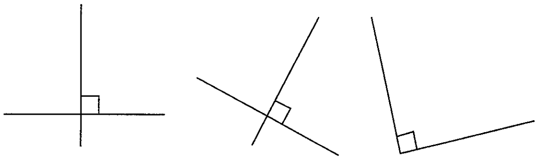
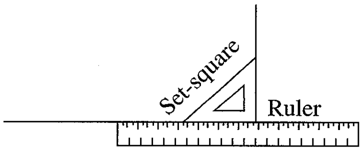


Fig. 9.86 Perpendicular lines

The diagram above shows *three pairs* of *perpendicular lines. Each pair* of *lines intersect* at *right-angles.* The symbol ⊥ indicates that *a pair* of *lines* are *perpendicular.*

Drawing Perpendicular Lines

One way of *drawing perpendicular lines* is by using a *ruler* and *set square* as shown in Fig. 9.87 below.



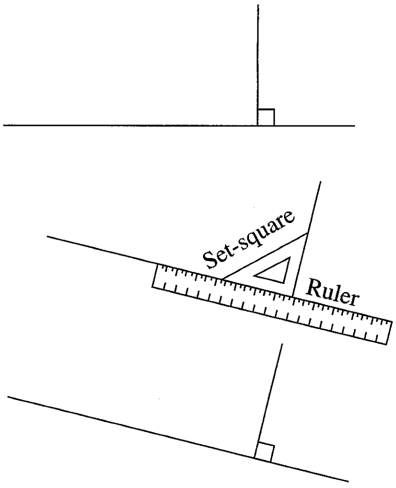


Fig. 9.87 Perpendicular lines

Two lines are drawn using the *upper edge* of the *ruler* and the *side* of the *set square* that is *perpendicular* to the *ruler.*

Constructing a Line Segment

To *construct* a *line segment* of a *given length* we use a *ruler* and *compasses.*

Example 14

Construct a line segment AB of length 6.5 cm.

Solution

Construction

First *draw* a *straight line greater* than 6.5 cm. Then *mark off* the *point A* to the *left* of the *line. Open* your *compasses* to an *opening* of 6.5 cm. Using *A* as *centre, construct* an *arc* to *intersect* the *straight line* at *B*. Then *AB is* the required *line segment of length* 6.5 cm.

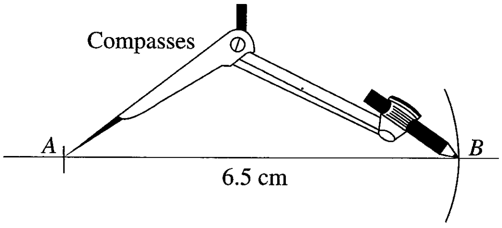


Fig. 9.88 Line segment

NOTE: We *measure line segments* using a *divider* as *illustrated* in Fig. 9.89. We *open* our *divider* from one *endpoint* to another (e.g. from *A to B*) and then *measure* the *separation of the divider* using a *ruler*. The *length of the line segment* is *equal* to the *value* of the *separation of the divider.*

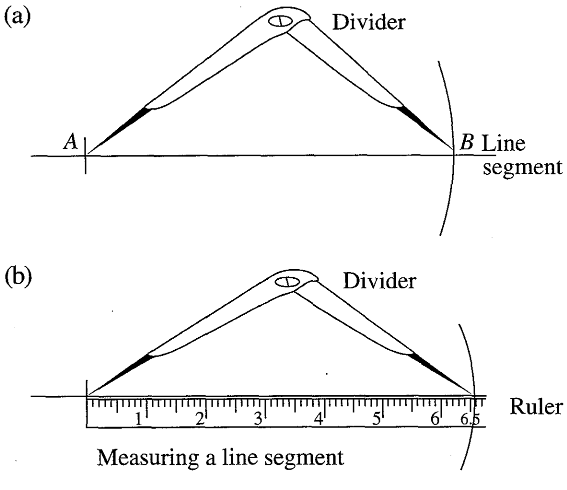


Fig. 9.89 Measuring a line segment

Bisecting a Line Segment

In order to *bisect* a *line segment* we need to *construct* its *perpendicular bisector*. The *perpendicular bisector* of a *line segment cuts* it at *right angles* and *divides* the *line segment* into *two halves.*

Example 15

Bisect a line segment PQ of length 5.6 cm.

Solution

Construction

First *construct* the *line segment PQ* of length 5.6 cm. Then *open* your *compasses* to *more* than *half the length of PQ*. With *P and Q as centres*, *construct two pairs of arcs* to *intersect above* and *below* the *line segment* at *A* and *B*, respectively. Now *draw* a *straight line* passing *through* the *points A and B* to *intersect PQ at X. AB* is the *perpendicular bisector* of the *line segment PQ*. By *measurement*, $PX = QX = 2.8$ cm and $P\hat{X}A = Q\hat{X}A = 90°$.

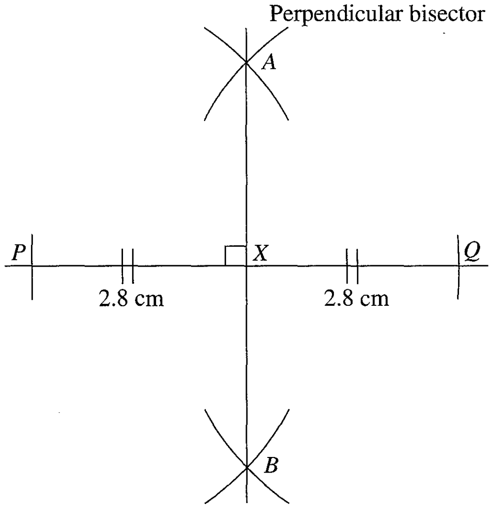


Fig. 9.90 Bisecting a line segment

Bisecting an Angle

The *bisector* of an *angle divides* the *angle* into *two halves.*

Example 16

Bisect angle BAC of magnitude 60°.

Solution

Construction

Open your *compasses* to a suitable *separation* and with *A* as *centre*, *construct* an *arc* to *intersect* the *arms AB* and *AC* at *X* and *Y*, respectively. Then using *X* and *Y* as *centres*, *construct two arcs* to *intersect above XY* at *Z*. Now *draw* a *straight line* passing *through* the *points A and Z. AZ* is the *bisector* of the *angle BAC*. By *measurement*, $B\hat{A}Z = C\hat{A}Z = 30°$.

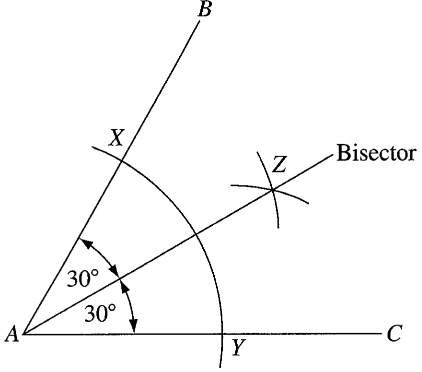


Fig. 9.91 Bisecting an angle

Exercise 9c

1. Use your protractor to measure each of the following angles:

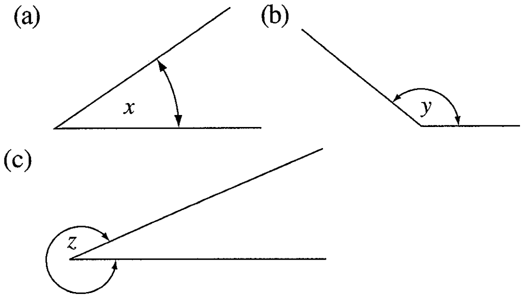


Fig. 9.92 Angles

2. Use your protractor to measure each of the following angles correct to the nearest degree:

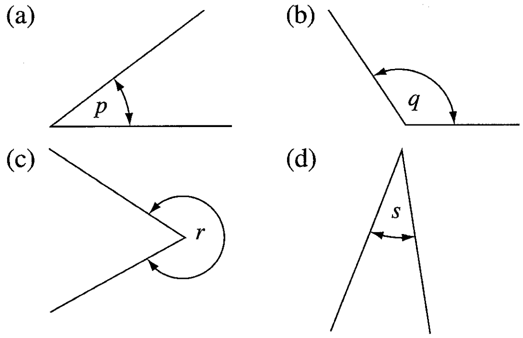
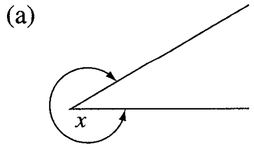


Fig. 9.93 Angles

3. Determine the magnitude of each of the following angles:



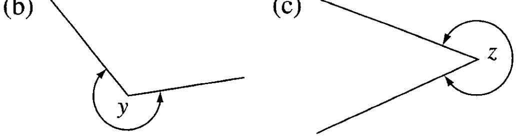


Fig. 9.94 Angles

4. Determine the size of each of the following angles to the nearest degree:

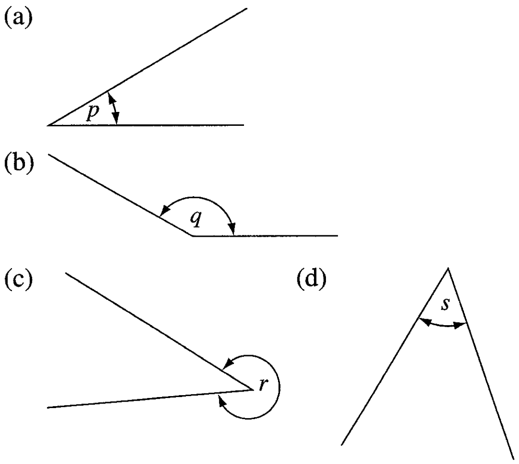


Fig. 9.95 Angles

5. Determine the magnitude of each of the following angles:

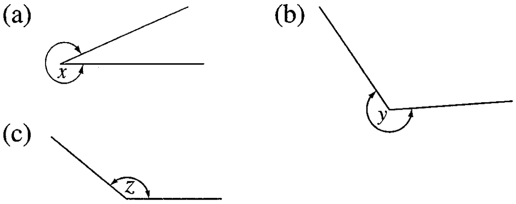


Fig. 9.96 Angles

6. Determine the size of each of the following angles:

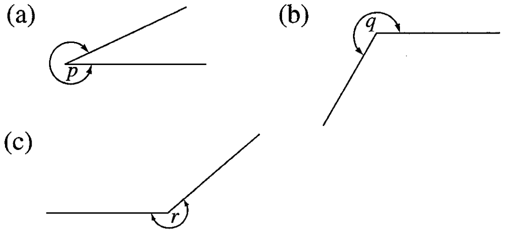


Fig. 9.97 Angles

7. Use your protractor to draw each of the following angles:
(a) 30° (b) 150° (c) 290°

8. Use your protractor to draw each of the following angles accurately:
(a) 45° (b) 145° (c) 225°

9. Use your protractor to draw each of the following angles accurately:
(a) 54° (b) 137° (c) 274°

10. Use your protractor to draw each of the following angles accurately:
(a) 29° (b) 164° (c) 128°

11. Use your protractor to draw each of the following angles accurately:
(a) 67° (b) 129° (c) 316°

12. Use your protractor to draw each of the following angles accurately:
(a) 83° (b) 178° (c) 293°

13. Use your protractor to draw each of the following angles accurately:
(a) 70.5° (b) 160.5° (c) 290.5°

14. Use your protractor to draw each of the following angles accurately:
(a) 35.5° (b) 135.5° (c) 285.5°

15. Use your protractor to draw each of the following angles accurately:
(a) 68.5° (b) 147.5° (c) 312.5°

16. Draw two parallel lines.

17. Draw three parallel lines.

18. Draw four parallel lines.

19. Draw five parallel lines.

20. Draw six parallel lines.

21. Draw two perpendicular lines.

22. Draw two perpendicular lines that are inclined differently from those above.

23. Draw two perpendicular lines that are inclined differently from those above.

24. Draw two perpendicular lines that are inclined differently from those above.

25. Draw two perpendicular lines that are inclined differently from those above.

26. Construct a line segment *AB* of length 7 cm.

27. Construct a line segment *PQ* of length 8 cm.

28. Construct a line segment *LM* of length 9 cm.

29. Construct a line segment *RS* of length 10.5 cm.

30. Construct a line segment *WX* of length 11.5 cm.

31. Construct a line segment *YZ* of length 12.5 cm.

32. Construct a line segment *CD* of length 7.6 cm.

33. Construct a line segment *UV* of length 8.4 cm.

34. Construct a line segment *XY* of length 10.7 cm.

35. Bisect a line segment *AB* of length 8 cm. Measure and state the distance between the end points and the perpendicular bisector.

36. Bisect a line segment *CD* of length 10 cm. Measure and state the distance between the end points and the perpendicular bisector.

37. Bisect a line segment *EF* of length 12 cm. Measure and state the distance between the end points and the perpendicular bisector.

38. Bisect a line segment *KL* of length 8.6 cm. Measure and state the distance between the end points and the perpendicular bisector.

39. Bisect a line segment *LM* of length 9.4 cm. Measure and state the distance between the end points and the perpendicular bisector.

40. Bisect a line segment *MN* of length 10.2 cm. Measure and state the distance between the end points and the perpendicular bisector.

41. Bisect a line segment *WX* of length 6.5 cm. Measure and state the distance between the end points and the perpendicular bisector.

42. Bisect a line segment *XY* of length 9.5 cm. Measure and state the distance between the end points and the perpendicular bisector.

43. Bisect a line segment *YZ* of length 12.5 cm. Measure and state the distance between the end points and the perpendicular bisector.

44. Bisect angle *ABC* of magnitude 36°. Measure and state the size of the two angles formed.

45. Bisect angle *LMN* of magnitude 48°. Measure and state the size of the two angles formed.

46. Bisect angle *XYZ* of magnitude 62°. Measure and state the size of the two angles formed.

47. Bisect angle *KLM* of magnitude 53.2°. Measure and state the size of the two angles formed.

48. Bisect angle *PQR* of magnitude 75.6°. Measure and state the size of the two angles formed.

49. Bisect angle *TUV* of magnitude 87.8°. Measure and state the size of the two angles formed.

50. Bisect angle *DEF* of magnitude 29.5°. Measure and state the size of the two angles formed.

51. Bisect angle *STU* of magnitude 47.5°. Measure and state the size of the two angles formed.

52. Bisect angle *WXY* of magnitude 78.5°. Measure and state the size of the two angles formed.

Constructing Angles of 90°, 45° and 22.5°

(a) *To construct an angle of* 90°, *we construct a perpendicular from a point which is situated on a line segment.*

Construction

First *draw* a *line segment AB of line l. Open* your *compasses* to a *suitable separation.* Using *A* as *centre*, *construct an arc* to *intersect* the *line l at X and Y.* Then *open* your *compasses* to *more* than *half the distance of XY.* Using *X* and *Y* as *centres, construct two arcs to intersect above* the *line l* at *C*. Now *draw a straight line* passing *through the points A and C.* We have finally *constructed angle BAC of magnitude* 90°.

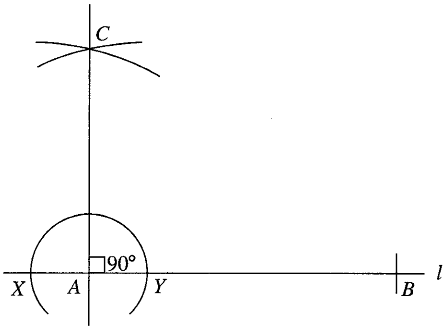


Fig. 9.98 Constructing an angle of 90° from a point on the line segment

(b) *We can also construct a perpendicular to a line segment from a point which is not situated on the line.*
Assuming that we are *given* a *line segment AB of line l* and a *point P above* the *line segment AB.*

Construction

With *centre P* and a *suitable compasses separation. construct* an *arc* to *intersect* the *line segment AB* at *C and D.* We now *construct* the *perpendicular bisector of CD* using *C* and *D* as *centres* and a *compasses separation* which is *more than half CD.* That is, we *draw arcs above* and *below* the *line segment AB* to *intersect at Q and R.* Now *draw* a *straight line* passing *through* the *points P, Q* and *R*. Then the *line PR* is *perpendicular* to the *line segment AB.*

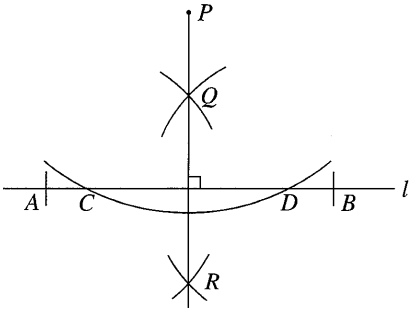


Fig. 9.99 Constructing an angle of 90° to a line segment from a point

To *construct* an *angle of* 45°, we now *bisect* the *angle of magnitude* 90°. This *method* is *illustrated* in Fig. 9.100 below.

Construction

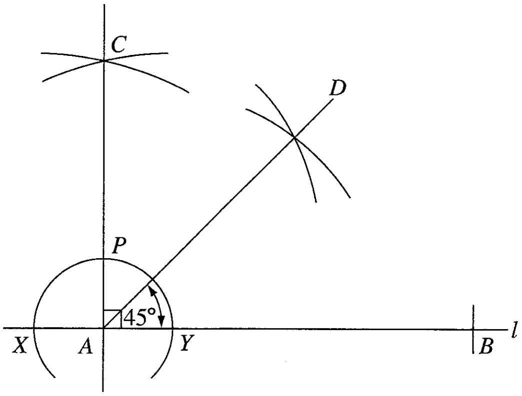


Fig. 9.100 Constructing an angle of 45°

Using *P* and *Y* as *centres, bisect angle BAC* = 90°. Then *angle BAD* is our *angle of magnitude* 45°.

To *construct* an *angle of* 22.5°, we now *bisect* the *angle of magnitude* 45°. This *method* is *illustrated* in Fig. 9.101.

Construction

Using *Q* and *Y* as *centres, bisect angle BAD* = 45°. Then *angle BAE* is our *angle of magnitude* 22.5°.

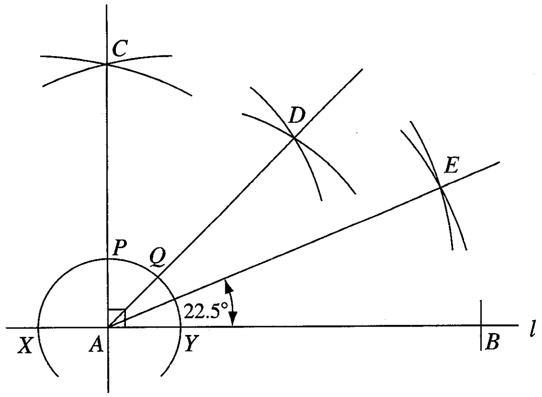


Fig. 9.101 Constructing an angle of 22.5°

Constructing Angles of 60°, 30° and 15°

To *construct* an angle of 60°.

Construction

We first *draw* a *line segment AB* of *line l*. Then using *A* as *centre* and a *suitable compasses separation*, *construct* an *arc above* the *line l* to *intersect* the *line segment AB* at *X*. With *X* as *centre* and the *same compasses separation*, *construct* a *second arc* to *intersect* the *first arc* at *C*. Now *draw* a *straight line* passing *through* the *points A and C*. We have finally *constructed angle BAC of magnitude* 60°.

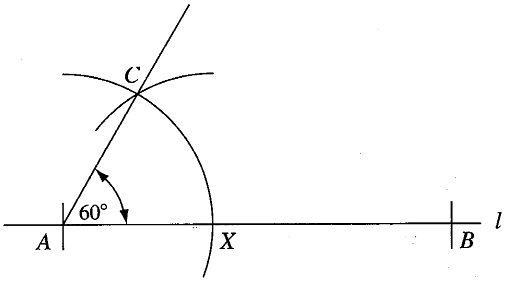


Fig. 9.102 Constructing an angle of 60°

To *construct* an *angle of* 30°, we now *bisect* the *angle of magnitude* 60°. This *method* is *illustrated* in Fig. 9.103.

Construction

Using *C* and *X* as *centres*, *bisect angle BAC* = 60°. Then *angle BAD* is our *angle of magnitude* 30°.

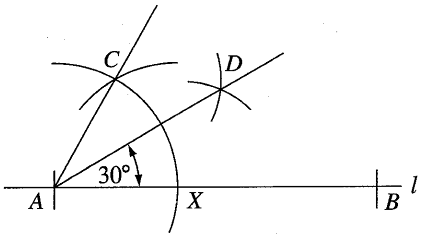


Fig. 9.103 Constructing an angle of 30°

To *construct* an *angle of* 15°, we now *bisect* the *angle of magnitude* 30°. This *method* is *illustrated* in Fig. 9.104 below.

Construction

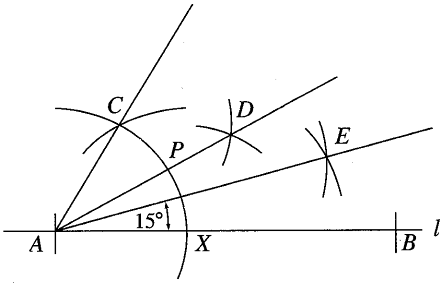


Fig. 9.104 Constructing an angle of 15°

Using *P* and *X* as *centres*, *bisect angle BAD* = 30°. Then *angle BAE* is our *angle of magnitude* 15°.

$$\textbf{NOTE: }\left.\begin{array}{l}180° - 90° = 90°\\ 180° - 45° = 135°\\ 180° - 22.5° = 157.5°\\ 180° - 11.25° = 168.75°\\ 180° - 60° = 120°\\ 180° - 30° = 150°\\ 180° - 15° = 165°\\ 180° - 7.5° = 172.5°\end{array}\right\}\text{ obtuse angles}$$

The above facts allow us to *construct* the *obtuse angles* 135°, 157.5°, 168.75°, 120°, 150°, 165° and 172.5° by an *indirect method*. For example: In order to *construct* the *obtuse angle of* 157.5°, we *construct* an *acute angle of* 22.5° on a *straight line*. The *adjacent angle* to the 22.5° will then be the *obtuse angle of magnitude* 157.5°. This fact is illustrated below in Fig. 9.105.

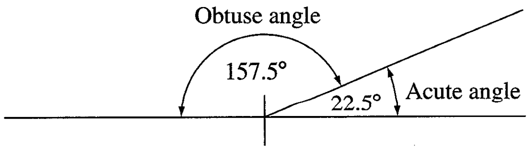


Fig. 9.105 Angle 157.5°

Bisecting an Angle Continuously

A simple method of *bisecting an angle continuously* is illustrated in Fig. 9.106 below. In this *example* we start by *bisecting* a 90° *angle.*

Construction

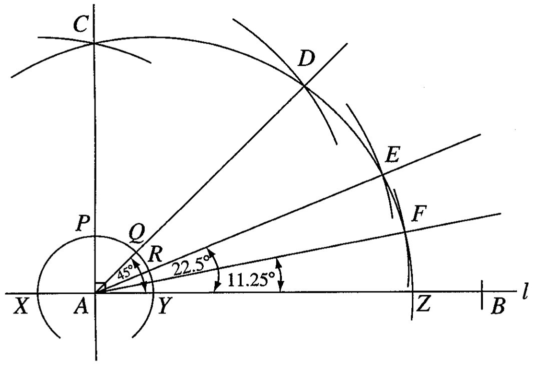


Fig. 9.106 Bisecting a 90° angle continuously

First draw a line segment *AB* of line *l*. Open your compasses to a suitable separation and using *A* as centre, construct an arc to intersect the line *l* at *X* and *Y*. Open your compasses to a larger separation and with centre *X*, construct an arc above the line *l*. We use this same compasses separation from here on. With *Y* as centre construct a second arc to intersect the first arc at *C*, and the line *l* at *Z*. Now draw a straight line passing through the points *A* and *C*. Hence *angle BAC* = 90°.

With *P* as centre, construct an arc to intersect the arc *CZ* at *D*. Draw a straight line passing through the points *A* and *D*. Hence *angle BAD* = 45°.

With *Q* as centre, construct an arc to intersect the arc *CZ* at *E*. Draw a straight line passing through the points *A* and *E*. Hence *angle BAE* = 22.5°.

With *R* as centre, construct an arc to intersect the arc *CZ* at *F*. Draw a straight line passing through the points *A* and *F*. Hence *angle BAF* = 11.25°.

In the second *example* we start by *bisecting a* 60° *angle*. The *method* is illustrated in Fig. 9.107.

Construction

First draw a line segment *AB* of line *l*. Open your compasses to a suitable separation and using *A* as centre, construct an arc to intersect the line segment *AB* at *X*. With the same compasses separation and using *X* as centre, construct a second arc to intersect the first arc at *C*. Draw a straight line passing through the points *A* and *C*. Hence *angle BAC* = 60°.

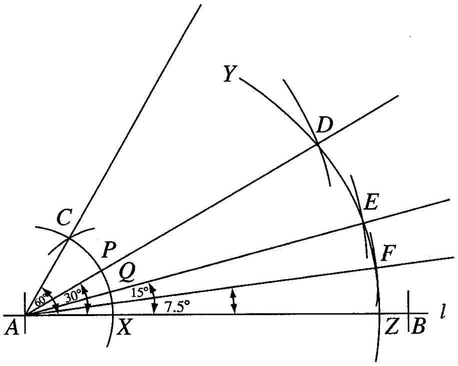


Fig. 9.107 Bisecting a 60° angle continuously

Open your compasses to a larger separation and with centre *X*, construct an arc *YZ* to intersect the line segment *AB* at *Z*. We use this same compasses separation from here on. With centre *C*, construct an arc to intersect the arc *YZ* at *D*.

Hence *angle BAD* = 30°.

With centre *P*, construct an arc to intersect the arc *YZ* at *E*. Hence *angle BAE* = 15°.

With *Q* as centre construct an arc to intersect the arc *YZ* at *F*. Hence *angle BAF* = 7.5°.

The above *procedures* can be used to *bisect any angle continuously.*

Example 17

Draw an angle of 80° and bisect it continuously to obtain a 10° angle.

Solution

The *construction* is shown in Fig. 9.108 below.

Construction

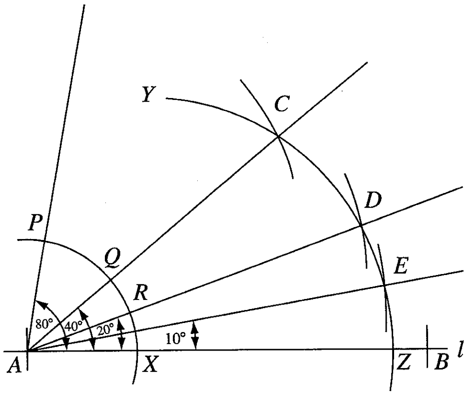


Fig. 9.108 Bisecting an 80° angle continuously

Exercise 9d

Using ruler and compasses only:

1. Construct an angle of magnitude 90°.
2. Construct an angle of size 45°.
3. Construct an angle of 22.5°.
4. Construct an angle of magnitude 11.25°.
5. Construct an angle of magnitude 60°.
6. Construct an angle of size 30°.
7. Construct an angle of 15°.
8. Construct an angle of magnitude 7.5°.
9. Construct an obtuse angle of magnitude 135°.
10. Construct an obtuse angle of size 120°.
11. Construct an obtuse angle of 150°.
12. Construct an obtuse angle of magnitude 165°.
13. Construct an obtuse angle of size 157.5°.
14. Construct an obtuse angle of 168.75°.
15. Construct an obtuse angle of 172.5°.
16. Construct an angle of 90° and bisect it continuously to obtain an 11.25° angle.
17. Construct an angle of 60° and bisect it continuously to obtain a 7.5° angle.

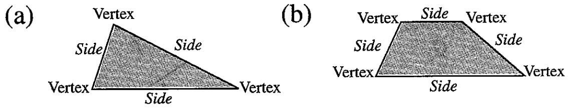
Using ruler, compasses and protractor only:

18. Draw an angle of 80° and bisect it continuously to obtain a 20° angle.
19. Draw an angle of 86° and bisect it continuously to obtain a 21.5° angle.
20. Draw an angle of 75° and bisect it continuously to obtain an 18.75° angle.
21. Draw an angle of 70° and bisect it continuously to obtain a 17.5° angle.
22. Draw an angle of 80° and bisect it continuously to obtain a 10° angle.
23. Draw an angle of 86° and bisect it continuously to obtain a 10.75° angle.
24. Draw an angle of 75° and bisect it continuously to obtain a 9.375° angle.
25. Draw an angle of 70° and bisect it continuously to obtain a 8.75° angle.

Plane and Polygon

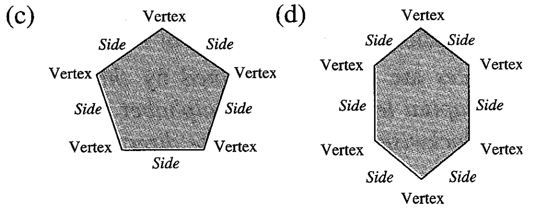
A *plane* is defined as a *flat surface* with *no thickness*. For example: The top of a desk, the surface of a blackboard, a page of a book and the surface of a pane of glass.

A *polygon* is defined as a *plane shape* (or *figure*) *bounded* by *three or more straight lines*. The *line segments* are called the *sides* of the *polygon*. And the *common end-points* of the *sides* (i.e. the *point* where *two sides meet*) are called *vertices*. Some examples of *polygons* are:



3 sides and 3 vertices: 3-sided polygon

4 sides and 4 vertices: 4-sided polygon



5 sides and 5 vertices: 5-sided polygon

6 sides and 6 vertices: 6-sided polygon

Fig. 9.109 Polygons

Triangle

A *triangle* is a *plane shape* (or *figure*) *bounded* by *three straight lines.*

A *triangle* can also be defined as a *three-sided polygon.*

An example of a *triangle* can be seen in Fig. 9.110 below.

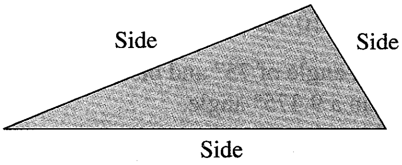
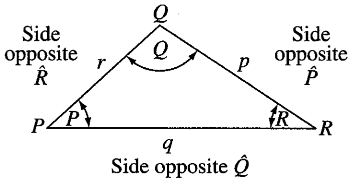


Fig. 9.110 Triangle

A *triangle* has *no thickness.*

Elements of a Triangle

(a)



or

(b)

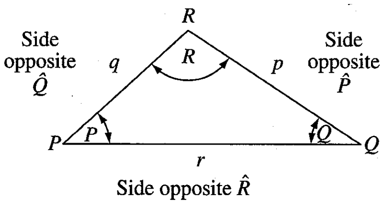


Fig. 9.111 Triangles

Fig. 9.111 indicates the *six elements* of a *triangle.*

The *vertices* are normally *denoted* by *three consecutive capital letters* from the *alphabet* and *written* in a *clockwise* or *anti-clockwise direction* in the *diagram.*

For example: P, Q and R.

An *angle* is *denoted* by the *same capital letter* as its *vertex.* Thus:

angle $P = \hat{P} = Q\hat{P}R = R\hat{P}Q$

angle $Q = \hat{Q} = P\hat{Q}R = R\hat{Q}P$ and

angle $R = \hat{R} = P\hat{R}Q = Q\hat{R}P$.

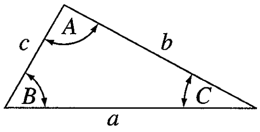
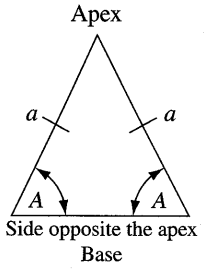
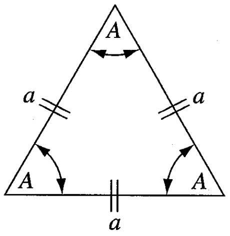
Angles P, Q and R are *interior angles* of the *triangle* PQR. And a *side* is *denoted* by the *common letter* of the *opposite angle.* Thus: $p = QR$, $q = PR$ and $r = PQ$.

Types of Triangle

Triangles can be *classified* according to their *sides* or according to their *angles.*

Triangles can be *classified* according to their *sides* into *three basic types.*

Table 9.1

Classification of Triangles	
By Sides	*Properties*
 *Scalene triangle*	1. *No two sides equal.* 2. *No two angles equal.*
 *Isosceles triangle*	1. *Two sides* are *equal.* 2. *Two angles* are *equal.* 3. The *vertex* where the *two equal sides meet* is *called* an *apex.* 4. An *equal angle* is *formed* by an *equal side* and the *side opposite* the *apex*, *called* the *base.*
 *Equilateral triangle*	1. *All three sides* are *equal.* 2. All *three angles* are *equal.* Hence *each angle* is *equal to* 60°. That is $\hat{A} = 60°$.

NOTE: An *equilateral triangle* is considered to be a *special kind* of *isosceles triangle.* That is an *equilateral triangle* is an *isosceles triangle* with *three equal sides* or *three equal angles.*

Thus: {*equilateral* $\triangle s$} $\subset$ {*isosceles* $\triangle s$}

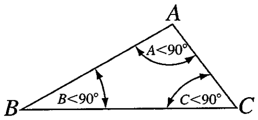
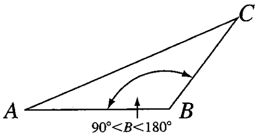
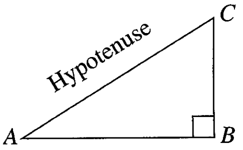
where the symbol $\triangle$ means 'triangle'.

The dash | on the two sides of the isosceles triangle indicates two equal sides.

The dashes || on the three sides of the equilateral triangle indicates three equal sides.

Triangles can be *classified* according to their *angles* into *three basic types*:

Table 9.2

Classification of Triangles	
By Angles	*Properties*
 *Acute-angled triangle*	*Each angle* is an *acute angle.*
 *Obtuse-angled triangle*	*One angle* is an *obtuse angle.*
 *Right-angled triangle*	*One angle* is *a right angle.*

NOTE: The *side* that is *opposite* the *right angle* in a *right-angled triangle* is called the *hypotenuse*. The *hypotenuse* is also the *longest side.*

Hypotenuse can be abbreviated to '*hyp*'.

Angle Properties of a Triangle

There are *four theorems* that we need to look at under this heading.

THEOREM 1: *The sum of the three interior angles of a triangle is equal to* 180° (*or* 2 *right angles*).

Class Activity

Take a *ruler* and a *pencil* and *draw* your *own triangle.* Now take your *protractor* and *measure each angle.* After you have obtained the *magnitudes of the three angles—sum them. What* do you *observe*?

Example:

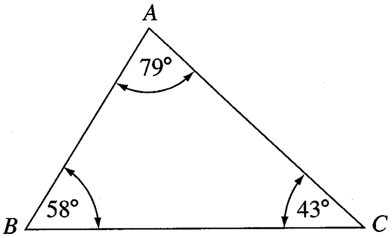


Fig. 9.112 Triangle

By *measurement*:

$\hat{A} = 79°$, $\hat{B} = 58°$ and $\hat{C} = 43°$.

So the *sum* of the *interior angles* of the *triangle ABC*,

$$\begin{aligned} S &= \hat{A} + \hat{B} + \hat{C} \\ &= 79° + 58° + 43° \\ &= 180° \\ &= 2 \text{ rt. } \angle\text{s} \end{aligned}$$

It has been *proved* that the *sum of the interior angles* of a *polygon* with *n sides* is $(2n - 4)$ *right angles* or $90° (2n - 4)$ or $180° (n - 2)$.

Thus the *sum* of the *interior angles* of a *triangle* $(n = 3)$,

$$\begin{aligned} S &= (2n - 4) \text{ rt. } \angle\text{s} \\ &= (2 \times 3 - 4) \text{ rt. } \angle\text{s} \\ &= (6 - 4) \text{ rt. } \angle\text{s} \\ &= 2 \text{ rt. } \angle\text{s} \end{aligned}$$

Or

$$\begin{aligned} S &= 90° (2n - 4) \\ &= 90° (2 \times 3 - 4) \\ &= 90° (6 - 4) \\ &= 90° \times 2 \\ &= 180° \end{aligned}$$

Or

$$\begin{aligned} S &= 180° (n - 2) \\ &= 180° (3 - 2) \\ &= 180° \times 1 \\ &= 180° \end{aligned}$$

THEOREM 2:

(a) *If one side of a triangle is longer than another side, then the angle opposite the longer side is greater than the angle opposite the shorter side.*

(b) *If one angle of a triangle is greater than another angle, then the side opposite the greater angle is longer than the side opposite the smaller angle.*

In other words, in *any triangle* the *longer side* is *opposite* the *greater angle* and the *shorter side* is *opposite* the *smaller angle*. And vice versa.

This *theorem* can be very useful as a *check* on your *answer* when *calculating* an *angle* or the *length* of a *side* of a *triangle*.

Class Activity

Measure the *lengths* of the *three sides* of the *triangle* that you had *drawn previously. Compare* the *size* of *each angle* with the *length* of its *opposite side. Compare each* of the *three pairs of measurements. Write* the *size* of the *three angles* in *ascending order*, then *write* the *lengths* of the *three sides* in *ascending order*. What *conclusion* did you come to? Does it *support* the *stated theorem*?

Example:

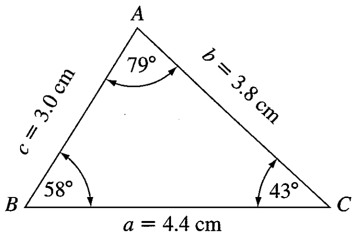


Fig. 9.113 Triangle

By *measurement*:

$a = 4.4$ cm, $b = 3.8$ cm and $c = 3.0$ cm

Table 9.3

Angle	*Length of opposite side*
$\hat{A} = 79°$	$a = 4.4$ cm
$\hat{B} = 58°$	$b = 3.8$ cm
$\hat{C} = 43°$	$c = 3.0$ cm

The *angles* in *ascending order*:

43° ($\hat{C}$) < 58° ($\hat{B}$) < 79° ($\hat{A}$).

The *sides* in *ascending order*:

3.0 cm (c) < 3.8 cm (b) < 4.4 cm (a)

Obviously the *theorem* is *supported* by the *data recorded.*

THEOREM 3: *If any side of a triangle is produced,* (i.e. extended) *then the exterior angle so formed is equal to the sum of the two interior opposite angles.*

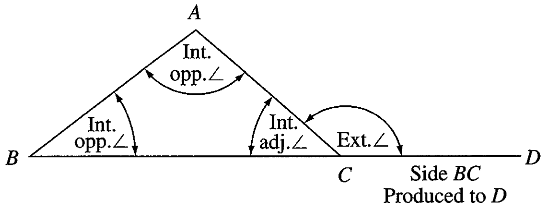


Fig. 9.114 Triangle

In Fig. 9.114 above, the *side BC* is *produced to D.*

$A\hat{C}D$ = The *exterior angle*

$A\hat{C}B$ = The *interior adjacent angle*

$\hat{A} = B\hat{A}C$ = The *interior opposite angle*, and

$\hat{B} = A\hat{B}C$ = The *interior opposite angle.*

Class Activity

Using the *triangle* that you had *drawn* previously, *produce* the *side BC to X. Measure* angle *ACX. Add angle A to angle B* and *compare* the *result with angle ACX. What* do you *observe*?

Now *produce AB to Y. Measure angle CBY. Sum angles A and C* and *compare* the *result with angle CBY. What* do you *observe*?

Finally, *produce CA to Z. Measure angle BAZ. Total angle B and angle C* and *compare* the *result with angle BAZ. What* do you *observe*?

Do your *results* in the *three different cases support* the *stated theorem*?

Example:

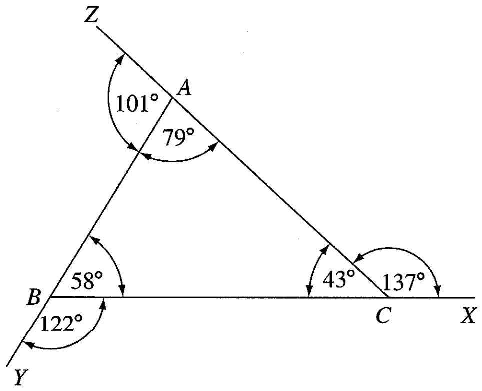


Fig. 9.115 Triangle

By *measurement*:

The *exterior angle* $ACX = 137°$

The *sum* of the *two*
interior opposite angles $= \hat{A} + \hat{B} = 79° + 58°$
$= 137°$

Hence $A\hat{C}X = \hat{A} + \hat{B} = 137°$.

The *exterior angle* $C\hat{B}Y = 122°$

The *sum* of the *two*
interior opposite angles $= \hat{A} + \hat{C} = 79° + 43°$
$= 122°$

Hence $C\hat{B}Y = \hat{A} + \hat{C} = 122°$.

The *exterior angle* $B\hat{A}Z = 101°$

The *sum* of the *two*
interior opposite angles $= \hat{B} + \hat{C} = 58° + 43°$
$= 101°$

Hence $B\hat{A}Z = \hat{B} + \hat{C} = 101°$.

Obviously, the *theorem* is *supported* by the *results*.

NOTE: *It can be seen that the sum of the exterior angle of a triangle and its interior adjacent angle is always equal to* 180°. This is so since the *sum* of the *adjacent angles* on a *straight line* is *equal to* 180°.

Thus $A\hat{C}X + \hat{C} = 137° + 43° = 180°$.
$C\hat{B}Y + \hat{B} = 122° + 58° = 180°$.

And $B\hat{A}Z + \hat{A} = 101° + 79° = 180°$.

THEOREM 4: *The sum of the three exterior angles of a triangle is equal to* 360° (*or* 4 *right angles*).

Class Activity

Using the *triangle* that you had *drawn* previously, *sum* the *three exterior angles*. *What* do you *observe*?

Example:

By *measurement*:

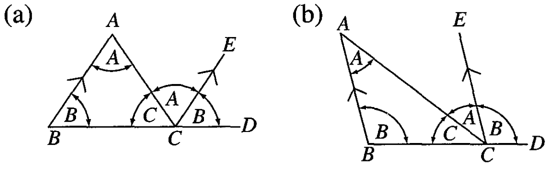
$A\hat{C}X = 137°$, $C\hat{B}Y = 122°$ and $B\hat{A}Z = 101°$.

So the *sum* of the *exterior angles*
of the *triangle ABC*, $S = A\hat{C}X + C\hat{B}Y + B\hat{A}Z$
$= 137° + 122° + 101°$
$= 360°$
$= 4 \text{ rt. } \angle s$

Obviously the *theorem* is *supported* by the *results*.

Proof of Theorems

Given a triangle ABC, prove Theorem 1 *and Theorem* 3.



Acute-angled △*ABC* Obtuse-angled △*ABC*

Fig. 9.116 Triangles

Considering an *acute-angled triangle ABC* and an *obtuse-angled triangle ABC*, *produce* the *side BC to D*. Then *draw CE parallel* to *BA*.

Now $A\hat{C}E = B\hat{A}C = \hat{A}$ (*alt.* $\angle s$)

And $E\hat{C}D = A\hat{B}C = \hat{B}$ (*corres.* $\angle s$)

Thus $A\hat{C}B + A\hat{C}E + E\hat{C}D = \hat{C} + \hat{A} + \hat{B}$
$= 180°$
$= 2 \text{ rt. } \angle s$
($\angle s$ *on a straight line*)

Also $B\hat{A}C + A\hat{B}C + A\hat{C}B = \hat{A} + \hat{B} + \hat{C}$
$= 180°$
$= 2 \text{ rt. } \angle s$

Hence *Theorem* 1 *is proved*.

Now $A\hat{C}D = A\hat{C}E + E\hat{C}D = B\hat{A}C + A\hat{B}C = \hat{A} + \hat{B}$

Hence *Theorem* 3 *is proved*.

Example 18

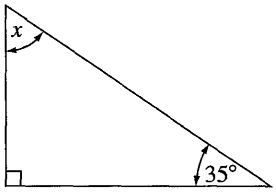


Fig. 9.117 Right-angled triangle

Calculate the size of angle x.

Solution

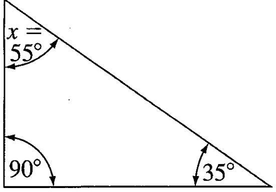


Fig. 9.117 Right-angled triangle

Now $\hat{x} + 35° + 90° = 180°$ (*sum of* $\angle$*s of a* $\triangle$)

So $\hat{x} + 125° = 180°$

i.e. $\hat{x} = 180° - 125°$

$\therefore$ $\hat{x} = 55°$

Hence *angle x* is 55°.

Alternatively $\hat{x} + 35° = 90°$ (*comp.* $\angle$*s*)

So $\hat{x} = 90° - 35°$

$\therefore$ $\hat{x} = 55°$

Hence *angle x* is 55°.

Example 19

Determine the size of angle x.

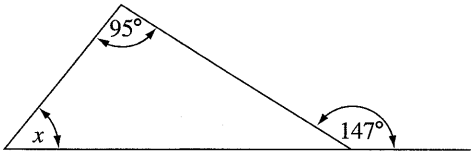


Fig. 9.118 Triangle

Solution

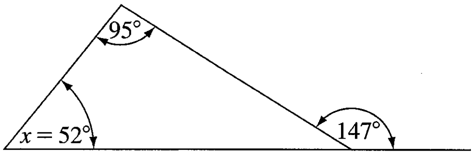


Fig. 9.118 Triangle

Now $\hat{x} + 95° = 147°$ (*ext.* $\angle$ = *sum of* 2 *int. opp* $\angle$*s*)

So $\hat{x} = 147° - 95°$

i.e. $\hat{x} = 52°$

Hence *angle x* is 52°.

Exercise 9e

1. Calculate the size of angle x, giving a reason for your answer.

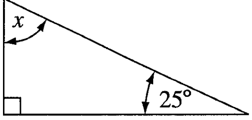


Fig. 9.119 Triangle

2. Calculate the size of angle y, giving a reason for your answer.

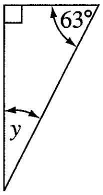


Fig. 9.120 Triangle

3. Determine the magnitude of angle z, giving a reason for your answer.

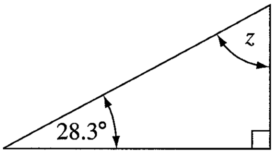


Fig. 9.121 Triangle

4. Determine the magnitude of angle x, giving a reason for your answer.

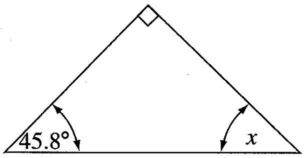


Fig. 9.122 Triangle

5.

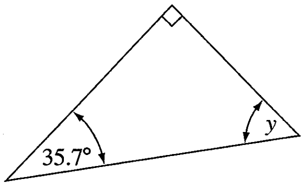


Fig. 9.123 Triangle

Calculate the magnitude of angle y, giving a reason for your answer.

6. Evaluate angles x and y, if angle x is twice angle y. Give a reason for your answers.

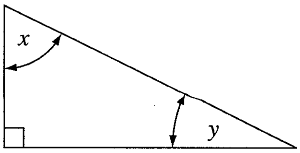


Fig. 9.124 Triangle

7. Evaluate angles p and q, giving reasons for your answers.

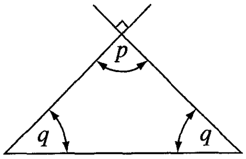


Fig. 9.125 Triangle

8. Evaluate angle x, giving a reason for your answer.

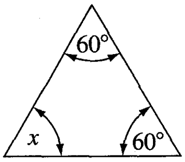


Fig. 9.126 Triangle

9. Determine the size of angle BAC, giving a reason for your answer.

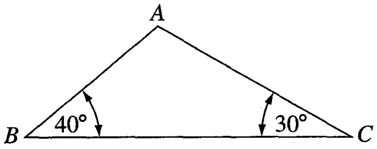


Fig. 9.127 Triangle

10. Calculate the magnitude of angle LNM, giving a reason for your answer.

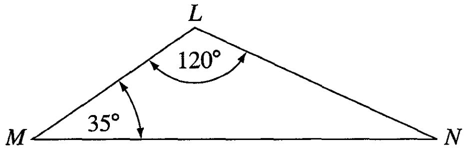


Fig. 9.128 Triangle

11. State the size of angle x, giving a reason for your answer.

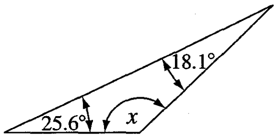


Fig. 9.129 Triangle

12. Determine the magnitude of angle y, giving a reason for your answer.

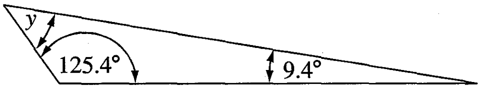


Fig. 9.130 Triangle

13. Evaluate angles d and e, giving reasons for your answers.

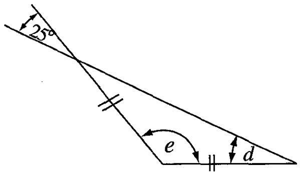


Fig. 9.131 Triangle

14. Evaluate angles d and e, giving reasons for your answers.

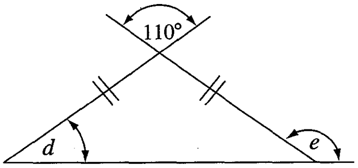


Fig. 9.132 Triangle

15. Evaluate angles s and r, giving reasons for your answers.

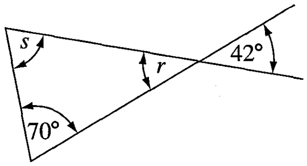


Fig. 9.133 Triangle

16. Evaluate angles x and y, giving reasons for your answers.

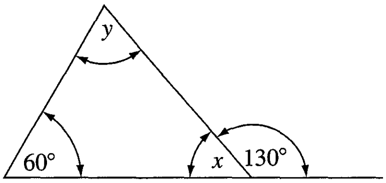


Fig. 9.134 Triangle

17.

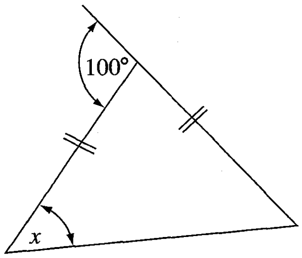


Fig. 9.135 Triangle

Calculate the magnitude of angle x, giving reasons for your answer.

18. Calculate the magnitudes of angles *x* and *y*, giving reasons for your answers.

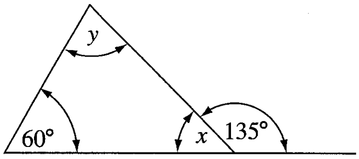


Fig. 9.136 Triangle

19. Determine the size of angle *ACE*, giving a reason for your answer.

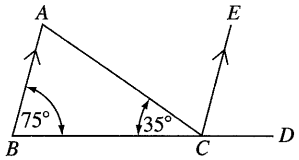


Fig. 9.137 Triangle

20.

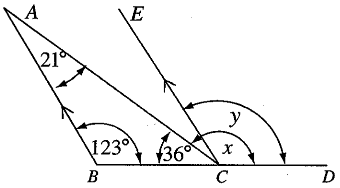


Fig. 9.138 Triangle

Evaluate angles *x* and *y*, giving reasons for your answers.

21.

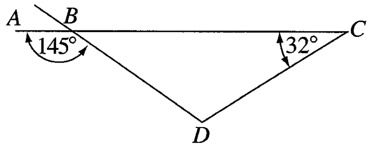


Fig. 9.139 Triangle

State the size of angle *CDB*, giving a reason for your answer.

22.

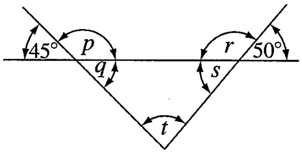


Fig. 9.140 Triangle

Evaluate the angles *p*, *q*, *r*, *s* and *t*. Give reasons for your answers.

23. Calculate the sizes of the angles marked *x* and *y*.

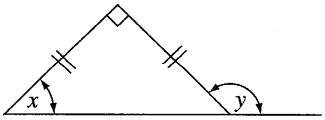


Fig. 9.141 Triangle

24. Determine the size of the angle marked *t*.

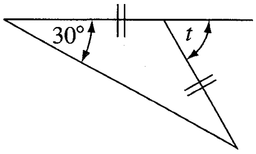


Fig. 9.142 Triangle

Constructing a Unique Triangle

A *unique triangle* is defined when we know for the triangle, *any set* of the following sets of *elements:*

(i) *Three sides.*

(ii) *Two sides* and the *angle included* by these two sides.

(iii) *One side* and *two angles.*

(iv) *Right angle*, *hypotenuse* and a *side.*

Hence a *unique triangle* can be *drawn* or *constructed* when we are *given any set of the sets of elements* stated above.

Always *draw* a *rough sketch* of the *triangle* to be *constructed*, before starting to *construct* the *actual triangle.* In this way you would have a fairly good idea of the *shape* of the *triangle* to be *constructed.*

Given Three Sides

Example 20

(a) Using ruler and compasses only, construct the triangle ABC, with AB = 6.5 cm, AC = 4.0 cm and BC = 5.0 cm. Show all construction lines clearly.

(b) Measure and state the size of angle ABC.

Solution

Construction

First *construct* the *line segment AB* = 6.5 cm. Then *set* your *compasses* to a *separation* of 4.0 cm using a *ruler*. With *centre A*, *construct an arc above* the *line segment AB*. Now *set* your *compasses* to a *separation* of 5.0 cm. Using *B* as *centre*, *construct* a *second arc* to *intersect* the *first arc* at *C*. *Draw straight lines* from *A to C* and from *B to C*. We have finally *constructed* the *triangle ABC*, with $AB = 6.5$ cm, $AC = 4.0$ cm and $BC = 5.0$ cm.

(a)

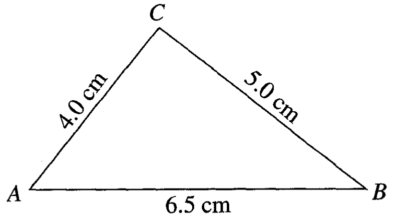


Fig. 9.143 Sketch of triangle

Above can be *seen* the *sketch* of the $\triangle ABC$ to be *constructed.*

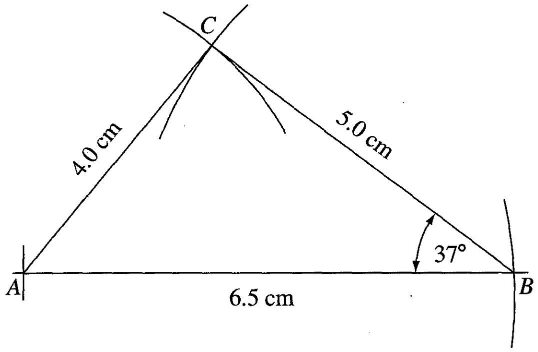


Fig. 9.143 Constructed triangle

(b) By *measurement*, the *size of angle ABC* = 37°.

Given Two Sides and the Included Angle

Example 21

(a) Using ruler and compasses only, construct the triangle PQR, with PQ = 4.9 cm, QR = 3.6 cm and angle PQR = 120°. Show all construction lines clearly.

(b) Measure and state the length of PR.

Solution

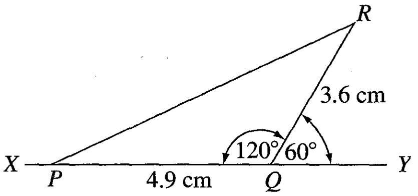


Fig. 9.144 Sketch of triangle

Above can be *seen* the *sketch* of $\triangle PQR$ to be *constructed.*

NOTE: In order to *construct* $P\hat{Q}R = 120°$ with your *compasses*, you must *construct* $R\hat{Q}Y = 60°$, since $180° - 60° = 120°$.

Construction

First *draw* the *line XY* and then *construct* the *line segment PQ* = 4.9 cm. Using *Q* as *centre*, *construct* a 60° *angle* on the *right-hand-side*. *Draw* a *straight line* passing *through* the *point Q* and the 60° *angle*. Now *set* your *compasses* to a *separation* of 3.6 cm and *construct* an *arc* to *intersect* the *line* at *R*. Then *draw* a *straight line joining* the *points P and R*. We have finally *constructed* the *triangle PQR*, with *PQ* = 4.9 cm, *QR* = 3.6 cm and *angle PQR* = 120°.

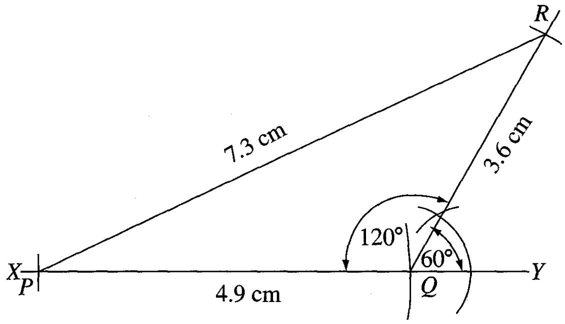


Fig. 9.144 Constructed triangle

(b) By *measurement* the *length of PR* = 7.3 cm.

Given One Side and Two Angles

Example 22

(a) Using ruler and compasses only, construct the triangle LMN, with LM = 4.9 cm, angle L = 45° and angle M = 30°. Show all construction lines clearly.

(b) Measure and state the magnitude of angle LNM.

Solution

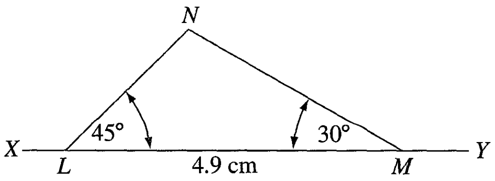


Fig. 9.145 Sketch of triangle

Above can be *seen* the *sketch* of △*LMN* to be *constructed.*

Construction

First *draw* a *line XY* and then *construct* the *line segment LM* = 4.9 cm. Use *L* as *centre*, and *construct* the 45° *angle* on the *right-hand-side*. *Draw* a *straight line* passing *through* the *point L* and the 45° *angle*. Now use *M* as *centre* and *construct* a 30° *angle* on the *left-hand-side*. *Draw* a *straight line* passing *through* the *point M* and the 30° *angle* to *intersect* the *last line* at *N*. Hence we have finally *constructed* the *triangle LMN*, with *LM* = 4.9 cm, *angle L* = 45° and *angle M* = 30°.

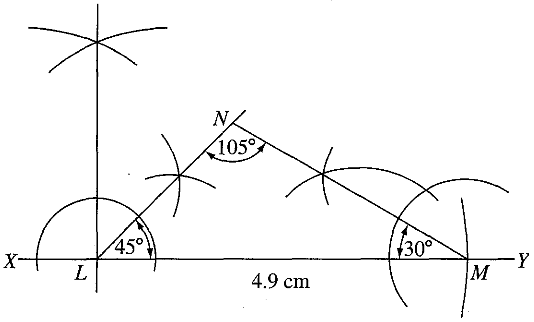


Fig. 9.145 Constructed triangle

(b) By *measurement*, the *magnitude of angle LNM* = 105°.

Given a Right Angle, Hypotenuse and a Side

Example 23

(a) Using rules and compasses only, construct the triangle XYZ, with angle XYZ = 90°, XZ = 6.6 cm and YZ = 3.8 cm. Show all construction lines clearly.

(b) Measure and state the length of XY.

Solution

(a)

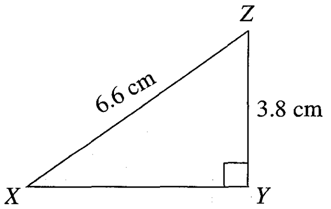


Fig. 9.146 Sketch of triangle

Above can be *seen* the *sketch* of the △*XYZ* to be *constructed.*

Construction

First *draw* a *straight line PQ*, then *mark off* the *point Y*. Now *construct* the 90° *angle* using *Y* as *centre*. *Draw* a *straight line* passing *through* the *point Y* and the 90° *angle*. Set your *compasses* to a *separation* of 3.8 cm and with *centre Y*, *construct an arc* to *cut* the *last line* at *Z*. Now set your *compasses* to a *separation* of 6.6 cm and with *centre Z*, *construct an arc* to *intersect* the *line PY* at *X*. *Draw* a *straight line joining* the *points X* and *Z*. We have finally *constructed* the *triangle XYZ*, with *angle XYZ* = 90°, *XZ* = 6.6 cm and *YZ* = 3.8 cm.

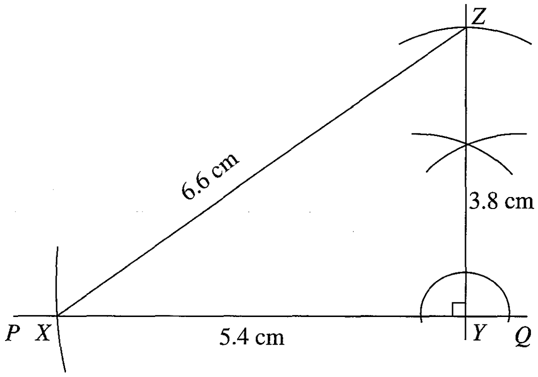


Fig. 9.146 Constructed triangle

(b) By *measurement*, the *length of XY* = 5.4 cm.

Exercise 9f

1. (a) Using ruler and compasses only, construct the triangle *ABC*, with *AB* = 10 cm, *BC* = 8 cm and *AC* = 6 cm. Show all construction lines clearly.

(b) Measure and state the size of angle *ACB*.

2. (a) Using ruler and compasses only, construct the triangle *KLM* in which *KL* = 12 cm, *LM* = 5 cm and *KM* = 9 cm. Show all construction lines clearly.

(b) Measure and state the magnitude of angle *KML*.

3. (a) Using ruler and compasses only, construct the △*PQR* in which *PQ* = 8 cm, *QR* = 16 cm and *PR* = 10 cm. Show all construction lines clearly.

(b) Measure and state the size of angle *PQR*.

4. (a) Using ruler and compasses only, construct the $\triangle TUV$, in which $TU = 12.5$ cm, $UV = 7.5$ cm and $TV = 10$ cm.
Show all construction lines clearly.
(b) Measure and state the size of angle TUV.

5. (a) Using ruler and compasses only, construct the triangle $\triangle PQR$, in which $PQ = 8.9$ cm, $QR = 6.5$ cm and $PR = 9.5$ cm.
Show all construction lines clearly.
(b) Measure and state the magnitude of angle QPR.

6. (a) Using ruler and compasses only, construct $\triangle ABC$, in which $AB = 7$ cm, $BC = 5$ cm and $A\hat{B}C = 60°$.
Show all construction lines clearly.
(b) Measure and state the length of AC.

7. (a) Using ruler and compasses only, construct $\triangle PQR$, in which $QR = 12$ cm, $PR = 15$ cm and $P\hat{R}Q = 60°$.
Show all construction lines clearly.
(b) Measure and state the length of PQ.

8. (a) Using ruler and compasses only, construct $\triangle LMN$, in which $LM = 7$ cm, $MN = 9$ cm and $LMN = 45°$.
Show all construction lines clearly.
(b) Measure and state the length of LN.

9. (a) Using ruler and compasses only, construct $\triangle KLM$, in which $MK = 12$ cm, $LM = 8$ cm and $K\hat{M}L = 30°$.
Show all construction lines clearly.
(b) Measure and state the length of KL.

10. (a) Construct a triangle PQR in which $p = 9$ cm, $r = 14$ cm and $Q = 39°$.
(b) Measure and state the length of side q.

11. (a) Construct a $\triangle ABC$ in which $a = 6$ cm, $c = 11$ cm and $\hat{B} = 145.8°$.
(b) Measure and state the magnitude of angle C.

12. (a) Construct $\triangle ABC$ in which $AB = 6.5$ cm, $BC = 7.5$ cm and $\hat{B} = 50°$.
(b) Measure and state the length of AC.

13. (a) Construct $\triangle ABC$ in which $BC = 7.9$ cm, $AC = 8.4$ cm and $A\hat{C}B = 125°$.
(b) Measure and state the length of AB.

14. (a) Using ruler and compasses only, construct the $\triangle ABC$, in which $AB = 12$ cm, $\hat{A} = 60°$ and $\hat{B} = 30°$. Show all construction lines clearly.
(b) Measure and state the magnitude of angle C.

15. (a) Using ruler and compasses only, construct the $\triangle PQR$, with $PQ = 8.5$ cm, $\hat{P} = 90°$ and $\hat{Q} = 60°$. Show all construction lines clearly.
(b) Measure and state the size of angle R.

16. (a) Construct $\triangle ABC$ in which $AB = 10$ cm, $\hat{A} = 60°$ and $\hat{B} = 20°$.
(b) Measure and state the magnitude of $\hat{C}$.

17. (a) Using ruler and compasses only, construct $\triangle PQR$, with $PQ = 11.5$ cm, $\hat{P} = 120°$ and $\hat{Q} = 30°$. Show all construction lines clearly.
(b) Measure and state the size of $\hat{R}$.

18. (a) Using ruler and compasses only, construct $\triangle LMN$, with $LM = 9.5$ cm, angle $L = 30°$ and angle $M = 45°$.
Show all construction lines clearly.
(b) Measure and state the size of angle MNL.

19. (a) Using ruler and compasses only, construct $\triangle PQR$, with $PQ = 10.9$ cm, $\hat{P} = 30°$ and $\hat{Q} = 120°$.
Show all construction lines clearly.
(b) Measure and state the size of $\hat{R}$.

20. (a) Using ruler and compasses only, construct $\triangle KLM$, with $KL = 11.5$ cm, $\hat{K} = 45°$ and $\hat{L} = 60°$.
Show all construction lines clearly.
(b) Measure and state the magnitude of $\hat{M}$.

21. (a) Construct a $\triangle ABC$ in which $AB = 9$ cm, $\hat{A} = 60°$ and $\hat{B} = 25°$.
(b) Measure and state the size of angle C.

22. (a) Construct a $\triangle LMN$ in which $ML = 9$ cm, $\hat{M} = 90°$ and $L = 25°$.
(b) Measure and state the magnitude of angle N.

23. (a) Using ruler and compasses only, construct the $\triangle ABC$, with $\hat{B} = 90°$, $AC = 10$ cm and $BC = 6$ cm.
Show all construction lines clearly.
(b) Measure and state the length of AB.

24. (a) Using ruler and compasses only, construct $\triangle PQR$, with $\hat{Q} = 90°$, $PR = 10$ cm and $QR = 8$ cm.

Show all construction lines clearly.

(b) Measure and state the magnitude of angle PRQ.

25. (a) Using ruler and compasses only, construct $\triangle ABC$, in which $A\hat{B}C = 90°$, $AC = 12$ cm and $AB = 6$ cm.
Show all construction lines clearly.

(b) Measure and state the size of angle BAC.

26. (a) Using ruler and compasses only, construct $\triangle KLM$, in which $K\hat{L}M = 90°$, $KM = 7.5$ cm and $ML = 6$ cm.
Show all construction lines clearly.

(b) Measure and state the length of KL.

27. (a) Using ruler and compasses only, construct $\triangle PQR$, in which $P\hat{Q}R = 90°$, $PR = 9$ cm and $PQ = 5.4$ cm.
Show all construction lines clearly.

(b) Measure and state the length of RQ.

28. (a) Using ruler and compasses only, construct $\triangle XYZ$, in which $X\hat{Y}Z = 90°$, $XZ = 10.5$ cm and $YZ = 8.4$ cm.
Show all construction lines clearly.

(b) Measure and state the length of XY.

29. Draw a triangle ABC in which $BC = 8$ cm, $AB = 5$ cm and angle $ABC = 50°$. State the length of AC. Through C, draw CD parallel to BA. If BC is produced to F, state the size of angle DCF.

30. Draw a triangle ABC in which $AB = 5$ cm, $AC = 4$ cm and angle $BAC = 40°$. State the length of BC. Through B, draw BD parallel to AC. If DB is produced to E, state the size of angle ABE.

31. Draw a triangle PQR in which angle $RPQ = 55°$, angle $PQR = 30°$ and $PQ = 9.5$ cm. State the length of PR. Through Q, draw QS parallel to PR. If PQ is produced to T, state the magnitude of angles RQS and SQT.

32. Draw a triangle KLM in which angle $KLM = 120°$, angle $KML = 30°$ and $ML = 8.5$ cm. State the length of MK. Through M, draw MN parallel to LK. If NM is produced to X, state the magnitude of angles KMN and LMX.

33. Draw a triangle XYZ in which angle $XYZ = 120°$, $XY = 7.4$ cm and $YZ = 6.5$ cm. State the length of XZ. Through Z, draw ZP parallel to XY. State the size of angle PZY.

34. Construct a triangle PQR such that $PQ = 9.5$ cm, $PR = 8$ cm and angle $QPR = 75°$. Construct also the perpendicular bisectors of PQ and PR to intersect at S. Measure and state the length of PS.
Show all construction lines clearly.

35. (a) Construct triangle XYZ with the dimensions given below.

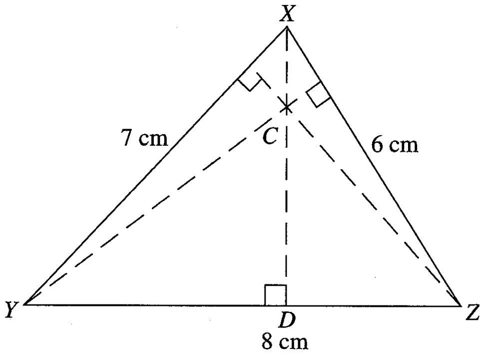


Fig. 9.147 Triangle

(b) Construct perpendiculars from the vertices X, Y and Z to the sides YZ, XZ and XY respectively. Let the perpendiculars intersect at the point C.

(c) Measure and state the lengths of XC and XD. This construction indicates that the three altitudes of a triangle are concurrent, that is, they all meet at a common point.

36. (a) Construct triangle ABC with the dimensions given below.

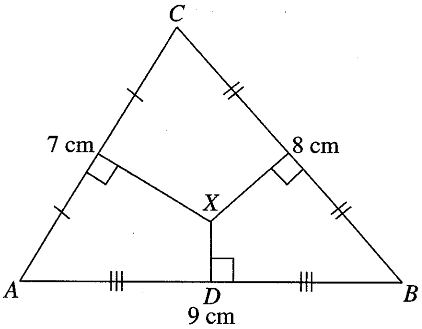


Fig. 9.148 Triangle

(b) Construct the perpendicular bisectors of the sides of the triangle ABC to intersect at the point X. Measure and state the length of XD.

(c) Using X as centre and XA as radius, construct a circle. What do you observe?

37. (a) Construct a triangle PQR, in which $PQ = 10$ cm, $PR = 8$ cm and $QR = 9$ cm.

(b) Construct the bisectors of the angles of the triangle to intersect at the point X.

(c) Let the angle bisectors meet the sides PQ, QR and PR at A, B and C respectively. Measure the lengths of XA, XB and XC. What do you observe?

(d) Using X as centre and XA as radius, construct a circle. What do you observe?

Properties of Congruent Triangles

We mentioned in the last section that a *unique triangle* is *defined* when we know for the *triangle* certain *sets of elements. All triangles* having for their *elements* one of these *sets of elements* will be *exactly* the *same* in *every respect*. That is, the *six elements of one triangle* will be *equal* to the *six corresponding elements* of any other *triangle* in this *set*, and the *triangles* will therefore be *equal* in *area*. In other words, if we *cut out a copy* of *one triangle* and *placed* it on *any other triangle* in this *set*, it will fit *exactly* once the *corresponding elements* are *together*. Such *triangles* are said to be *congruent* (or *identical*) *triangles*.

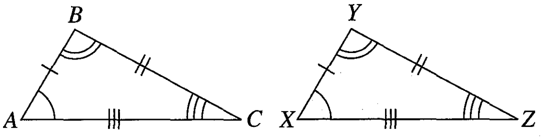


Fig. 9.149 Congruent triangles

Thus:

$AB = XY \qquad BC = YZ \qquad AC = XZ$

And $\hat{A} = \hat{X} \qquad \hat{B} = \hat{Y} \qquad \hat{C} = \hat{Z}$

The *six elements of the triangle ABC* are *equal* to the *six elements of the triangle XYZ*. So *the triangle ABC* is *exactly* the *same* as the *triangle XYZ*. Hence we write $\triangle ABC \equiv \triangle XYZ$.

The symbol $\equiv$ means 'is *congruent to*'.

NOTE: The *corresponding sides* of *two congruent triangles* are those *sides* which *lie opposite equal angles*. The *corresponding angles* of *two congruent triangles* are those *angles* which *lie opposite equal sides*.

Two triangles will be *congruent* if:

(i) *three sides* of *one triangle* are *equal* to the *corresponding three sides of the other triangle* (*S.S.S.*).

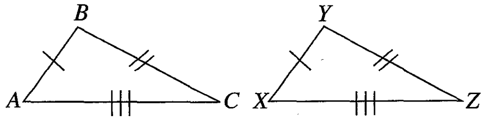


Fig. 9.150 Congruent triangles

(ii) *two sides* and the *included angle* of *one triangle* are *equal* to the *corresponding two sides* and *included angle* of the *other triangle* (*S.A.S.*).

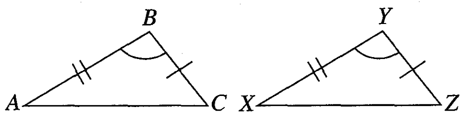


Fig. 9.151 Congruent triangles

(iii) *two angles* and a *side* of *one triangle* are *equal* to the *corresponding two angles* and *side* of the *other triangle* (*A.A.S.*).

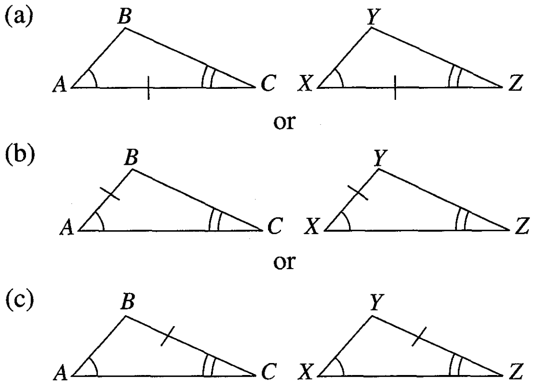


Fig. 9.152 Congruent triangles

(iv) *both triangles* are *right-angled*, and the *hypotenuse* and a *side* of *one triangle* are *equal* to the *hypotenuse* and *corresponding side* of the *other triangle* (*R.H.S.*).

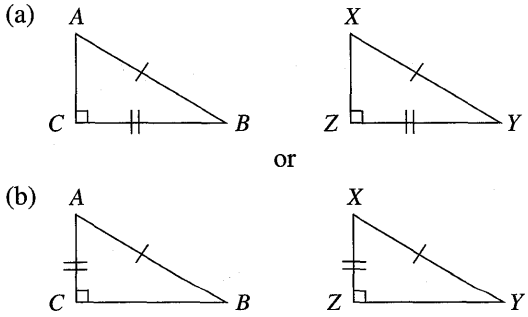


Fig. 9.153 Congruent triangles

Given the Measures of Three Sides

Example 24

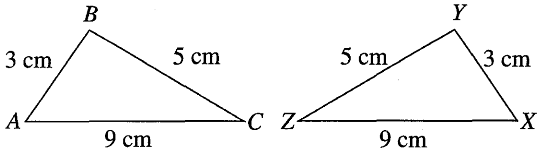


Fig. 9.154 Triangles

Given the triangles ABC and XYZ in Fig. 9.154, prove whether or not the two triangles are congruent.

Solution

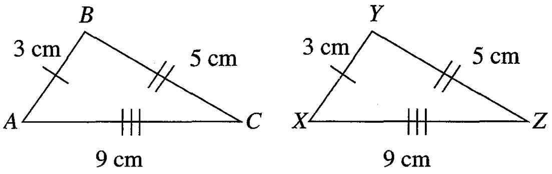


Fig. 9.154 Congruent triangles

Considering $\triangle$*s ABC and XYZ*:

$AB = XY = 3$ cm (*corresponding sides equal*)

$BC = YZ = 5$ cm (*corresponding sides equal*)

$AC = XZ = 9$ cm (*corresponding sides equal*).

Hence $\triangle ABC \equiv \triangle XYZ$ (*S.S.S.*).

So the *two triangles* are *congruent.*

Given the Measures of Two Sides and the Included Angle

Example 25

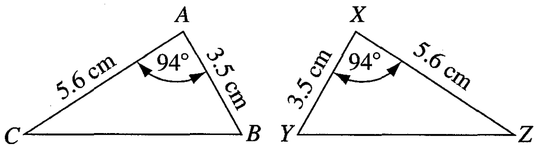


Fig. 9.155 Triangles

Given the triangles ABC and XYZ in Fig. 9.155, prove whether or not the two triangles are congruent.

Solution

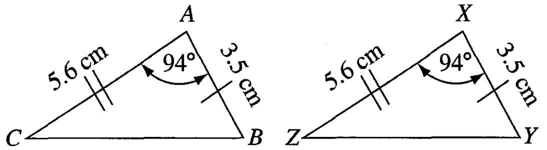


Fig. 9.155 Congruent triangles

Considering $\triangle$*s ABC and XYZ*:

$AB = XY = 3.5$ cm (*corresponding sides equal*)

$\hat{A} = \hat{X} = 94°$ (*included angle equal*)

$AC = XZ = 5.6$ cm (*corresponding sides equal*).

Hence $\triangle ABC \equiv \triangle XYZ$ (*S.A.S.*).

So the *two triangles* are *congruent.*

Given the Measures of Two Angles and a side

Example 26

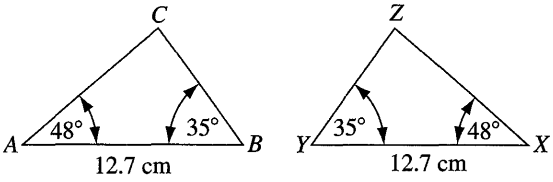


Fig. 9.156 Triangles

Given the triangles ABC and XYZ in Fig. 9.156, prove whether or not the two triangles are congruent.

Solution

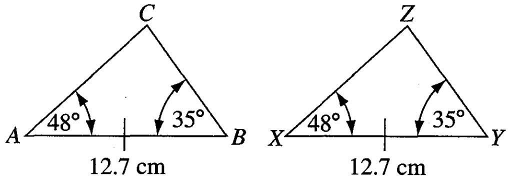


Fig. 9.156 Congruent triangles

Considering $\triangle$*s ABC and XYZ*:

$\hat{A} = \hat{X} = 48°$ (*corresponding angles equal*)

$\hat{B} = \hat{Y} = 35°$ (*corresponding angles equal*)

$AB = XY = 12.7$ cm (*corresponding sides equal*).

Hence $\triangle ABC \equiv \triangle XYZ$ (*A.A.S.*).

So the *two triangles* are *congruent.*

Given the Measures of a Right Angle, Hypotenuse and a Side

Example 27

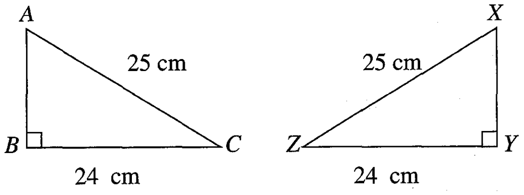


Fig. 9.157 Triangles

Given the triangles ABC and XYZ in Fig. 9.157, prove whether or not the two triangles are congruent.

Solution

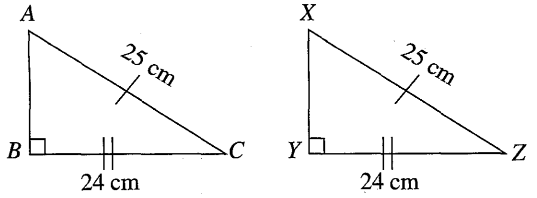


Fig. 9.157 Congruent triangles

Considering $\triangle$*s ABC and XYZ*:

$\hat{B} = \hat{Y} = 90°$ (*right angles given*)

$AC = XZ = 25$ cm (*hypotenuses equal*)

$BC = YZ = 24$ cm (*corresponding sides equal*).

Hence $\triangle ABC \equiv \triangle XYZ$ (*R.H.S.*).

So the *two triangles* are *congruent*.

Properties of Isosceles and Equilateral Triangles

Two very important *properties of an isosceles triangle* and hence of an *equilateral triangle* are *derived* when the *apex angle* (i.e. the angle formed by the two equal sides) is *bisected*.

Example 28

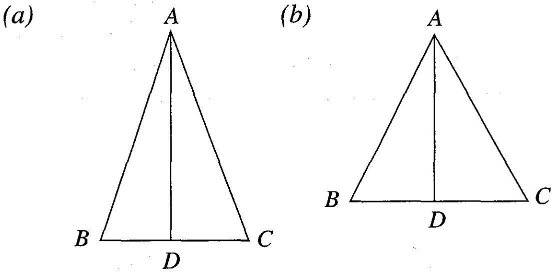


Fig. 9.158 Isosceles triangle Equilateral triangle

ABC is an isosceles triangles or an equilateral triangle in which AB = AC. Angle A is bisected by a straight line which meets the side BC at D.

Prove that BD = CD, and that AD is perpendicular to BC.

Solution

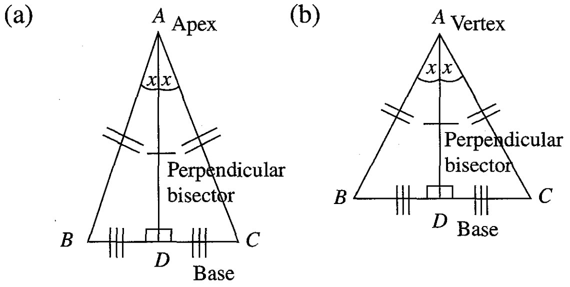


Fig. 9.158 Isosceles triangle Equilateral triangle

Considering the $\triangle$*s ABD* and *ACD*:

$AD = AD$ (*common side*)

$B\hat{A}D = C\hat{A}D = \hat{x}$ (since $\hat{A}$ *is bisected*)

$AB = AC$ (given)

Thus $\triangle ABD \equiv \triangle ACD$ (*S.A.S.*)

Hence $BD = CD$.

And $A\hat{D}B = A\hat{D}C$.

So $A\hat{D}B = A\hat{D}C = \frac{180°}{2} = 90°$ ($\angle$s *on a st. line*)

Hence *AD is perpendicular BC.*

From the previous *example* it can be *seen* that when a *bisector* is *drawn* from the *apex angle* to the *unequal side* (or *base*) of an *isosceles triangle*:

(i) The *apex angle* is *bisected, i.e.* $B\hat{A}D = C\hat{A}D$.

(ii) The *unequal side* (or *base*) is *bisected*, i.e. $BD = CD$.

(iii) The *bisector is perpendicular* to the *unequal side* (or *base*). i.e. $A\hat{D}B = A\hat{D}C = 90°$. Hence the *bisector AD of isosceles triangle ABC* is called a *perpendicular bisector* (or *mediator*).

Exercise 9g

1. Prove whether or not the two triangles are congruent.

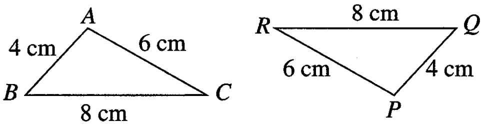


Fig. 9.159 Triangles

2. Prove whether or not the two triangles are congruent.

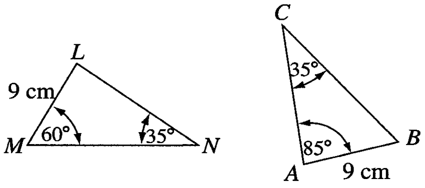


Fig. 9.160 Triangles

3. Prove whether or not the two triangles are congruent.

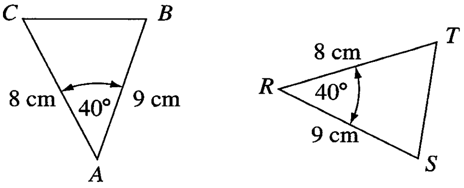


Fig. 9.161 Triangles

4. Prove whether or not the two triangles are congruent.

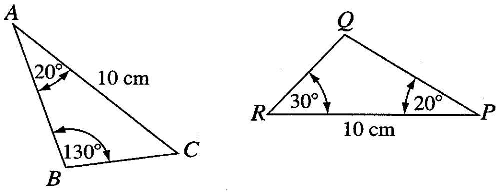


Fig. 9.162 Triangles

5. Prove whether or not the two triangles are congruent.

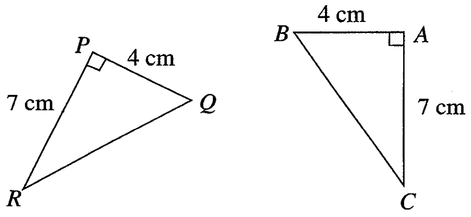


Fig. 9.163 Triangles

6. Prove whether or not the two triangles are congruent.

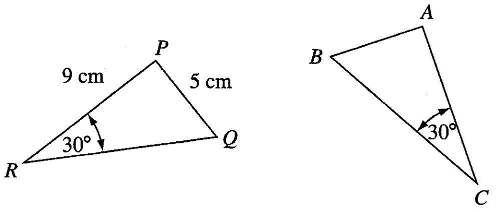


Fig. 9.164 Triangles

7. Prove whether or not the two triangles are congruent.

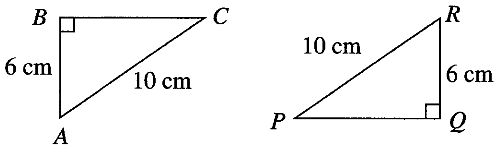


Fig. 9.165 Triangles

8. Prove whether or not the two triangles are congruent.

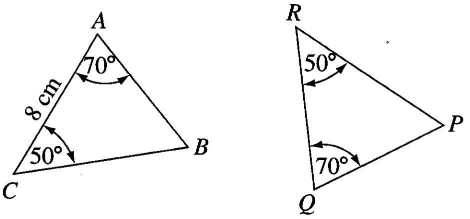


Fig. 9.166 Triangles

9. Prove whether or not the two triangles are congruent.

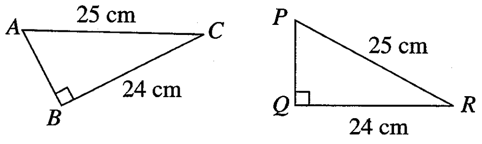


Fig. 9.167 Triangles

10. Prove whether or not the two triangles are congruent.

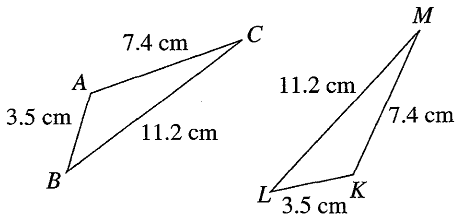


Fig. 9.168 Triangles

11. Prove whether or not the two triangles are congruent.

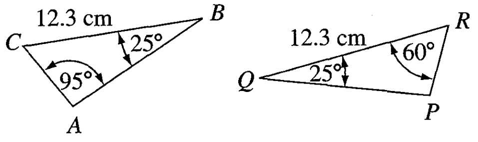


Fig. 9.169 Triangles

12.

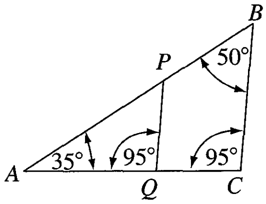


Fig. 9.170 Triangles

Prove whether or not the triangles *ABC* and *APQ* are congruent or not.

13.

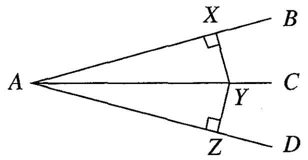


Fig. 9.171 Triangles

In Fig. 9.171, angle *BAD* is bisected by the straight line *AC*, and *XY* = *ZY*. Prove that *AX* = *AZ*.

14.

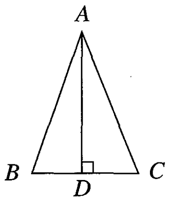


Fig. 9.172 Triangles

In Fig 9.172, triangle *ABC* is such that *BD* = *CD*. Prove that *AB* = *AC*.

15.

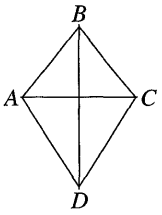


Fig. 9.173 Triangles

In Fig 9.173, *ABC* and *ADC* are isosceles triangles. Prove that triangle *ABD* is congruent to triangle *CBD*.

16.

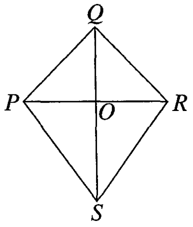


Fig. 9.174 Triangles

In Fig. 9.174, *PQR* and *PSR* are isosceles triangles. Prove that *QS* bisects the angles at *Q* and at *S*.

17.

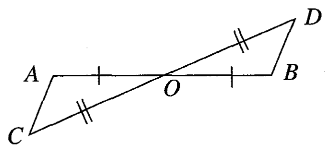


Fig. 9.175 Triangles

In Fig. 9.175, the straight lines bisect each other at *O*. Prove that *AC* = *BD*.

18.

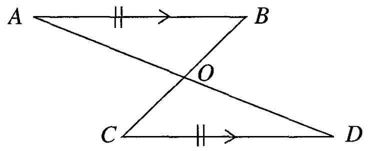


Fig. 9.176 Triangles

In Fig. 9.176, the straight lines *AD* and *BC* intersect at *O* in such a way that *AB* = *CD* and *AB*//*CD*. Prove that *AO* = *DO* and *BO* = *CO*.

19.

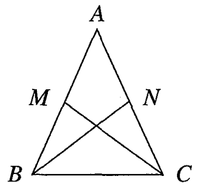


Fig. 9.177 Triangles

In Fig. 9.177, *ABC* is an isosceles triangle such that *M* is the mid-point of *AB* and *N* is the midpoint of *AC*. Prove that *NC* = *MB*.

20.

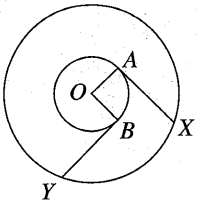


Fig. 9.178 Circles and triangles

In Fig. 9.178, the two circles shown are concentric (i.e. the both have the same centre *O*). *A* and *B* are two points on the inner circle such that *AX* is perpendicular to *OA* and *BY* is perpendicular to *OB*. Prove that *AX* = *BY*.

21.

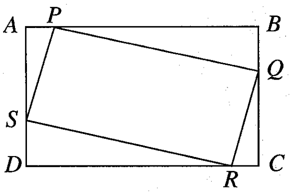


Fig. 9.179 Rectangle and triangles

In Fig. 9.179, *ABCD* is a rectangle such that *AS* = *CQ* and *BP* = *DR*.

Prove that:

(a) $\triangle PAS \equiv \triangle RCQ$

(b) $\triangle QBP \equiv \triangle SDR$

(c) *PS* = *RQ*

(d) *PQ* = *RS*

22.

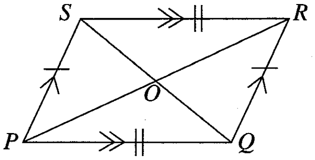


Fig. 9.180 Triangles

In Fig. 9.180, *PQRS* is such that its opposite sides are equal and parallel. *POR* and *QOS* are straight lines.

Prove that:

(a) $\triangle PQS \equiv \triangle RSQ$

(b) $\triangle PQR \equiv \triangle RSP$

23.

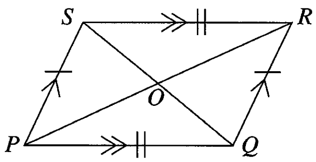


Fig. 9.181 Triangles

In Fig. 9.181, *PQRS* is such that its opposite sides are equal and parallel. *POR* and *QOS* are straight lines.

Prove that:

(a) (i) $\triangle ROS \equiv \triangle POQ$ (ii) *RO* = *PO*
(iii) *SO* = *QO*

(b) (i) $\triangle POS \equiv \triangle ROQ$ (ii) *PO* = *RO*
(iii) *QO* = *SO*

24.

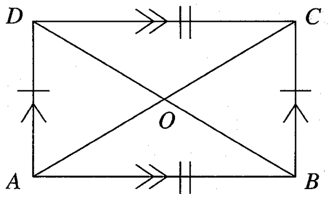


Fig. 9.182 Triangles

In Fig 9.182, *ABCD* is such that its opposite sides are equal and parallel. *AOC* and *BOD* are straight lines.

Prove that:

(a) angle *DOC* = angle *BOA*

(b) angle *AOD* = angle *COB*.

25.

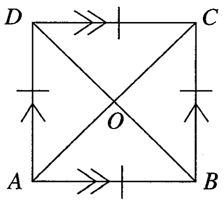


Fig. 9.183 Triangles

In Fig. 9.183, *ABCD* is such that its opposite sides are parallel and all four sides are equal. *AOC* and *BOD* are straight lines.

Prove that:

(a) *BD* bisects angle *B* and angle *D*.

(b) *AC* bisects angle *A* and angle *C*.

(c) *AC* and *BD* are perpendicular.

Similar Triangles

By *deduction*, we can see that *any number of triangles* can be *equi-angular*, that is, have the *same corresponding angles equal*. This *fact* is *illustrated* in Fig. 9.184 below.

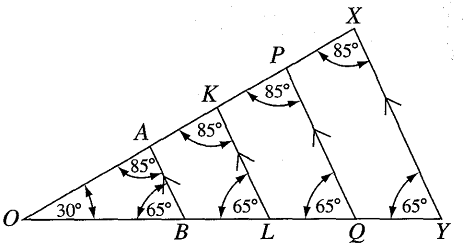


Fig. 9.184 Equi-angular triangles

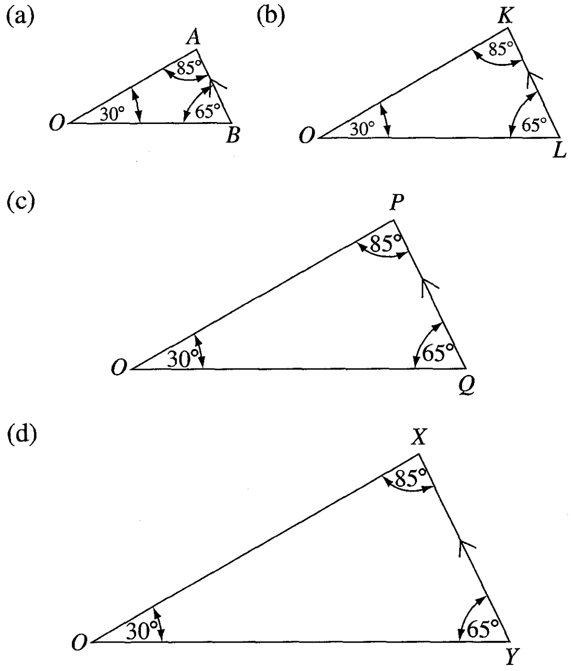


Fig. 9.184 Equi-angular triangles

In Fig. 9.184 above, *each* of the *four equi-angular triangles* has the *corresponding angles equal.* However, these *equi-angular triangles* are *far* from being *congruent. Equi-angular triangles* are said to be *similar. Similar triangles* are said to have *same shape.*

Properties of Similar Triangles

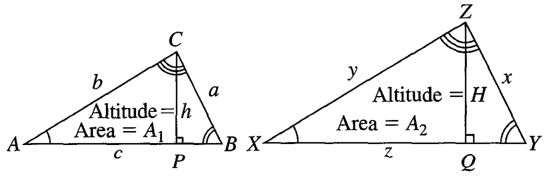


Fig. 9.185 Similar triangles

In $\triangle$s ABC and XYZ shown in Fig. 9.185:

$$\hat{A} = \hat{X},\ \hat{B} = \hat{Y} \text{ and } \hat{C} = \hat{Z}$$

Hence the *two triangles* are *similar*.

Thus we write $\dfrac{\triangle XYZ}{\triangle ABC}$ are *similar*. Or we can write $\triangle ABC$ and $\triangle XYZ$ are similar.

When *two triangles* are *similar*, the *length* of a side of *one triangle* is *k times* the *corresponding length* of the side of the *other triangle. k* is a constant called the scale factor.

(i) Since the *two triangles* are *similar*, then:

$$\frac{YZ}{BC} = \frac{XZ}{AC} = \frac{XY}{AB} = \frac{ZQ}{CP} \quad = k$$

$$\text{or } YZ{:}BC = XZ{:}AC = XY{:}AB = ZQ{:}CP \quad = k$$

$$\text{i.e.}\quad \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{H}{h} \quad = k$$

$$\text{or}\quad x{:}a = y{:}b = z{:}c = H{:}h \quad = k$$

Thus: $YZ = k \cdot BC$, $XZ = k \cdot AC$, $XY = k \cdot AB$ and $ZQ = k \cdot CP$

i.e. $x = k \cdot a$, $y = k \cdot b$, $z = k \cdot c$ and $H = k \cdot h$

When *two triangles* are *similar*:

1. the *area* of *one triangle* is k^2 *times* the *area* of the *other triangle.*
2. the *square* of the *length* of a side of *one triangle* is k^2 *times* the *square* of the *corresponding length* of the side of the *other triangle.* k is a constant called the scale factor.

(ii)
$$\frac{\textit{The area of } \triangle XYZ}{\textit{The area of } \triangle ABC} = \frac{YZ^2}{BC^2} = \frac{XZ^2}{AC^2} = \frac{XY^2}{AB^2} = \frac{ZQ^2}{CP^2} = k^2$$

or *The area of* $\triangle XYZ$:*The area of* $\triangle ABC$

$$= YZ^2{:}BC^2 = XZ^2{:}AC^2 = XY^2{:}AB^2$$

$$= ZQ^2{:}CP^2 = k^2$$

$$\text{i.e.}\quad \frac{A_2}{A_1} = \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{H^2}{h^2} = k^2$$

or $A_2{:}A_1 = x^2{:}a^2 = y^2{:}b^2 = z^2{:}c^2 = H^2{:}h^2 = k^2$

Thus:

The area of $\triangle XYZ = k^2 \cdot$ *The area of* $\triangle ABC$,
$YZ^2 = k^2 \cdot BC^2, XZ^2 = k^2 \cdot AC^2, XY^2 = k^2 \cdot AB^2$ and $ZQ^2 = k^2 \cdot CP^2$

i.e. $A_2 = k^2 \cdot A_1, x^2 = k^2 \cdot a^2, y^2 = k^2 \cdot b^2, z^2 = k^2 \cdot c^2$ and $H^2 = k^2 \cdot h^2$

Example 29

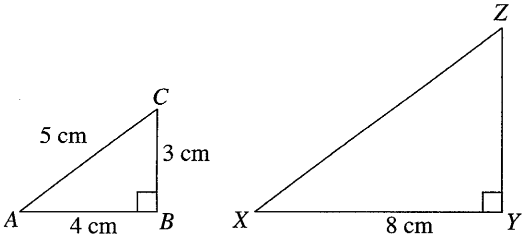


Fig. 9.186 Similar triangles

In Fig. 9.186, the △s ABC and XYZ are similar.

(a) Calculate the scale factor, k.

(b) Hence, determine the length of:

(i) YZ (ii) XZ

(c) If the area of △ABC is 6 cm², calculate the area of △XYZ.

(d) Use ratios to determine the length of:

(i) YZ (ii) XZ

Solution

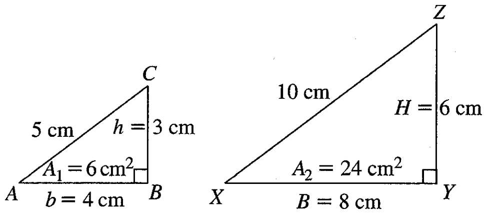


Fig. 9.186 Similar triangles

(a) The *scale factor*, $k = \dfrac{XY}{AB} = \dfrac{8\ \cancel{cm}}{4\ \cancel{cm}} = 2$

(b) (i) The *length of YZ* $= k \cdot BC = 2 \times 3\text{ cm}$
$= 6\text{ cm}$

(ii) The *length of XZ* $= k \cdot AC = 2 \times 5\text{ cm}$
$= 10\text{ cm}$

(c) The *area of* △*XYZ* $= k^2 \cdot$ *The area of* △*ABC*
$= 2^2 \times 6\text{ cm}^2$
$= 4 \times 6\text{ cm}^2$
$= 24\text{ cm}^2$

(d) (i) Now $\dfrac{YZ}{BC} = \dfrac{XY}{AB}$

So $\dfrac{YZ}{3\text{ cm}} = \dfrac{8\ \cancel{cm}}{4\ \cancel{cm}} = 2$

i.e. $YZ = 2 \times 3\text{ cm}$

∴ $YZ = 6\text{ cm}$

(ii) Now $\dfrac{XZ}{AC} = \dfrac{XY}{AB}$

So $\dfrac{XZ}{5\text{ cm}} = \dfrac{8\ \cancel{cm}}{4\ \cancel{cm}} = 2$

i.e. $XZ = 2 \times 5\text{ cm}$

∴ $XZ = 10\text{ cm}$

NOTE:

The *area of* △*ABC*, $A_1 = \dfrac{1}{2}bh$

$= \dfrac{1}{2} \times 4\text{ cm} \times 3\text{ cm}$

$= 2\text{ cm} \times 3\text{ cm}$

$= 6\text{ cm}^2$

And the *area of* △*XYZ*, $A_2 = \dfrac{1}{2}BH$

$= \dfrac{1}{2} \times 8\text{ cm} \times 6\text{ cm}$

$= 4\text{ cm} \times 6\text{ cm}$

$= 24\text{ cm}^2$

Calculating the area of a triangle *using* the *similar triangles method* is consistent with the results obtained when using the formula,

$A = \dfrac{1}{2} \times \text{base} \times \text{altitude}.$

Exercise 9h

1. Prove that the two triangles are similar.

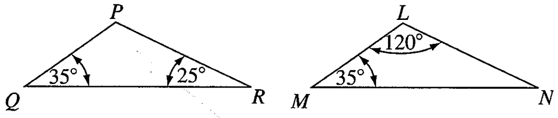


Fig. 9.187 Triangles

2. Prove that the two triangles are similar.

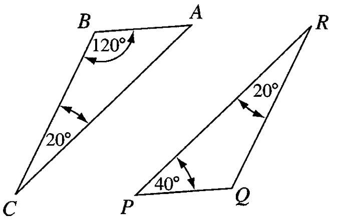


Fig. 9.188 Triangles

3. Prove that the two triangles are similar.

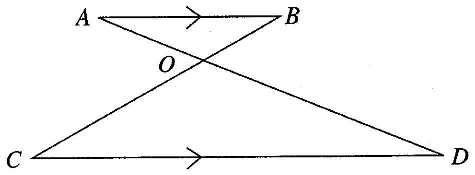


Fig. 9.189 Triangles

4. Prove that the two triangles are similar.

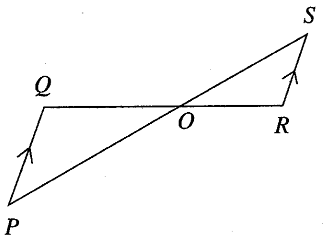


Fig. 9.190 Triangles

5. Prove that the two triangle are similar.

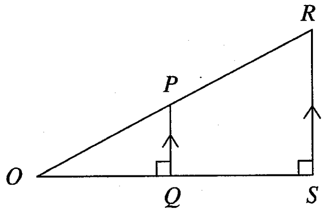


Fig. 9.191 Triangles

6.

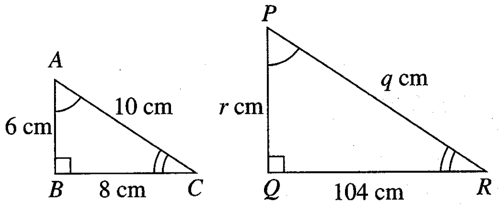


Fig. 9.192 Triangles

(a) Prove that the two triangles ABC and PQR are similar.

(b) Hence, calculate the value of the scale factor k.

(c) Use ratios to evaluate side:

(i) r in cm (ii) q in cm.

(d) Given that area of $\triangle ABC$ is 24 cm^2, determine the area of $\triangle PQR$.

7.

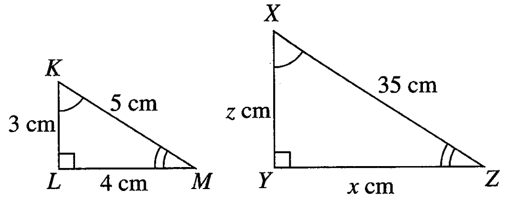


Fig. 9.193 Triangles

(a) Prove that the $\triangle$s KLM and XYZ are similar.

(b) Hence, calculate the value of the scale factor k.

(c) Use ratios to evaluate side:

(i) x in cm (ii) z in cm.

(d) Given that area of $\triangle KLM$ is 6 cm^2, calculate the area of $\triangle XYZ$.

8.

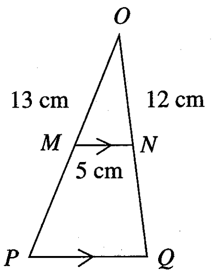


Fig. 9.194 Triangles

In Fig. 9.194, M and N are the mid-points of OP and OQ respectively.

(a) Prove that the triangles MON and POQ are similar.

(b) Then calculate the value of the scale factor k.

(c) Hence, determine the length:

(i) OP (ii) OQ (iii) PQ

(d) If the area of $\triangle MON$ is 30 cm^2, calculate the area of $\triangle POQ$.

9.

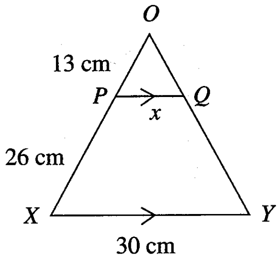


Fig. 9.195 Triangles

(a) Prove that $\triangle$s POQ and XOY are similar.

(b) Use ratios to determine the value of x.

(c) Given that the area of $\triangle XOY$ is 540 cm^2, calculate the area of $\triangle POQ$.

10.

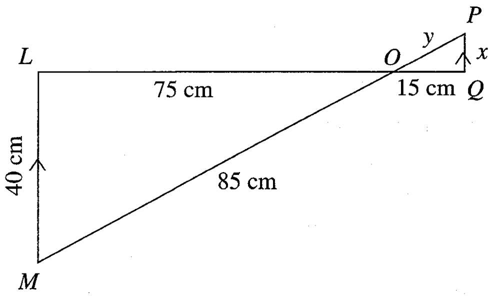


Fig. 9.196 Triangles

(a) Prove that $\triangle$s POQ and MOL are similar.

(b) Use ratios to determine the value of:
  (i) x in cm (ii) y in cm.

(c) Given that the area of $\triangle MOL$ is 1 500 cm^2, calculate the area of $\triangle POQ$.

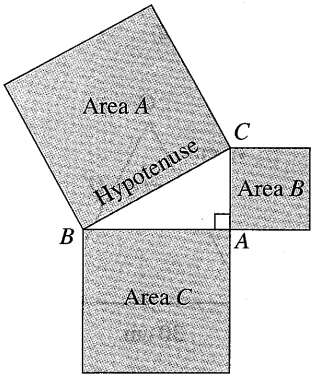
Pythagoras' Theorem

Pythagoras' theorem is a fundamental and very important theorem in Mathematics.

Pythagoras' theorem states that in *any right-angled triangle*, the *square on the hypotenuse* is *equal* to the *sum of the squares* on the *two other sides.*

Thus:

(a)



(b)

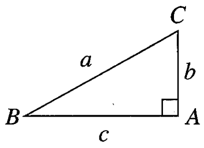


Fig. 9.197 Right-angled triangle

The *area A* = *The area B* + *The area C.*

or $BC^2 = AC^2 + AB^2$

or $a^2 = b^2 + c^2$.

Values of a, b and c which *form right-angled triangles* are called *Pythagorean triples*. Some *Pythagorean triples* worth remembering are:

{3, 4, 5}, {5, 12, 13}, {7, 24, 25}, {8, 15, 17}, {12, 35, 37} and {20, 21, 29}.

Using the *Pythagorean triple* 3, 4 and 5, we get the following *diagram*:

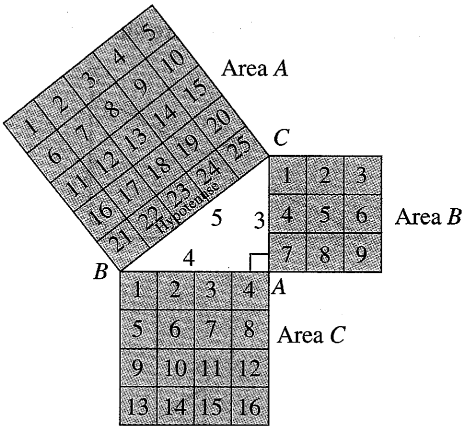


Fig. 9.198 Right-angled triangle

From Fig. 9.198:

The *area A* = 25 square units

The *area B* = 9 square units

The *area C* = 16 square units

So the *area B* + *The area C* = (9 + 16) square units
= 25 square units

Hence the *area A* = *The area B* + *The area C*
= 25 square units

So the *result* is *consistent* with what is *expected* according to *Pythagoras' theorem.*

Example 30

(a) In $\triangle ABC$, $\hat{B} = 90°$, $a = 8.3$ cm and $c = 5.2$ cm. Evaluate b.

(b) In $\triangle PQR$, $\hat{P} = 90°$, $p = 12.5$ cm and $r = 6.1$ cm. Evaluate q.

(c) In $\triangle KLM$, $\hat{M} = 90°$, $m = 13.4$ cm and $k = 9.7$ cm. Evaluate l.

Solution

(a)

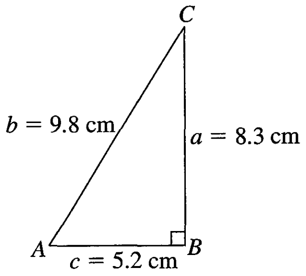


Fig. 9.199 Right-angled triangle

Considering the *right-angled triangle ABC* and using *Pythagoras' theorem*:

$$AC^2 = BC^2 + AB^2$$

So $b^2 = a^2 + c^2$

$= (8.3 \text{ cm})^2 + (5.2 \text{ cm})^2$

$= 68.89 \text{ cm}^2 + 27.04 \text{ cm}^2$

$= 95.93 \text{ cm}^2$

$\therefore \quad b = \sqrt{95.93 \text{ cm}^2}$

$= 9.79$ cm

$= 9.8$ cm (*correct* to 1 *decimal place*).

Hence $b = 9.8$ cm.

(b)

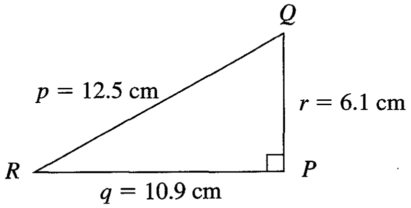


Fig. 9.200 Right-angled triangle

Considering the *rt. ∠ed.* $\triangle PQR$ and using *Pythagoras' theorem*:

$$QR^2 = PQ^2 + PR^2$$

So $\quad p^2 = r^2 + q^2$

And $\quad q^2 = p^2 - r^2$

$= (12.5 \text{ cm})^2 - (6.1 \text{ cm})^2$

$= 156.25 \text{ cm}^2 - 37.21 \text{ cm}^2$

$= 119.04 \text{ cm}^2$

$\therefore \quad q = \sqrt{119.04 \text{ cm}^2}$

$= 10.91$ cm

$= 10.9$ cm (*correct* to 1 *decimal place*)

Hence $q = 10.9$ cm.

(c)

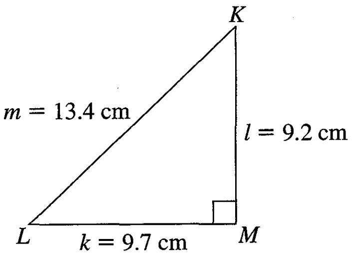


Fig. 9.201 Right-angled triangle

Considering the *rt. ∠ed.* △ and using *Pythagoras' theorem*:

$$m^2 = l^2 + k^2$$

So $\quad l^2 = m^2 - k^2$

$= (13.4 \text{ cm})^2 - (9.7 \text{ cm})^2$

$= 179.56 \text{ cm}^2 - 94.09 \text{ cm}^2$

$= 85.47 \text{ cm}^2$

$\therefore \quad l = \sqrt{85.47 \text{ cm}^2}$

$= 9.24$ cm

$= 9.2$ cm (*correct* to 1 *decimal place*)

Hence $l = 9.2$ cm.

Example 31

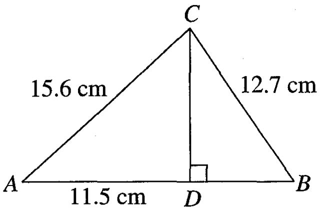


Fig. 9.202 Triangle

In triangle ABC above, $AC = 15.6$ cm, $AD = 11.5$ cm, $BC = 12.7$ cm and CD is perpendicular to AB.

Calculate the length of:

(a) CD *(b) AB*

Solution

(a)

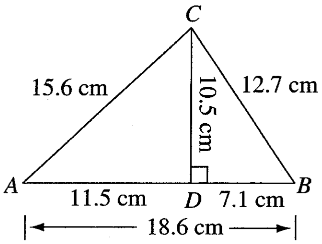


Fig. 9.202 Triangle

Considering the *rt. ∠ed.* $\triangle ACD$ and using *Pythagoras' theorem*:

$$AC^2 = CD^2 + AD^2$$

So $CD^2 = AC^2 - AD^2$

$= (15.6 \text{ cm})^2 - (11.5 \text{ cm})^2$

$= 243.36 \text{ cm}^2 - 132.25 \text{ cm}^2$

$= 111.11 \text{ cm}^2$

$\therefore$ $CD = \sqrt{111.11 \text{ cm}^2}$

$= 10.54 \text{ cm}$

$= 10.5 \text{ cm}$ (*correct* to 1 *decimal place*)

Hence CD is 10.5 cm in length.

(b) Considering the *rt. ∠ed.* $\triangle BCD$ and using *Pythagoras' theorem*:

$$BC^2 = CD^2 + BD^2$$

So $BD^2 = BC^2 - CD^2$

$= (12.7 \text{ cm})^2 - (10.5 \text{ cm})^2$

$= 161.29 \text{ cm}^2 - 110.25 \text{ cm}^2$

$= 51.04 \text{ cm}^2$

$\therefore$ $BD = \sqrt{51.04 \text{ cm}^2}$

$= 7.14 \text{ cm}$

$= 7.1 \text{ cm}$ (*correct* to 1 *decimal place*)

And $AB = AD + BD$

$= (11.5 + 7.1) \text{ cm}$

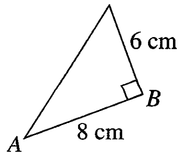
$= 18.6 \text{ cm}$

Hence AB is 18.6 cm in length.

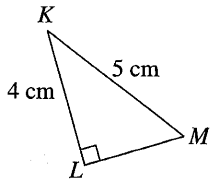
Exercise 9i

1. Calculate the length of each of the following unknown sides:

(a)



(b)



(c)

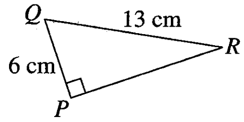


Fig. 9.203 Triangles

2.

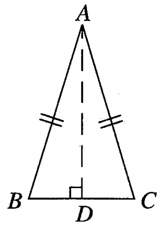


Fig. 9.204 Triangle

In $\triangle ABC$ above, $AB = AC = 24$ cm and $BC = 14$ cm. Determine the altitude of the isosceles triangle.

3.

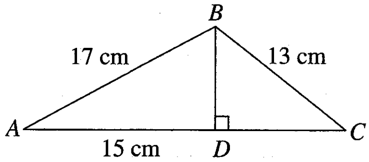


Fig. 9.205 Triangle

In $\triangle ABC$ above, $AB = 17$ cm, $AD = 15$ cm, $BC = 13$ cm and BD is perpendicular to AC. Calculate the length of: (a) BD (b) AC

4.

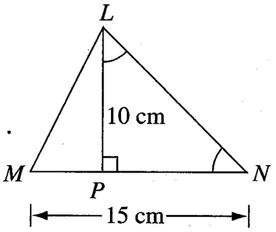


Fig. 9.206 Triangle

In $\triangle LMN$ above, $MN = 15$ cm. LP is perpendicular to MN. $LP = 10$ cm and angle PLN = angle PNL.

(a) State the length of PN. Give a reason for your answer.

(b) Calculate the length of: (i) LN (ii) LM

5.

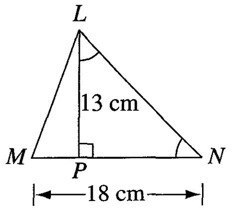


Fig. 9.207 Triangle

In triangle *LMN* above, $MN = 18$ cm. *LP* is perpendicular to *MN*. $LP = 13$ cm and angle $PLN =$ angle *PNL*.

(a) State the length of *PN*. Give a reason for your answer.

(b) Calculate the length of: (i) *LM* (ii) *LN*

6.

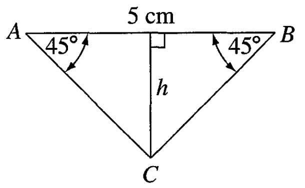


Fig. 9.208 Triangle

In $\triangle ABC$ above, $AB = 5$ cm, angle $A =$ angle $B = 45°$ and the altitude $= h$.

(a) Determine the value of *h*.

(b) Hence calculate the length of *AC*.

7.

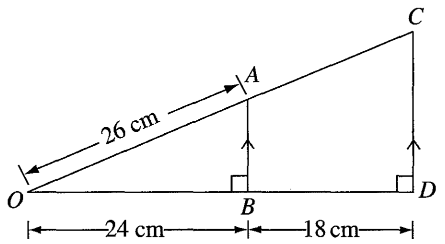


Fig. 9.209 Triangle

In the figure above, $OA = 26$ cm, $OB = 24$ cm and $BD = 18$ cm.

(a) Calculate the length of *AB*.

(b) Determine the length of *CD*.

8. A vertical tower *AB* which is 15.6 m high was built on level ground. The distance of a point *C* on the ground from the base of the tower is 9.5 m. Calculate the distance from the top of the tower to the point *C*.

9. A point *R* on level ground is situated 5.4 m from the base of a vertical tree. The distance from the top of the tree *P* to the point *R* is 12.5 m.
Calculate the height of the tree *PQ*.

10. The height of a vertical lamp-post *XY* which was placed in level ground is 10.5 m. The distance from the top of the lamp-post *X* to a point *Z* on the ground is 15.4 m. Calculate the distance of the point *Z* from the base of the lamp-post *Y*.

11.

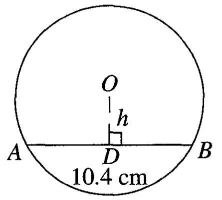


Fig. 9.210 Circle

A circle with centre *O* has a radius of 7 cm. The length of a chord *AB* is 10.4 cm. Determine the distance of the chord from the centre of the circle.

12. The slant height of a cone is 15 cm and the base radius is 6 cm. Calculate the height of the cone *h*.

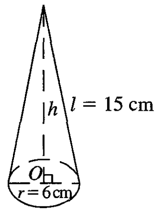


Fig. 9.211 Cone

13.

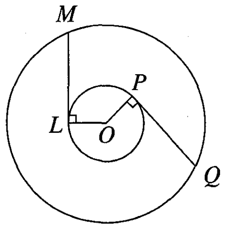


Fig. 9.212 Concentric circles

The figure above shows two circles with their centres at *O*. The radius of the smaller circle is 6 cm, $\angle MLO = 90°$, $\angle OPQ = 90°$ and $ML = 8$ cm.

(a) Determine the length of *PQ*, stating reasons.

(b) Calculate the radius of the larger circle.

14.

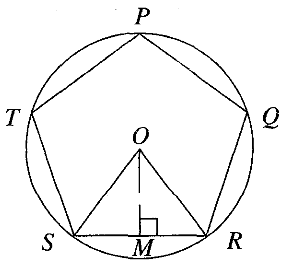


Fig. 9.213 Pentagon

PQRST is a regular pentagon (i.e. a plane figure with 5 equal sides) inscribed in a circle centre *O*. radius 37 cm. *M* is the mid-point of *RS* and *MO* = 29.9 cm.

(a) Calculate: (i) *MR* (ii) *RS*

(b) Hence, determine the perimeter of the pentagon.

15.

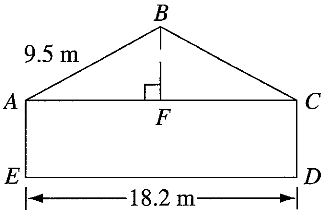


Fig. 9.214 End of a house

Fig. 9.214 illustrates the cross-section of a building of width *DE* = 18.2 m and roof slope *AB* = 9.5 m. Calculate the altitude *BF* of the roof.

Quadrilateral

A *quadrilateral* is a *plane shape* (or *figure*) *bounded* by *four straight lines*.

A *quadrilateral* can also be defined as a *four-sided polygon*.

An example of a *quadrilateral* can be seen in Fig. 9.215.

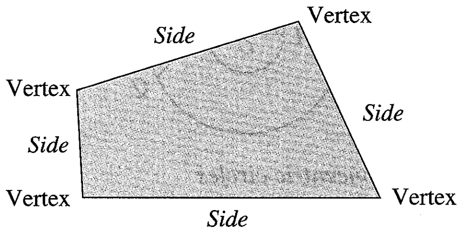
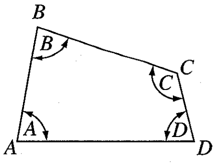


Fig. 9.215 Quadrilateral

A *quadrilateral* has no *thickness*.

Elements of a Quadrilateral

(a)



Or

(b)

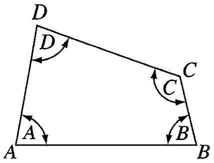


Fig. 9.216 Quadrilaterals

Fig. 9.216 indicates the *eight elements* of a *quadrilateral.*

The *vertices* are normally *denoted* by *four consecutive capital letters* from the *alphabet* and *written* in a *clockwise* or *anti-clockwise direction* in the *diagram*. For example: *A*, *B*, *C* and *D*.

An *angle* is *denoted* by the *same capital letter* as its *vertex*. Thus:

angle $A = \hat{A} = B\hat{A}D = D\hat{A}B$,

angle $B = \hat{B} = A\hat{B}C = C\hat{B}A$,

angle $C = \hat{C} = B\hat{C}D = D\hat{C}B$ and

angle $D = \hat{D} = A\hat{D}C = C\hat{D}A$.

Angles A, *B*, *C* and *D* are *interior angles* of the *quadrilateral ABCD.*

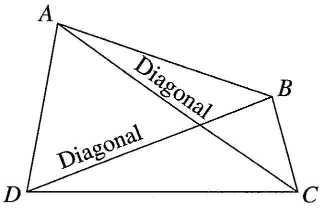


Fig. 9.217 Quadrilateral

Vertices on the *same side* are called *adjacent vertices*. For example: *A* and *B* are *adjacent vertices*. *A* and *D* are *adjacent vertices*.

Vertices directly across are called *opposite vertices*. For example: *A* and *C* are *opposite vertices*. *B* and *D* are *opposite vertices*.

Line segments joining *opposite vertices* are called *diagonals*. For example: *AC* is a *diagonal*. *BD* is a *diagonal.*

Two sides sharing a *common vertex* are called *adjacent sides*. For example: *AB* and *BC* are *adjacent sides*. *AD* and *DC* are *adjacent sides*.

Sides directly across are called *opposite sides*. For example: *AB* and *DC* are *opposite sides*. *AD* and *BC* are *opposite sides*.

Two angles sharing a *common side* are called *adjacent angles*. For example: $\hat{A}$ and $\hat{B}$ are *adjacent angles*. $\hat{A}$ and $\hat{D}$ are *adjacent angles*.

Angles directly across are called *opposite angles*. For example: $\hat{A}$ and $\hat{C}$ are *opposite angles*. $\hat{B}$ and $\hat{D}$ are *opposite angles*.

Types of Quadrilateral

Quadrilaterals can be *classified* according to certain *unique properties*.

Table 9.4

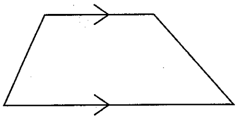
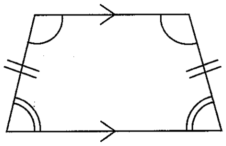
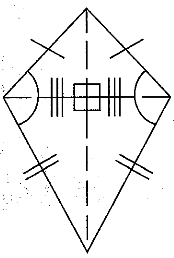
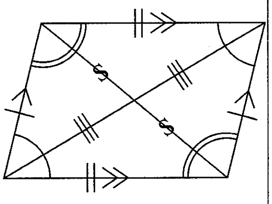
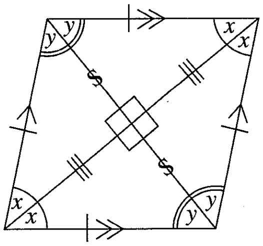
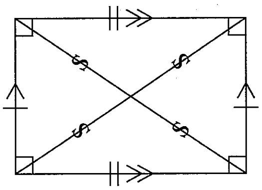
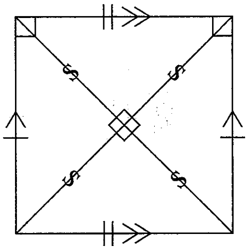
Classification of Quadrilaterals	
Quadrilateral	*Properties*
 *Trapezium*	1. *One pair* of *opposite sides parallel.*
 *Isosceles trapezium* (special case)	1. *One pair* of *opposite sides parallel.* 2. *The pair* of *non-parallel sides* are *equal.* 3. *Two pairs* of *equal adjacent angles.*
 *Kite*	1. *Two pairs* of *equal adjacent sides.* 2. *One pairs* of *equal opposite angles.* 3. *Diagonals intersect* at *right angles.* 4. *One diagonal* is *bisected* by the *other diagonal.* 5. *Two pairs* of *congruent triangles* are *formed* by the *diagonals.*
 *Parallelogram*	1. *Opposite sides* are *parallel.* 2. *Opposite sides* are *equal.* 3. *Opposite angles* are *equal.* 4. *Diagonals bisect each other.* 5. *Two pairs* of *congruent triangles are formed* by the *diagonals.*

Table 9.4 Continued

Classification of Quadrilaterals	
Quadrilateral	*Properties*
 *Rhombus*	1. *Opposite sides* are *parallel.* 2. *All four sides* are *equal.* 3. *Opposite angles* are *equal.* 4. *Diagonals bisect* each other at *right angles.* 5. *Diagonals bisect* the *angles* at the *vertices.* 6. *Four congruent triangles* are formed by the *diagonals.*
 *Rectangle*	1. *Opposite sides* are *parallel.* 2. *Opposite sides* are *equal.* 3. *All four angles* are *right angles.* 4. *Diagonals bisect each other.* 5. *Diagonals* are *equal* in *length.* 6. *Two pairs* of *congruent triangles* are *formed* by the *diagonals.*
 *Square*	1. *Opposite sides* are *parallel.* 2. *All four sides* are *equal.* 3. *All four angles* are *right angles.* 4. *Diagonals bisect* each other at *right angles.* 5. *Diagonals bisect* the *angles* at the *vertices.* Hence *each angle formed* is *equal to* 45°. 6. *Diagonals* are *equal* in *length.* 7. *Four congruent triangles* are *formed* by the *diagonals.*

From the *properties of quadrilaterals* stated above it can be *seen* that:

1. A *square* is a *rectangle* with *equal sides.*
2. A *square* is a *rhombus* with *each angle equal* to 90°.

3. A *square* is a *parallelogram* with *equal sides* and with *each angle equal* to 90°.
4. A *rectangle* is a *parallelogram* with *each angle equal* to 90°.
5. A *rhombus* is a *parallelogram* with *equal sides.*

These *facts* are illustrated in Fig. 9.218 below.

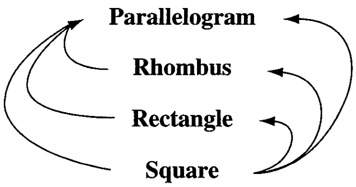


Fig. 9.218 Quadrilaterals

We can also *represent* the above stated *facts* on a *Venn diagram* as shown below.

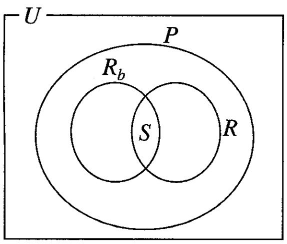


Fig. 9.219 Venn diagram

Where $U = \{\text{quadrilaterals}\}$

$S = \{\text{squares}\}$

$R = \{\text{rectangles}\}$

$R_h = \{\text{rhombuses}\}$

and $P = \{\text{parallelograms}\}$.

For *simplicity* then, we can *define*:

(i) a *parallelogram* as a *quadrilateral* in which *opposite sides are parallel.* Thus $P \subset U$.

(ii) a *rhombus* as a *parallelogram* in which *all its sides are equal.* Thus $R_h \subset P$.

(iii) a *rectangle* as a *parallelogram* having *four right angles*. Thus $R \subset P$.

(iv) (a) a *square* as a *rectangle* in which *all* its *sides are equal.* Thus $S \subset R$.

  (b) a *square* as a *rhombus* having *four right angles*. Thus $S \subset R_h$ *and* $R_h \cap R = S$.

These *facts* can all be *seen* in the *Venn* diagram above.

For *simplicity*, we can *define*:

(i) a *trapezium* as a *quadrilateral* with a pair of *parallel sides.*

(ii) a *kite* as a *quadrilateral* with *two pairs* of *equal adjacent sides.*

Angle Properties of a Quadrilateral

There are *two theorems* that we need to look at under this heading.

THEOREM 1: *The sum of the four interior angles of a quadrilateral is equal to* 360° (*or* 4 *right angles*).

Class Activity

Take a *ruler* and *pencil* and *draw* your *own quadrilateral.* Now take your *protractor* and *measure each angle.* After you have obtained the *magnitudes of the four angles—sum them. What* do you *observe*?

Example:

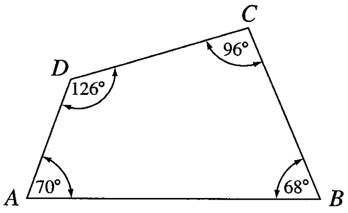


Fig. 9.220 Quadrilateral

By *measurement*:

$\hat{A} = 70°$, $\hat{B} = 68°$, $\hat{C} = 96°$ and $\hat{D} = 126°$.

So the *sum* of the *interior angles* of the *quadrilateral ABCD*, $S = \hat{A} + \hat{B} + \hat{C} + \hat{D}$

$$= 70° + 68° + 96° + 126°$$

$$= 360°$$

$$= 4 \text{ rt. } \angle\text{s}$$

Alternatively, the *quadrilateral ABCD* can be *divided* into *two triangles* by *joining opposite vertices* as shown in Fig. 9.221 below.

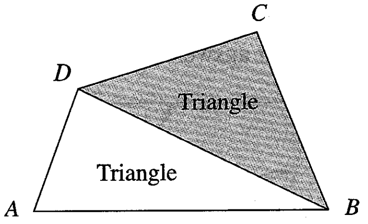


Fig. 9.221 Triangles

Hence the *sum* of the *interior angles* of the *quadrilateral ABCD, S* = The *sum* of the *interior angles* of *triangles ABD and BCD*

$= 2 \times 180°$

$= 360°$

Alternatively, the *sum* of the *interior angles* of a *quadrilateral* ($n = 4$), $S = (2n - 4)$ rt. ∠s

$= (2 \times 4 - 4)$ rt. ∠s

$= (8 - 4)$ rt. ∠s

$= 4$ rt. ∠s

THEOREM 2: *The sum of the four exterior angles of a quadrilateral is equal to 360° (or 4 right angles).*

Class Activity

Using the *quadrilateral* that you had *drawn* previously, *produce sides AB to W, BC to X, CD to Y and DA to Z.* Then use your *protractor* to *measure* the *four exterior angles, CBW, DCX, ADY and BAZ.* Now *sum* the *four exterior angles. What* do you *observe?*

Example:

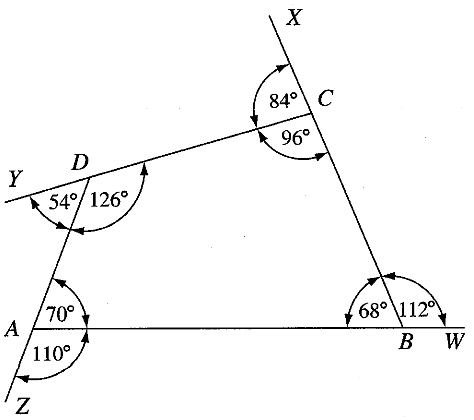


Fig. 9.222 Quadrilateral

By *measurement*:

$C\hat{B}W = 112°$, $D\hat{C}X = 84°$, $A\hat{D}Y = 54°$ and $B\hat{A}Z = 110°$.

The *sum* of the *exterior angles* of the *quadrilateral ABCD*, $S = C\hat{B}W + D\hat{C}X + A\hat{D}Y + B\hat{A}Z$

$= 112° + 84° + 54° + 110°$

$= 360°$

$= 4$ rt. ∠s

NOTE: It can be *seen* that the *sum* of an *exterior angle* of a *quadrilateral* and its *interior adjacent angle* is always *equal to* 180°.

This is so since the *sum* of the *adjacent angles* on a *straight line* is *equal to* 180°.

Thus $B\hat{A}Z + \hat{A} = 110° + 70° = 180°$

$C\hat{B}W + \hat{B} = 112° + 68° = 180°$

$D\hat{C}X + \hat{C} = 84° + 96° = 180°$

and $A\hat{D}Y + \hat{D} = 54° + 126° = 180°$

Example 32

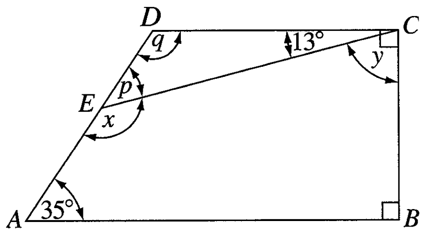


Fig. 9.223 Quadrilateral

Calculate each of the unknown angles in Fig. 9.223 above. Give a reason for each of your answers.

Solution

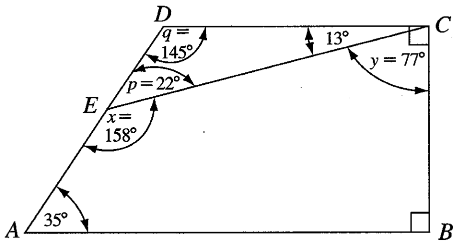


Fig. 9.223 Quadrilateral

Considering the *right angle BCD*:

$\hat{y} + 13° = 90°$ (*given*)

So $\hat{y} = 90° - 13° = 77°$

Considering the *quadrilateral ABCE*:

$\hat{x} + 77^\circ + 90^\circ + 35^\circ = 360^\circ$ ($\angle$*s of a quad.*)

So $\hat{x} + 202^\circ = 360^\circ$

i.e. $\hat{x} = 360^\circ - 202^\circ = 158^\circ$

Considering the *straight line AED*:

$\hat{p} + 158^\circ = 180^\circ$ ($\angle$*s on a straight line*)

So $\hat{p} = 180^\circ - 158^\circ = 22^\circ$

Considering the *triangle CDE*:

$\hat{q} + 13^\circ + 22^\circ = 180^\circ$ ($\angle$*s of a* $\triangle$)

So $\hat{q} + 35^\circ = 180^\circ$

i.e. $\hat{q} = 180^\circ - 35^\circ = 145^\circ$

Hence $\hat{x} = 158^\circ$, $\hat{y} = 77^\circ$, $\hat{p} = 22^\circ$ and $\hat{q} = 145^\circ$.

Exercise 9j

1. Calculate the size of angle x. Give a reason for your answer.

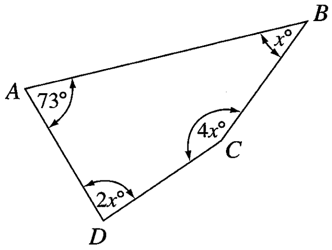


Fig. 9.224 Quadrilateral

2. Determine the magnitude of angle d. Give a reason for your answer.

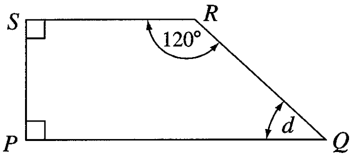


Fig. 9.225 Quadrilateral

3. Calculate the sizes of angles d and e. Give reasons for your answers.

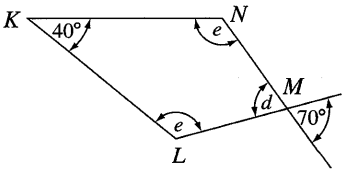


Fig. 9.226 Quadrilateral

4. Calculate the magnitude of angle d. Give a reason for your answer.

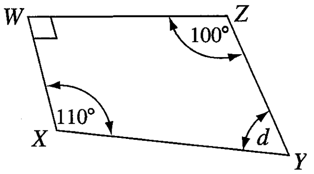


Fig. 9.227 Quadrilateral

5. Determine the sizes of angles a and b. Give reasons for your answers.

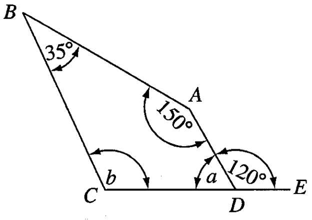


Fig. 9.228 Quadrilateral

6. Calculate the magnitude of angle d. Give a reason for your answer.

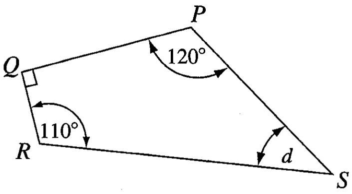


Fig. 9.229 Quadrilateral

7. Evaluate angles e and f. Give reasons for your answers.

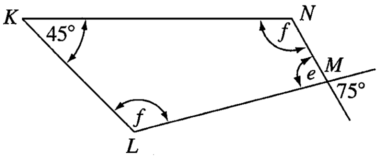


Fig. 9.230 Quadrilateral

8. Evaluate angles x and y. Give reasons for your answers.

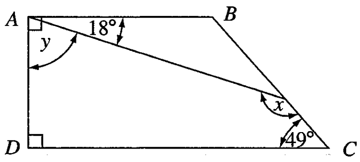


Fig. 9.231 Quadrilateral

9. Determine the sizes of angles x and y. Give reasons for your answers.

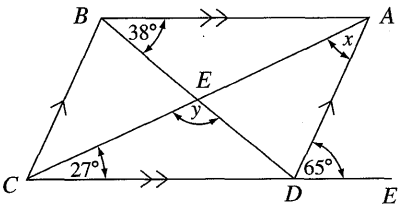


Fig. 9.232 Quadrilateral

10. Determine the magnitude of each of the unknown angles. Give reasons for each of your answers.

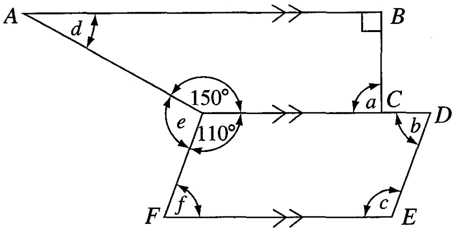


Fig. 9.233 Quadrilateral

11. Calculate the size of angle x. Give a reason for your answer.

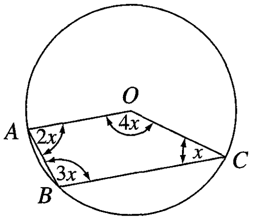


Fig. 9.234 Quadrilateral

12. Determine the magnitude of angle x. Give reasons for your answer.

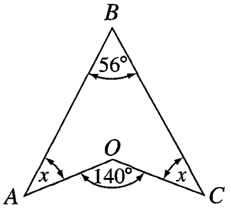


Fig. 9.235 Quadrilateral (arrowhead)

13. Calculate the size of angle x. Give a reason for your answer.

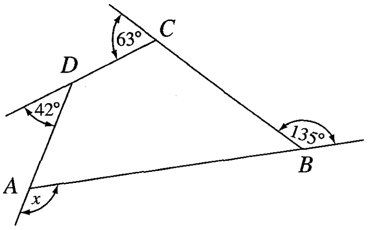


Fig. 9.236 Quadrilateral

14. Calculate the magnitude of angle y. Give a reason for your answer.

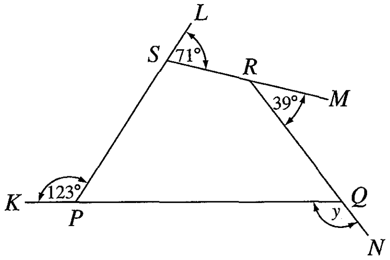


Fig. 9.237 Quadrilateral

15. State the size of the unknown angles. Give a reason for each of your answers.

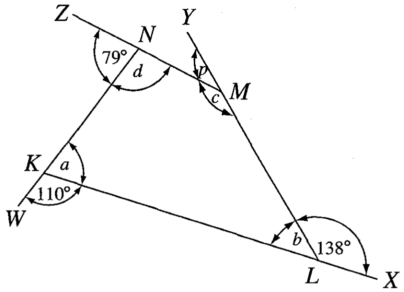


Fig. 9.238 Quadrilateral

16. Given that one angle of a parallelogram is 75°, determine the magnitude of the adjacent angle.

17. Given that one angle of a rhombus is 60°, determine the size of the adjacent angle.

18. Given that one angle of a parallelogram is 125°, state the magnitude of the adjacent angle.

19. Given that one angle of a rhombus is 85°, calculate the size of the adjacent angle.

Constructing a Unique Quadrilateral

Before starting the *actual construction*, always *draw* a *rough sketch* of the *quadrilateral* and *mark* the *given elements* on it.

Rectangle

Example 33

(a) *Using ruler and compasses only, construct the rectangle ABCD, with adjacent sides AB = 5.6 cm and AD = 3.1 cm.*
Show all construction lines clearly.

(b) *Measure and state the lengths of the diagonals AC and BD.*
State your observation.

(c) *Let the point of intersection of the diagonals be represented by 0.*
Measure and state the length of:
(i) AO (ii) BO (iii) CO (iv) DO
State your observation.

(d) *Examine:*
- *(i) △s AOB and COD*
- *(ii) △s AOD and COB.*

State your observations.

Solution

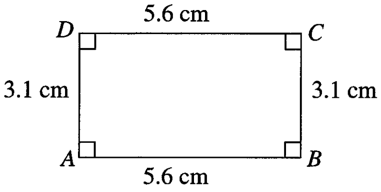


Fig. 9.239 Sketch of rectangle

Above can be *seen* the *sketch* of the *rectangle ABCD* which is to be *constructed.*

Construction

(a) First *draw* a *line k*, then *construct* the *line segment AB* = 5.6 cm. Now *construct perpendiculars* from the *points A* and *B*. Set your *compasses* to a *radius of* 3.1 cm, then using *A* and *B* as *centres*, *construct arcs* to *intersect* the *perpendiculars* at *D* and *C* respectively. Now *draw a straight line joining* the *points D and C*. We have finally *constructed* the *rectangle ABCD*, with *AB* = *DC* = 5.6 cm and *AD* = *BC* = 3.1 cm.

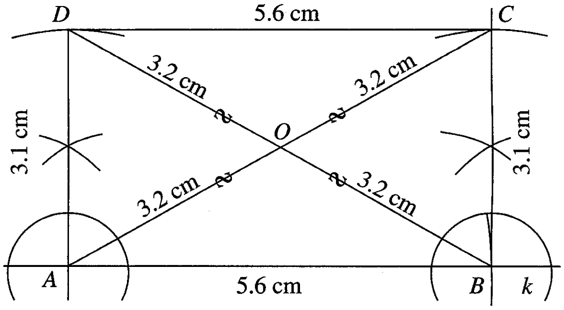


Fig. 9.239 Constructed rectangle

Above can be *seen* the *constructed rectangle ABCD.*

(b) *Draw* the *diagonals AC* and *BD.*

By *measurement*:

The *length* of the *diagonal AC* = 6.4 cm.

The *length* of the *diagonal BD* = 6.4 cm.

So *AC* = *BD* = 6.4 cm.

Hence the *diagonals* are *equal* in *length.*

(c) By *measurement*:

- (i) The *length* of *AO* = 3.2 cm.
- (ii) The *length* of *BO* = 3.2 cm.
- (iii) The *length* of *CO* = 3.2 cm.
- (iv) The *length* of *DO* = 3.2 cm.

So *AO* = *BO* = *CO* = *DO* = 3.2 cm.

Hence the *diagonals bisect each other.*

(d) (i) Now $\triangle AOB \equiv \triangle COD$ (S.S.S.)
(ii) Now $\triangle AOD \equiv \triangle COB$ (S.S.S.)

Hence *two pairs* of *congruent triangles* are *formed* by the *diagonals.*

Square

The *procedures* for *constructing* a *square* are *exactly* the *same* as for a *rectangle—except* for the fact that, in the case of a *square*, *all four sides* are *equal.*

Example 34

(a) *Using ruler and compasses only, construct the square ABCD, with AB = 5 cm.*
Show all construction lines clearly.

(b) Measure and state the lengths of the diagonals AC and BD.
State your observation.

(c) Let the point of intersection of the diagonals be represented by O.
Measure and state the length of:
- *(i) AO*
- *(ii) BO*
- *(iii) CO*
- *(iv) DO*
- *(v) Measure and state the magnitude of angle AOB. State your observation.*

(d) Measure and state the magnitude of angle OAB. State your observation.

(e) Examine $\triangle$s AOB, COB, COD and AOD. State your observation.

Solution

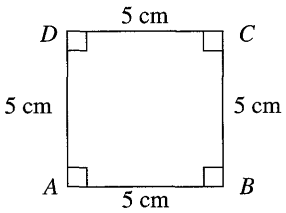


Fig. 9.240 Sketch of square

Above can be *seen* the *sketch* of the *square ABCD*, which is to be *constructed.*

Construction

(a) In *constructing* the *square*, the *radius* of the *compasses* is set to 5 cm to *construct* its *sides*.

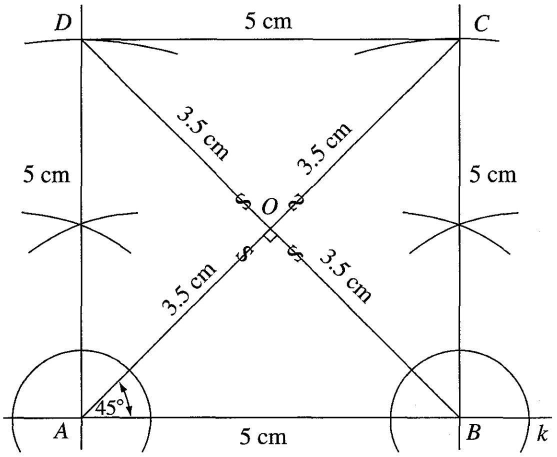


Fig. 9.240 Constructed square

Above can be *seen* the *constructed square ABCD*.

(b) *Draw* the *diagonals AC* and *BD*.

By *measurement*:

The *length* of the *diagonal* $AC = 7$ cm.

The *length* of the *diagonal* $BD = 7$ cm.

So $AC = BD = 7$ cm.

Hence the *diagonals* are *equal* in *length.*

(c) By *measurement*:

(i) The *length* of $AO = 3.5$ cm.

(ii) The *length* of $BO = 3.5$ cm.

(iii) The *length* of $CO = 3.5$ cm.

(iv) The *length* of $DO = 3.5$ cm.

So $AO = BO = CO = DO = 3.5$ cm.

(v) The *magnitude* of *angle* $AOB = 90° = 1$ rt. $\angle$.

Hence the *diagonals bisect each other* at *right angles*.

(d) By *measurement*:

The *magnitude* of *angle* $OAB = 45°$.

Hence the *diagonals bisect* the *angles* at the *vertices.*

(e) Now $\triangle AOB \equiv \triangle COB \equiv \triangle COD \equiv \triangle AOD$ (S.S.S.)

Hence *four congruent triangles* are *formed* by the *diagonals.*

Parallelogram

The *procedures* for *constructing* a *parallelogram* are *exactly* the *same* as for a *rectangle, except* for the fact that, in the case of a *parallelogram*, the *adjacent angles* are now *not right angles.*

Example 35

(a) Using ruler and compasses only, construct the parallelogram PQRS, with
$PQ = 5.6$ cm,
$PS = 3.4$ cm and angle $SPQ = 60°$.
Show all construction lines clearly.

(b) Let the point of intersection of the diagonals be represented by O.
Measure and state the length of:
(i) PO (ii) QO (iii) RO (iv) SO
State your observation.

(c) *Examine:*

(i) *$\triangle$s POQ and ROS*

(ii) *$\triangle$s POS and ROQ*

State your observation.

Solution

(a)

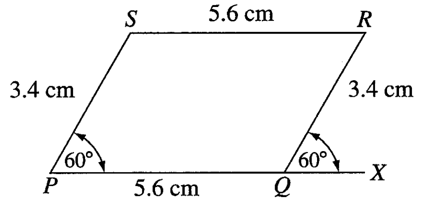


Fig. 9.241 Sketch of parallellogram

Above can be *seen* the *sketch* of the *parallelogram PQRS*, which is to be *constructed.*

Construction

In *constructing* the *parallelogram,* $\angle SPQ = \angle RQX = 60°$ (*corres.* $\angle s$).

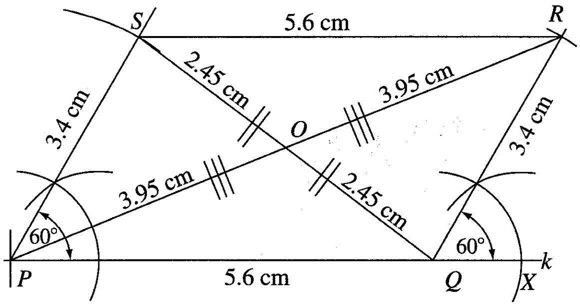


Fig. 9.241 Constructed parallelogram

Above can be *seen* the *constructed parallelogram PQRS.*

(b) By *measurement*:

(i) The *length* of $PO = 3.95$ cm.

(ii) The *length* of $QO = 2.45$ cm.

(iii) The *length* of $RO = 3.95$ cm.

(iv) The *length* of $SO = 2.45$ cm.

So $PO = RO = 3.95$ cm.

And $QO = SO = 2.45$ cm.

Hence the *diagonals bisect each other.*

(c) (i) Now $\triangle POQ \equiv \triangle ROS$ (*S.S.S.*)

(ii) Now $\triangle POS \equiv \triangle ROQ$ (*S.S.S.*)

Hence *two pairs* of *congruent triangles* are *formed* by the *diagonals.*

Rhombus

The *procedures* for *constructing* a *rhombus* are *exactly* the *same* as for a *parallelogram, except* for the fact that, in the case of a *rhombus, all four sides* are *equal.*

Example 36

(a) *Using ruler and compasses only, construct a rhombus PQRS with $PQ = 4.5$ cm and $\angle SPQ = 60°$. Show all construction lines clearly.*

(b) *Let the point of intersection of the diagonals be represented by O.*
Measure and state the length of:

(i) PO (ii) QO (iii) RO (iv) SO

(v) Measure and state the magnitude of angle POQ. State your observation.

(c) *Measure and state the magnitude of angles*
(i) OPQ (ii) OQP.
State your observation.

(d) *Examine $\triangle$s POQ, ROQ, ROS and POS. State your observation.*

Solution

(a)

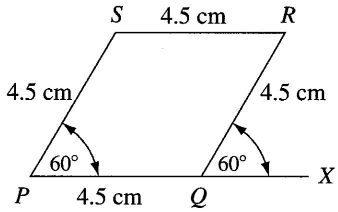


Fig. 9.242 Sketch of rhombus

Above can be *seen* the *sketch* of the *rhombus PQRS*, which is to be *constructed.*

Construction

In *constructing* the *rhombus*, $\angle SPQ = \angle RQX = 60°$ (*corres.* $\angle s$). And the *radius* of the *compasses* is set to 4.5 cm to *construct* its *sides.*

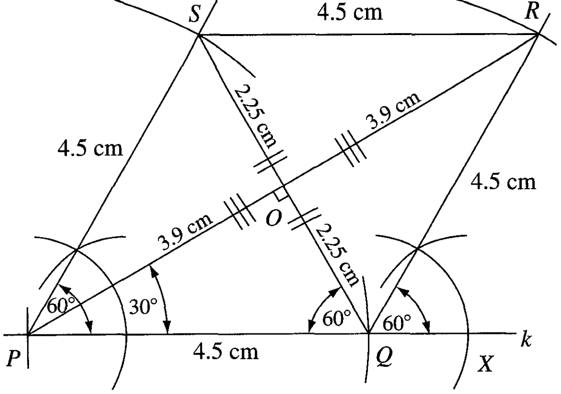


Fig. 9.242 Constructed Rhombus

Above can be *seen* the *constructed rhombus PQRS.*

(b) By *measurement*:

- (i) The *length* of $PO = 3.9$ cm.
- (ii) The *length* of $QO = 2.25$ cm.
- (iii) The *length* of $RO = 3.9$ cm.
- (iv) The *length* of $SO = 2.25$ cm.

So $PO = RO = 3.9$ cm.

And $QO = SO = 2.25$ cm.

- (v) The *magnitude* of *angle* $POQ = 90° =$ 1 rt. ∠.

Hence the *diagonals bisect each other* at *right angles.*

(c) By *measurement*:

- (i) The *magnitude* of *angle* $OPQ = 30°$.
- (ii) The *magnitude* of *angle* $OQP = 60°$.

Hence the *diagonals bisect* the *angles* at the *vertices.*

(d) Now $\triangle POQ \equiv \triangle ROQ \equiv \triangle ROS \equiv \triangle POS$ (*S.S.S*)

Hence *four congruent triangles* are *formed* by the *diagonals.*

Kite

Example 37

(a) Using ruler and compasses only, construct a kite KLMN in which KL = KN = 3.25 cm, KM = 6.5 cm and LN = 3.9 cm. Show all construction lines clearly.

(b) Measure and state the magnitude of angle:

(i) KLM (ii) KNM.

State your observation.

(c) Measure and state the length of:

(i) LM (ii) NM

(d) Let the point of intersection of the diagonals be represented by O.

Examine:

(i) ∆s KOL and KON

(ii) ∆s LOM and NOM

State your observations.

Solution

(a)

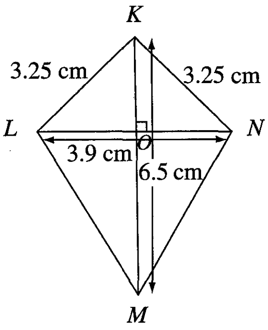


Fig. 9.243 Sketch of kite

Above can be *seen* the *sketch* of the *kite KLMN*, which is to be *constructed.*

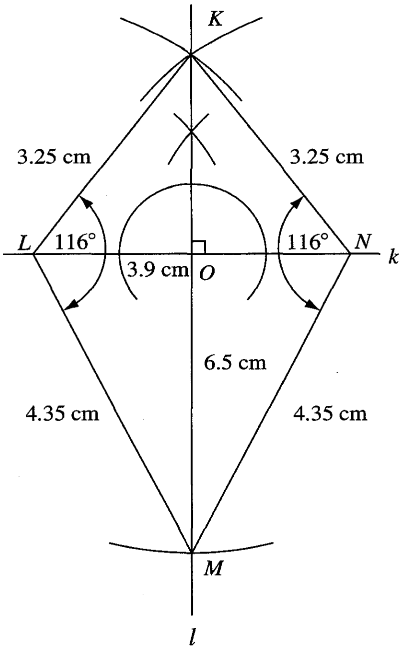


Fig. 9.243 Constructed kite

Construction

First *draw* a *line k*, then *construct* the *line segment LN* = 3.9 cm. Now *construct* the *perpendicular* *bisector* of *LN*. *Set* your *compasses* to a *radius* of 3.25 cm, then using *L* and *N* as *centres, construct two arcs to intersect* the *perpendicular bisector above LN* at *K*. *Draw straight lines* from the *point K to L*, and the *point K to N*. Then *set* your *compasses* to a *radius* of 6.5 cm and using *K* as *centre*, *construct an arc* to *intersect* the *perpendicular bisector below LN* at *M*. Draw *straight lines joining* the *points L* and *M*, and the *points N* and *M*. We have finally *constructed* the *kite KLMN* in which $KL = KN = 3.25$ cm, $KM = 6.5$ cm and $LN = 3.9$ cm.

(b) By *measurement:*
  (i) The *magnitude* of *angle* $KLM = 116°$.
  (ii) The *magnitude* of *angle* $KNM = 116°$.
  So $\angle KLM = \angle KNM = 116°$.

  Hence there is *one pair* of *equal opposite angles.*

(c) By *measurement*:
  (i) The *length* of $LM = 4.35$ cm.
  (ii) The *length* of $NM = 4.35$ cm.

(d) (i) Now $\triangle KOL \equiv \triangle KON$ (*S.S.S.*)
  (ii) And $\triangle LOM \equiv \triangle NOM$ (*S.S.S.*)

  Hence *two pairs* of *congruent triangles* are *formed* by the *diagonals.*

Exercise 9k

1. (a) Using ruler and compasses only, construct the rectangle *ABCD*, with adjacent sides $AB = 8.5$ cm and $AD = 5.4$ cm. Show all construction lines clearly.
  (b) Measure and state the lengths of the diagonals.

2. (a) Using ruler and compasses only, construct the rectangle *PQRS*, with adjacent sides $PQ = 9.3$ cm and $PS = 6.5$ cm. Show all construction lines clearly.
  (b) Measure and state the lengths of the diagonals.

3. (a) Using ruler and compasses only, construct the rectangle *KLMN*, with adjacent sides $KL = 7.9$ cm and $LM = 4.3$ cm. Show all construction lines clearly.
  (b) Measure and state the lengths of the diagonals.

4. (a) Using ruler and compasses only, construct the rectangle *WXYZ*, with adjacent sides $WX = 10.5$ cm and $XY = 7.2$ cm. Show all construction lines clearly.
  (b) Measure and state the lengths of the diagonals.

5. (a) Using ruler and compasses only, construct the rectangle *JKLM*, with adjacent sides $JK = 11.3$ cm and $KL = 8.1$ cm. Show all construction lines clearly.
  (b) Measure and state the lengths of the diagonals.

6. (a) Using ruler and compasses only, construct the square *ABCD*, with $AB = 7.5$ cm. Show all construction lines clearly.
  (b) Measure and state the lengths of the diagonals.

7. (a) Using ruler and compasses only, construct the square *PQRS*, with $PQ = 8.4$ cm. Show all construction lines clearly.
  (b) Measure and state the lengths of the diagonals.

8. (a) Using ruler and compasses only, construct the square *KLMN*, with $KL = 9.1$ cm. Show all construction lines clearly.
  (b) Measure and state the lengths of the diagonals.

9. (a) Using ruler and compasses only, construct the square *JKLM*, with $JK = 10.6$ cm. Show all construction lines clearly.
  (b) Measure and state the lengths of the diagonals.

10. (a) Using ruler and compasses only, construct the square *WXYZ*, with $WX = 11.7$ cm. Show all construction lines clearly.
  (b) Measure and state the lengths of the diagonals.

11. (a) Using ruler and compasses only, construct the Parallelogram *PQRS*, with $PQ = 12$ cm, $PS = 6.5$ cm and $\hat{P} = 60°$. Show all construction lines clearly.
  (b) Measure and state the length of the diagonal *QS*.

12. (a) Using ruler and compasses only, construct a parallelogram *ABCD*, Such that $AB = 9.5$ cm, $AD = 6.7$ cm and the angle $DAB = 60°$. All construction lines must be clearly shown.
  (b) Measure and state the lengths of *BD* in centimetres.

13. (a) Using ruler and compasses only, construct a parallelogram $ABCD$, Such that $AB = 10.5$ cm, $AD = 7.3$ cm and the angle $DAB = 60°$.
All construction lines must be clearly shown.
(b) Measure and state the length of BD in centimetres.

14. (a) Using ruler and compasses only, construct a parallelogram $PQRS$, Such that $PS = 7$ cm, $PQ = 4$ cm and angle $QPS = 60°$.
All construction lines must be clearly shown.
(b) Measure and state the length of QS.

15. (a) Construct a parallelogram $PQRS$, Such that $PS = 7$ cm, $PQ = 3$ cm and angle $QPS = 54.6°$.
(b) Measure and state the magnitude of the angle PSQ.

16. (a) Construct the parallelogram $PQRS$, Such that $PQ = 6$ cm, $PS = 13.6$ cm and angle $QPS = 42.5°$.
(b) Measure and state the size of the angle PSQ.

17. (a) Construct the parallelogram $PQRS$, Such that $PQ = 8$ cm, $PS = 15$ cm and $QPS = 52.7°$.
(b) Measure and state the magnitude of angle PSQ.

18. (a) Using ruler and compasses only, construct a rhombus $PQRS$, such that $PQ = 5.8$ cm and angle $QPS = 60°$.
(b) Measure and state the length of QS.

19. (a) Using ruler and compasses only, construct a rhombus $KLMN$, such that $KL = 6.5$ cm, and angle $LKN = 120°$.
(b) Measure and state the length of NL.

20. (a) Using ruler and compasses only, construct a rhombus $ABCD$, such that $AB = 7.3$ cm, and angle $BAD = 45°$.
(b) Measure and state the magnitude of angle ABD.

21. (a) Using ruler and compasses only, construct a rhombus $WXYZ$, such that $WX = 8.5$ cm and angle $XWZ = 150°$.
(b) Measure and state the magnitude of angle WZX.

22. (a) Construct a rhombus $ABCD$, such that $AB = 9.2$ cm, and angle $DAB = 120°$.
(b) Measure and state the size of angle ADB.

23. (a) Using ruler and compasses only, construct a kite $KLMN$, in which $KL = LM = 5$ cm, $KM = 6$ cm and $LN = 9$ cm.
Show all construction lines clearly.
(b) Measure and state the magnitude of angle:
(i) KNL (ii) KMN

24. (a) Using ruler and compasses only, construct a kite $PQRS$, in which $PQ = PS = 7.2$ cm, $PR = 8$ cm and $QS = 9$ cm.
Show all construction lines clearly.
(b) Measure and state the magnitude of angle:
(i) SPR (ii) QSR

25. (a) Using ruler and compasses only, construct a kite $ABCD$, such that $AD = DC = 7.5$ cm, $AC = 9$ cm and $BD = 14$ cm.
All construction lines must be clearly shown.
(b) Measure and state the length of:
(i) AB (ii) BC

26. (a) Using ruler and compasses only, construct a kite $PQRS$, such that $PS = RS = 6$ cm, $PR = 9.6$ cm and $QS = 9$ cm.
All construction lines must be clearly shown.
(b) Measure and state the length of:
(i) PQ (ii) RQ

27. (a) Using ruler and compasses only, construct a kite $KLMN$, in which $KL = ML = 6.3$ cm, $KM = 7.2$ cm and $NL = 10$ cm.
All construction lines must be clearly shown.
(b) Measure and state the magnitude of angle:
(i) LKN (ii) LMN

28. (a) Using ruler and compasses only, construct a quadrilateral $ABCD$ in which $AB = AD = 9$ cm, $BC = 6$ cm, angle $BAD = 60°$ and angle $ABC = 90°$.
(b) Measure and state:
(i) the length of DC
(ii) the size of angle BCD.

29. Draw accurate scale drawings of the following, using a scale of 1 cm to represent 1 m.

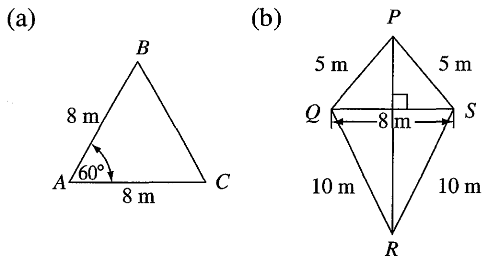


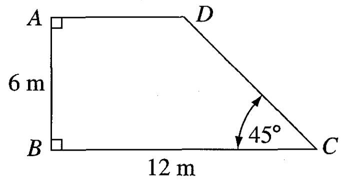
Fig. 9.244 Triangle *Kite*

Measure and state the length of *BC* in metres.

Measure and state the length of *PR* in metres.

30. Draw accurate scale drawings of the following, using a scale of 1 cm to represent 2 m.

(a) Measure and state the length of *AD* in metres.



Trapezium

(b) Measure and state the length of *KM* in metres.

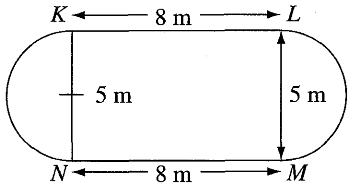


Fig. 9.245 Compound figure

31. Without your protractor, construct a triangle *OAB* in which angle $OAB = 60°$ and $OA = AB = 5$ cm.

Hence construct the rhombus *OABF*. On the same figure, draw a circle with centre *O* passing through the points *A*, *B* and *F*.

Polygon

As previously mentioned, a *polygon* is defined as a *plane shape* (or *figure*) *bounded* by *three or more straight sides*.

An example of a *polygon* can be seen in Fig. 9.246 below.

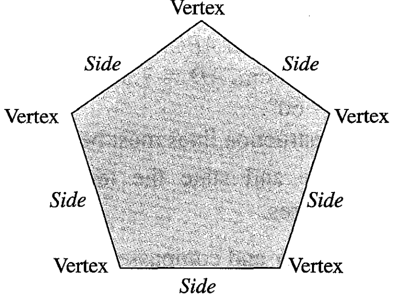


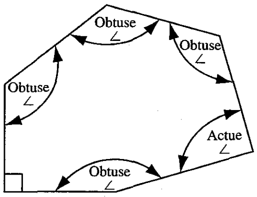
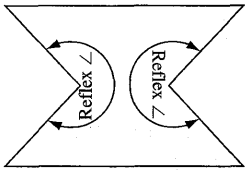
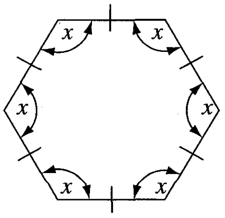
Fig. 9.246 Polygon

A *polygon* has *no thickness.*

Types of Polygon

Polygons can be *classified* according to their *angles* into *three basic types*:

Table 9.5

Classification of Polygons	
By Angles	*Properties*
 *Convex polygon*	*Each interior angle* is *less than* 180°. That is, $0° < \theta < 180°$. So an *interior angle* θ can be *acute-angled, right-angled* or *obtuse-angled*. For example: An *acute-angled triangle, a right-angled triangle, an obtuse-angled triangle.*
 *Re-entrant polygon*	*One or more* of its *interior angles* is *greater than* 180°. That is $180° < \theta <$ 360°. So *one or more interior angle* can be *reflex-angled.*
 *Regular polygon*	1. *All interior angles* are *equal.* 2. *All sides* are *equal.* For example: An *equilateral triangle.*

NOTE: A *shape* that is *curved outwards* is said to be *convex.*

A *curve or surface* that *bulges* away from *an internal point of reference* is said to be *convex.*

For example: The *domes* of some *places* of *worship* are *convex* towards the *sky.*

A *convex figure* is one in which the *line joining any two points* on the *figure* stays *inside* the *figure* and *does not extend outside* of it.

For example:

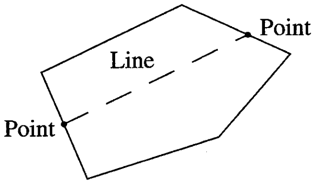


Fig. 9.247 Convex figure (polygon)

Figures which are *not convex* are said to be *re-entrant.*

For example:

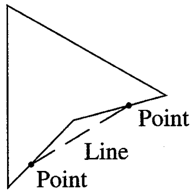


Fig. 9.248 Re-entrant figure (polygon)

Angle Properties of a Polygon

There are *two theorems* that we need to look at under this heading.

THEOREM 1: *The sum of the interior angles of an n-sided polygon is* (2*n* − 4) *right angles or* 90°(2*n* − 4) *or* 180°(*n* − 2).

THEOREM 2: *The sum of the exterior angles of an n-sided polygon is* 4 *right angles or* 360°.

We have already *investigated* these *two theorems* for a *triangle* and a *quadrilateral.*

Class Activity

Draw polygons with the *stated number of sides* in Table 9.6. Then *choose one vertex* and from it *draw diagonals* until *no more* can be *drawn.* Hence *fill* in the *missing information* in Table 9.6 and *confirm* the *given information.*

Table 9.6

Name of polygon	*Number of vertices*	*Number of sides*	*Number of triangles obtained*	*Sum of interior angles*
Triangle	3	3	1	1 × 180° = 180°
Quadrilateral	4	4	2	2 × 180° = 360°
Pentagon		5		× 180° = 540°
Hexagon		6		× 180° = 720°
Heptagon		7		× 180° = 900°
Octagon		8		× 180° = 1 080°
Nonagon		9		× 180° = 1 260°
Decagon		10		× 180° = 1 440°
Undecagon		11		× 180° = 1 620°
Dodecagon		12		× 180° = 1 800°
Icosagon		20		× 180° = 3 240°
n-agon	*n*	*n*	*n* − 2	(*n* − 2) × 180° = 180°(*n* − 2)

Class Activity

Using the *polygons drawn* in the last *class activity*, *produce* the *sides* of *each polygon* in a *cyclic order.*

Thus:

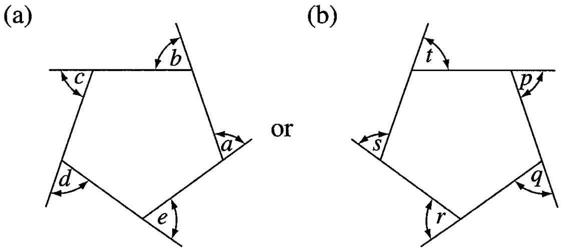


Fig. 9.249 Polygons

Now use a *protractor* to *measure* the *exterior angles* for *each polygon* and then *sum them.* Hence *complete* Table 9.7 below. *What* do you *observe*? Are the results those that you *would expect*?

Table 9.7

Name of polygon	*Number of vertices*	*Number of sides*	*Number of interior angles*	*Number of exterior angles*	*Sum of exterior angles*
Triangle	3	3	3	3	360°
Quadrilateral	4	4	4	4	360°

Example 38

(a) Determine the sum of the interior-angles of a polygon with thirteen sides.
(i) in right angles (ii) in degrees.

(b) If the polygon is regular, calculate the size of each
(i) interior angle (ii) exterior angle.

Solution

(a) (i) The *sum* of the *interior angles* of a *polygon* with 13-*sides* (i.e. $n = 13$), $S = (2n - 4)$ rt. ∠s

$= (2 \times 13 - 4)$ rt. ∠s

$= (26 - 4)$ rt. ∠s

$= 22$ rt. ∠s

Hence the *sum* of the *interior angles* of the 13-sided polygon is 22 rt. ∠s.

(ii) The *sum* of the *interior angles* of a *polygon* with 13-*sides* (i.e. $n = 13$), $S = 180°(n - 2)$

$= 180°(13 - 2)$

$= 180° \times 11$

$= 1\,980°$

or $S = 90°(2n - 4)$

$= 90°(2 \times 13 - 4)$

$= 90°(26 - 4)$

$= 90° \times 22$

$= 1\,980°$

Hence the *sum* of the *interior angles* of the 13-sided polygon is 1 980°.

(b) (i) *Each interior angle* of the 13-*sided regular polygon* $= \frac{S}{n}$

$= \frac{1\,980°}{13}$

$= 152.3°$ (*correct to 1 decimal place*)

Hence *each interior angle* of the polygon is 152.3°

(ii) *Each exterior angle* of the 13-*sided regular polygon* $= 180° -$ *The interior angle*

$= 180° - 152.3°$

$= 27.7°$

Hence *each exterior angle* of the polygon is 27.7°

Example 39

How many sides has a regular polygon if each interior angle is 156°?

Solution

Each interior angle of the *n-sided regular polygon* $= \frac{S}{n} = \frac{180°(n - 2)}{n}$

And *each interior angle* of the *n-sided regular polygon* = 156°

Thus $\frac{180°(n - 2)}{n} = 156°$

So $180(n - 2) = 156 \times n$

And $180n - 360 = 156n$

i.e. $180n - 156n = 360$

$\therefore \quad 24n = 360$

$\Rightarrow \quad n = \dfrac{360}{24} = 15$

Hence the regular polygon has 15 *sides*.

Alternative Method

The *exterior angle* of the *polygon* = 180° − *The interior angle*

= 180° − 156°

= 24°

And the *sum* of the *exterior angles* of the *polygon* = 360°

So the *number of sides* of the *regular polygon* $= \dfrac{360}{24} = 15$

Hence the regular polygon has 15 *sides*.

Example 40

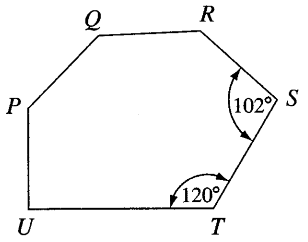


Fig. 9.250 Polygon

PQRSTU is a hexagon with ∠S = 102°, ∠T = 120°, ∠U = 90° and ∠P = ∠Q = ∠R.

(a) Calculate the magnitude of angle Q.

(b) QR is produced to X. Determine the size of the exterior angle SRX.

Solution

(a)

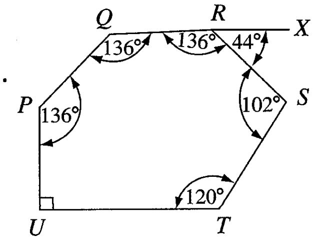


Fig. 9.250 Polygon

The *sum* of the *interior angles* of a *hexagon* (*n* = 6) = 180°(*n* − 2)

= 180°(6 − 2)

= 180° × 4

= 720°

The *sum* of the *three* given *interior angles* $= \hat{S} + \hat{T} + \hat{U}$

= 102° + 120° + 90°

= 312°

So the *sum* of the *three unknown interior angles* = 720° − 312° = 408°

$\therefore \quad \angle Q = \dfrac{408°}{3°} = 136°$

Hence the *magnitude of angle Q* is 136°.

(b) The *size* of the *exterior angle SRX* = 180° − *The interior angle*

= 180° − 136°

= 44°

Hence the *size* of the *exterior angle SRX* is 44°.

Exercise 91

1. (a) Determine the sum of the interior angles of a polygon with 18 sides
   (i) in right angles (ii) in degrees.
   (b) If the polygon is regular, calculate the size of each
   (i) interior angle (ii) exterior angle.

2. (a) Determine the sum of the interior angles of a polygon with 19 sides
   (i) in right angles (ii) in degrees.
   (b) If the polygon is regular, calculate the magnitude of each
   (i) interior angle (ii) exterior angle.

3. (a) Calculate the sum of the interior angles of a polygon with 21 sides
   (i) in right angles (ii) in degrees.
   (b) Given that the polygon is regular, calculate the size of each
   (i) interior angle (ii) exterior angle.

4. (a) Calculate the sum of the interior angles of a polygon with 25 sides
   (i) in right angles (ii) in degrees.
   (b) Given that the polygon is regular, calculate the magnitude of each
   (i) interior angle (ii) exterior angle.

5. (a) Determine the sum of the interior angles of a polygon with 30 sides
   (i) in right angles (ii) in degrees.
   (b) Given that the polygon is regular, evaluate the size of each
   (i) interior angle (ii) exterior angle.

6. How many sides has a regular polygon if each interior angle is 140°.

7. Determine the number of sides of a regular polygon if each interior angle is 135°.

8. Calculate the number of sides of a regular polygon if each interior angle is 165°.

9. Determine the number of sides of a regular polygon if each exterior angle is 30°.

10. Evaluate the number of sides of a regular polygon if each exterior angle is 14.4°.

11. A pentagon has interior angles of 90° and 150°. If the remaining angles are equal, calculate the size of each unknown interior angle.

12. A hexagon has interior angles of 95° and 175°. The remaining angles are equal. Calculate the size of each unknown interior angle.

13. A heptagon has interior angles of 130°, 145° and 165°. Determine the magnitude of each unknown interior angle, given that they are equal.

14. A nonagon has interior angles of 147°, 178° 146° and 193°. Calculate the magnitude of each unknown interior angle, given that they are equal.

15. A decagon has interior angles of 100°, 115°, 120°, 125° and 130°. The remaining angles are equal. Calculate the size of each unknown interior angle.

16. In a regular polygon, each interior angle is greater than each exterior angle by 90°. Calculate the number of sides of the polygon.

17.

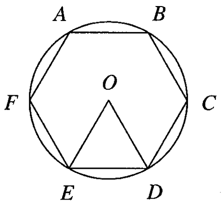


Fig. 9.251 Polygon

ABCDEF is a regular hexagon inscribed in a circle centre O, radius 10 cm, as shown in the diagram above.

(a) Calculate the angle DOE (in degrees)
(b) Determine DE
(c) Hence, state the perimeter of the hexagon.

18. A polygon has n sides. Two of its angles are right-angles. Each of the remaining angles is equal to 150°. Calculate n.

19. Calculate the sum of the interior angles in degrees of a convex polygon with 9 sides.

20. In a regular polygon each interior angle is greater by 120° than each exterior angle. Calculate the number of sides of the polygon.

21. Calculate the exterior angle of a regular polygon in which the interior angle is five times the exterior angle. Hence state the number of sides in the polygon.

22.

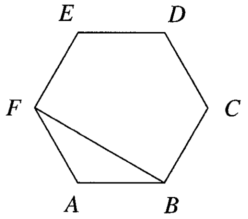


Fig. 9.252 Polygon

In the diagram above, *ABCDEF* represents a regular hexagon.
Calculate the size of
(a) angle BCD (b) angle ABF.

Areas: Triangle, Trapezium, Parallelogram and Rectangle

Plane figures are said to be *between* the *same parallels* when they are *situated* as *shown* in Fig. 9.253 on the next page.

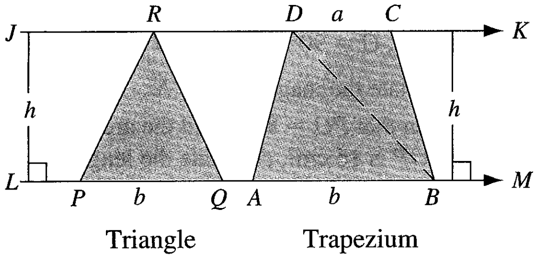


Fig. 9.253 Plane figures between the same parallels

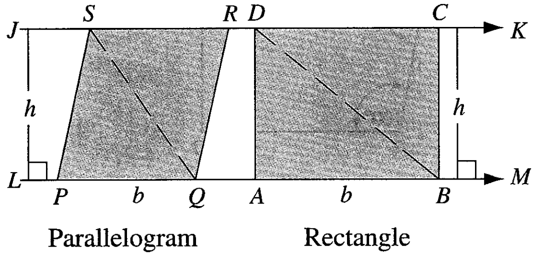


Fig. 9.253 Plane figures between the same parallels

In Fig. 9.253 above, the *plane figures: triangle, trapezium, parallelogram* and *rectangle*, each have base *b* and the *same altitude h*, since they are *situated between two parallel lines JK* and *LM*, *h units apart.*

If we accept that the *area* of *triangle PQR* = $\frac{1}{2}bh$ then the *area* of the

$$\textit{trapezium ABCD} = \begin{matrix}\text{The area of}\\ \triangle BCD\end{matrix} + \begin{matrix}\text{The area of}\\ \triangle ABD\end{matrix}$$
$$= \frac{1}{2}ah + \frac{1}{2}bh$$
$$= \frac{1}{2}(a + b)h$$

where *a* and *b* are the *lengths* of the *parallel sides* and *h* is the *altitude* of the *trapezium*.

$$\begin{matrix}\text{The } \textit{area} \text{ of}\\ \textit{parallelogram PQRS}\end{matrix} = \begin{matrix}\text{The area of}\\ \triangle QRS\end{matrix} + \begin{matrix}\text{The area of}\\ \triangle PQS\end{matrix}$$
$$= \frac{1}{2}bh + \frac{1}{2}bh$$
$$= bh$$

where *b* is the *base* and *h* is the *altitude* of the *parallelogram.*

$$\begin{matrix}\text{The } \textit{area} \text{ of}\\ \textit{rectangle ABCD}\end{matrix} = \begin{matrix}\text{The area of}\\ \triangle BCD\end{matrix} + \begin{matrix}\text{The area of}\\ \triangle ABD\end{matrix}$$
$$= \frac{1}{2}bh + \frac{1}{2}bh$$
$$= bh$$

where *b* is the *length* and *h* is the *width* of the *rectangle.*

Example 41

Given that the area of a triangle is 18 cm², calculate its base if its altitude is 5 cm.

Solution

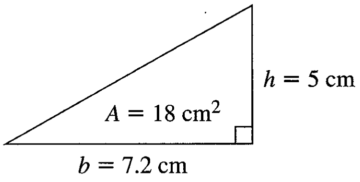


Fig. 9.254 Triangle

The *area* of the *triangle*, $A = \frac{1}{2}bh$

$$= \frac{1}{2}b \times 5 \text{ cm}^2$$
$$= \frac{5}{2}b \text{ cm}^2$$

And the *area* of the *triangle*, $A = 18 \text{ cm}^2$

Thus $\frac{5}{2}b = 18$

So $b = 18 \times \frac{2}{5}$

$$= 3.6 \times 2$$
$$= 7.2 \text{ cm}$$

Hence the *base* of the *triangle* is 7.2 cm.

Areas: Triangle, Rhombus and Square

In Fig. 9.255 below, the *plane figures: triangle, rhombus* and *square*, each have *base b* and the *same altitude h*, since they are *situated between two parallel lines JK* and *LM, h units apart.*

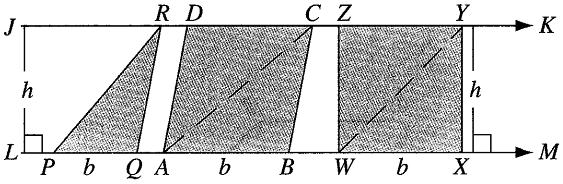


Fig. 9.255 Plane figures between the same parallels

If we accept that the *area* of *triangle PQR* $= \frac{1}{2}bh$ then the *area* of the

$$\textit{rhombus ABCD} = \begin{array}{c}\text{The area of}\\ \triangle BCD\end{array} + \begin{array}{c}\text{The area of}\\ \triangle ABD\end{array}$$

$$= \frac{1}{2}bh + \frac{1}{2}bh$$

$$= bh$$

where b is the *base* and h is the *altitude* of the *rhombus.*

$$\begin{array}{l}\text{The } \textit{area} \text{ of the}\\ \textit{square WXYZ}\end{array} = \begin{array}{c}\text{The area of}\\ \triangle WXY\end{array} + \begin{array}{c}\text{The area of}\\ \triangle WZY\end{array}$$

$$= \frac{1}{2}bh + \frac{1}{2}bh$$

$$= bh$$

where $b = h =$ the *length* of the *square.*

Example 42

Given that the area of a rhombus is 5.0 cm², with base 2.5 cm, calculate its altitude.

Solution

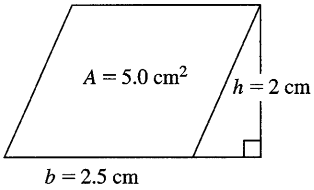


Fig. 9.256 Rhombus

The *area* of the *rhombus*, $A = bh = 2.5 \times h$ cm²
$= 2.5h$ cm²

And the *area* of the *rhombus*, $A = 5.0$ cm²

Thus $2.5h = 5.0$

So $h = \frac{5.0}{2.5} = \frac{50}{25} = 2$ cm

Hence the *altitude* of the *rhombus* is 2 cm.

Exercise 9m

1.

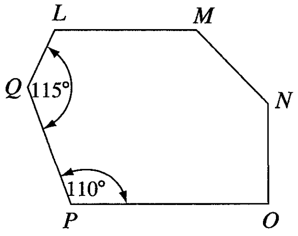


Fig. 9.257 Polygon

$LMNOPQ$ is a hexagon with $\hat{Q} = 115°$, $\hat{P} = 110°$, $\hat{O} = 90°$ and $\hat{L} = \hat{M} = \hat{N}$.

(a) Calculate the magnitude of $\hat{N}$.

(b) Given that $PO = 8$ cm and the area of $\triangle NOP$ is 16 cm², calculate the length of PN in centimetre.

2.

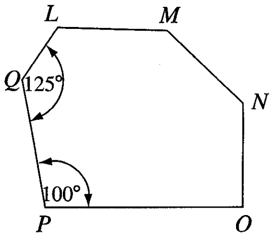


Fig. 9.258 Polygon

$LMNOPQ$ is a hexagon with $\hat{Q} = 125°$, $\hat{P} = 100°$, $\hat{O} = 90°$ and $\hat{L} = \hat{M} = \hat{N}$.

(a) Calculate the magnitude of $\hat{M}$.

(b) Given that $PO = 6$ cm and the area of $\triangle NOP$ is 24 cm², calculate the length of PN in centimetre.

3.

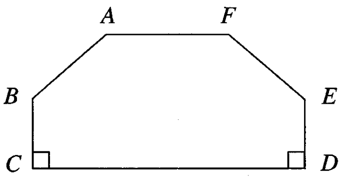


Fig. 9.259 Polygon

$ABCDEF$ is a hexagon with $\hat{C} = \hat{D} = 90°$, $\hat{A} = \hat{F} = 165°$ and $\hat{B} = \hat{E}$.

(a) Calculate the magnitude of $\hat{B}$.

(b) Given that $CD = 12$ cm and the area of $\triangle CDE$ is 18 cm², calculate the length of ED.

4.

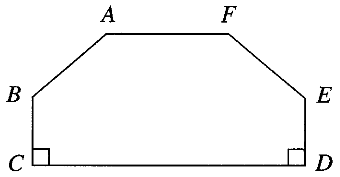


Fig. 9.260 Polygon

$ABCDEF$ is a hexagon with $\hat{C} = \hat{D} = 90°$, $\hat{A} = \hat{F} = 155°$ and $\hat{B} = \hat{E}$.

(a) Calculate the magnitude of $\hat{E}$.

(b) Given that $CD = 18$ cm and the area of $\triangle BCD$ is 45 cm², calculate the length of BC.

5.

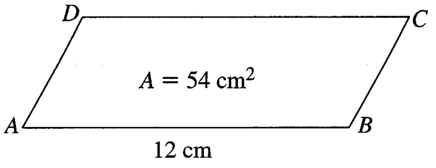


Fig. 9.261 Parallelogram

ABCD is a parallelogram of area 54 cm^2. If *AB* = 12 cm, calculate the altitude of the parallelogram *ABCD*.

6. The area of a parallelogram of altitude 6.2 cm is 93 cm^2. Calculate the base of the parallelogram.

7.

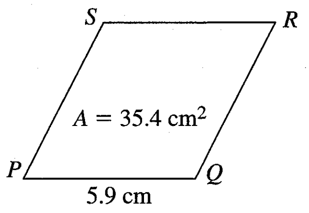


Fig. 9.262 Rhombus

PQRS is a rhombus of area 35.4 cm^2. If *PQ* = 5.9 cm, calculate the altitude of the rhombus *PQRS*.

8. The area of a rhombus of altitude of 7 cm is 45.5 cm^2. Calculate the base of the rhombus.

9.

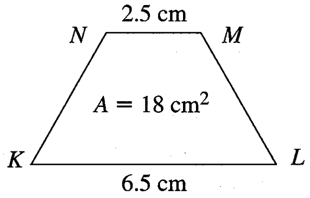


Fig. 9.263 Trapezium

KLMN is a trapezium of area 18 cm^2. If *KL* = 6.5 cm and *NM* = 2.5 cm, calculate the altitude of the trapezium *KLMN*.

10.

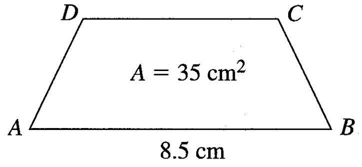


Fig. 9.264 Trapezium

ABCD is a trapezium of area 35 cm^2. The altitude of the trapezium is 5 cm and *AB* = 8.5 cm, calculate the length of *DC*.

Circle

A *circle* is defined as a *plane curve* formed by the *set of all points* which are a *given fixed distance* from a *fixed point*. The *fixed distance* is called the *radius* and is *denoted* by the letter *r*. The *fixed point* is called the *centre* and is *denoted* by the letter *O*. These properties are illustrated in Fig. 9.265 below.

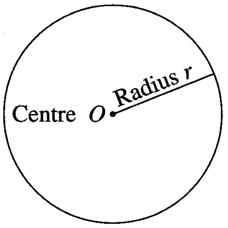


Fig. 9.265 Circle

A *circle* has no *thickness*.

Properties of a Circle

The *properties of a circle* are defined below.

Circumference

The *circumference of a circle* is used to mean both the *boundary line of a circle*, and the *length of the boundary line* (i.e. the *perimeter of the circle* or the *distance around the circle*). The *circumference of a circle* is *represented* by the letter *C*. These properties are illustrated in Fig. 9.266 below.

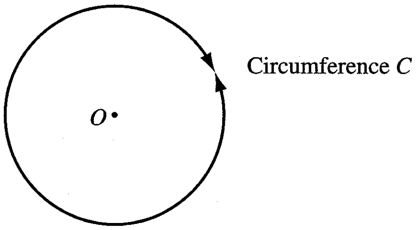
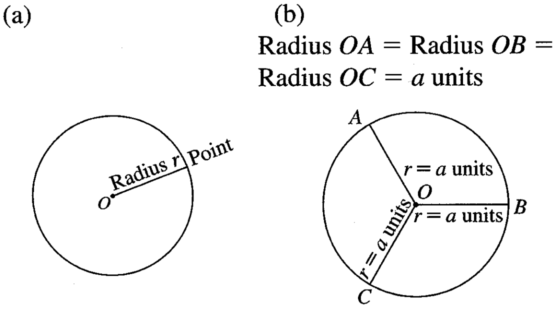


Fig. 9.266 Circumference of a circle

Radius

A *radius of a circle* is a *straight line drawn* from the *centre* to any *point* on the *circumference*. All *radii* of the *same circle* are *equal.* These properties are illustrated in Fig. 9.267 below.

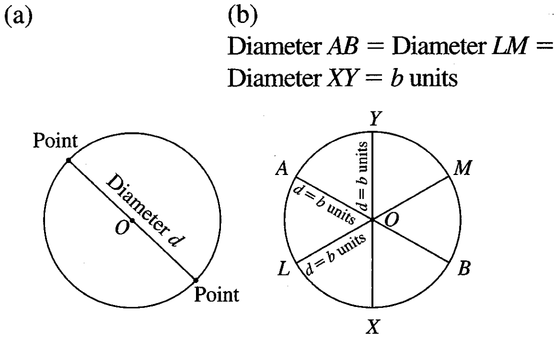

(a)

(b) Radius OA = Radius OB = Radius OC = a units

Fig. 9.267 Radii of a circle

Diameter

A *diameter of a circle* is any *straight line* that *joins two points* on the *circumference* and *passes through* the *centre*. The *diameter of a circle* is *represented* by the letter *d*. Thus the *diameter of a circle* is *twice its radius*, that is, $d = 2r$. *All diameters* of the *same circle* are *equal.* These properties are illustrated in Fig. 9.268 below.


(a)

(b) Diameter AB = Diameter LM = Diameter XY = b units

Fig. 9.268 Diameters of a circle

Chord

A *chord of a circle* is a *straight line joining* any *two points* on the *circumference. All chords* which are the *same perpendicular distance* away from the *centre of a circle* are *equal* in length . The *diameter of a circle* is a *special chord.* These properties are illustrated in Fig. 9.269 below.

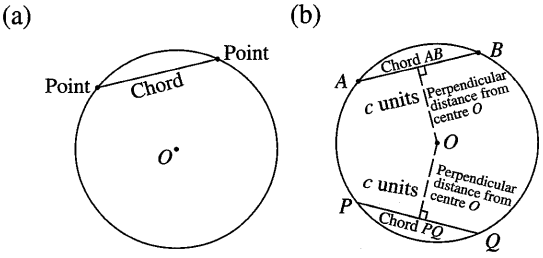


Fig. 9.269 Chords of a circle

Arc

An *arc of a circle* is any *part* of the *circumference.* The *length* of an *arc of a circle* is *represented* by the letter *l*. These properties are illustrated in Fig. 9.270 below.

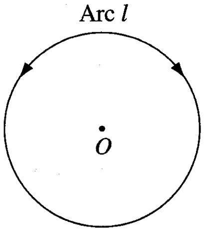


Fig. 9.270 Arc of a circle

Segments

A *segment of a circle* is a *plane figure bounded* by a *chord* and *one arc formed* by the *chord.* Usually *two distinct segments of a circle exist* at the *same time*—a *minor segment* and a *major segment.* These properties are illustrated in Fig. 9.271 below.

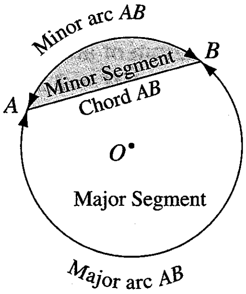


Fig. 9.271 Segments of a cirlce

Semi-circles

If the *chord forms* a *diameter*, then the *circle* is *divided* into *two equal segments* and each is called a *semi-circle*. These properties are illustrated in Fig. 9.272 below.

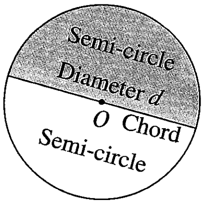
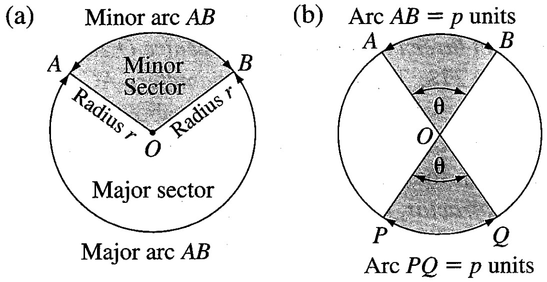


Fig. 9.272 Semi-circles

Sectors

A *sector of a circle* is a *plane figure bounded* by *two radii* and *one arc formed* by the *radii*. Usually *two distinct sectors of a circle exist* at the *same time*—a *minor sector* and a *major sector. All sectors of a circle* with *equal arc lengths* or *equal sector angles* are *equal* in area. These properties are illustrated in Fig. 9.273 below.

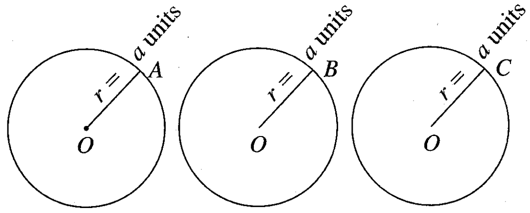


Arc AB = Arc $PQ = p$ units
Sector angle $= A\hat{O}B$
$= P\hat{O}B = \theta$
Sector AOB = Sector POQ

Fig. 9.273 Sectors of a circle

Equal Circles

Equal circles are *circles* with *equal radii*. This property is illustrated in Fig. 9.274 below.



Radius OA = Radius OB = Radius $OC = a$ units

Fig. 9.274 Equal circles

Concentric Circles

Concentric circles are *circles* which have the *same centre*. This property is illustrated in Fig. 9.275 below.

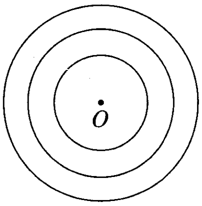


Fig. 9.275 Concentric circles

Angle Properties of a Circle

There are *five theorems* that we need to look at under this heading.

THEOREM 1: *The angle at the centre of a circle is twice the angle at the circumference standing on the same arc (or chord).*

Class Activity

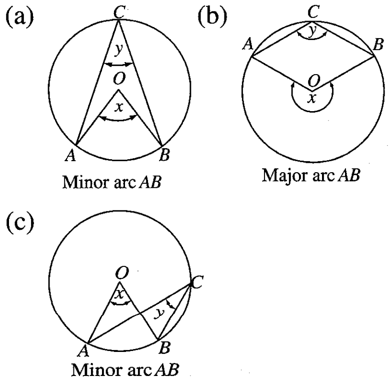


Fig. 9.276 Circles

Set your *compasses* to a *radius of* 5 cm and *construct three circles*, each with *centre O* as shown in Fig. 9.276. Then *draw* the *radii AO* and *BO*, similar to those *shown* in the *diagram* above, using a *ruler*. Now *draw* the *chords AC* and *BC*. Using your *protractor*, *measure* and *state* the *size* of *each pair of angles AOB* and *ACB*. What is the *relationship* between *each pair of angles AOB* and *ACB*?

THEORY:

AB is the *arc of the circle* under consideration.

Then the *angle at the centre* of the *circle O*, *standing* on the *arc AB* $= A\hat{O}B$

And the *angle* at the *circumference standing* on the *same arc AB* $= A\hat{C}B$

Thus $A\hat{O}B = 2 \cdot A\hat{C}B$

(∠ *at centre* = 2 · ∠ *at circum.*)

or $x = 2y$

Example 43

If $A\hat{O}B = 96°$, *determine* $A\hat{C}B$. *Give a reason for your answer.*

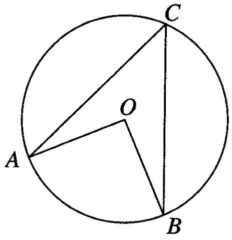


Fig. 9.277 Circle

Solution

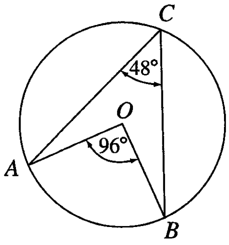


Fig. 9.277 Circle

Now $A\hat{O}B = 2 \cdot A\hat{C}B$

(∠ *at centre* = 2 · ∠ *at circum.*)

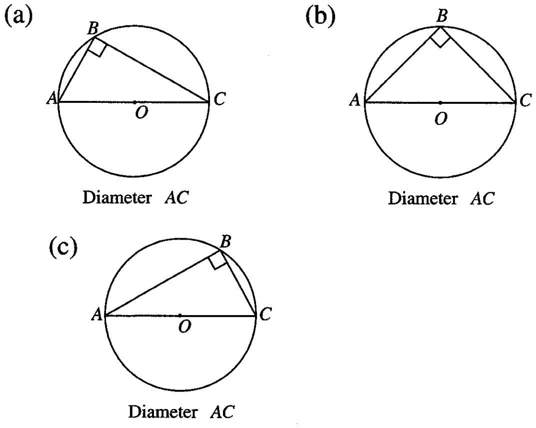
So $96° = 2 \cdot A\hat{C}B$

i.e. $A\hat{C}B = \dfrac{96°}{2} = 48°$

Hence $A\hat{C}B$ is 48°.

THEOREM 2: *The angle in a semi-circle is a right angle.*

Class Activity



(a) Diameter *AC*

(b) Diameter *AC*

(c) Diameter *AC*

Fig. 9.278 Circles

Set your *compasses* to a *radius of* 4.5 cm and *construct three circles*, each with *centre O* as shown in Fig. 9.278. Then *draw* the *diameters AC*, *similar* to those shown in the *diagram* above, using a *ruler*. Now *draw* the *straight lines AB* and *CB* to the *point B* on the *circumference of each circle*. Using your *protractor*, *measure* and *state* the *magnitude of each angle ABC*. *What* do you *observe*?

THEORY:

Since *AC* is the *arc of the circle* under consideration.

Then the *angle at the centre* of the *circle O*, *standing* on the *arc AC* $= A\hat{O}C$

= 2rt. ∠s (*straight* ∠)

And the *angle* at the *circumference standing* on the *same arc AC* $= A\hat{B}C$

Thus $A\hat{O}C = 2 \cdot A\hat{B}C$

So $2 \text{ rt. } \angle\text{s} = 2 \cdot A\hat{B}C$

i.e. $A\hat{B}C = \dfrac{2 \text{ rt. } \angle\text{s}}{2} = 1 \text{ rt. } \angle$

(∠ *in a semi-circle*)

Example 44

Determine the length of the diameter AC, stating reasons for your answer.

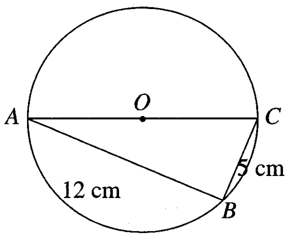


Fig. 9.279 Circle

Solution

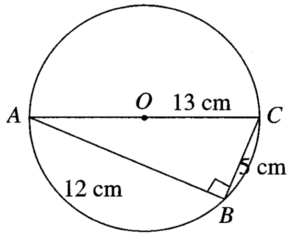


Fig. 9.279 Circle

Now $A\hat{B}C = 90°$ (*∠ in a semi-circle*).

So *triangle ABC* is *right-angled* at *B*.

Considering the *right-angled ABC* and using *Pythagoras' theorem*:

$$AC^2 = AB^2 = BC^2$$
$$= (12\text{ cm})^2 + (5\text{ cm})^2$$
$$= 144\text{ cm}^2 + 25\text{ cm}^2$$
$$= 169\text{ m}^2$$

So $AC = \sqrt{169\text{ cm}^2} = 13\text{ cm}$

Hence the *length* of the *diameter AC* is 13 cm.

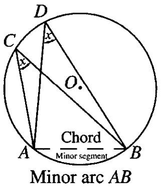
THEOREM 3: *Angles at the circumference of a circle standing on the same arc (or chord) are equal.*

or

Angles in the same segment of a circle are equal.

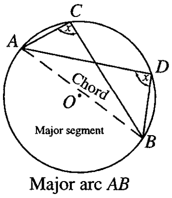
Class Activity

(a)



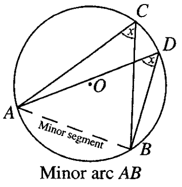
Minor arc *AB*

(b)



Major arc *AB*

(c)



Minor arc *AB*

Fig. 9.280 Circles

Set your *compasses* to a *radius of* 5 cm and *construct three circles*, each with *centre O* as shown in Fig. 9.280. Then *mark* off the *points*, *A*, *B*, *C* and *D* on your *circles*. Now use your *ruler* to *draw straight lines* from *A to C*, *B to C*, *A to D*, and *B to D*. Using your *protractor*, *measure each pair of angles ACB* and *ADB*. What is the *relationship* between *each pair of angles ACB* and *ADB*?

THEORY: *Angles ACB* and *ADB* are *angles* at the *circumference of a circle standing* on the *same arc* (*or chord*) *AB*.

or

Angles ACB and *ADB* are *angles* in the *same segment of a circle*, since *AB* is a *chord*.

Thus $A\hat{C}B = A\hat{D}B$ (*∠s on the same arc*)

or $A\hat{C}B = A\hat{D}B$ (*∠s in the same segment*)

Example 45

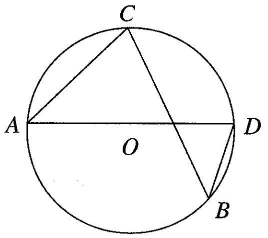


Fig. 9.281 Circle

If angle ACB = 59°, determine the size of angle ADB. State a reason for your answer.

Solution

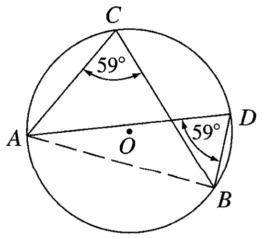


Fig. 9.281 Circle

Given that $A\hat{C}B = 59^\circ$

Then $A\hat{D}B = A\hat{C}B = 59^\circ$ (*∠s on the same arc*)

or $A\hat{D}B = A\hat{C}B = 59^\circ$ (*∠s in the same segment*)

THEOREM 4: *The opposite angles of a cyclic quadrilateral are supplementary.*

Class Activity

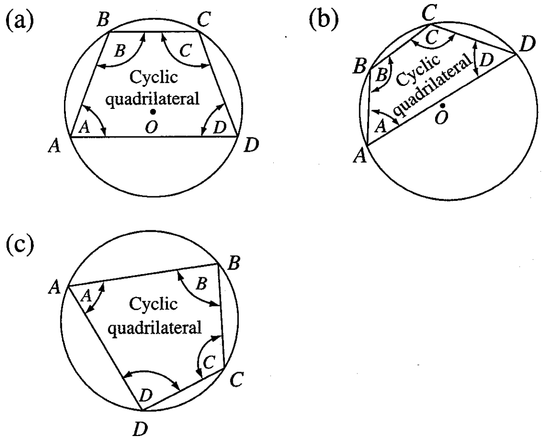


Fig. 9.282 Cyclic quadrilaterals

A cyclic quadrilateral is a *quadrilateral* having its *four vertices lying on the circumference* of a *circle* as shown in Fig. 9.282 above.

Set your *compasses* to a *radius of 4* cm and *draw three circles*, each with *centre O* as shown in Fig. 9.282 above. Then using your *ruler, draw three cyclic quadrilaterals ABCD similar* to those *shown* in the *diagram* above. Using your *protractor, measure* the *angles A, B, C and D*. Now *sum each pair of opposite angles,* that is, *A and C*, and *B and D*. *What* do you *observe*?

THEORY: *Angles A and C are opposite angles* of the *cyclic quadrilaterals ABCD.*

Thus $\hat{A} + \hat{C} = 180^\circ$ (*opposite ∠s supplementary*)

Also *angles $\hat{B}$ and $\hat{D}$* are *opposite angles* of the *cyclic quadrilateral ABCD.*

Thus $\hat{B} + \hat{D} = 180^\circ$ (*opposite ∠s supplementary*).

Example 46

Without measuring, determine the value of the unknown angles in the circle, giving reasons for your answers.

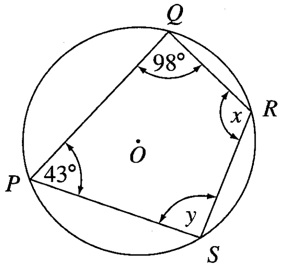


Fig. 9.283 Cyclic quadrilateral

Solution

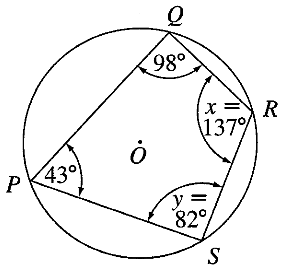


Fig. 9.283 Cyclic quadrilateral

Now $\hat{x} + 43^\circ = 180^\circ$ (*opp. ∠s supp.*)

So $\hat{x} = 180^\circ - 43^\circ$

i.e. $\hat{x} = 137^\circ$

Now $\hat{y} + 98^\circ = 180^\circ$ (*opp. ∠s supp.*)

So $\hat{y} = 180^\circ - 98^\circ$

i.e. $\hat{y} = 82^\circ$

Hence $\hat{x} = 137^\circ$ and $\hat{y} = 82^\circ$.

THEOREM 5: *The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.*

Class Activity

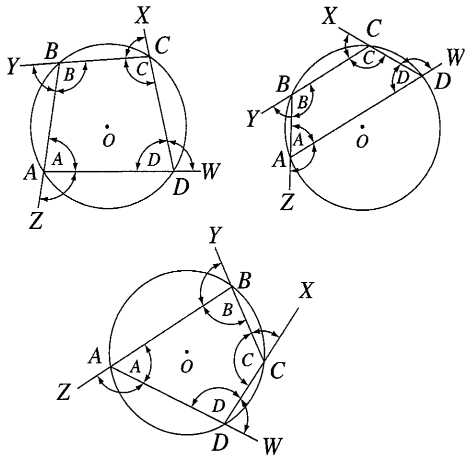


Fig. 9.284 Cyclic quadrilaterals

Using your *cyclic quadrilaterals ABCD* (Fig. 9.282), *produce* the *side AD to W, DC to X, CB to Y* and *BA to Z* as shown in Fig. 9.284. Now *measure each set of exterior angles:* $W\hat{D}C$, $X\hat{C}B$, $Y\hat{B}A$ and $Z\hat{A}D$. *Compare each exterior angle* with its *interior opposite angle. What* do you *observe*?

THEORY:

Now $W\hat{D}C = \hat{B}$ (*ext.* ∠ = *int. opp.* ∠)

$X\hat{C}B = \hat{A}$ (*ext.* ∠ = *int. opp.* ∠)

$Y\hat{B}A = \hat{D}$ (*ext.* ∠ = *int. opp.* ∠)

and $Z\hat{A}D = \hat{C}$ (*ext.* ∠ = *int. opp.* ∠)

Example 47

Without measuring, determine the magnitude of each of the marked angles in the circle, stating reasons for your answers.

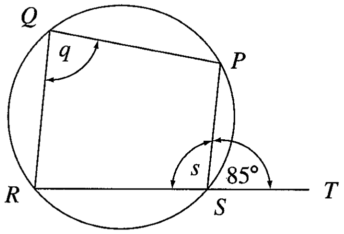


Fig. 9.285 Cyclic quadrilateral

Solution

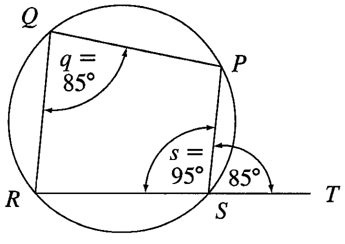


Fig. 9.285 Cyclic quadrilateral

Now $\hat{q} = 85°$ (*ext.* ∠ = *int. opp.* ∠)

And $\hat{s} + \hat{q} = 180°$ (*opp.* ∠*s supp.*)

So $\hat{s} + 85° = 180°$

i.e. $\hat{s} = 180° - 85°$

∴ $\hat{s} = 95°$

Hence $\hat{q} = 85°$ and $\hat{s} = 95°$.

Alternative Method

Now $\hat{s} + 85° = 180°$ (∠*s on a straight line*).

So $\hat{s} = 180° - 85°$

i.e. $\hat{s} = 95°$

Now $\hat{q} + \hat{s} = 180°$ (*opp.* ∠*s supp.*)

So $\hat{q} + 95° = 180°$

i.e. $\hat{q} = 180° - 95°$

∴ $\hat{q} = 85°$

Hence $\hat{q} = 85°$ and $\hat{s} = 95°$.

Exercise 9n

1. If reflex ∠*AOB* = 210°, calculate ∠*ACB*, giving a reason for your answer.

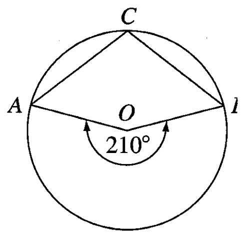


Fig. 9.286 Circle

2. If obtuse $\angle AOB = 96°$, determine $\angle ACB$, giving a reason for your answer.

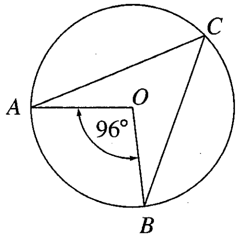


Fig. 9.287 Circle

3. If $A\hat{C}B = 47°$, calculate $A\hat{O}B$, stating a reason for your answer.

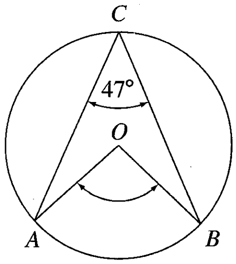


Fig. 9.288 Circle

4. If $\angle AOB = 70°$, evaluate $\angle ACB$, stating a reason for your answer.

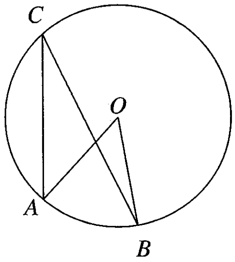


Fig. 9.289 Circle

5. If $\angle ADB = 108°$, determine reflex angle AOB, giving a reason for your answer.

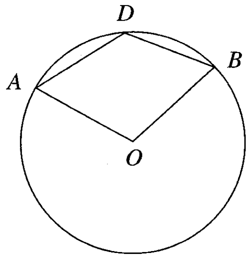


Fig. 9.290 Circle

6. Determine the magnitude of angle ACB, giving a reason for your answer.

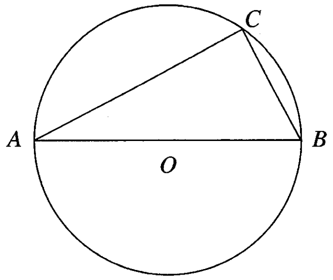


Fig. 9.291 Circle

7. Determine the length of the diameter BC, giving reasons for your answer.

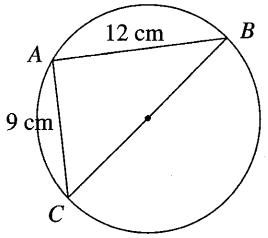


Fig. 9.292 Circle

8. If AB is a diameter, evaluate AC, stating reasons for your answer.

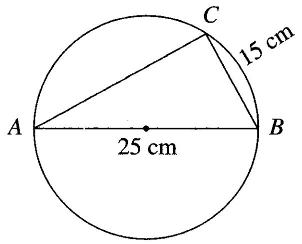


Fig. 9.293 Circle

9. Calculate the hypotenuse of each of the following triangles, giving reasons for each of your answers.

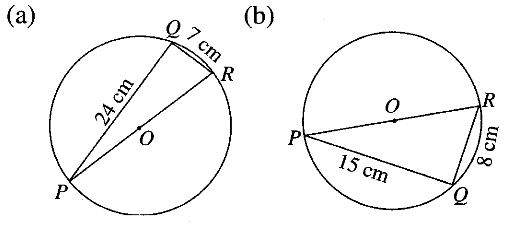

(a) (b)

Fig. 9.294 Circles

10. Calculate the unknown side of each of the following triangles, giving reasons for each of your answers.

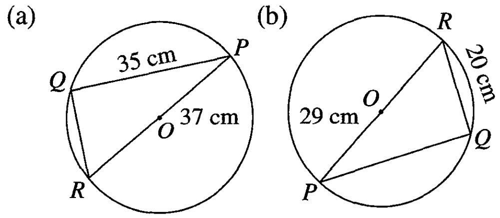


Fig. 9.295 Circles

11. Calculate the size of angles x and y, giving reasons for your answers.

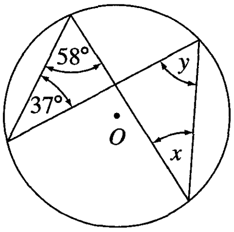


Fig. 9.296 Circle

12. If angle $APB = 55°$, determine the magnitude of angle AQB, stating a reason for your answer.

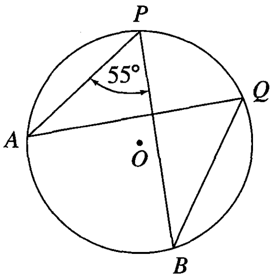


Fig. 9.297 Circle

13. Evaluate the angles x and y shown, stating reasons for your answers.

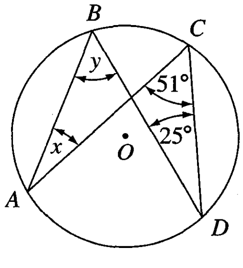


Fig. 9.298 Circle

14. Evaluate the angles x and y shown, giving reasons for your answers.

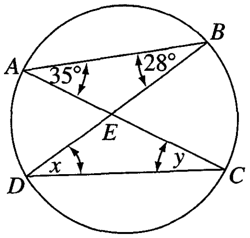


Fig. 9.299 Circle

15. If $\angle AQB = 21°$, determine the magnitude of $\angle APB$, stating a reason for your answer.

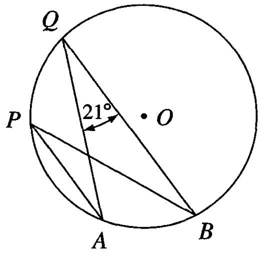


Fig. 9.300 Circle

16. In the cyclic quadrilateral $ABCD$, angle $ADC = 39°$. Calculate angle ABC, giving a reason for your answer.

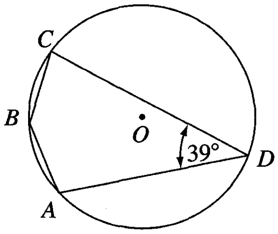


Fig. 9.301 Cyclic quadrilateral

17. In the cyclic quadrilateral $PQRS$, $\angle QPS = 48°$ and $\angle PQR = 97°$. Calculate $\angle QRS$ and $\angle RSP$, stating reasons for your answers.

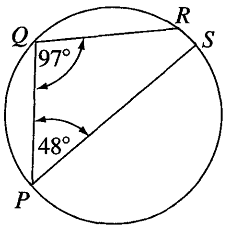


Fig. 9.302 Cyclic quadrilateral

18. In the cyclic quadrilateral *KLMN*, $\hat{L} = 95^\circ$ and $\hat{M} = 108^\circ$. Evaluate *K* and *N*, giving reasons for your answers.

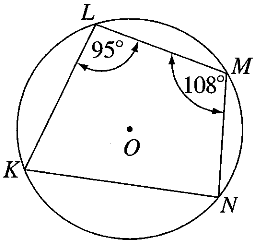


Fig. 9.303 Cyclic quadrilateral

19. In the cyclic quadrilateral *WXYZ*, $\hat{W} = 35^\circ$, and $\hat{X} = 109^\circ$. Determine $\hat{Y}$ and $\hat{Z}$, stating reasons for your answers.

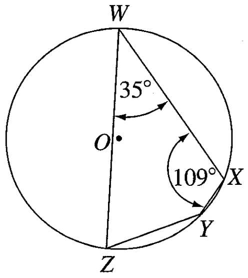


Fig. 9.304 Cyclic quadrilateral

20. In the cyclic quadrilateral *PQRS*, $\angle Q = 115^\circ$ and $\angle R = 33^\circ$. Evaluate $\angle P$ and $\angle S$, giving reasons for your answer.

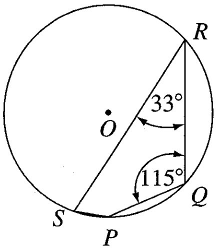


Fig. 9.305 Cyclic quadrilateral

21. In the cyclic quadrilateral *ABCD*, angle *CDE* = 93°. Determine the magnitude of angle *ABC*, giving a reason for your answer.

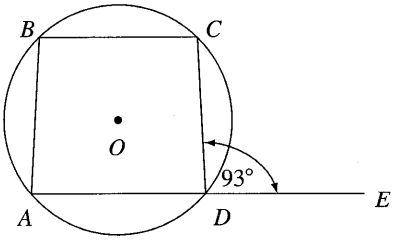


Fig. 9.306 Cyclic quadrilateral

22. In the cyclic quadrilateral *PQRS*, angle *PQR* = 105°. Evaluate angle *RST*, giving a reason for your answer.

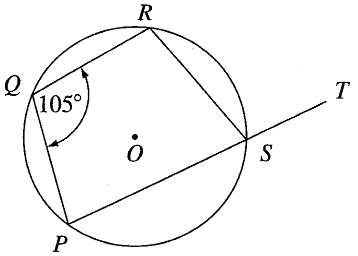


Fig. 9.307 Cyclic quadrilateral

23. In the cyclic quadrilateral *KLMN*, angle *KLM* = 123°. Evaluate angles *XNK* and *KNM*, stating reasons for your answers.

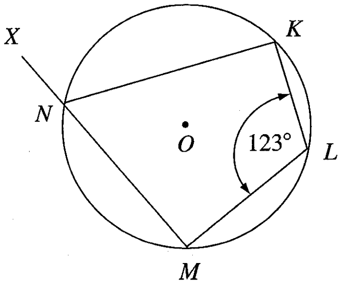


Fig. 9.308 Cyclic quadrilateral

24. In the cyclic quadrilateral *ABCD*, $\angle ABC = 87^\circ$. Evaluate angle *ADE*, giving a reason for your answer.

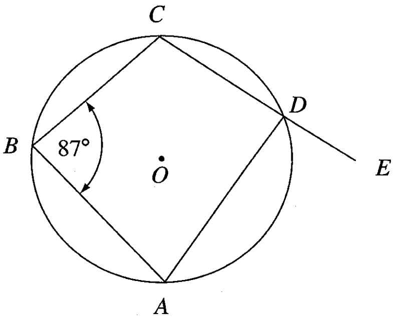


Fig. 9.309 Cyclic quadrilateral

25. In the cyclic quadrilateral *WXYZ*, angle *AZW* = 125°. Evaluate angle *WXY*, giving a reason for your answer.

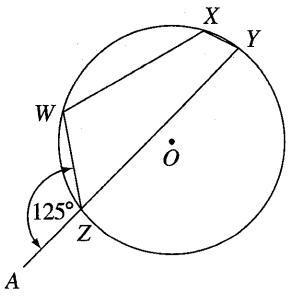


Fig. 9.310 Cyclic quadrilateral

26. Calculate the magnitude of each marked angle in the following circles, giving a reason for each of your answers.

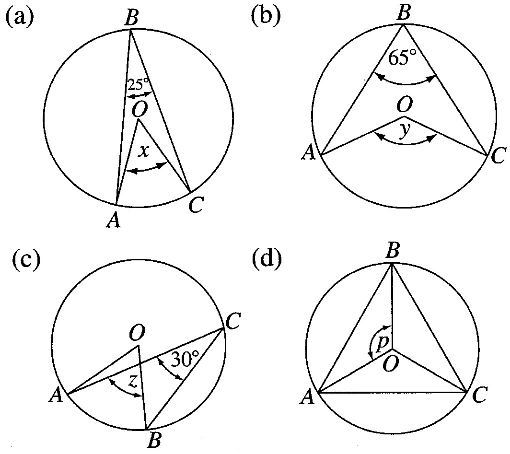


Fig. 9.311 Circles

27. Determine the magnitude of each marked angle in the following circles, giving a reason for each of your answers.

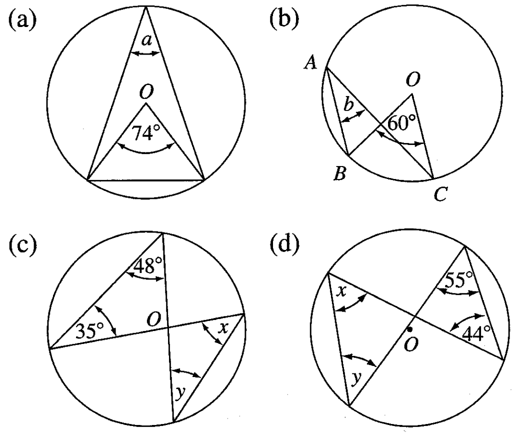


Fig. 9.312 Circles

28. Calculate the magnitude of each marked angle in the following circles, stating a reason for each of your answers.

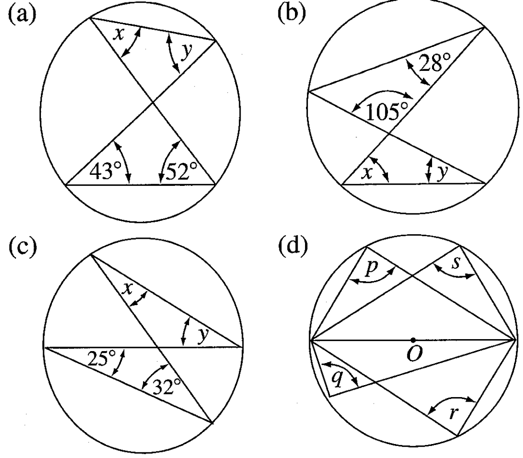


Fig. 9.313 Circles

29. Without measuring, determine the value of the letter representing an angle in each circle, giving a reason for each of your answers.

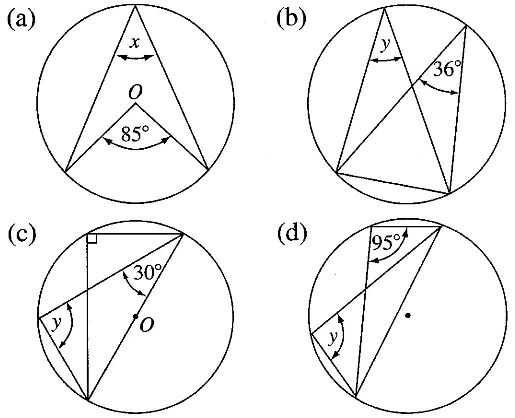


Fig. 9.314 Circles

30. Without measuring, determine the value of each letter representing an angle in the circles, giving a reason for each of your answers.

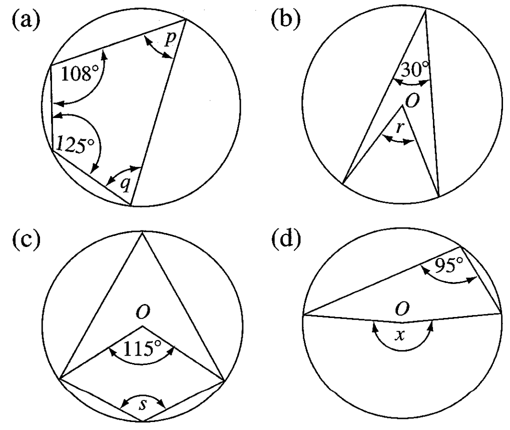


Fig. 9.315 Circles

31. Without measuring, calculate the magnitude of each letter representing an angle in the circles, giving a reason for each of your answers.

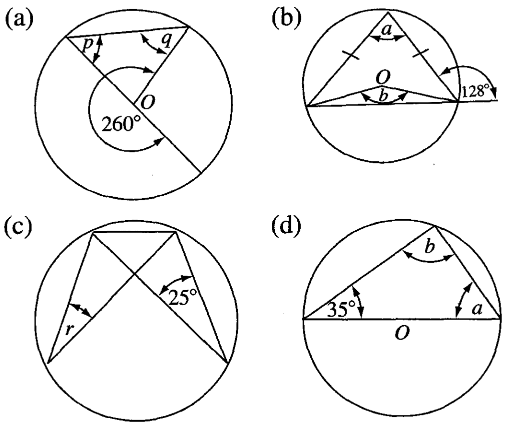
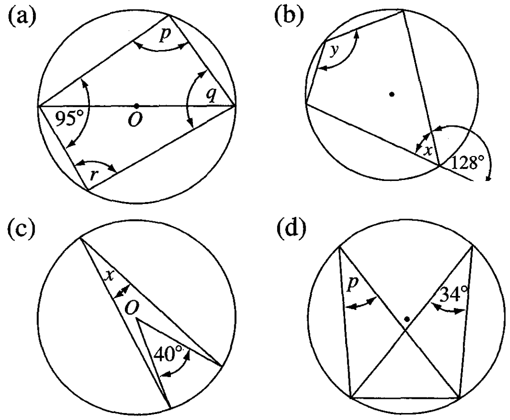


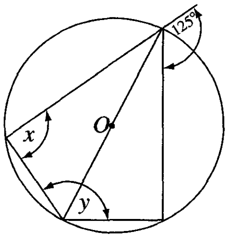
Fig. 9.316 Circles

32. Without measuring, determine the magnitude of each letter representing an angle in the circles, giving a reason for each of your answers.

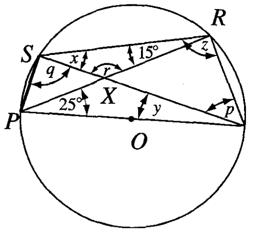


33. Without measuring, evaluate the magnitude of each letter representing an angle in the circles, giving a reason for each of your answers.

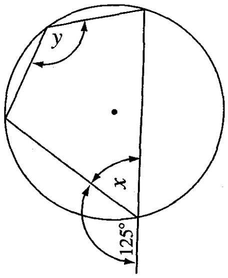
(a)



(b)



(c)



(d)

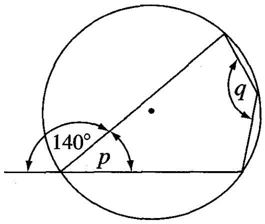


Fig. 9.318 Circles

34. Calculate the magnitude of each letter representing an angle in the circles, giving a reason for each of your answers.

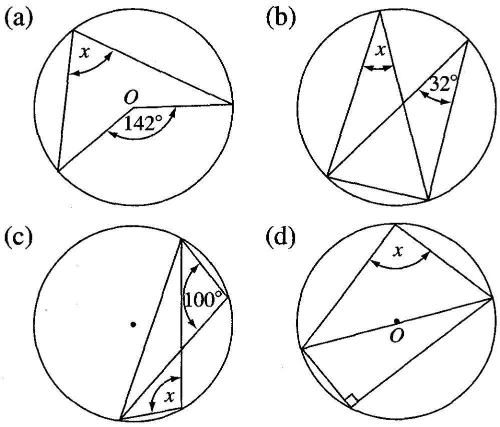


Fig. 9.319 Circles

35. Calculate the magnitude of each letter representing an angle in the circles, giving a reason for each of your answers.

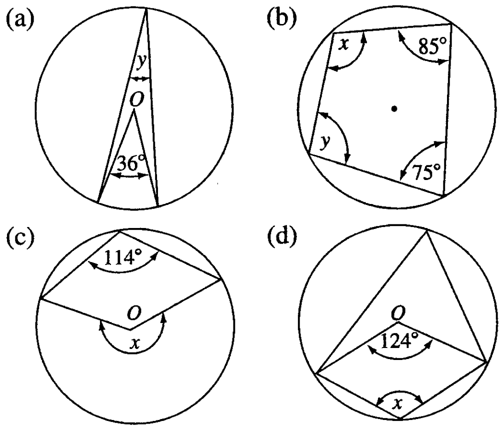


Fig. 9.320 Circles

36. Evaluate the magnitude of each letter representing an angle in the circles, giving a reason for each of your answers.

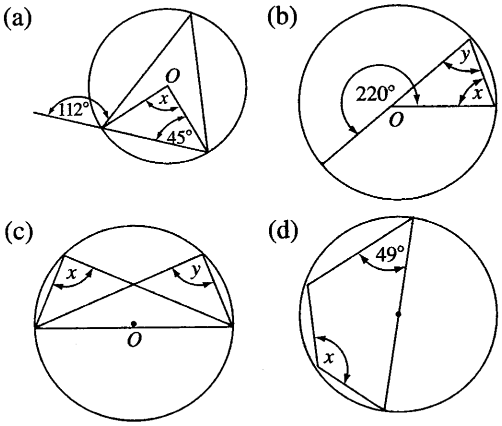


Fig. 9.321 Circles

37. Determine the magnitude of each letter representing an angle in the circles, giving a reason for each of your answers.

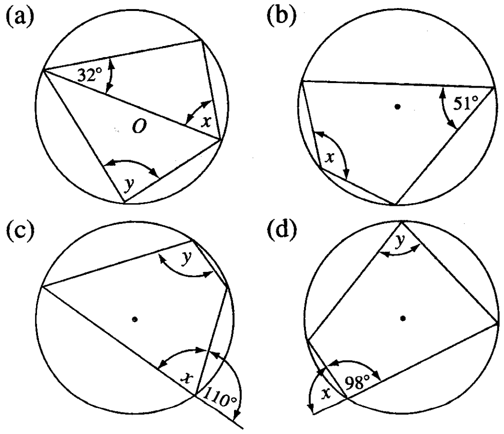


Fig. 9.322 Circles

38. Without measuring, determine the magnitude of each letter representing an angle in the circles, giving a reason for each of your answers.

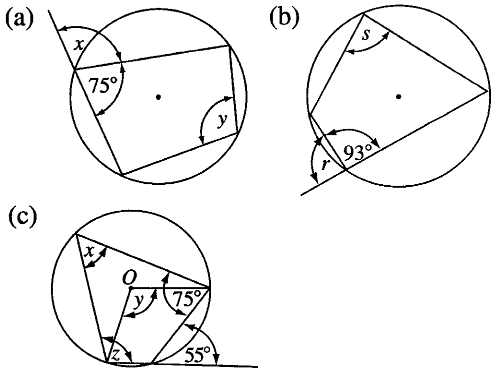


Fig. 9.323 Circles

39. Calculate, the size of each marked angle in the circles, stating a reason for each of your answers.

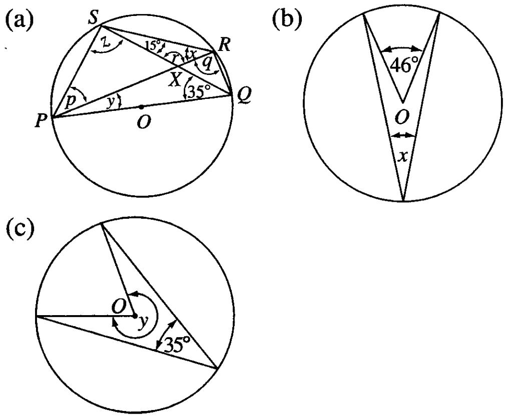


Fig. 9.324 Circles

40. *ABC* is an isosceles triangle inscribed in a circle whose centre is *O*. If $\angle AOB = 74°$, determine $\angle BAC$, giving reasons for your answers.

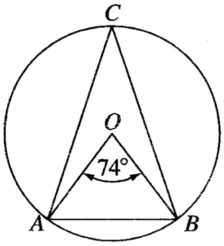


Fig. 9.325 Circle

41. If $A\hat{O}B = 84°$, calculate $A\hat{E}B$, stating a reason for your answer.

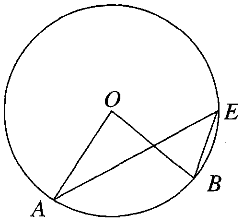


Fig. 9.326 Circle

42. If angle $ABD = 76°$ and angle $OBD = 90°$, evaluate angle AOB, giving reasons for your answers.

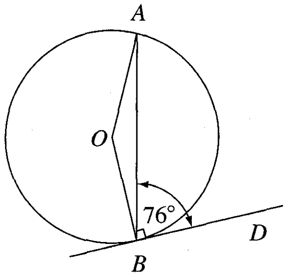


Fig. 9.327 Circle

43. Determine the marked angles in the following diagrams, stating reasons for your answers:

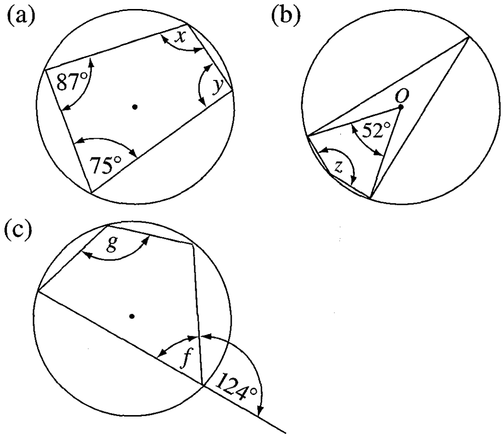


Fig. 9.328 Circles

Solids

A *solid* is defined as *anything* that occupies *space*. For example: yourself, a book, a car and a house. As in the case of plane figures, a *solid* has *both* a *length* and a *breadth*. However, it also has a *thickness* (or *height* or *depth*). Thus a *solid* is *three dimensional*. Most *solids* are *irregular* in *shape*. For example: a stone, a heap of dirt and a speck of dust. However there are also solids which are non-irregular in *shape*. For example: a glass, a box and a leaf. Some *common non-irregular solids* are shown in Fig. 9.329 below.

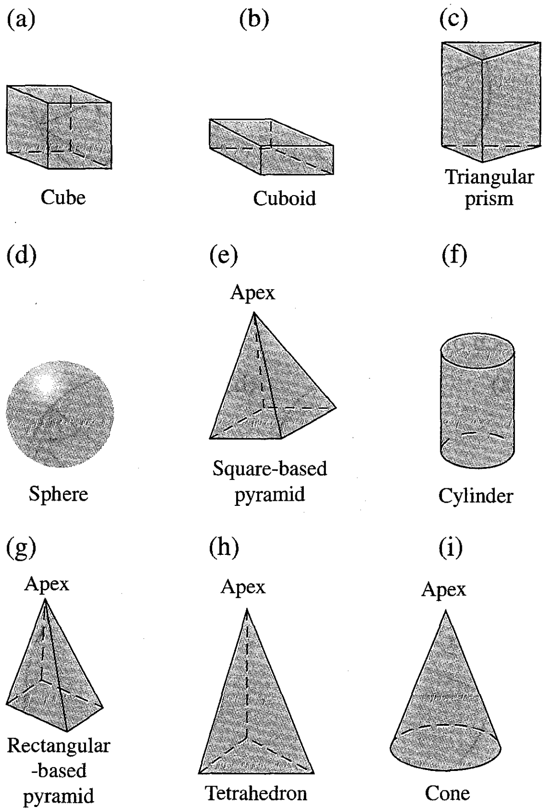


Fig. 9.329 Common non-irregular solids

A *polyhedron* is a *solid shape* with *flat sides*, where the *flat sides* (or *faces*) are *all polygons.*

For example: The *cube* and the *tetrahedron.*

A *prism* is a *polyhedron* having throughout its *length*, the *same end* (or *cross-section*).

For example: A *cube* and a *triangular prism.*

A *pyramid* is a *polyhedron* with a *base* in the *shape* of a *polygon*, the *other faces* being *triangles* meeting at a *common vertex* called the *apex*. For example: A *tetrahedron.*

Solids are *bounded* by *surfaces* called *faces*. These *surfaces* are *two kinds—plane* (or *flat*) and *curved.* The *faces* of a *cube* are all *plane surfaces*. The *surface* of a *sphere* is *curved.* A *cylinder* has both *plane* and *curved surfaces.*

Surfaces are *bounded* by *lines* and *intersect* at *lines*. These *lines* are either *straight* or *curved*. When *two surfaces intersect* they *meet* at an *edge*. Thus:

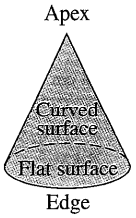


Fig. 9.330 Edge

Lines intersect at *points*. The *meeting place* of *two edges* is called a *point* or *vertex*. Thus:

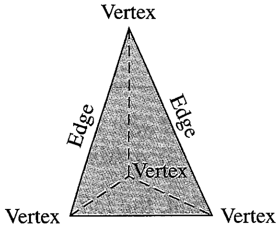


Fig. 9.331 Point or vertex

Examine the *solids* in Fig. 9.329 and *complete* the following table:

Table 9.8

Solid	*No. of faces (F)*	*No. of edges (E)*	*No. of vertices (V)*
Cube	6	12	8
Cuboid			
Triangular prism			
Cylinder			
Square-based pyramid			
Rectangular-based pyramid			
Tetrahedron			
Cone			

Euler's formula which *relates* the *number of faces*, *vertices* and *edges* in a *polyhedron* states:

$$F + V = E + 2$$

For the *cube*:

$$F + V = 6 + 8 = 14$$

$$E + 2 = 12 + 2 = 14$$

Thus $F + V = E + 2 = 14$

Net: Two Dimensional Representation of a Solid

One way of *illustrating* a *solid* is by *drawing* a *two dimensional representation* of it called a *net*. The *net* of a *solid* is a *plane shape*, which when *cut out* and *folded*, can be *made* into the *solid shape*. For example: a *shape* is said to *form* the *net* of a *cube*, if it could be *cut out* and *folded* up to *form a cube*. Fig. 9.332 illustrates some *possible nets* of a *cube*.

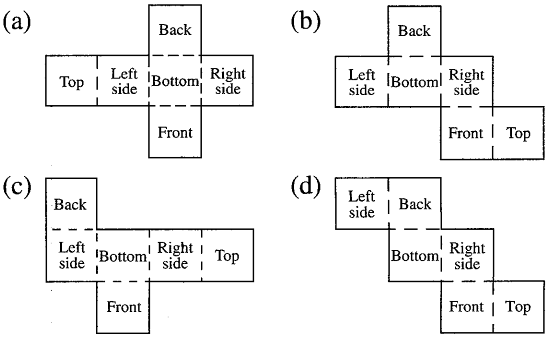


Fig. 9.332 Nets of a cube

From Fig. 9.332, it can be *seen* that it is *possible* for a *solid* to have *more than one net*. This is the *case* when the *polyhedron* is *regular*. That is, when *all the faces* of a *polyhedron* are *congruent*. For example: a *regular tetrahedron* has *four triangular faces* which are *congruent equilateral triangles*. There are only *five regular polyhedra*: the regular *tetrahedron*, the *cube* the regular *octahedron* (*each* of its 8 *faces* in an *equilateral triangle*), the regular *dodecahedron* (*each* of its 12 *faces* is a *regular pentagon*) and the regular icosahedron (*each* of its 20 *faces* is an *equilateral triangle*).

If the *nets* shown previously are *drawn* on *Bristol board* (i.e. *stiff paper*), *cut out* and *folded* appropriately along the *dotted lines*, then each net will *form* a *cube*. *Each net* consists of 6 *congruent squares*, since a *cube* has 6 *congruent faces*.

Normally, *nets* are *drawn* on *graph paper* using *ruler*, *compasses*, *set square* and *protractor*. When *graph paper* is used, it helps in the *drawing* of *parallel* and *perpendicular lines*, and to find *mid-points*. Always *draw* a *rough sketch* of the *net* to be *constructed*, *before starting* to *construct* the *actual net*.

Example 48

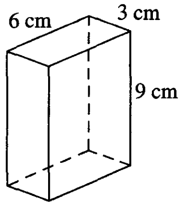


Fig. 9.333 Cuboid or rectangular prism

Sketch a net of the cuboid or rectangular prism shown in Fig. 9.333.

Solution

Scale: 1 ≡ 3

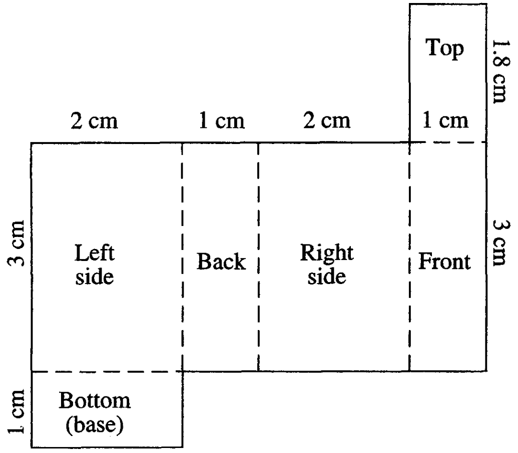


Fig. 9.334 Sketch of the net of the cuboid or rectangular prism

From Fig. 9.333, the *left side* and the *right side* of the *cuboid* are *congruent rectangles* of *dimensions* 6 cm by 9 cm, the *front* and the *back* are *congruent rectangles* of *dimensions* 3 cm by 9 cm, and the *top* and the *bottom* are *congruent rectangles* of *dimensions* 6 cm by 3 cm. These facts are represented in the *net* of the *cuboid* shown in Fig. 9.334 above, where a *scale* of 1 cm *to represent* 3 cm was used.

Example 49

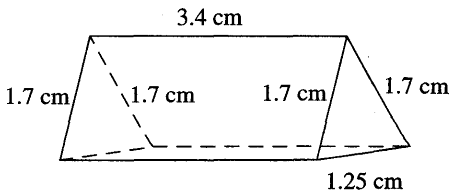


Fig. 9.335 Triangular prism

Draw an accurate full size diagram of the net for the triangular prism shown in Fig. 9.335.

Solution

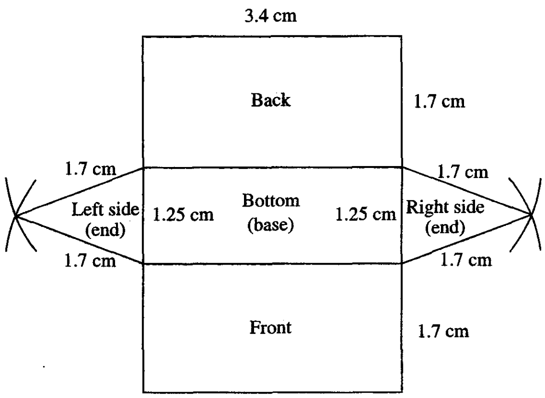


Fig. 9.336 Accurate full size diagram of the net of the triangular prism

From it, can be seen that, the *left side* and the *right side* of the *triangular prism* are *congruent triangles* with *given dimensions*, and the *back* and *front* are *congruent rectangles* with *given dimensions*, and the *base is a rectangle of given dimensions*. These facts are represented in the *accurate full size diagram* of the *net* of the *triangular prism* shown in Fig. 9.336 above.

Example 50

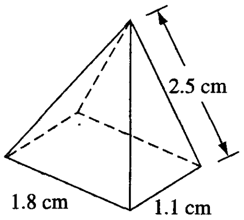


Fig. 9.337 Rectangle-based pyramid

Construct an accurate full size diagram of the net for the rectangle-based pyramid shown in Fig. 9.337.

Solution

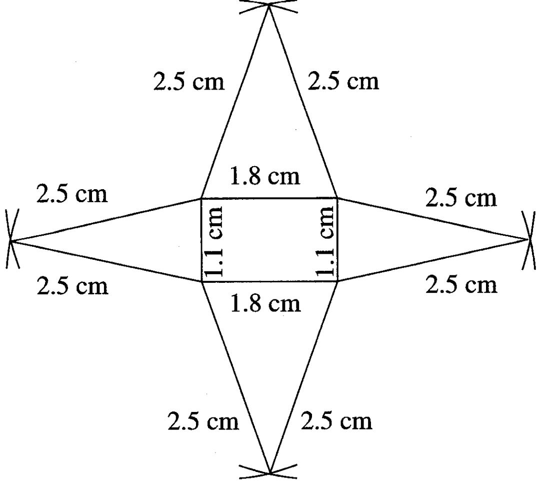


Fig. 9.338 Constructed accurate full size diagram of the net of the rectangle-based pyramid

From it, can be seen that, the *left side* and the *right side* of the *rectangle-based pyramid* are *congruent triangles* of *given dimensions*, the *back* and the *front* are *congruent triangles* of *given dimensions*, and the *base* is a *rectangle* of *given dimensions*. These facts are represented in the *constructed accurate full size diagram* of the *net* of the *rectangle-based pyramid* shown in Fig. 9.338 above.

Example 51

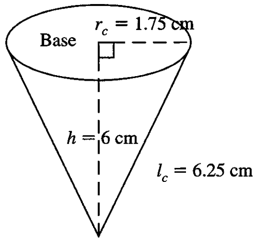


Fig. 9.339 Cone

Sketch a net for the cone shown in Fig. 9.339

Solution

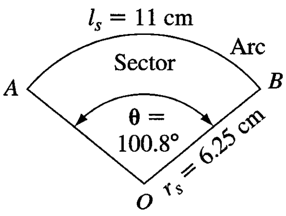


Fig. 9.340 Sketch of the net of the cone

The *circumference* of the *base* of the *cone*, $C = 2\pi r_c$

$$= 2 \times \frac{22}{7} \times 1.75 \text{ cm}$$

$$= 11 \text{ cm}$$

So the *length* of the *arc* of the *net* of the *cone*, $l_s = C = 11$ cm

The *radius* of the *sector* forming the *net* of the *cone*, $r_s = l_c = 6.25$ cm

We now need to know the *sector angle* that represents an *arc* of *length* 11 cm and *radius* 6.25 cm.

The *length* of the *arc*, $l_s = 2\pi r_s \frac{\theta}{360}$

So $11 \text{ cm} = 2 \times \frac{22}{7} \times 6.25 \text{ cm} \times \frac{\theta}{360}$

And the *sector angle*, $\theta = \frac{11\,\cancel{cm} \times 7 \times 360}{2 \times 22 \times 6.25\,\cancel{cm}}$

$$= 100.8°$$

Thus in order to *construct* the *net* of the *cone*, we *construct* a *sector AOB* of *radius* 6.25 cm and *sector angle* 100.8°. The *sketch* of the *net of the cone* is shown in Fig. 9.340.

NOTE:

Since $C = l_s = 2\pi r_c$

$$= 2 \times \frac{22}{7} \times 1.75 \text{ cm}$$

And $l_s = 2\pi r_s \frac{\theta}{360}$

$$= 2 \times \frac{22}{7} \times 6.25 \text{ cm} \times \frac{\theta}{360}$$

Then

$$\cancel{2} \times \frac{\cancel{22}}{\cancel{7}} \times 6.25\,\cancel{cm} \times \frac{\theta}{360} = \cancel{2} \times \frac{\cancel{22}}{\cancel{7}} \times 1.75\,\cancel{cm}$$

So $6.25 \times \frac{\theta}{360} = 1.75$

i.e. $\theta = \frac{1.75 \times 360}{6.25}$

∴ $\theta = 100.8°$

When the *sector* is *cut out* and *folded*, the *radius of the sector* will become the *slant height of the cone*, and the *length of the arc* will become the *circumference of the base of the cone.*

Exercise 9o

1. Sketch the net of a cube.
2. Sketch the net of a cuboid.
3. Sketch the net of a triangular prism.
4. Sketch the net of a cylinder.
5. Sketch the net of a square-based pyramid.
6. Sketch the net of a rectangle-based pyramid.
7. Sketch the net of a tetrahedron.
8. Sketch the net of a cone.
9. Draw the net for a cube with edge 12 cm, using a scale of 1 cm to represent 4 cm.
10. Draw the net for a cuboid with length 18 cm, width 9 cm and height 6 cm, using a scale of 1 cm to represent 3 cm.
11. Draw the net for a triangular prism with triangular ends of dimensions 12, 12 and 8 cm; and rectangular base with dimensions 20 cm by 8 cm. Use a scale of $1 \equiv 4$.
12. Draw the net for a cylinder of height 25 cm and radius 3.5 cm. Use a scale of $1 \equiv 5$ and take π as $\frac{22}{7}$
13. Draw the net for a square-based pyramid with base of edge 36 cm and slant height of length 48 cm. Use a scale of $1 \equiv 6$.
14. Draw the net for a rectangle-based pyramid with base of dimensions 28 cm by 35 cm and slant height of length 49 cm. Use a scale of 1:7.
15. Draw the net for a tetrahedron with base of dimensions 16, 24 and 32 cm; and slant height of length 48 cm. Use a scale of 1:8.
16. Draw the net for a cone of base radius 27 cm and slant height of length 45 cm. Use a scale of 1:9 and take π as $\frac{22}{7}$
17. Construct an accurate full size diagram of the net for a cube with edge 5 cm.
18. Construct an accurate full size diagram of the net for a cuboid with length 5 cm, width 4 cm and height 9 cm.
19. Construct an accurate full size diagram of the net for a triangular prism, with triangular ends of dimensions 5, 5 and 4 cm; and rectangular base with dimensions 10 cm by 4 cm.
20. Construct an accurate full size diagram of the net for a cylinder with height 7 cm and radius 1.75 cm.
21. Construct an accurate full size diagram of the net for a square-based pyramid with base of edge 4.5 cm and slant height of length 7.5 cm.
22. Construct an accurate full size diagram of the net for a rectangle-based pyramid with base of dimensions 5.8 cm by 3.9 cm and slant height of length 9.5 cm.
23. Construct an accurate full size diagram of the net for a tetrahedron with base of dimensions 5.7, 4.5 and 3.8 cm; and slant height of length 7.9 cm.
24. Construct an accurate full size diagram of the net for a cone with base radius 1.4 cm and slant height of length 5 cm.

Plan and Elevations

Architects and *contractors* use *scale drawings* extensively to show the *layout of a building* or some other *structure* to be built. In order to show a *complete picture* of the *structure* to be constructed, *different views* of the *structure* must be *drawn.*

Oblique Projection

We normally represent a *solid* in *two dimensions* by *drawing* an *oblique projection* of it. In *drawing* the *oblique projection* of a *solid*, the *front elevation forms* the *front face. Other lines* are *drawn* at 45° to give *depth.* Normally *lengths* along the *vertical* and *horizontal* axes are *full length*, and *lengths along* the 45° *axis* are *half length.*

Fig. 9.344 shows an *oblique projection* of a *cube.*

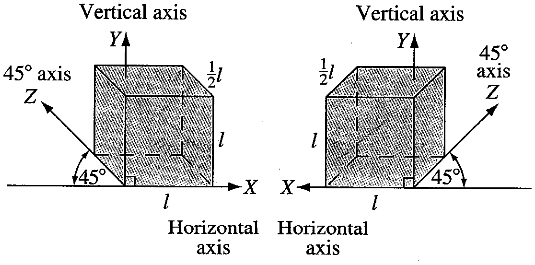


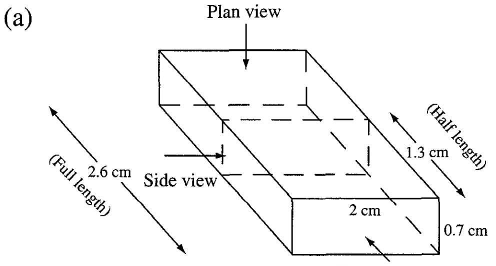
Fig. 9.341 Oblique projection of a cube

However this *method* of *representation* of a *solid* has its *limitations*, as it does *not* give an *accurate* idea of its *shape* and *dimensions*.

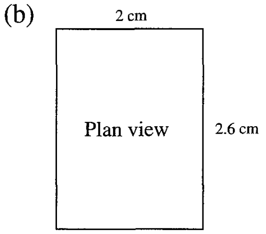
Plan and Elevations

The *plan and elevation* of a *solid* give a *more accurate idea* of its *shape* and *dimensions*.

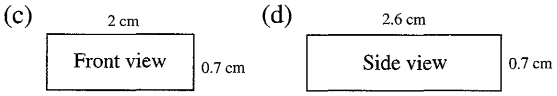
Views projected onto a *horizontal plane* are called *plans*, and *views projected* onto a *vertical plane* are called *elevations*. The *plan and elevations* of a *cuboid* are shown in Fig. 9.342 below.



Oblique projection
of a cuboid



Plan



Front elevation

Side elevation

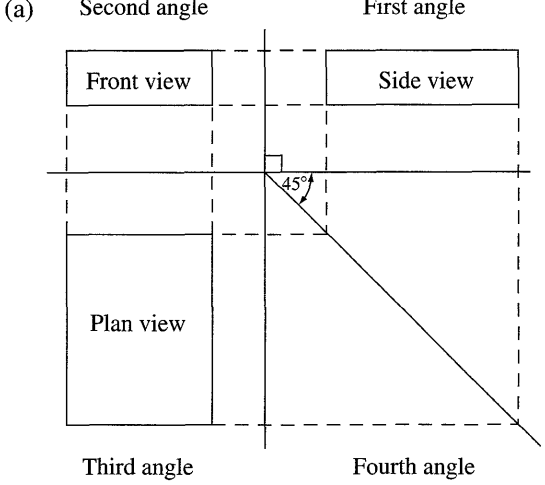
Fig. 9.342 Plan and elevations of a solid

The *plan* of a *solid* is the *view* when *looking vertically downwards*. The *map* of a *country* is a *plan*.

The *front elevation* of a *solid* is the *view* when *looking horizontally in front.*

The *side elevation* of a *solid* is the *view* when *looking horizontally at one side.*

The *cuboid* can be *drawn* in either *first angle projection* or *third angle projection* as shown in Fig. 9.343 below.



First angle projection of cuboid

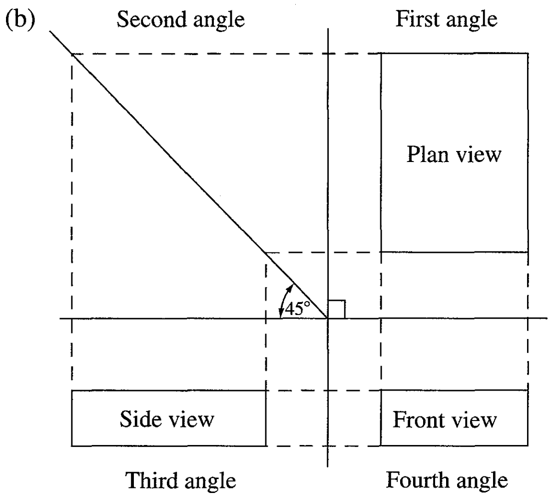


Fig. 9.343 Third angle projection of cuboid

To *draw* the *cuboid* in *first angle projection*, *first draw* the *plan* at the *bottom left corner* of the paper and *then* the *front elevation above it*. So the *side elevation* will be in the *top right corner*, that is, in the *first angle*.

To *draw* the *cuboid* in *third angle projection*, *first draw* the *plan* at the *top right corner* of the paper and *then* the *front elevation below it*. So the *side elevation* will be in the *bottom left corner*, that is, in the *third angle*.

The *dashed lines* in Fig. 9.343 above are called *projection lines* or *construction lines*.

Isometric Projection

We can also *represent* a *solid* in *two dimensions* by *drawing* an *isometric projection* of it. In *drawing* the *isometric projection* of a *solid*, the *isometric axes OX, OY and OZ* are *drawn* at 120° *to each other. All lengths* along the *isometric axes* will be *full-scale*, but *not so along diagonals. Isometric projections* are *more difficult* to *draw* than *oblique projections.*

Again this *method* of *representation* of a *solid* has its *limitations*, as it *does not* given an *accurate idea* of its *shape* and *dimensions.*

Example 52

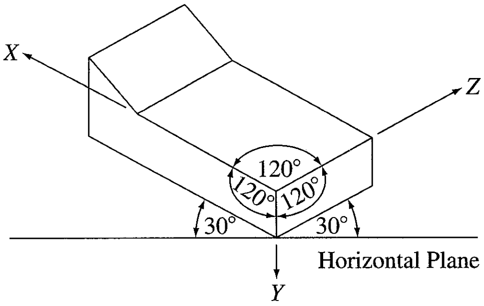


Fig. 9.344 Isometric projection of a solid

Draw the plan, front elevation and side elevation of the solid shown in Fig. 9.344.

Solution

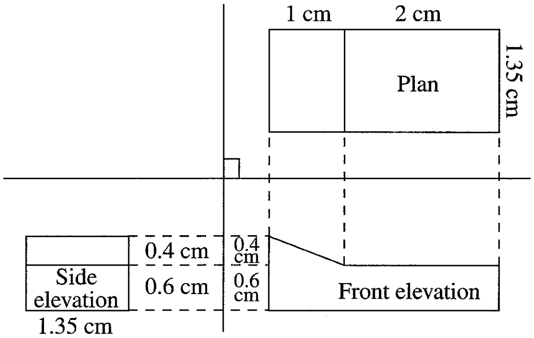


Fig. 9.345 Plan, front elevation and side elevation of solid

Fig. 9.345 shows the *plan, front elevation* and *side elevation* of the *solid* as a *composite diagram.* Alternatively, the *plan, front elevation* and *side elevation* can be *drawn separately* as shown in Fig. 9.342.

Example 53

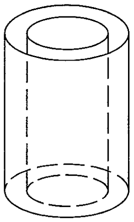


Fig. 9.346 Cylindrical tube

Fig. 9.346 shows a cylindrical tube with an inner radius of 1.0 cm and outer radius 1.2 cm. If the length of the tube is 3.4 cm long, draw a full-scale plan, front elevation and side elevation for the tube.

Solution

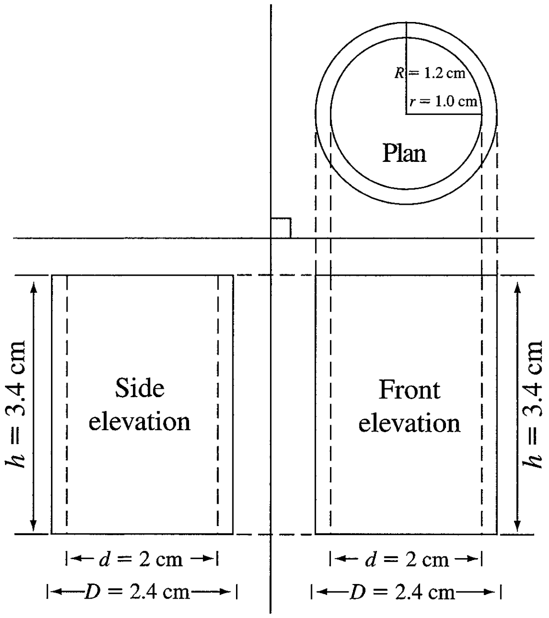


Fig. 9.347 Plan, front elevation and side elevation of cylindrical tube

Fig. 9.347 shows the *plan, front elevation* and *side elevation* of the *cylindrical tube* with the *given dimensions.* The *dashed lines* in the *diagram* represent *hidden lines* in the *solid.*

Example 54

Sketch the plan, front elevation and side elevation representing the following solid, using the scale 1:6.

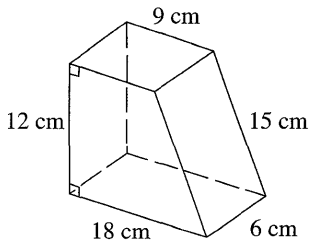


Fig. 9.348 Solid

Solution

Scale: 1 ≡ 6

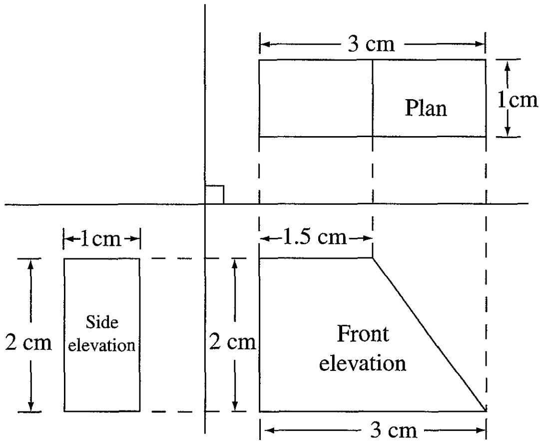


Fig. 9.349 Plan, front elevation and side elevation of solid

Fig. 9.349 shows the *plan*, *front elevation* and *side elevation* of the *solid*, using the *given scale.*

Exercise 9p

Draw the plan, front elevation and side elevation of each of the following solids:

1.

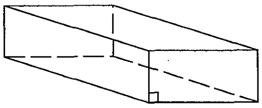


Fig. 9.350 Cuboid

2.

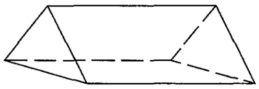


Fig. 9.351 Triangular prism

3.

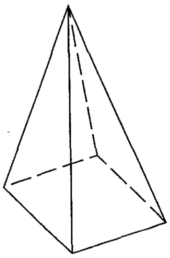


Fig. 9.352 Square-based pyramid

4.

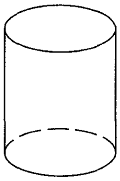


Fig. 9.353 Cylinder

5.

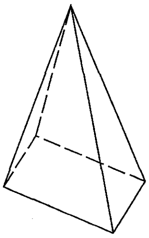


Fig. 9.354 Rectangular-based triangle

6.

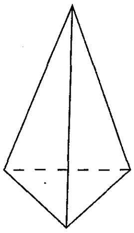


Fig. 9.355 Tetrahedron

7.

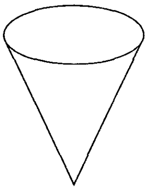


Fig. 9.356 Cone

8.

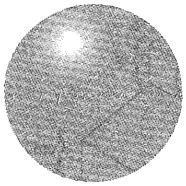


Fig. 9.357 Sphere

Draw a full-scale plan, front elevation and side elevation for each of the following solids:

9.

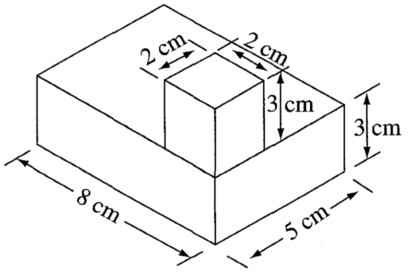


Fig. 9.358 Solid

10.

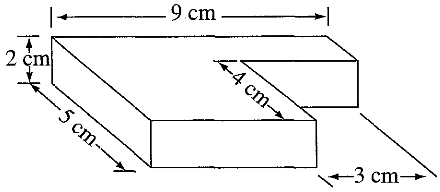


Fig. 9.359 Solid

11.

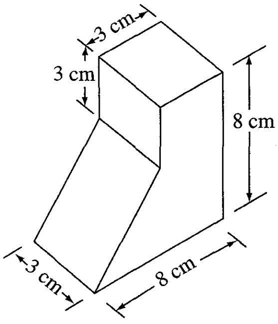


Fig. 9.360 Solid

12.

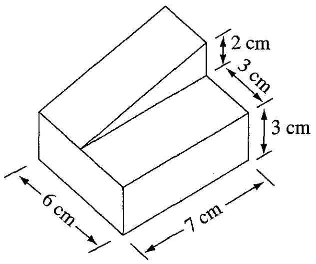


Fig. 9.361 Solid

13.

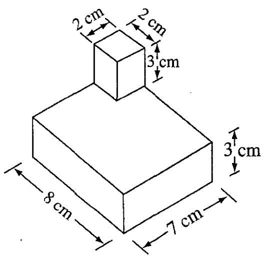


Fig. 9.362 Solid

14.

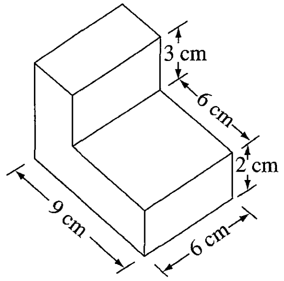


Fig. 9.363 Solid

15.

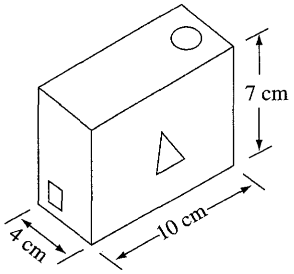


Fig. 9.364 Solid

16.

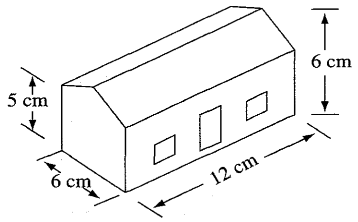
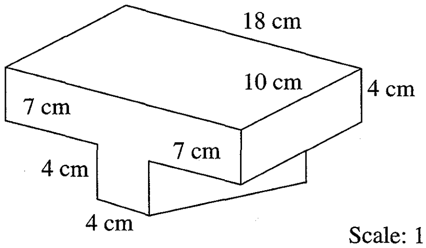


Fig. 9.365 Solid

Sketch the plan, front elevation and side elevation representing the following solids, using the given scales:

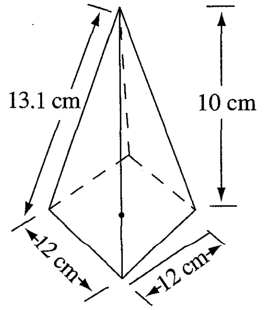
17.



Scale: 1 ≡ 2

Fig. 9.366 Solid

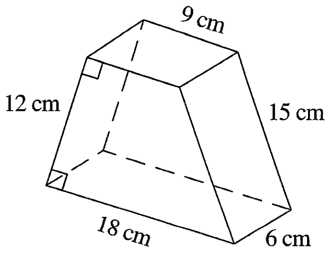
18.



Scale: 1 ≡ 4

Fig. 9.367 Solid

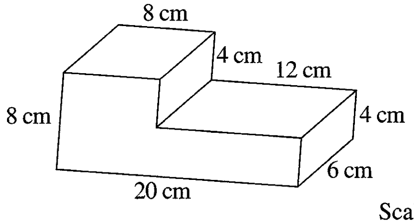
19.



Scale: 1 ≡ 5

Fig. 9.368 Solid

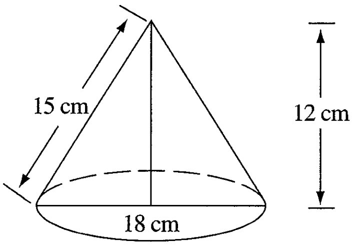
20.



Scale: 1 ≡ 2

Fig. 9.369 Solid

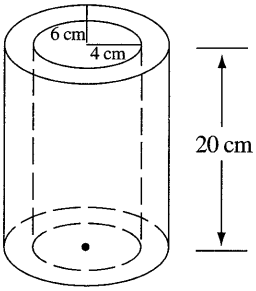
21.



Scale: 1 ≡ 3

Fig. 9.370 Solid

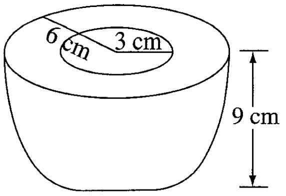
22.



Scale: 1 ≡ 4

Fig. 9.371 Solid

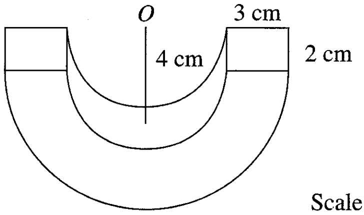
23.



Scale: 1 ≡ 3

Fig. 9.372 Solid

24.

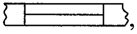
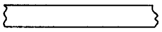


Scale: 1 ≡ 2

Fig. 9.373 Solid

Plan

Before any well-designed house is *constructed*, a *plan* must be *drawn* showing the *layout* for each *floor*. The *plan* will show the *number* and *size* of each *room*, *window*, *door* and *patio* (or *verandah*) et cetera.

In a *plan*, the *symbol* used to represent a *window* is , the *symbol* used to represent a *wall* is  and the *symbol* used to represent a *door* is . The *symbol* that is used to represent a *door* also *indicates* the *direction* in which the *door opens* and the *side* which *opens*.

Exercise 9q

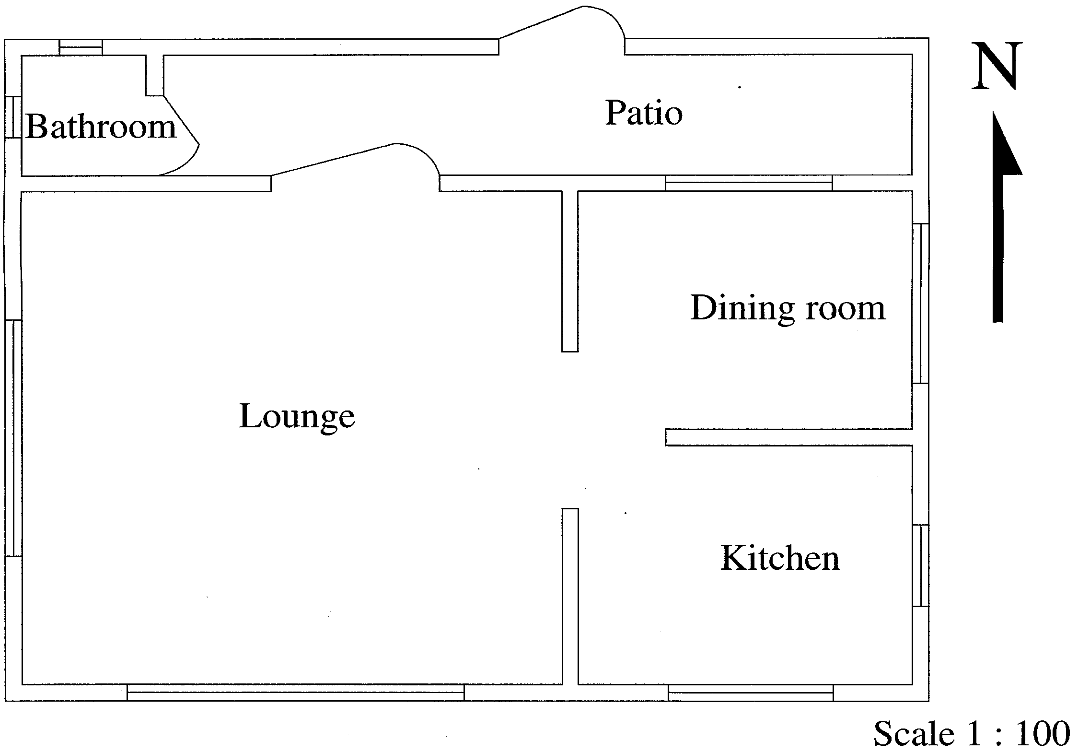


Fig. 9.374 Plan

Fig. 9.374 shows the plan of the ground floor of a house. Use the plan to answer the following questions.

1. (a) How many windows are there in the
   - (i) lounge
   - (ii) dining room
   - (iii) kitchen
   - (iv) bathroom?

   (b) Hence state the number of windows on the ground floor.
2. How many doors are there on the ground floor?
3. Using the spacing between walls, measure and state the width of the door (in cm) of
   - (a) the lounge
   - (b) the patio
   - (c) the bathroom.
4. Using the inner measurements, state the length and width (in cm) of
   - (a) the lounge
   - (b) the dining room
   - (c) the kitchen
   - (d) the patio
   - (e) the bathroom.
5. Measure and state the width of the windows (in cm) in
   - (a) the lounge
   - (b) the dining room
   - (c) the kitchen
   - (d) the bathroom.
6. How thick are the walls of the house (in cm)?
7. How many metres does 1 cm on the plan represent?
8. (a) Calculate the area of the lounge.
   - (b) How many square metres of carpet are required to cover the lounge?
   - (c) If the carpet costs $29.00 per square metre, determine the cost of carpeting the lounge.
9. (a) Determine the area of the dining room.
   - (b) How many square metres of terrazzo are needed to cover the dining room?
   - (c) If the terrazzo costs $37.50 per square metre, calculate the cost of terrazzoing the dining room.

10. (a) Evaluate the area of the kitchen.

(b) How many square ceramic tiles of length 0.25 m are required to cover the kitchen?

(c) If the cost per ceramic tile is $2.25, estimate the cost of tiling the kitchen.

11. (a) Determine the area of the patio.

(b) How many clay tiles of length 0.3 m and width 0.1 m are needed to cover the patio.

(c) If the cost per clay tile is $0.95, find the cost of tiling the patio.

12. (a) Calculate the area of the bathroom.

(b) How many square tiles of length 0.15 m are required to cover the bathroom.

(c) If the cost per tile is $1.60, calculate the cost of tiling the bathroom.

13. Draw a simple plan of your classroom.

14. Draw a plan for a block of classes in your school.

15. Draw a plan for your home or the place where you live.

C.X.C. Past Paper Questions

The following supplementary questions were taken from C.X.C. Past Papers.

Exercise 9r

1. Using ruler and compasses only, construct a triangle *ABC* with $\angle A = 60°$, $\angle B = 45°$ and $AB = 10$ cm. Find by measurement the length of *BC* in centimetres.

Question 3(i). C.X.C.(Basic). June 1981.

2. Draw a triangle *ABC* in which $BC = 6$ cm, $AB = 4$ cm and angle $ABC = 50°$. State the length of *AC*. Through *C* draw *CD* parallel to *BA*. If *BC* is produced to *F*, state the size of angle *DCF*.

Question 7(a). C.X.C.(Basic). June 1987.

3. Using ruler and compasses only,

(i) Construct a triangle *ABC* in which $AB = 8.5$ cm, angle $ABC = 60°$ and $BC = 7.0$ cm.

(ii) Measure and state the length of *AC*.

(iii) Construct *BD*, the perpendicular from *B* to *AC*.

(Note: All construction lines must be clearly shown.)

Question 6(a). C.X.C.(Basic). June 1991.

4. (i) **Using ruler and compasses only,** construct a trapezium *ABCD* in which $AB = 10$ cm, angle $BAD = 60°$, $AD = 6.5$ cm, $DC = 8$ cm and *DC* is parallel to *AB*.

(ii) Measure and state the length of *BC* in centimetres.

Question 3(b). C.X.C.(Basic). June 1992.

5. (i) **Using ruler and compasses only,** construct a quadrilateral *ABCD* in which $AB = AD = 6$ cm, $BC = 4$ cm, angle $BAD = 60°$ and angle $ABC = 90°$.

(ii) Measure and state
- the length of *DC*
- the size of angle *ADC*.

Question 4(a). C.X.C.(General). June 1992.

6. *VMNPQ* is a pyramid on a square base *MNPQ* of side 40 cm.

(i) Draw a diagram to represent the pyramid. Clearly label the vertices.

(ii) Draw a plan of the pyramid, viewed from above. State the scale used.

(iii) The height of the pyramid is 20 cm. Show that the length of the sloping edge *VM* is $20\sqrt{3}$ cm.

Question 11(b). C.X.C.(General). June 1994.

CHAPTER 10

Geometry: Symmetry and Transformations 1

This chapter will teach you about

- symmetry and its types: translational, reflective and rotational
- transformations: translation reflection, rotation and enlargement
- translation, image under a translation and inverse translation
- reflection, image under a reflection and inverse reflection
- rotation, image under a rotation and inverse rotation
- enlargement, image under an enlargement and inverse enlargement

Symmetry

Symmetry is the *kind of pattern* that a *shape* has. It *deals* with the *exact matching* of a *position* or *form* about a *point, line* or *plane*. The symmetry of a plane figure describes how, under certain rules of movement, the plane figure fits exactly onto itself. There are *three types of symmetry* that a *plane figure* can have:

(i) *Translational symmetry.*
(ii) *Line symmetry* (or *reflection symmetry*).
(iii) *Rotational symmetry.*

Translational Symmetry

A *movement* along a *straight line without turning* is called a *translation.* A *plane figure* is said to have *translational symmetry* if it can be *translated* and *still look the same.*

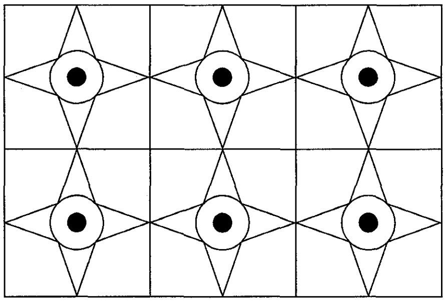


Fig. 10.1 Tiles

The *tile pattern* in Fig. 10.1 has *translational symmetry*, since it can be *moved horizontally*, *vertically* or *obliquely* and *still look the same.* All *repeating patterns* have *translational symmetry.*

A translation is a *movement* of a *certain distance* in a *stated direction* and is therefore described by a *vector.* A *vector* is a *quantity* that has *both* a *magnitude* and a *direction.* When a *pattern* is *translated* and *still looks the same*, then the *vector* is called the *translational vector.*

Some samples of *translational vectors* for the *tile pattern* shown in Fig. 10.1 are:

(a)

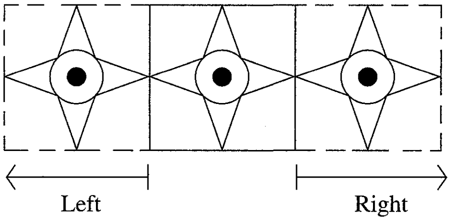


Fig. 10.2 Tile

If *each tile* is *translated one unit length* to the *left*, or *one unit length* to the *right*, then the *tile pattern still looks the same.*

(b)

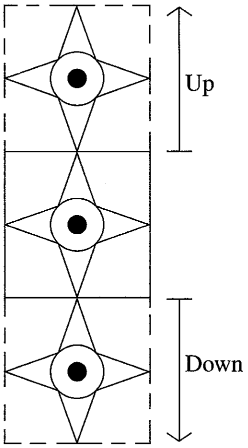


Fig. 10.3 Tile

If *each tile* is *translated one unit width up*, or *one unit width down*, then the *tile pattern still looks the same.*

(c)

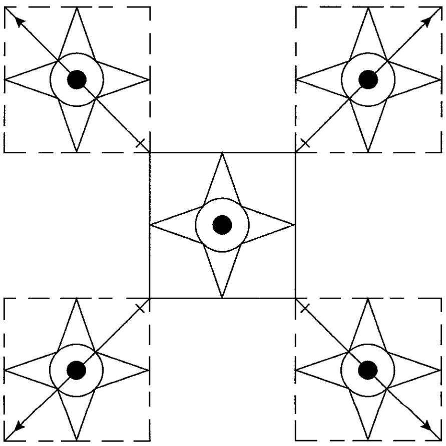


Fig. 10.4 Tile

If *each tile* is *translated one unit diagonally* in any of the *four directions* as shown in Fig. 10.4, then the *tile pattern still looks the same.*

Exercise 10a

State which of the following patterns have translational symmetry.

1.

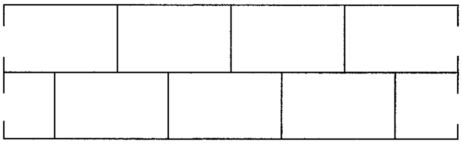


Fig. 10.5 Pattern

2.

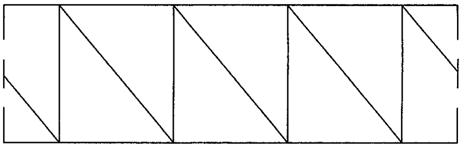


Fig. 10.6 Pattern

3.

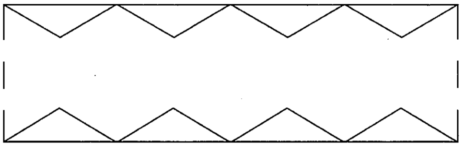


Fig. 10.7 Pattern

4.

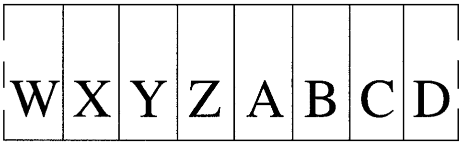


Fig. 10.8 Pattern

5.

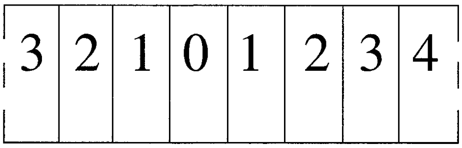


Fig. 10.9 Pattern

Each of the following patterns has translational symmetry. Measure and state the smallest translational vector in each case.

6.

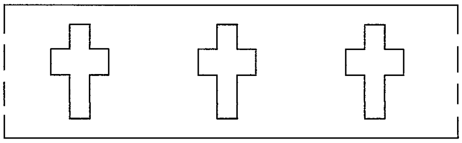


Fig. 10.10 Pattern

7.

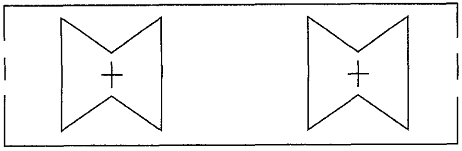


Fig. 10.11 Pattern

8.

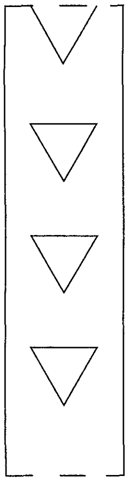


Fig. 10.12 Pattern

9.

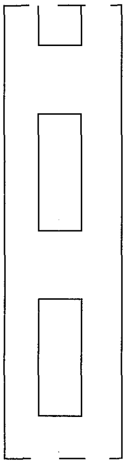


Fig. 10.13 Pattern

For the pattern in Fig. 10.14, measure and state the smallest

(a) horizontal translational vector

(b) vertical translational vector

(c) oblique translational vector

10.

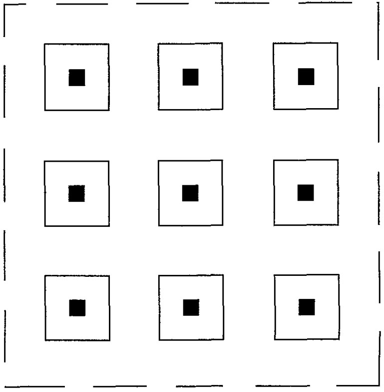


Fig. 10.14 Pattern

Continue each of the following patterns across the graph page provided

11.

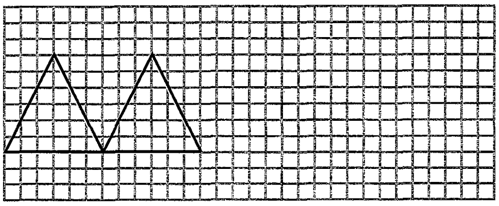


Fig. 10.15 Pattern

12.

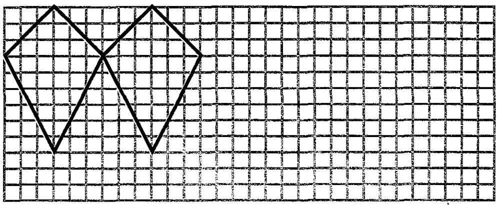


Fig. 10.16 Pattern

13.

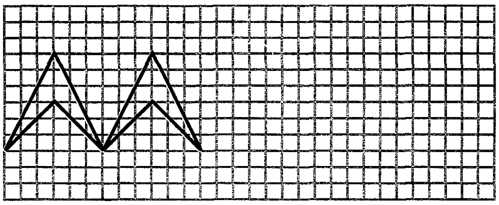


Fig. 10.17 Pattern

14.

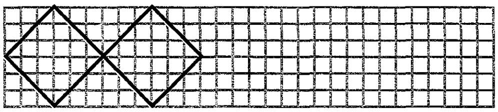


Fig. 10.18 Pattern

15.

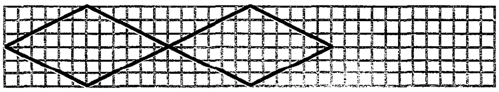


Fig. 10.19 Pattern

16.

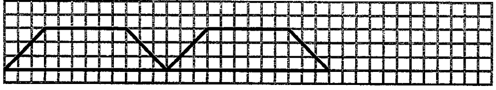


Fig. 10.20 Pattern

17.

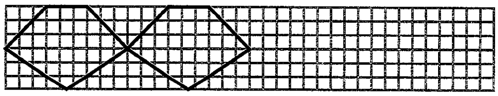


Fig. 10.21 Pattern

18.

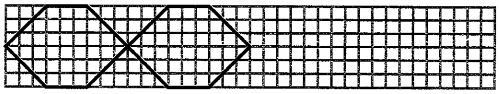


Fig. 10.22 Pattern

Continue each of the following patterns. Fill all the space provided.

19.

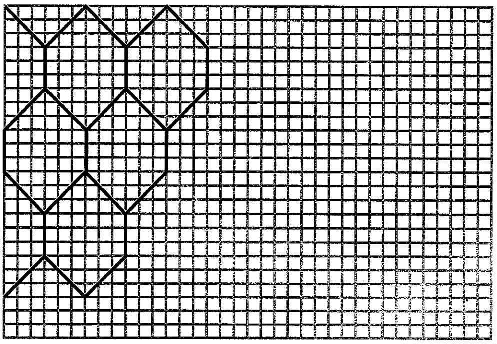


Fig. 10.23 Pattern

20.

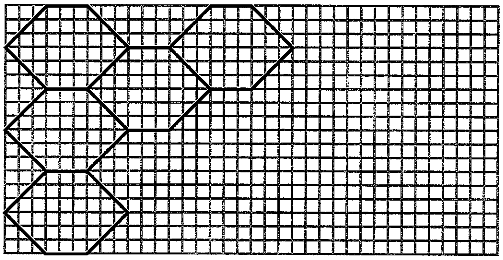


Fig. 10.24 Pattern

21.

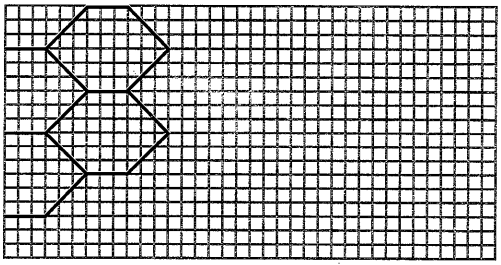
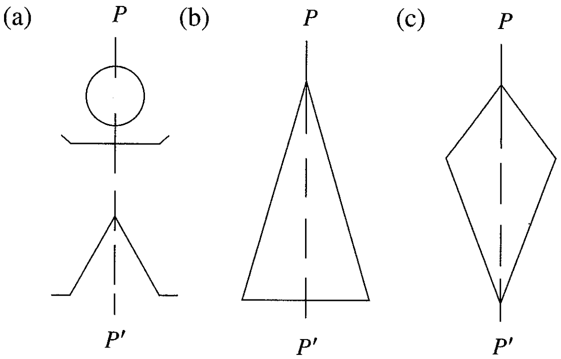


Fig. 10.25 Pattern

Line Symmetry (or Reflective Symmetry)

The *line of symmetry* (or the *axis of symmetry*) of a *plane figure* is a *line* which can be used as a *fold*, so that *one half* of the *shape covers* the *other half exactly*. A *plane figure* can have *one or more lines of symmetry* (or *axes of symmetry*). Thus:

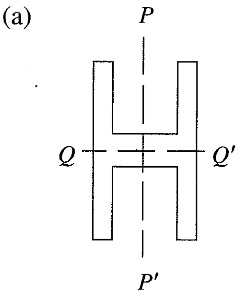


Plane figure Isosceles triangle Kite

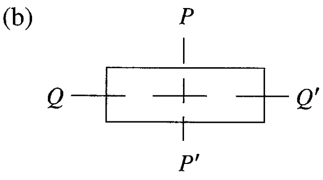
Fig. 10.26 One line of symmetry

The *shapes* shown in Fig. 10.26 are *symmetrical about* the l*ine PP'* only. Hence the *shapes* are said to have only *one line of symmetry* (or *line symmetry of order* 1). *Plane figures* with *one line of symmetry* are said to illustrate *bilateral symmetry*. One *half* of *each shape* is an *exact copy* of the *other half*. So we call *each half* a *mirror image* of the *other half*. Hence *line symmetry* is also called *reflective symmetry* (or *mirror symmetry*), since *a mirror reflection on the line* will *produce* the *whole shape*.

The following *plane figures* have *two lines of symmetry* (or *line symmetry of order* 2).

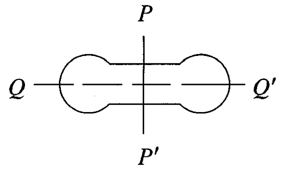


Letter *H*



Rectangle

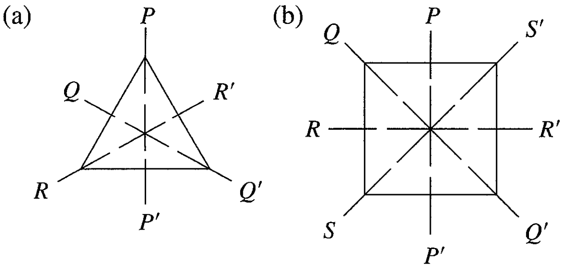
(c)



Plane figure

Fig. 10.27 Two lines of symmetry

The following *plane figures* have *three or more lines of symmetry*.



Equilateral triangle
Three lines of symmetry

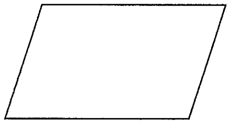
Square
Four lines of symmetry

Fig. 10.28

As can be *seen* from the examples above, *a line of symmetry* (or an *axis of symmetry*) can be *vertical*, *horizontal* or *oblique* (i.e. *sloping*).

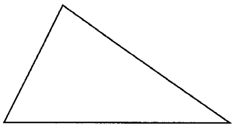
There are also *plane figures* with *no line of symmetry*, as can be seen in Fig. 10.29 below.

(a)



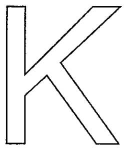
Parallelogram

(b)



Scalene triangle

(c)



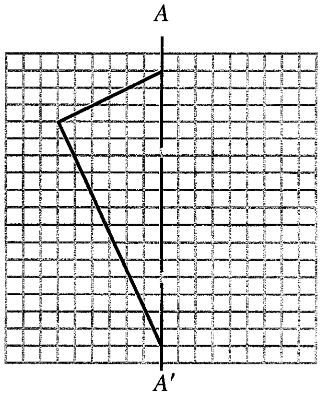
Letter K

Fig. 10.29 No line of symmetry

Exercise 10b

Complete each of the following symmetrical shapes. The broken lines indicate the lines of symmetry.

1. (a)



(b)

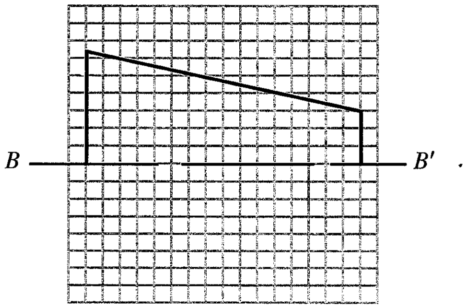
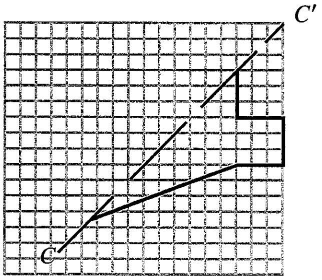


Fig. 10.30 Symmetrical shapes

2. (a)



(b)

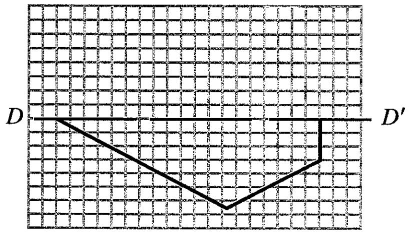


Fig. 10.31 Symmetrical shapes

3.

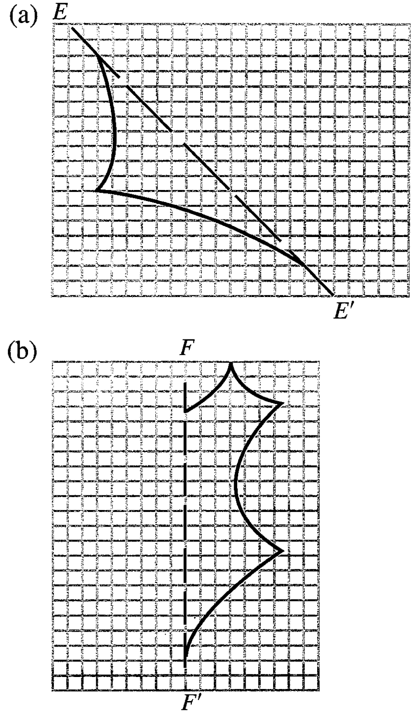


Fig. 10.32 Symmetrical shapes

4.

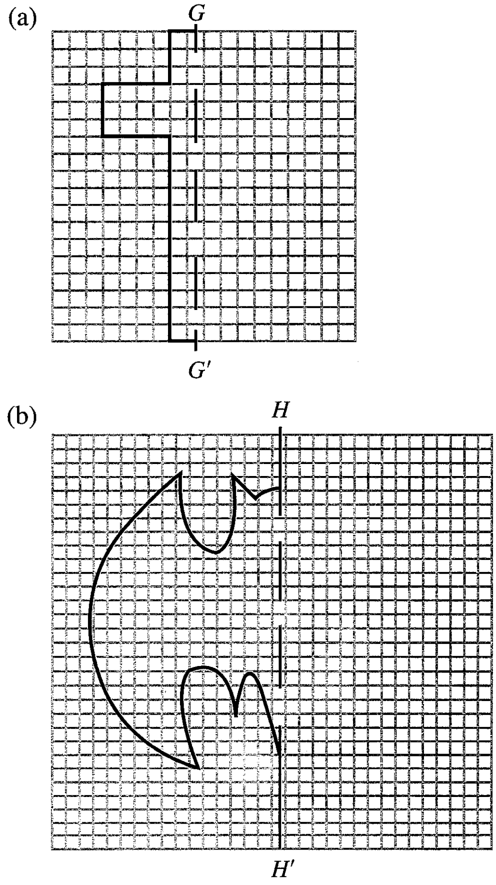


Fig. 10.33 Symmetrical shapes

5.

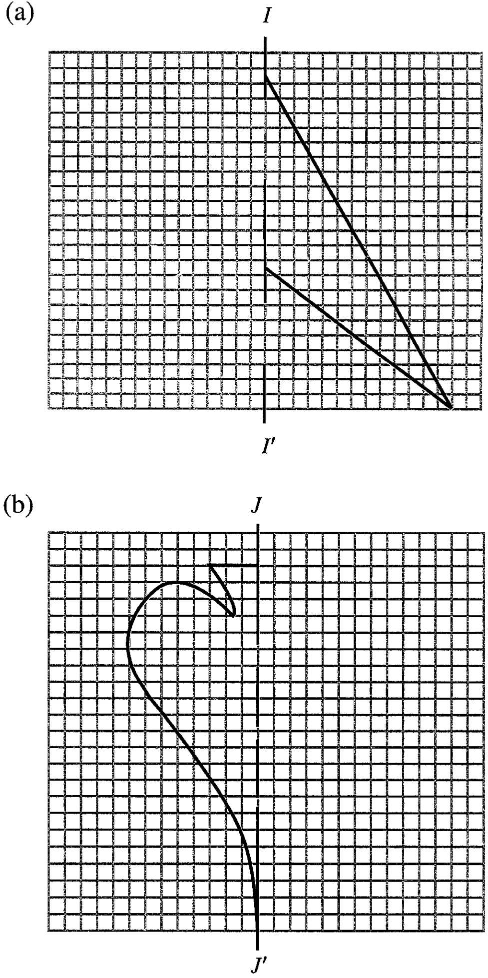


Fig. 10.34 Symmetrical shapes

Some of the following figures have mirror symmetry. For those which are symmetrical, draw the mirror line(s).

6.

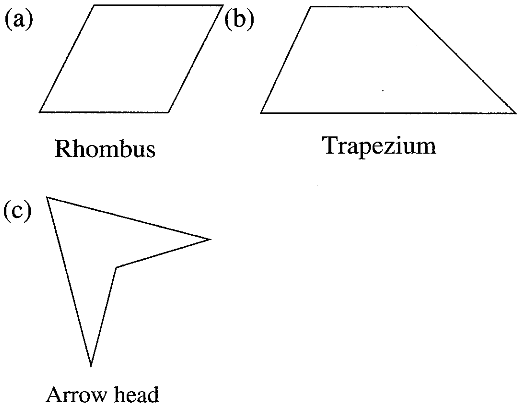
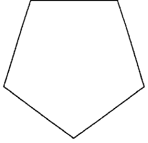


Fig. 10.35 Plane figures

7. (a)

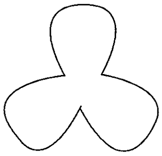
Isosceles trapezium

(b)



Regular pentagon

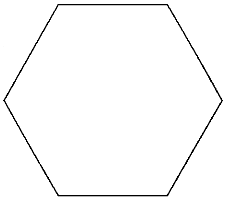
(c)



Three-sided petal

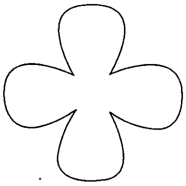
Fig. 10.36 Plane figures

8. (a)



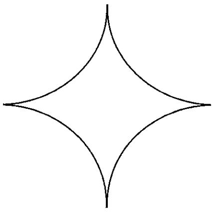
Regular hexagon

(b)



Four-sided petal

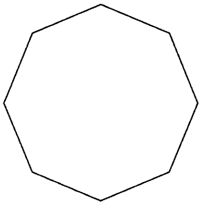
(c)



Plane figure

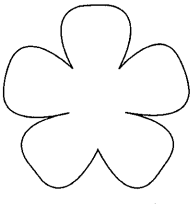
Fig. 10.37

9. (a)



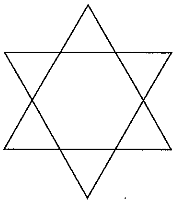
Regular octagon

(b)



Five-sided petal

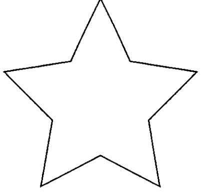
(c)



Six-pointed star

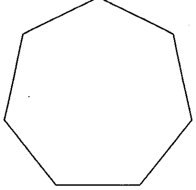
Fig. 10.38 Plane figures

10. (a)



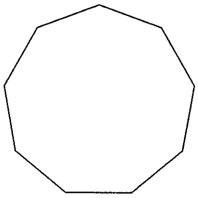
Five-pointed star

(b)



Regular heptagon

(c)



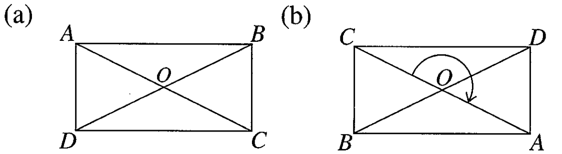
Regular nonagon

Fig. 10.39 Plane figures

Rotational Symmetry

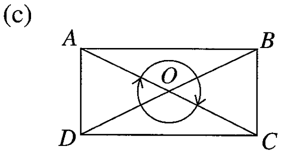
A *plane figure* is said to have *rotational symmetry* of a certain *order*, if the *plane figure maps onto itself* (i.e. *coincides* with *itself*) under *rotations* through stated *angles* about a *common centre*.

All *plane figures* have a *rotational symmetry* of *order* 1, since a *rotation* about its *centre* through 360° will *map* it *onto itself*. Hence we *do not* consider *rotational symmetry of order* 1. Thus a *scalene triangle does not* possess *rotational symmetry*.



Original position

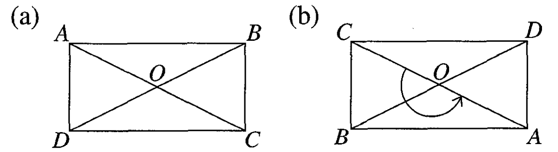
Rotated clockwise about *O* through 180°



Rotated clockwise about *O* through 360°

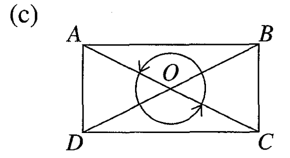
Fig. 10.40 Rectangle

Or



Original position

Rotated anti-clockwise about O through 180°

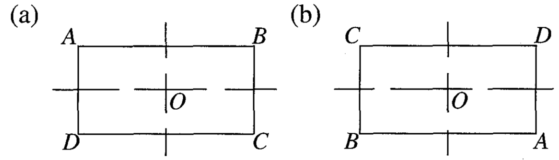


Rotated anti-clockwise about O through 360°

Fig. 10.41 Rectangle

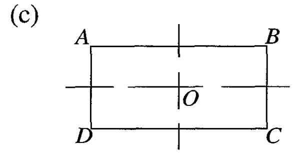
In Figures 10.40 and 10.41, the *rectangle is mapped onto itself* when it is *rotated* about O (the *point of intersection* of its *diagonals*) through *angles* of 180° and 360°, *clockwise* or *anti-clockwise.*

Hence the *rectangle* is said to have *rotational symmetry of order* 2. And the *point O* is referred to as the *centre of rotational symmetry.*



Original position

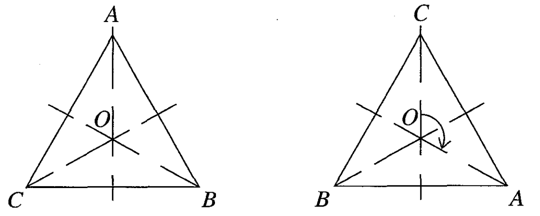
Rotation through 180° clockwise or anti-clockwise about O



Rotation through 360° clockwise or anti-clockwise about O

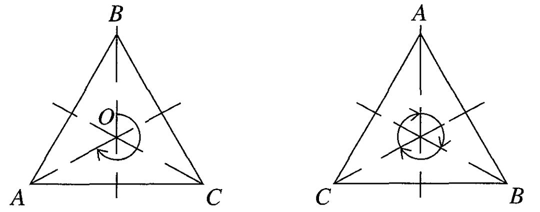
Fig. 10.42 Rectangle

In Fig. 10.42, the *rectangle* is *mapped onto itself* when it is reflected in either of the two lines of symmetry, or when it is *rotated about O* (the *point of intersection* of its *lines of symmetry*) through *angles* of 180° and 360°, *clockwise* or *anti-clockwise.* Hence the *rectangle* has *line symmetry of order* 2 and *rotational symmetry of order* 2. The *point O* is referred to as the *centre of rotational symmetry.*



Original position

Rotation through 120° clockwise about O

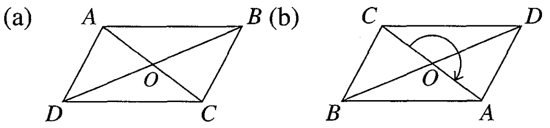


Rotation through 240° clockwise about O

Rotation through 360° clockwise about O

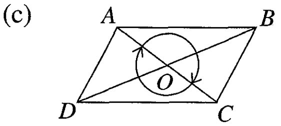
Fig. 10.43 Equilateral triangle

In Fig. 10.43 the *equilateral triangle* is *mapped onto itself* when it is reflected in any of the three lines of symmetry or when it is *rotated about O* (the *point of intersection* of its *lines of symmetry*) through *angles* of 120°, 240° and 360° clockwise. (*Rotation* through angles of 120°, 240° and 360° *anti-clockwise* will give the *same results*). Hence the *equilateral triangle ABC* has *line symmetry of order* 3 and *rotational symmetry of order* 3.



Original position

Rotation through 180° clockwise about O



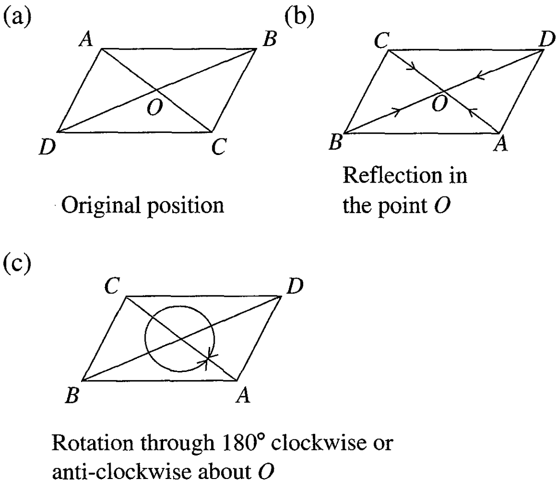
Rotation through 360° clockwise about O

Fig. 10.44 Parallelogram

In Fig. 10.44, the *parallelogram* is *mapped onto itself* when it is *rotated about O* (the *point of intersection* of its *diagonals*) through *angles* of 180° and 360° *clockwise.* (*Rotation* through *angles* of 180° and 360° *anti-clockwise* will give the *same results*). Hence the *parallelogram* has *rotational symmetry of order* 2, although it has *no line of symmetry.*

Point Symmetry

A *plane figure* is said to have *point symmetry*, if the *plane figure maps onto itself* (i.e. *coincides* with *itself*) after reflection in the point. Point symmetry is equivalent to a *rotation through* 180° about the same *point*. Hence, if a *plane figure* possess *rotational symmetry of order* 2, it also has *point symmetry*.



Original position

Reflection in the point O

Rotation through 180° clockwise or anti-clockwise about O

Fig. 10.45 Parallelogram

In Fig. 10.45(b), the parallelogram is mapped onto itself, when it is reflected in the point O (the point of intersection of its diagonals).

In Fig. 10.45(c), the *parallelogram* is *mapped onto itself*, when it is *rotated about O* (the *point of intersection* of its *diagonals*) through an *angle* of 180° *clockwise* or *anti-clockwise*.

A parallelogram has point symmetry.

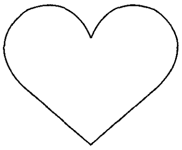
Exercise 10c

For each of the following plane figures, state

(a) the order of rotation symmetry and

(b) which shape has point symmetry.

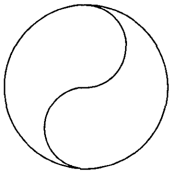
1.



Heart

Fig. 10.46

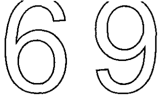
2.



Ying and yang

Fig. 10.47

3.



Numerals

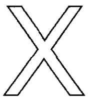
Fig. 10.48

4.

Letter

Fig. 10.49

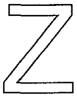
5.



Letter

Fig. 10.50

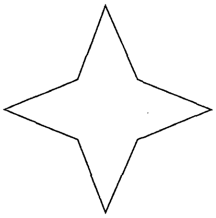
6.



Letter

Fig. 10.51

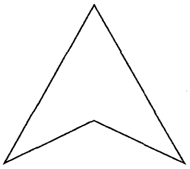
7.



Four-pointed star

Fig. 10.52

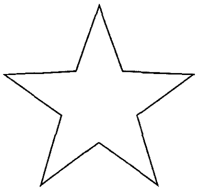
8.



Arrow head

Fig. 10.53

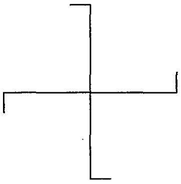
9.



Five-pointed star

Fig. 10.54

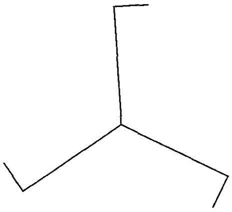
10.



Plane figure

Fig. 10.55

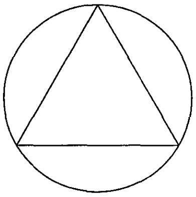
11.



Plane figure

Fig. 10.56

12.



Plane figure

Fig. 10.57

13. State which of the following plane figures have rotational symmetry.

(a) An isosceles triangles.

(b) A circle.

(c) A trapezium.
(d) A rhombus.
(e) A kite.
(f) A square.
(g) A right-angled triangle.

14. Investigate the properties of the regular polygons and then complete the table following.

Table 10.1

Regular polygon	*Number of lines of symmetry*	*Order of rotational symmetry*
Equilateral triangle	3	3
Square	4	4
Pentagon		
Hexagon		
Heptagon		
Octagon		
Nonagon		
Decagon		
Undecagon		
Dodecagon		

15. What can be deduced about the number of lines of symmetry of a regular polygon with n sides?

16. What can be deduced about the order of rotational symmetry of a regular polygon with n sides?

Transformations

A *transformation* is said to *describe* the *relation* between any *point* (or *object point* or *pre-image point*) and its *image point*. A *transformation* is a *one-to-one relation* (or *one-to-one mapping*) of *all points* on the *object onto corresponding points* on the *image*. The *object* under a *transformation* is the *plane figure* (or *point* in some cases) that is *undergoing* a *change in position*. The *image* under a *transformation* is the *plane figure* (or *point* in some cases) that *results* from a *change in position* of the *object* or *pre-image. The hierarchy of transformations* usually investigated is shown in Fig. 10.58 below.

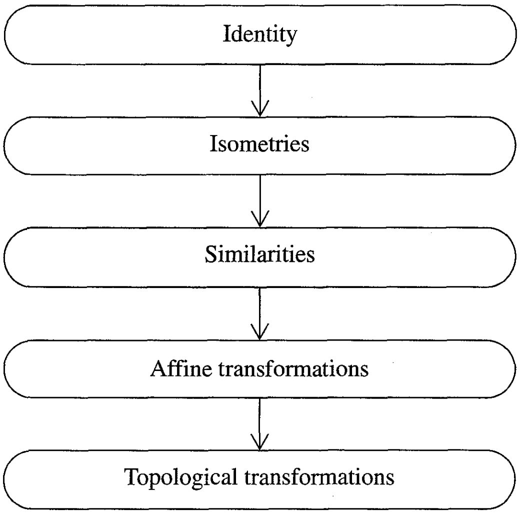


Fig. 10.58 Hierarchy of transformations

There are *six basic types of transformations*:
(i) *Translation*, represented by the letter T.
(ii) *Reflection*, represented by the letter M.
(iii) *Rotation*, represented by the letter R.
(iv) *Enlargement*, represented by the letter E.
(v) *Shear*, represented by the letter H.
(iv) *Stretch*, represented by the letter S.

The *identity* is placed the *highest* in the *hierarchy of transformations*, it requires, that, everything be *invariant* (i.e. *unchanged*) by a given *transformation*. Only *translations* and *rotations* are able to *satisfy* these *conditions*. So *translation* and *rotation* are both *congruency transformations*.

After *identity*, we have the *length preserving transformations*. *Translation*, *reflection* and *rotation* are known as the *isometries*, since they *preserve length*. *Translation*, *reflection* and *rotation* are all *congruency transformations*.

If we now allow *angles to be preserved*, although *lengths are not*, we introduce the *enlargement transformation. Enlargement* and the *isometries* are known as the *similarities*. So *enlargement* is a *similarity transformation*.

If further, we only require that *parallel lines be mapped onto parallel lines*, we introduce the *shears* and *stretches*. *All the transformations* mentioned above can now be classified as *affine transformations*.

Finally, if only the *order of points on a line* and the *order of a node* were the important *invariants*, we introduce the *topological transformations of networks*. However, we will *not* be *investigating* this *type of transformation.*

Translation

A *translation* is a *transformation* in which a *plane figure* slides along a *straight line* and changes its *position without turning. Each point moves* the *same distance* and in the *same direction.* Hence *all points* subjected to the *same translation undergo* the *same displacement.* So a *translation* is also referred to as a *displacement.* And the *transformation* is *completely defined* by the *displacement* of *any one point.*

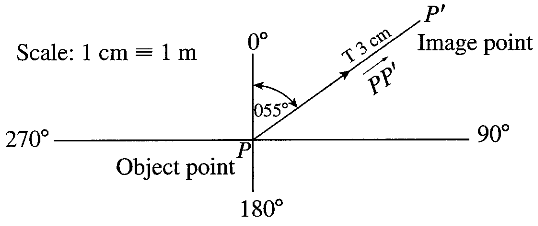


Fig. 10.59 Translation of a point P to P′

Fig. 10.59 shows the *translation* of an *object point P* (or a *pre-image point P*) to an *image point P′.*

P prime (i.e. *P′*) is the notation used to represent the *image* of the *object point P*. Under the *translation*, the *point P* moves a *distance* of 3 m in the *direction* 055°.

The *distance* the *point P moves* is *equal* to the *length* of the *line segment joining* the *point P* to its *image P′*, (i.e. the length of *PP′*). The *line segment* here is *drawn* to *scale.*

The *direction* in which the *point P moves* is from *P to P′* as indicated by the *arrow*. This direction is *equivalent* to the *bearing* of the *image P′ from* the *object P*, which is 055°.

The *translation* can be *symbolically* represented as follows:

$$T: P \rightarrow P' \text{ or } T(P) = P' \text{ or } P \xrightarrow{T} P'.$$

The *mapping notations* given above *describes* the *translation T.*

The *mapping notations*, $T: P \rightarrow P'$, $T(P) = P'$ and $P \xrightarrow{T} P'$, *describe* a *translation* of the *point P to P′* over a *distance equal* to the *length of PP′* and in a *direction from P to P′*, where *P′* is the *image of P under the translation* denoted by *T.*

Besides *mapping expressions*, a *translation* can also be described by a *displacement vector*, also known as *a vector of translation* (or a *translation vector*).

For the *translation* given above, the *displacement vector* that describes it is $\overrightarrow{PP'}$ or **PP′**.

The *placement* of the *letters* and the *arrow* in the *displacement vector* indicates the *direction* of the *translation*, i.e. *from P to P′*, while the *distance moved* is the *magnitude* of the *displacement* which is *denoted* by $|\overrightarrow{PP'}|$ or PP'. So $|\overrightarrow{PP'}| = PP' = 3$ m.

The *mapping notations* and the *displacement vector* are *equivalent statements.*

Properties of a Translation

When *studying* the different *types of transformations*, we try to *discover* which *properties* of a *figure* remain *unchanged after the transformation.* The *properties* of a *figure* which are *preserved* by a *transformation* are called the *invariants of the transformation.*

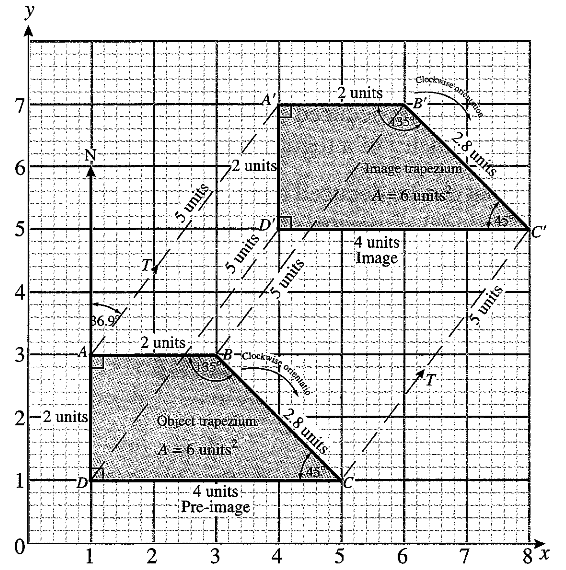


Fig. 10.60 Translation of a plane figure

Fig. 10.60 above shows the *image trapezium A′B′C′D′* of *trapezium ABCD* under a *translation T.*

The *points A*, *B*, *C* and *D* are *translated* under the same *translation T*, so

$T: A \rightarrow A'$, $T: B \rightarrow B'$, $T: C \rightarrow C'$ and
$T: D \rightarrow D'$ or
T: *Trapezium* $ABCD \rightarrow$ *Trapezium* $A'B'C'D'$.

The *line segments*, AA', BB', CC' and DD' are all *parallel* and *equal*, indicating that the *points A*, *B*, *C* and *D* moved the *same distance* and in the *same direction*. As a result the *corresponding displacement vectors* $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, $\overrightarrow{CC'}$ and $\overrightarrow{DD'}$ are all equal.

Thus:

$$\overrightarrow{AA'} = \overrightarrow{BB'} = \overrightarrow{CC'} = \overrightarrow{DD'}$$

or $$\mathbf{AA'} = \mathbf{BB'} = \mathbf{CC'} = \mathbf{DD'}.$$

So any one of the *four displacement vectors* $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, $\overrightarrow{CC'}$ or $\overrightarrow{DD'}$ *describes* the *translation T*. The *translation T* has a *magnitude* of 5 units and a *bearing of* 036.9°.

Hence we can *conclude* that:

(i) *All points subjected to the same translation undergo the same displacement. That is, each point move the same distance in the same direction.*

(ii) *Equal translations are translations over the same distance and in the same direction.*

In Fig. 10.60, the *line segments* $A'B'$, $B'C'$, $C'D'$ and $A'D'$ are the *images* of the *line segments* AB, BC, CD and AD *respectively*, under the *translation T*.

It can be *observed* that:

$AB = A'B' = 2$ units and $AB \,//\, A'B'$,
$BC = B'C' = 2.8$ units and $BC \,//\, B'C'$,
$CD = C'D' = 4$ units and $CD \,//\, C'D'$,
$AD = A'D' = 2$ units and $AD \,//\, A'D'$,
where the symbol // means '*is parallel to*'.

This is so since a *line segment* consists of a *set of points*, and *all points* under the *same translation* move the *same distance* in the *same direction*.

Hence we can *conclude* that:

(i) (a) *Under a translation, the image of any line segment is a line segment that is equal in length and parallel to the object line segment. So corresponding sides are equal and parallel.*

(b) *Translation preserves the distance between two points. Translation preserves lengths.*

From Fig. 10.60, it can be *observed* that:

$\hat{A} = \hat{A}' = 90°$, $\hat{B} = \hat{B}' = 135°$, $\hat{C} = \hat{C}' = 45°$ and $\hat{D} = \hat{D}' = 90°$.

So the *size of the angle remains unchanged* under the *translation*, since the *corresponding angles are equal.*

Hence we can *conclude* that:

(ii) *Translation preserves angles.*

In Fig. 10.60, it can be *seen* that:
$AB \,//\, DC$ and $A'B' \,//\, D'C'$.
So *parallel sides remain parallel* under the *translation*.

Hence we can *conclude* that:

(iii) *Translation preserves parallelism.*

From Fig. 10.60, it can be *observed* that:
The *area of trapezium* $ABCD$ = The *area of trapezium* $A'B'C'D'$
= 6 square units.

So *area* remains the *same* under the *translation*.

Hence we can *conclude* that:

(iv) *Translation preserves area.*

In Fig. 10.60, it can be *seen* that:

$AB{:}DC = 1{:}2$ and $A'B'{:}D'C' = 1{:}2$.

$AD{:}BC = 1{:}1.4$ and $A'D'{:}B'C' = 1{:}1.4$

So *ratios* are *unchanged* under the *translation*.

Hence we can *conclude that*:

(v) *Translation preserves ratios.*

From Fig. 10.60, it can be *observed* that:

The *order* of the *vertices* in the *object* is *ABCD*.

The *order* of the *vertices* in the *image* is $A'B'C'D'$.

So the *order* of the *vertices* remains the *same*.

Hence we can *conclude* that:

(vi) *Translation preserves the order of points.*

In Fig. 10.60, it can be seen that:

The *orientation* (or *sense*) of *trapezium ABCD* and *trapezium* $A'B'C'D'$ is *clockwise*. That is, *both figures* have the same *orientation* (or *sense*).

Hence we can *conclude* that:

(vii) *Translation preserves the orientation* (*or sense*) *of a figure.*

We can *summarize* the *properties of a translation* in a *table* as shown below.

Table 10.2

Property	Length	Angle	Parallelism	Area	Ratio	Order of Points	Orientation (or sense)
Invariant under translation	Yes	Yes	Yes	Yes	Yes	Yes	Yes

From the *properties of translations* discussed above, it follows that the *image A'B'C'D'* of figure *ABCD* under a *translation* is of the *same size* and *shape* as the *object ABCD.* Hence the *object ABCD* and the *image A'B'C'D'* are said to be *congruent*. Thus:
The *trapezium ABCD* ≡ the *trapezium A'B'C'D'*.
Hence *translation* is a *congruency transformation.*

We can *summarize* the *properties of a translation* as follows:

(i) *All points under the same translation move the same distance in the same direction.*

(ii) *Translation is a congruency transformation.*

Column Vector

Each *displacement vector* (or *translation vector*), *T* can be represented by a *column vector* (or a *column matrix*), $\begin{pmatrix} x \\ y \end{pmatrix}$.

Thus $T = \begin{pmatrix} x \\ y \end{pmatrix}$.

The *translation* $T = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$ *moves all points* in the plane 5 *units to the right* and then 7 *units upwards* as shown in fig. 10.61.

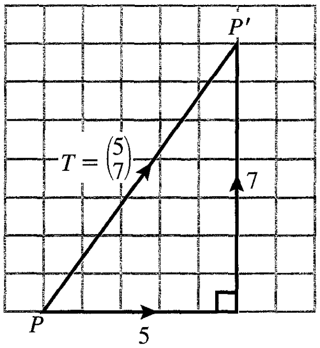


Fig. 10.61 Translation

The *translation* $T = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ *moves all points* in the plane 4 *units to the left* and then 6 *units upwards* as shown in Fig. 10.62.

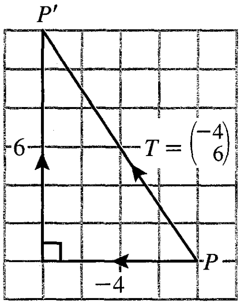


Fig. 10.62 Translation

The *translation* $T = \begin{pmatrix} -8 \\ -5 \end{pmatrix}$ *moves all points* in the plane 8 *units to the left* and then 5 *units downwards* as shown in Fig. 10.63.

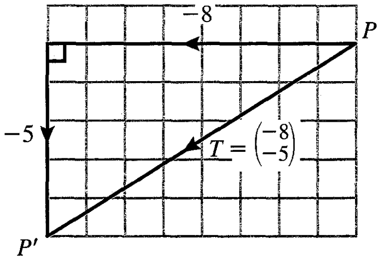


Fig. 10.63 Translation

The *translation* $T = \begin{pmatrix} 7 \\ -9 \end{pmatrix}$ *moves all points* in the plane 7 *units to the right* and then 9 *units downwards* as shown in Fig. 10.64.

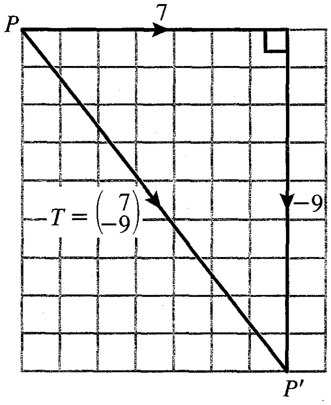


Fig. 10.64 Translation

Image Under a Translation

When the *pre-image* (or *object*) point $P(x, y)$ undergoes a *translation* (or *displacement*), $T = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ then it is *mapped onto* $P'(x', y') = P'(x + x_1, y + y_1)$.

That is T: $P(x, y) \rightarrow P'(x', y')$

or $\quad T$: $P(x, y) \rightarrow P'(x + x_1, y + y_1)$.

These properties are *illustrated* in Fig. 10.65, below.

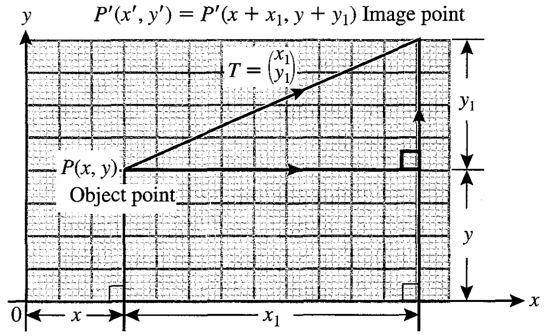


Fig. 10.65 Translation

Expressed as *column vectors* (or *column matrixes*):

$$\begin{array}{ccccc} P & & T & & P' \\ \begin{pmatrix} x \\ y \end{pmatrix} & + & \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} & = & \begin{pmatrix} x + x_1 \\ y + y_1 \end{pmatrix} \\ \textit{Object matrix} & + & \textit{Translation matrix} & = & \textit{Image matrix} \end{array}$$

From the matrix equation, it follows that:

$$\begin{array}{ccccc} T & & P' & & P \\ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} & = & \begin{pmatrix} x + x_1 \\ y + y_1 \end{pmatrix} & - & \begin{pmatrix} x \\ y \end{pmatrix} \\ \textit{Translation matrix} & = & \textit{Image matrix} & - & \textit{Object matrix} \end{array}$$

and

$$\begin{array}{ccccc} P & & P' & & T \\ \begin{pmatrix} x \\ y \end{pmatrix} & = & \begin{pmatrix} x + x_1 \\ y + y_1 \end{pmatrix} & - & \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \\ \textit{Object matrix} & = & \textit{Image matrix} & - & \textit{Translation matrix} \end{array}$$

Example 1

(a) Determine the image of the point A(4, 5) under the translation $T = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$.

(b) The points A(2, 1), B(4, 3) and C(3, 6) are vertices of a triangle ABC. What is the image of the triangle ABC under the translation $T = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$?

Solution

(a) The image of the point $A(4, 5)$ under the translation $T = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$ can be determined using a *graphical method.*

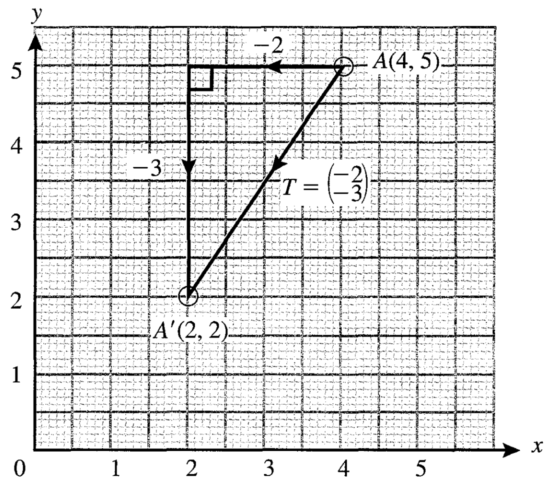


Fig. 10.66 Translation

From Fig. 10.66:

The *point* $A(4, 5)$ is *shifted* 2 *units to the left* and then 3 *units downwards.*

Hence the *image* of the *point* A is $A'(2, 2)$.

Alternative Method

(a) Alternatively, a *matrix method* can also be used to determine the image of the point $A(4, 5)$ under the translation $T = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$.

$$\begin{array}{cccccccc} & A & & T & & A' & & A' \\ \text{Now} & \begin{pmatrix} 4 \\ 5 \end{pmatrix} & + & \begin{pmatrix} -2 \\ -3 \end{pmatrix} & = & \begin{pmatrix} 4 - 2 \\ 5 - 3 \end{pmatrix} & = & \begin{pmatrix} 2 \\ 2 \end{pmatrix} \end{array}$$

So the *image* of the *point* A has *coordinates* $A'(2, 2)$.

(b) The image of the vertices of the triangle ABC under the translation $T = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ can be determined using a *graphical method.*

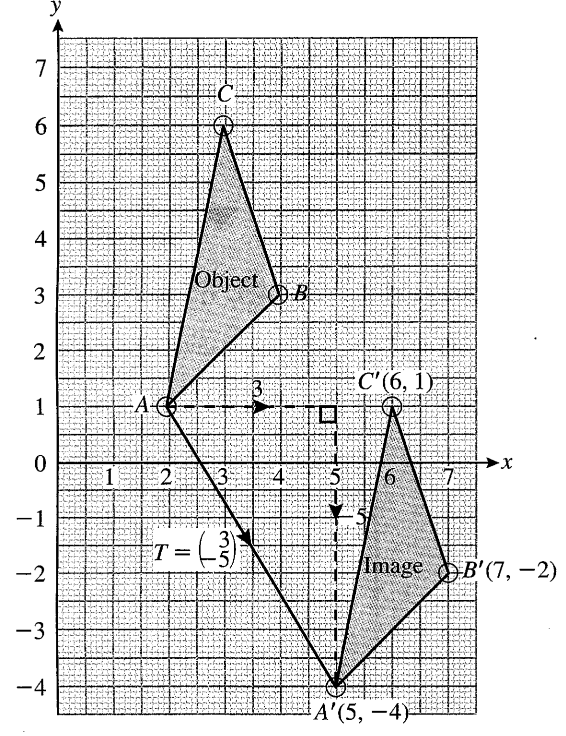


Fig. 10.67 Translation

From Fig. 10.67:

Each vertex is *shifted* 3 *units to the right* and then 5 *units downwards.*

Hence the *image* of the *point* $A(2, 1)$ is $A'(5, -4)$.

The *image* of the *point* $B(4, 3)$ is $B'(7, -2)$.

The *image* of the *point* $C(3, 6)$ is $C'(6, 1)$.

So the *image* of triangle ABC, *triangle* $A'B'C'$, has *vertices* $A'(5, -4)$, $B'(7, -2)$ and $C'(6, 1)$.

Alternative Method

(b) Alternatively, a *matrix method* can be used to determine the image of triangle ABC under the translation $T = \begin{pmatrix}3\\-5\end{pmatrix}$.

$$\text{Now } \overset{A}{\begin{pmatrix}2\\1\end{pmatrix}} + \overset{T}{\begin{pmatrix}3\\-5\end{pmatrix}} = \overset{A'}{\begin{pmatrix}2+3\\1-5\end{pmatrix}}$$

$$= \overset{A'}{\begin{pmatrix}5\\-4\end{pmatrix}}$$

$$\overset{B}{\begin{pmatrix}4\\3\end{pmatrix}} + \overset{T}{\begin{pmatrix}3\\-5\end{pmatrix}} = \overset{B'}{\begin{pmatrix}4+3\\3-5\end{pmatrix}}$$

$$= \overset{B'}{\begin{pmatrix}7\\-2\end{pmatrix}}$$

$$\text{and } \overset{C}{\begin{pmatrix}3\\6\end{pmatrix}} + \overset{T}{\begin{pmatrix}3\\-5\end{pmatrix}} = \overset{C'}{\begin{pmatrix}3+3\\6-5\end{pmatrix}}$$

$$= \overset{C'}{\begin{pmatrix}6\\1\end{pmatrix}}.$$

So the *image* of triangle ABC, *triangle* $A'B'C'$ has *vertices* $A'(5, -4)$, $B'(7, -2)$ and $C'(6, 1)$.

Example 2

(a) The point A(2, 3) is mapped onto the point A′(5, −2) under the translation $T = \begin{pmatrix}x_1\\y_1\end{pmatrix}$. *Determine the column vector that represents the translation T.*

(b) The vertices of parallelogram PQRS are P(1, 1), Q(5, 1), R(6, 3) and S(2, 3). Parallelogram PQRS is mapped onto parallelogram P′Q′R′S′, with vertices P′(−3, −4), Q′(1, −4), R′(2, −2) and S′(−2, −2). Determine the translation T that maps parallelogram PQRS onto parallelogram P′Q′R′S′.

Solution

(a) The column vector that represents the translation T can be determined using a *graphical method.*

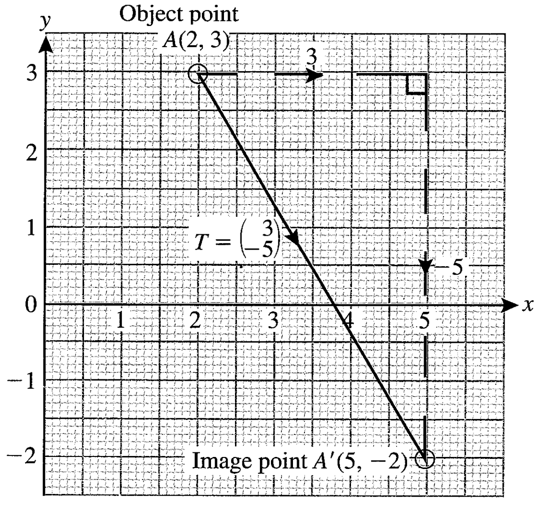


Fig. 10.68 Translation

From Fig. 10.68:
The *point* $A(2, 3)$ is mapped onto the *point* $A'(5, -2)$ by *moving* 3 *units to the right* and then 5 *units downwards.*

Hence the *column vector that* represents the *translation* is $T = \begin{pmatrix}3\\-5\end{pmatrix}$.

Alternative Method

(a) Alternatively, a *matrix method* can be used to determine the column vector that represents the translation $T = \begin{pmatrix}x_1\\y_1\end{pmatrix}$.

$$\text{Now } \overset{T}{\begin{pmatrix}x_1\\y_1\end{pmatrix}} = \overset{A'}{\begin{pmatrix}5\\-2\end{pmatrix}} - \overset{A}{\begin{pmatrix}2\\3\end{pmatrix}} = \overset{T}{\begin{pmatrix}5-2\\-2-3\end{pmatrix}} = \overset{T}{\begin{pmatrix}3\\-5\end{pmatrix}}$$

Hence the column vector that represents the translation is $T = \begin{pmatrix}3\\-5\end{pmatrix}$.

(b) The translation T that maps parallelogram $PQRS$ onto parallelogram $P'Q'R'S'$ can be determined using a *graphical method.*

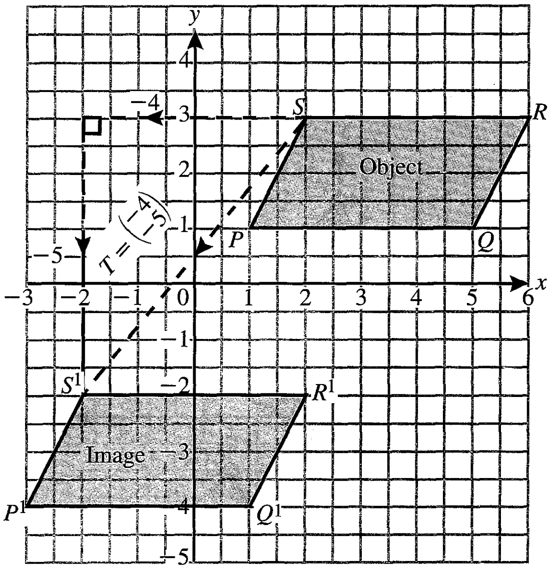


Fig. 10.69 Translation

From Fig. 10.69:
The *point* $S(2, 3)$ is *mapped onto* the *point* $S'(-2, -2)$ by *moving* 4 *units to the left* and then 5 *units downwards.*

Hence the *translation* that maps parallelogram $PQRS$ onto parallelogram $P'Q'R'S'$ is $T = \begin{pmatrix}-4\\-5\end{pmatrix}$.

Alternative Method

(b) Alternatively, a *matrix method* can be used to determine the translation T that maps parallelogram $PQRS$ onto parallelogram $P'Q'R'S'$.

$$\text{Now } \overset{T}{\begin{pmatrix}x_1\\y_1\end{pmatrix}} = \begin{array}{c}\begin{array}{cccc}P' & Q' & R' & S'\end{array}\\ \begin{pmatrix}-3 & 1 & 2 & -2\\ -4 & -4 & -2 & -2\end{pmatrix}\end{array} - \begin{array}{c}\begin{array}{cccc}P & Q & R & S\end{array}\\ \begin{pmatrix}1 & 5 & 6 & 2\\ 1 & 1 & 3 & 3\end{pmatrix}\end{array}$$

$$= \begin{array}{c}\begin{array}{cccc}T & T & T & T\end{array}\\ \begin{pmatrix}-3-1 & 1-5 & 2-6 & -2-2\\ -4-1 & -4-1 & -2-3 & -2-3\end{pmatrix}\end{array}$$

$$= \begin{array}{c}\begin{array}{cccc}T & T & T & T\end{array}\\ \begin{pmatrix}-4 & -4 & -4 & -4\\ -5 & -5 & -5 & -5\end{pmatrix}\end{array}$$

$$= \overset{T}{\begin{pmatrix}-4\\-5\end{pmatrix}}$$

Hence the *translation* that maps parallelogram $PQRS$ onto parallelogram $P'Q'R'S'$ is $T = \begin{pmatrix}-4\\-5\end{pmatrix}$.

Inverse Translation

The *inverse of a transformation* is that *transformation* that *returns* the *image of a point* or a *plane figure* to its *original position.* The *inverse of a translation* is another *translation.* It is obtained by *multiplying* the *column vector* representing the *translation* by -1.

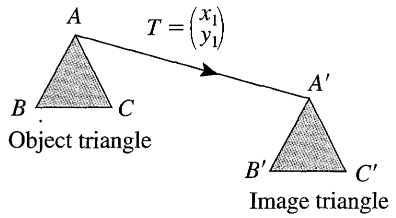


Fig. 10.70 Translation

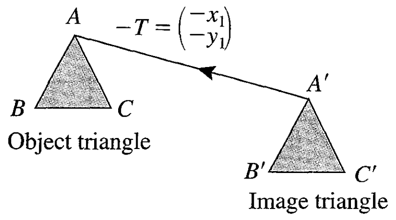


Fig. 10.71 Inverse translation

From Figs. 10.70 and 10.71 above:
The *translation, T maps triangle ABC onto triangle A′B′C′. The inverse translation −T, maps triangle A′B′C′ onto triangle ABC* since it *reverses* the

mapping. So the *inverse translation maps* the *image under* the *translation onto the object.*

The *translation T*, is defined by $\overrightarrow{AA'}$, and the *inverse translation* $-T$, is defined by $\overrightarrow{A'A}$. The *magnitude* of the *inverse translation* is the *same* as that of the *translation*; and the *direction* is *opposite* but on the *same line of motion* as the *translation*.

The *inverse translation* is used to *identify* the *object* that *gives rise* to an *image under a translation.*

Example 3

(a) Find the coordinates of the point P which is mapped onto P'(2, 3) under the translation $T = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

(b) Triangle ABC is mapped onto triangle A'B'C' with vertices A'(−3, −2), B'(3, 1) and C'(0, 4) under a translation $T = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$. *Determine the vertices of triangle ABC.*

Solution

(a) The object point P which is mapped onto the image point $P'(2, 3)$ under the translation $T = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ can be determined using a *graphical method.*

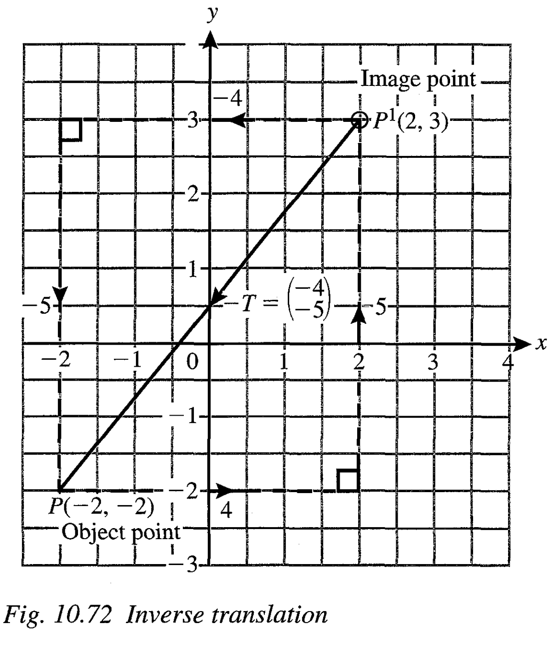


Fig. 10.72 Inverse translation

From Fig. 10.72:

The *point P* is *mapped onto* $P'(2, 3)$ by *moving* 4 *units to the right* and then 5 *units upwards.* So the *point* $P'(2, 3)$ is *mapped onto the point P* by *moving* 4 *units to the left* and then 5 *units downwards.* Hence the *inverse translation* that maps the point $P'(2, 3)$ onto point P is $-T = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$.

And the *coordinates* of the *point P* are $(-2, -2)$.

Alternative Method

(a) Alternatively, a *matrix method* can be used to determine the point P which is mapped onto the point $P'(2, 3)$ under the translation $T = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

$$\text{Now } \overset{P}{\begin{pmatrix} x \\ y \end{pmatrix}} = \overset{P'}{\begin{pmatrix} 2 \\ 3 \end{pmatrix}} - \overset{T}{\begin{pmatrix} 4 \\ 5 \end{pmatrix}} = \overset{P}{\begin{pmatrix} 2-4 \\ 3-5 \end{pmatrix}} = \overset{P}{\begin{pmatrix} -2 \\ -2 \end{pmatrix}}$$

Hence the *coordinates* of the *point P* are $(-2, -2)$.

(b) The vertices of triangle ABC which is mapped onto triangle $A'B'C'$ with vertices $A'(-3, -2)$, $B'(3, 1)$ and $C'(0, 4)$ under a translation $T = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ can be determined using a *graphical method.*

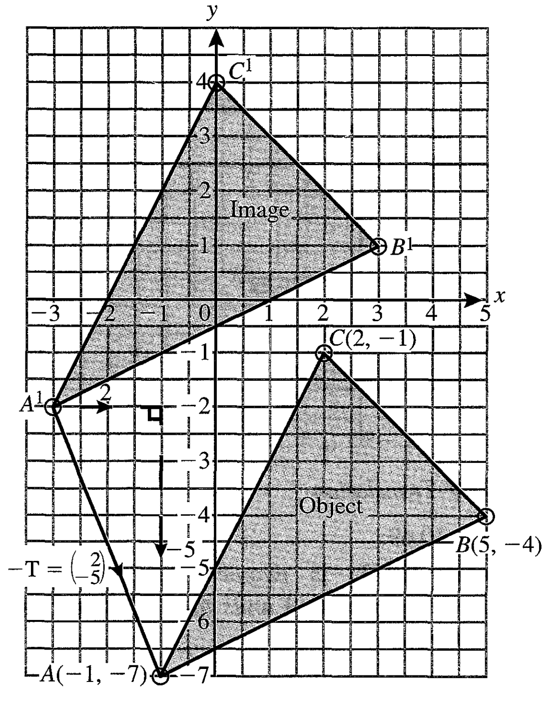


Fig. 10.73 Inverse translation

From Fig. 10.73:

The *point A* is *mapped onto A'*(−1, −7) by *moving* 2 *units to the left* and then 5 *units upwards*. So the *point A'*(−1, −7) is *mapped onto* the *point A* by *moving* 2 *units to the right* and then 5 *units downwards*.

Hence the *inverse translation* is $-T = \begin{pmatrix}2\\-5\end{pmatrix}$.

And the *vertices* of *triangle ABC* are $A(-1, -7)$, $B(5, -4)$ and $C(2, -1)$.

Alternative Method

(b) Alternatively, a *matrix method* can be used to determine the vertices of triangle ABC which is mapped onto triangle $A'B'C'$ with vertices $A'(-3, -2)$, $B'(3, 1)$ and $C'(0, 4)$ under the translation $T = \begin{pmatrix}-2\\5\end{pmatrix}$.

$$\text{Now}\ \overset{\begin{array}{ccc}A & B & C\end{array}}{\begin{pmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{pmatrix}} = \overset{\begin{array}{ccc}A' & B' & C'\end{array}}{\begin{pmatrix}-3 & 3 & 0\\ -2 & 1 & 4\end{pmatrix}} - \overset{T}{\begin{pmatrix}-2\\ 5\end{pmatrix}}$$

$$= \overset{\begin{array}{ccc}A & B & C\end{array}}{\begin{pmatrix}-3+2 & 3+2 & 0+2\\ -2-5 & 1-5 & 4-5\end{pmatrix}}$$

$$= \overset{\begin{array}{ccc}A & B & C\end{array}}{\begin{pmatrix}-1 & 5 & 2\\ -7 & -4 & -1\end{pmatrix}}$$

Hence the *vertices* of *triangle ABC* have *coordinates* $A(-1, -7)$, $B(5, -4)$ and $C(2, -1)$.

Exercise 10d

1. The points $A(2, 3)$, $B(4, 5)$ and $C(7, 3)$ are vertices of a triangle ABC. What is the image of the triangle under the translation $T = \begin{pmatrix}2\\3\end{pmatrix}$?

2. A quadrilateral $ABCD$ is such that A is the point (2, 1), B is the point (6, 1), C is the point (5, 3) and D is the point (4, 3).
If the image of $ABCD$ under the translation $T = \begin{pmatrix}-5\\2\end{pmatrix}$ is $A'B'C'D'$, find the coordinates of $A'B'C'D'$.

3. A parallelogram is defined by the points $A(2, 1)$, $B(3, 3)$, $C(6, 3)$ and $D(5, 1)$. Find the points of the image when the parallelogram undergoes a translation $T = \begin{pmatrix}2\\5\end{pmatrix}$.

4. Draw axes of x and y from −5 to 9. Draw $\triangle ABC$ with $A(-4, 5)$, $B(2, 3)$, $C(2, 5)$.
Translate $\triangle ABC$ using the vector $\begin{pmatrix}5\\1\end{pmatrix}$. Label this image $A'B'C'$. Then translate $\triangle A'B'C'$ using the vector $\begin{pmatrix}-1\\-3\end{pmatrix}$.
Label this new image $A''B''C''$. Give the vectors that describe the translations which map
(a) $\triangle ABC$ to $\triangle A''B''C''$
(b) $\triangle A''B''C''$ to $\triangle ABC$
(c) $\triangle A''B''C''$ to $\triangle A'B'C'$.

5. Draw axes for x and y from −5 to 7. Draw $\triangle ABC$ with $A(-3, 2)$, $B(1, 2)$, and $C(1, 5)$.
Translate $\triangle ABC$ using the vector $\begin{pmatrix}5\\2\end{pmatrix}$. Label this image $A'B'C'$. Then translate $\triangle A'B'C'$ using the vector $\begin{pmatrix}-1\\-3\end{pmatrix}$. Label this new image $A''B''C''$. Give the vectors that describe the translations which map
(a) $\triangle ABC$ to $\triangle A''B''C''$
(b) $\triangle A''B''C''$ to $\triangle ABC$
(c) $\triangle A''B''C''$ to $\triangle A'B'C'$.

6. Draw axes for x and y from −2 to 7. Draw $\triangle ABC$ with $A(-2, 4)$, $B(2, 3)$ and $C(1, 5)$.
Translate $\triangle ABC$ using the vector $\begin{pmatrix}4\\1\end{pmatrix}$. Label this image $A'B'C'$. Then translate $\triangle A'B'C'$ using the vector $\begin{pmatrix}-1\\-3\end{pmatrix}$. Label this new image $A''B''C''$. Give the vectors describing the translations which map
(a) $\triangle ABC$ to $\triangle A''B''C''$
(b) $\triangle A''B''C''$ to $\triangle ABC$
(c) $\triangle A''B''C''$ to $\triangle A'B'C'$.

7. Draw axes for x and y from −5 to 7. Draw $\triangle ABC$ with $A(-3, 2)$, $B(1, 2)$ and $C(1, 5)$.
Translate $\triangle ABC$ using the vector $\begin{pmatrix}3\\2\end{pmatrix}$. Label this image $\triangle A'B'C'$. Then translate $\triangle A'B'C'$ using the vector $\begin{pmatrix}-1\\-2\end{pmatrix}$. Label this new image $A''B''C''$.
Give the vectors describing the translations which map
(a) $\triangle ABC$ to $\triangle A''B''C''$
(b) $\triangle A''B''C''$ to $\triangle ABC$
(c) $\triangle A''B''C''$ to $\triangle A'B'C'$.

8. The diagram below shows a triangle ABC.

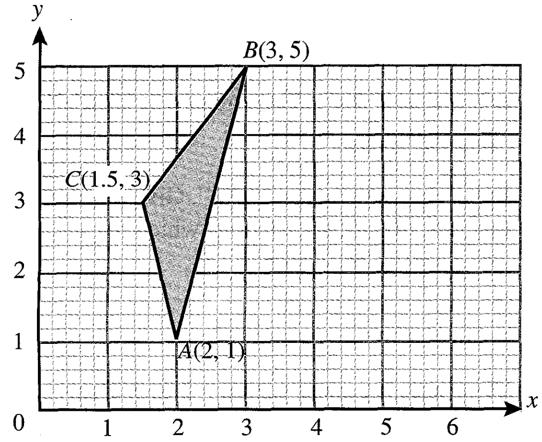


Fig. 10.74 Translation

(a) $\triangle ABC$ undergoes a translation $T = \begin{pmatrix}3\\-2\end{pmatrix}$. Determine the coordinates of the image of the vertices of the $\triangle ABC$, A', B' and C'.

(b) If the image of the $\triangle ABC$ under a translation is $A'(4.5, 0)$, $B'(5.5, 4)$ and $C'(4, 2)$, determine the matrix representing the translation in the form $T = \begin{pmatrix}x\\y\end{pmatrix}$.

9. The diagram below shows a triangle ABC.

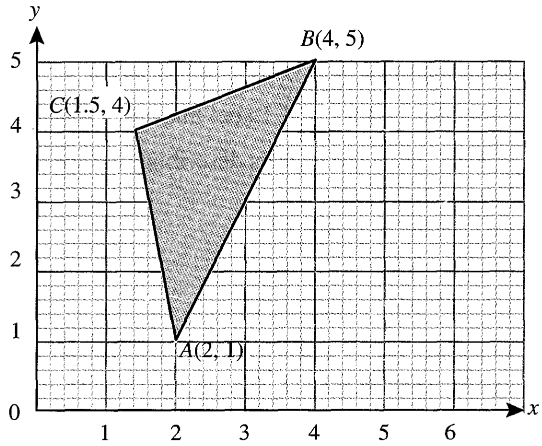


Fig. 10.75 Translation

(a) $\triangle ABC$ undergoes a translation $T = \begin{pmatrix}-3\\2\end{pmatrix}$. Determine the coordinates of the image of the vertices of the $\triangle ABC$, A', B' and C'.

(b) If the image of the $\triangle ABC$ under a translation is $A'(4, -2)$, $B'(6, 2)$ and $C'(3.5, 1)$, determine the matrix representing the translation in the form $T = \begin{pmatrix}x\\y\end{pmatrix}$.

10. Under the translation $T = \begin{pmatrix}-9\\5\end{pmatrix}$, triangle ABC is mapped onto $\triangle A'B'C'$ with coordinates $(2, -3)$, $(4, -1)$ and $(5, -6)$, respectively. What are the coordinates of A, B and C?

11. Under the translation $T = \begin{pmatrix}-4\\-6\end{pmatrix}$, figure $KLMN$ is mapped onto figure $K'L'M'N'$ with vertices $(-3, -2)$, $(-2, -4)$, $(-3, -5)$ and $(0, -3)$, respectively. Find the coordinates of the vertices of figure $KLMN$.

12. Quadrilateral $ABCD$ is mapped onto quadrilateral $A'B'C'D'$ with vertices $(-4, 1)$, $(-3, 4)$, $(0, 3)$ and $(-2, 0)$, respectively under the translation $T = \begin{pmatrix}0\\5\end{pmatrix}$. Find the coordinates of A, B, C and D.

Reflection

A *reflection* is a *way* of *transforming* a *shape* as a *plane mirror* does at home. In a *plane*, the *result* of *reflecting* an *object* in a *mirror line* (or an *axis of reflection*) is called its *mirror image*. The *object* and *image* are *symmetrical about* the *mirror line.*

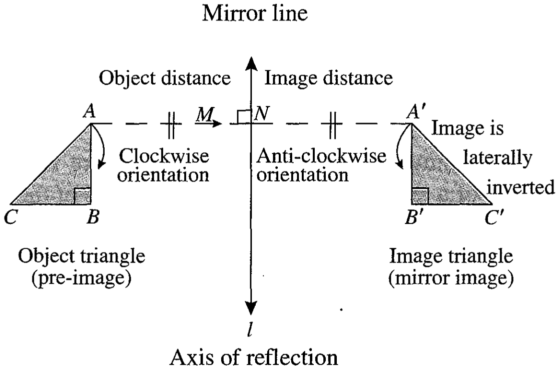


Fig. 10.76 Reflection of a plane figure

From Fig. 10.76:

If A *and* A' are *corresponding points*, then the *mirror line* ℓ is a *mediator of* AA'. That is, a *line joining any point* A to its *image* A' will always be *perpendicular* to the *mirror line* ℓ and the *distances of* A and A' from *the mirror* will line be *equal.* So $\hat{N} = 90°$ and $AN = A'N$ (i.e. $AA' = 2 \cdot AN = 2 \cdot A'N$). It can also be seen that the *orientation* of the *object triangle* is *clockwise*, while the *orientation* of the *image triangle* is *anti-clockwise*. We say that the *image* of a *plane figure under* a *reflection* is *laterally inverted.* The *reflection* of the *point A* can be *denoted* by:

$M: A \rightarrow A'$ or $M(A) = A'$ or $A \overset{M}{\rightarrow} A'$.

Since the *mirror line* is represented by the *letter* ℓ, then the *reflection* of the *point A* can also be *denoted* by:

$M_\ell: A \rightarrow A'$ or $M_\ell(A) = A'$ or $A \overset{M_\ell}{\rightarrow} A'$.

The *reflection* of *triangle ABC* can be *denoted* by:

$M_\ell: \triangle ABC \rightarrow \triangle A'B'C'$ or

$M_\ell(\triangle ABC) = \triangle A'B'C'$ or $\triangle ABC \overset{M_\ell}{\rightarrow} \triangle A'B'C'$.

The *mapping notations* $M_\ell: \triangle ABC \rightarrow A'B'C'$,

$M_\ell(\triangle ABC) = \triangle A'B'C'$ and $\triangle ABC \overset{M_\ell}{\rightarrow} \triangle A'B'C'$, *describe* a *reflection* of the *triangle ABC* in a *mirror line* ℓ, where $\triangle A'B'C'$ is the *image of* $\triangle ABC$ under the *reflection* denoted by M_ℓ.

Properties of a Reflection

Here we *investigate* the *properties of a figure* which are *invariant* under a *reflection.*

Fig. 10.77 below shows the *image trapezium* $P'Q'R'S'$ of a *trapezium PQRS* under a *reflection M* in a *mirror line* ℓ.

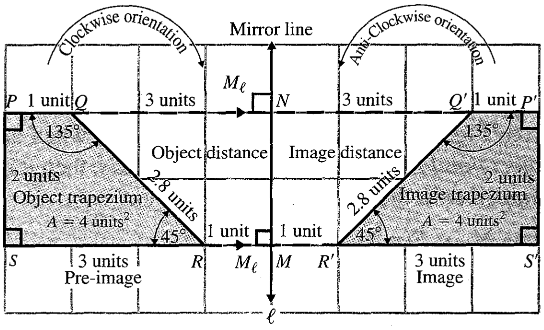


Fig. 10.77 Reflection of a plane figure

Thus $M_\ell: P \rightarrow P'$, $M_\ell: Q \rightarrow Q'$, $M_\ell: R \rightarrow R'$ and $M_\ell: S \rightarrow S'$

Or M_ℓ: *Trapezium PQRS* $\rightarrow$ *Trapezium* $P'Q'R'S'$.

From Fig. 10.77 it can be *seen* that:

(i) $PP' \perp$ *mirror line* ℓ, $QQ' \perp$ *mirror line* ℓ, $RR' \perp$ *mirror line* ℓ and $SS' \perp$ *mirror line* ℓ.

(ii) $PN = P'N = 4$ units (i.e. $PP' = 2 \cdot PN = 2 \cdot P'N = 8$ units).

$QN = Q'N = 3$ units (i.e. $QQ' = 2 \cdot QN = 2 \cdot Q'N = 6$ units).

$RM = R'M = 1$ unit (i.e. $RR' = 2 \cdot RM = 2 \cdot R'M = 2$ units).

$SM = S'M = 4$ units (i.e. $SS' = 2 \cdot SM = 2 \cdot S'M = 8$ units).

Hence we can *conclude* that:

(i) The *mirror line (or the axis of reflection) is the perpendicular bisector (or mediator) of the line segment joining an object point and its image point.*

(ii) *The perpendicular distance of the object point from the mirror line is equal to the perpendicular distance of its image point from the mirror line.*

That is:

The object distance = The image distance

(iii) (a) *The distance of the image from the object is twice the object distance.*

(b) *The distance of the image from the object is twice the image distance.*

From Fig. 10.77 it can be *seen* that:

$PQ = P'Q' = 1$ unit, $QR = Q'R' = 2.8$ units, $RS = R'S' = 3$ units and $PS = P'S' = 2$ units.

So *corresponding sides* are *equal* under the *reflection.* Hence we can *conclude* that:

(i) *Reflection preserves the distance between two points. Reflection preserves lengths.*

In Fig. 10.77 it can be *observed* that:

$\hat{P} = \hat{P}' = 90°$, $\hat{Q} = \hat{Q}' = 135°$, $\hat{R} = \hat{R}' = 45°$ and $\hat{S} = \hat{S}' = 90°$.

So *corresponding angles* are *equal* under the *reflection.*

Hence we can *conclude* that:

(ii) *Reflection preserves angles.*

From Fig. 10.77 it can be *seen* that:

$PQ \parallel SR$ and $P'Q' \parallel S'R'$.

So *parallel sides remain parallel* under the *reflection.*

Hence we can *conclude* that:

(iii) *Reflection preserves Parallelism.*

In Fig. 10.77 it can be *observed* that:

The *area of trapezium PQRS* = The *area of trapezium* $P'Q'R'S'$ = 4 square units.

So *area remains* the *same* under the *reflection.*

Hence we can *conclude* that:

(iv) *Reflection preserves area.*

From Fig. 10.77 it can be *seen* that:

$PQ{:}SR = 1{:}3$ and $P'Q'{:}S'R' = 1{:}3$.

$PS{:}QR = 1{:}1.4$ and $P'S'{:}Q'R' = 1{:}1.4$

So *ratios* are *unchanged* under the *reflection.*

Hence we can *conclude* that:

(v) *Reflection preserves ratios.*

In Fig. 10.77 it can be *observed* that:

The *order* of the *vertices* in the *object* is *PQRS.*

The *order* of the *vertices* in the *image* is $P'Q'R'S'$.

So the *order* of the *vertices* remain the *same.*

Hence we can *conclude* that:

(vi) *Reflection preserves the order of points.*

From Fig. 10.77 it can be *seen* that:

The *orientation* (or *sense*) of the *object* is *clockwise.*

The *orientation* (or *sense*) of the *image* is *anti-clockwise.*

So the *orientation* (or *sense*) is *changed* under the *reflection.*

Hence we can *conclude* that:

(vii) *Reflection does not preserve orientation* (*or sense*) *of a figure.*

We can *summarize* the *properties of a reflection* in a *table* as shown below.

Table 10.3

Invariant property	Length	Angle	Parallelism	Area	Ratio	Order of points	Orientation or sense
Invariant under reflection	Yes	Yes	Yes	Yes	Yes	Yes	No

From the *properties of reflections* discussed above, it follows that the *image* $P'Q'R'S'$ of figure *PQRS* under the *reflection* is of the *same size* and *shape* as the *object PQRS*. Hence the *object PQRS* and the *image* $P'Q'R'S'$ are said to be *congruent.*

Thus:

The *trapezium PQRS* ≡ The *trapezium* $P'Q'R'S'$. Hence *reflection* is a *congruency transformation.*

We can *summarize* the *properties of a reflection* as follows:

(i) *The object distance = The image distance.*
(ii) *The image is laterally inverted.*
(iii) *Reflection is a congruency transformation.*

Image Under a Reflection

Here we will *investigate* the *graphical method* of obtaining the *image* of a *plane figure* under a *specified reflection.*

The *image* of a *plane figure* under a *reflection* is obtained by *determining* the *image* of its *vertices* under the *reflection* and then *joining* the *image vertices* in a *clockwise* or *anti-clockwise direction.*

REFLECTION IN THE *X*-AXIS

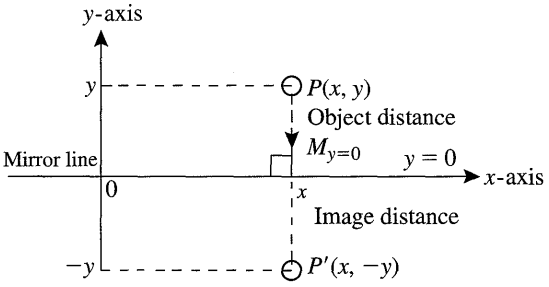


Fig. 10.78 Reflection of a point in the x-axis

From Fig. 10.78 above, it can be *seen* that when an *object point* $P(x, y)$ is *reflected* in the *x-axis*, then it is *mapped onto* the *image point* $P'(x, -y)$.

Thus $M_{y=0}$: $P(x, y) \rightarrow P'(x, -y)$

or M_{0x}: $P(x, y) \rightarrow P'(x, -y)$.

So under a *reflection* in the *x-axis* (i.e. the *mirror line* $y = 0$), the *x-coordinate* remains *unchanged* (*invarient*), but the *y-coordinate* is *multiplied by* −1 (i.e. *changes its sign*).

Once we can find the *image of a point*, then we can find the *image of a vertex* and hence the *image of any plane figure* under a *reflection* in the *x-axis.*

Example 4

Draw parallelogram ABCD with vertices A(2, 1), B(3, 3), C(6, 3) and D(5, 1).

Parallelogram A′B′C′D′ is the image of parallelogram ABCD under a reflection in the x-axis. Draw parallelogram A′B′C′D′ and state the coordinates of its vertices.

Solution

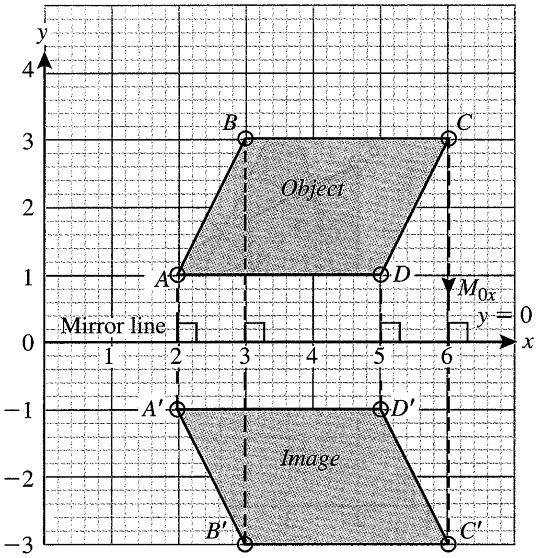


Fig. 10.79 Reflection of a plane figure in the x-axis

Parallelogram ABCD with the stated *vertices* was *drawn* on graph paper.

Parallelogram ABCD was then *reflected* in the *x-axis* and the *image parallelogram A′B′C′D′* was obtained.

From Fig. 10.79:

The *coordinates* of the *vertices* of *parallelogram A′B′C′D′* are:

$A'(2, -1)$, $B'(3, -3)$, $C'(6, -3)$ and $D'(5, -1)$.

Alternative Method

Since M_{0x}: $P(x, y) \rightarrow P'(x, -y)$,

M_{0x}: $A(2, 1) \rightarrow A'(2, -1)$,

M_{0x}: $B(3, 3) \rightarrow B'(3, -3)$,

M_{0x}: $C(6, 3) \rightarrow C'(6, -3)$

and M_{0x}: $D(5, 1) \rightarrow D'(5, -1)$.

Hence the *coordinates* of the *vertices* of *parallelogram A′B′C′D′* are:

$A'(2, -1)$, $B'(3, -3)$, $C'(6, -3)$ and $D'(5, -1)$.

REFLECTION IN THE Y-AXIS

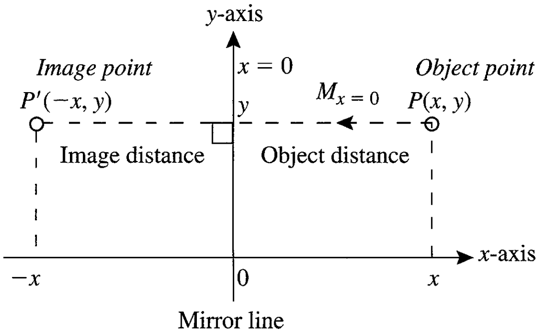


Fig. 10.80 Reflection of a point in the y-axis

From Fig. 10.80 above, it can be *seen* that when an *object point* $P(x, y)$ is *reflected* in the *y-axis*, then it is *mapped onto* the *image point* $P'(-x, y)$.

Thus $M_{y=0}$: $P(x, y) \rightarrow P'(x, -y)$

or M_{0x}: $P(x, y) \rightarrow P'(x, -y)$.

So under a *reflection* in the *y-axis* (i.e. the *mirror line* $x = 0$), the *x-coordinate* is *multiplied by* -1 (i.e. *changes its sign*), but the *y-coordinate* remains *unchanged* (*invariant*).

Once we can find the *image of a point*, then we can find the *image of a vertex* and hence the *image of any plane figure* under a *reflection* in the *y-axis*.

Example 5

ABC is a triangle with coordinates (−4, 3), (−1, 0) and (−3, −2) respectively. What is the image of triangle ABC under a reflection in the y-axis?

Solution

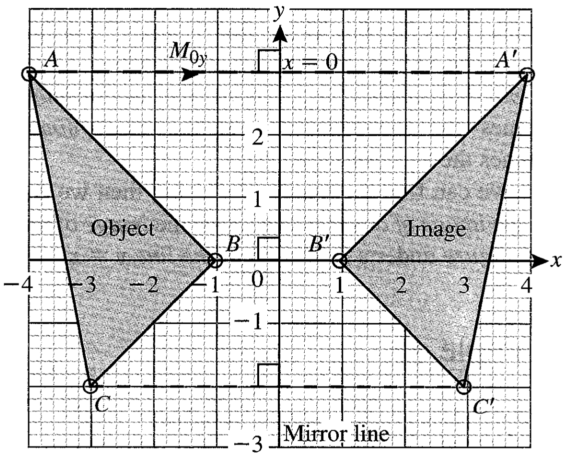


Fig. 10.81 Reflection of a plane figure in the y-axis

$\triangle ABC$ with the stated *vertices* was *drawn* on graph paper.

$\triangle ABC$ was then *reflected* in the *y-axis* and the *image triangle* $A'B'C'$ was obtained.

From Fig. 10.81:
The *coordinates* of the *vertices* of $\triangle A'B'C'$ are: $A'(4, 3)$, $B'(1, 0)$ and $C'(3, -2)$.

Alternative Method

Since $M_{0y}: P(x, y) \rightarrow P'(-x, y)$,

$M_{0y}: A(-4, 3) \rightarrow A'(4, 3)$,

$M_{0y}: B(-1, 0) \rightarrow B'(1, 0)$

and $M_{0y}: C(-3, -2) \rightarrow C'(3, -2)$.

Hence the *image* of the triangle ABC under a *reflection in the y-axis* is given by the *vertices*: $A'(4, 3)$, $B'(1, 0)$ and $C'(3, -2)$.

REFLECTION IN THE LINE $y = x$

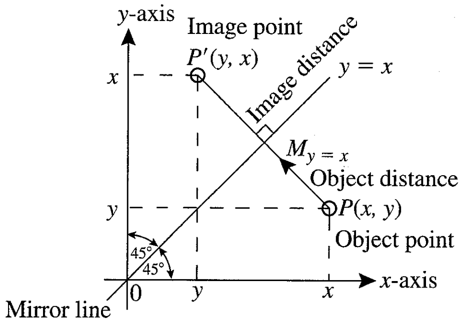


Fig. 10.82 Reflection of a point in the line $y = x$

From Fig. 10.82 above, it can be *seen* that when an *object point* $P(x, y)$ is *reflected* in the *line* $y = x$, then it is *mapped onto* the *image point* $P'(y, x)$.
Thus $M_{y=x}: P(x, y) \rightarrow P'(y, x)$.
So under a *reflection* in the *line* $y = x$ (i.e. a *line inclined at* 45° to the *top* of the *positive x-axis* and *passing through* the *origin*), the *x-coordinate* becomes the *y-coordinate* and the *y-coordinate* becomes the *x-coordinate*.
Once we can find the *image of a point*, then we can find the *image of a vertex* and hence the *image of any plane figure* under a *reflection in the line* $y = x$.

Example 6

The point $A(2, 2)$, $B(5, 2)$ and $C(3, 6)$ are vertices of a triangle ABC. The triangle is reflected in the line which passes through the points $O(0, 0)$ and $Q(3, 3)$ to produce triangle $A'B'C'$.

(a) (i) Draw on graph paper the triangles ABC and $A'B'C'$

(ii) State the coordinates of A', B' and C'.

(b) State the single transformation that maps triangle ABC onto triangle $A'B'C'$.

Solution

(a)

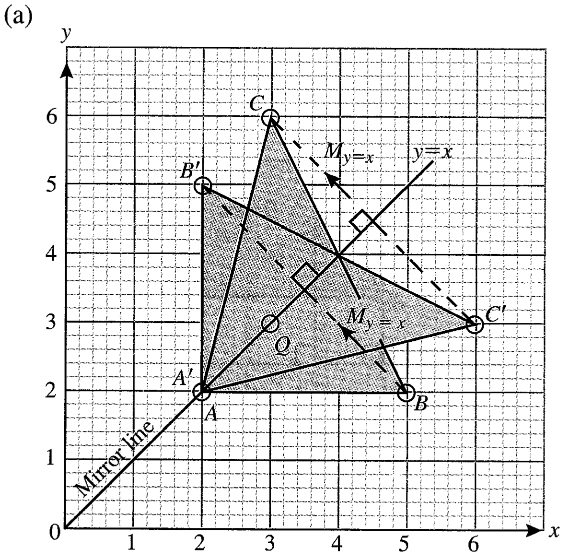


Fig. 10.83 Reflection of a plane figure in the line $y = x$

In Fig. 10.83, the *triangle ABC* and *its image*, *triangle $A'B'C'$*, were *drawn* on graph paper.

Note that the *point A* situated *on the mirror line* is *invariant* under the *reflection*. That is, $A(2, 2) = A'(2, 2)$. *All points situated on a mirror line are invariant under a reflection.* This is so because the *mirror line* is an *invariant line*.

(ii) The *coordinates* of A', B' and C' are: $A'(2, 2)$, $B'(2, 5)$ and $C'(6, 3)$.

(b) From above it can be *seen* that:

$A(2, 2) \rightarrow A'(2, 2)$

$B(5, 2) \rightarrow B'(2, 5)$

$C(3, 6) \rightarrow C'(6, 3)$.

That is $P(x, y) \rightarrow P'(y, x)$.
Hence the *single transformation* that maps $\triangle ABC$ onto $\triangle A'B'C'$ is a *reflection in the line* $y = x$.

REFLECTION IN THE LINE $y = -x$

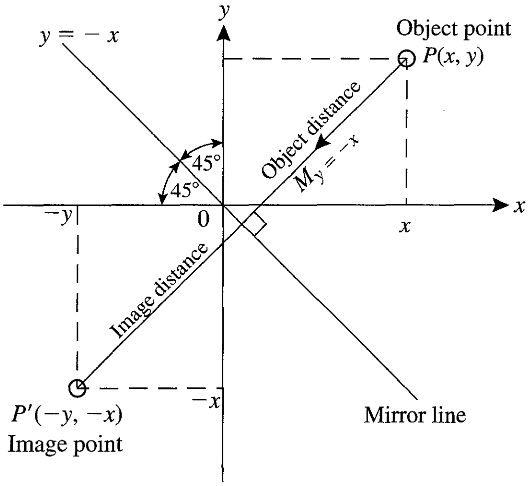


Fig. 10.84 Reflection of a point in the line $y = -x$

From Fig. 10.84 above, it can be *seen* that when an *object point* $P(x, y)$ is *reflected* in the *line* $y = -x$, then it is *mapped onto* the *image point* $P'(-y, -x)$.

Thus $M_{y=-x}$: $P(x, y) \rightarrow P'(-y, -x)$.

So under a *reflection* in the *line* $y = -x$ (i.e. a *line inclined at* 45° to the *top* of the *negative x-axis* and *passing through* the *origin*), the *x-coordinate multiplied by* -1 becomes the *y-coordinate* and the *y-coordinate multiplied by* -1 becomes the *x-coordinate*.

Once we can find the *image of a point*, then we can find the *image of a vertex* and hence the *image of any plane figure* under a *reflection in the line* $y = -x$.

Example 7

The points $A(-2, 2)$, $B(-5, 2)$ and $C(-4, 0)$ are vertices of a triangle ABC. The triangle is reflected in the line which passes through the points $O(0, 0)$ and $Q(-3, 3)$ to produce triangle $A'B'C'$.

(a) (i) Draw on graph paper the triangle ABC and $A'B'C'$.

(ii) State the coordinates of A', B' and C.

(b) State the single transformation that maps triangle ABC onto triangle $A'B'C'$.

Solution

(a) (i)

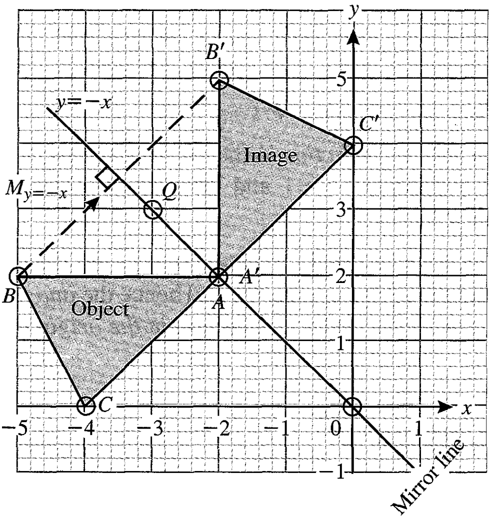


Fig. 10.85 Reflection of a plane figure in the line $y = -x$

In Fig. 10.85, the *triangle ABC* and its *image, triangle $A'B'C'$*, were *drawn* on graph paper. Note that the *point A* situated *on the mirror line* is *invariant* under the *reflection*. That is, $A(-2, 2) = A'(-2, 2)$. *All points situated on a mirror line are invariant under a reflection.*

(ii) The *coordinates* of A', B' and C' are:

$A'(-2, 2)$, $B'(-2, 5)$ and $C'(0, 4)$.

(b) From above it can be *seen* that:

$$A(-2, 2) \rightarrow A'(-2, 2)$$
$$B(-5, 2) \rightarrow B'(-2, 5)$$
$$C(-4, 0) \rightarrow C'(0, 4).$$

That is $P(x, y) \rightarrow P'(-y, -x)$.

Hence the *single transformation* that maps triangle ABC onto triangle $A'B'C'$ is a *reflection in the line* $y = -x$.

REFLECTION IN THE ORIGIN

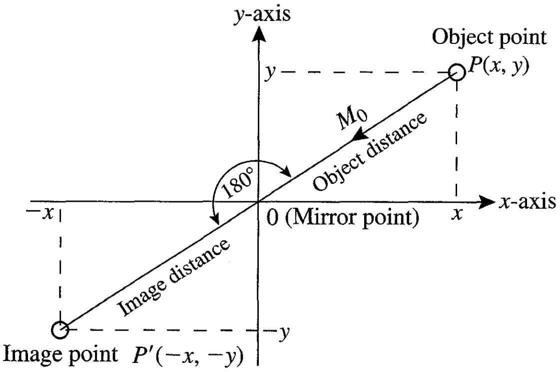


Fig. 10.86 Reflections of a point in the origin

From Fig. 10.86, it can be *seen* that when an *object point* $P(x, y)$ is *reflected* in the *origin* 0, then it is *mapped onto* the *image point* $P'(-x, -y)$.

Thus M_0: $P(x, y) \rightarrow P'(-x, -y)$.

So under a *reflection* in the *origin* 0, the *x-coordinate* is *multiplied by* -1 and the *y-coordinate* is *multiplied by* -1.

Once we can find the *image of a point*, then we can find the *image of a vertex* and hence the *image of any plane figure* under a *reflection in the origin* 0.

Example 8

Draw $\triangle PQR$ with vertices $P(2, 3)$, $Q(5, 1)$ and $R(1, 1)$ on graph paper. $\triangle P'Q'R'$ is the image of $\triangle PQR$ under a reflection in the origin. Draw $\triangle P'Q'R'$ and state the coordinates of its vertices.

Solution

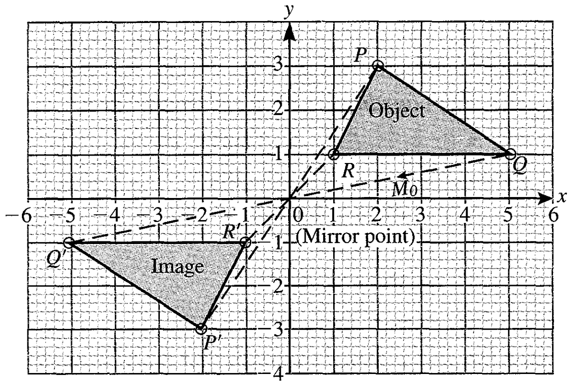


Fig. 10.87 Reflection of a plane figure in the origin

$\triangle PQR$ with the stated *vertices* was *drawn* on graph paper using the given *scale*.

$\triangle PQR$ was then *reflected* in the *origin* and the *image* $\triangle P'Q'R'$ was obtained.

From Fig. 10.87:

The *coordinates* of the *vertices* of $\triangle P'Q'R'$ are:

$P'(-2, -3)$, $Q'(-5, -1)$ and $R'(-1, -1)$.

Alternative Method

Since M_0: $P(x, y) \rightarrow P'(-x, -y)$,

M_0: $P(2, 3) \rightarrow P'(-2, -3)$,

M_0: $Q(5, 1) \rightarrow Q'(-5, -1)$

and M_0: $R(1, 1) \rightarrow R'(-1, -1)$.

Hence the *coordinates* of the *vertices* of are:

$P'(-2, -3)$, $Q'(-5, -1)$ and $R'(-1, -1)$.

REFLECTION IN THE LINE $x = b$

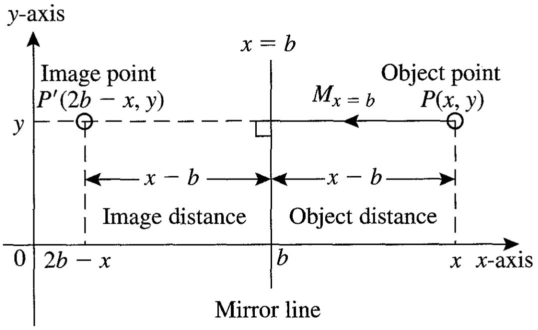


Fig. 10.88 Reflection of a point in the line $x = b$

From Fig. 10.88, it can be *seen* that the *object point* $P(x, y)$ is *reflected* in the *mirror line* $x = b$, where the *line* $x = b$ is a *straight line parallel* to the *y-axis*.

Under *reflection* in the *line* $x = b$:

The *object distance* $= (x - b)$ *units*.

The *image distance* $= (x - b)$ *units*,

since the *image distance* is *equal* to the *object distance* under *reflection* in a *line*.

Hence the *distance* of the *image from the object* $= 2(x - b)$ *units*.

so the *x-coordinate* of the *image*,

$x' = x - 2(x - b) = x - 2x + 2b = 2b - x$.

And the *y-coordinate* of the *image*, $y' = y$.

Thus $M_{x=b}$: $P(x, y) \rightarrow P'(2b - x, y)$.

Hence, under a *reflection* in the *line* $x = b$, $(x, y) \rightarrow (2b - x, y)$.

Example 9

PQRS is a quadrilateral with vertices $P(-8, 2)$, $Q(-7, 6)$, $R(-5, 3)$ and $S(-6, 1)$. Under reflection in the line $x = -3$, the image of PQRS is $P'Q'R'S'$. Find the coordinates P', Q', R' and S'. Illustrate the transformation on graph paper.

Solution

Using $M_{x=b}$: $P(x, y) \rightarrow P'(2b - x, y)$,

then $M_{x=-3}$: $P(x, y) \rightarrow P'(-6-x, y)$,

since $2b = 2(-3) = -6$.

Also $M_{x=-3}$: $P(-8, 2) \rightarrow P'(-6 - [-8], 2)$
$= P'(-6 + 8, 2)$
$= P'(2, 2).$

$M_{x=-3}$: $Q(-7, 6) \rightarrow Q'(-6 - [-7], 6)$
$= Q'(-6 + 7, 6)$
$= Q'(1, 6).$

$M_{x=-3}$: $R(-5, 3) \rightarrow R'(-6-[-5], 3)$
$= R'(-6 + 5, 3)$
$= R'(-1, 3).$

and $M_{x=-3}$: $S(-6, 1) \rightarrow S'(-6-[-6], 1)$
$= S'(-6 + 6, 1)$
$= S'(0, 1).$

Hence the *coordinates* of P', Q', R' and S' are:
$P'(2, 2)$, $Q'(1, 6)$, $R'(-1, 3)$ and $S'(0, 1)$.

The *transformation* can be seen *illustrated* in Fig. 10.89.

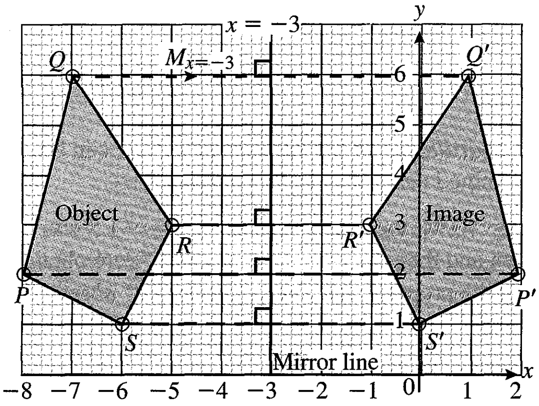


Fig. 10.89 Reflection of a plane figure in the line $x = -3$

This *problem* could also have been *solved* using a *graphical method.*

REFLECTION IN THE LINE $y = c$

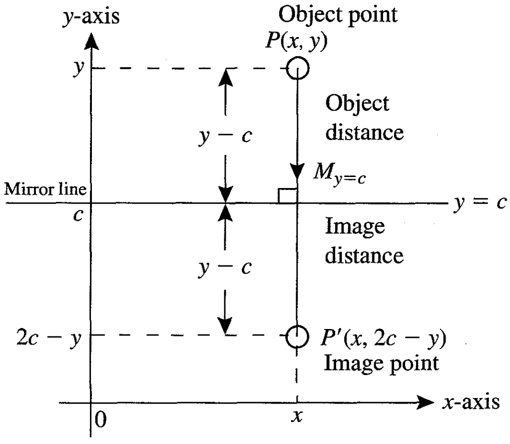


Fig. 10.90 Reflection of a point in the line $y = c$

From Fig. 10.90 it can be *seen* that the *object point* $P(x, y)$ is *reflected* in the *mirror line* $y = c$, where the *line* $y = c$ is a *straight line parallel* to the *x-axis.*

Under *reflection* in the *line* $y = c$:

The *object distance* $= (y - c)$ *units.*

So the *image distance* $= (y - c)$ *units,*

since the *image distance* is *equal* to the *object distance* under *reflection* in a *line.*

Hence the *distance* of the *image from the object* $= 2(y - c)$ *units.*

So the *x-coordinate* of the *image*, $x' = x$.

And the *y-coordinate* of the *image*,

$y' = y - 2(y - c) = y - 2y + 2c = 2c - y$.

Thus $M_{y=c}$: $P(x, y) \rightarrow P'(x, 2c - y)$.

Hence, under a *reflection* in the *line* $y = c$,

$(x, y) \rightarrow (x, 2c - y)$.

Example 10

Triangle PQR with vertices P(2, 1), Q(5, 3) and R(7, 1) is reflected in the line $y = -1$. The image of triangle PQR is triangle P'Q'R'. Find the coordinates of P', Q' and R'. Illustrate the transformation on graph paper.

Solution

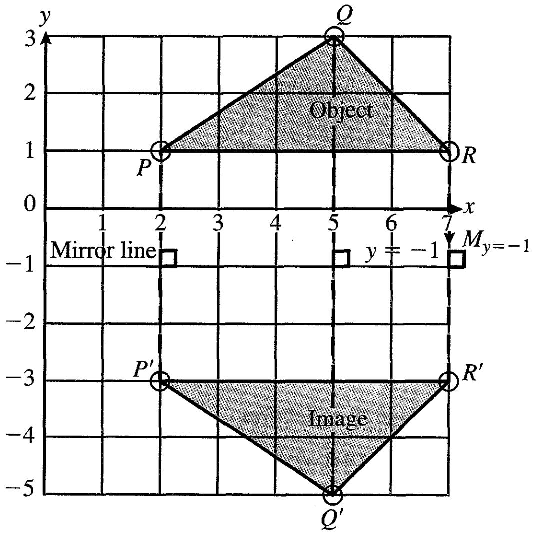


Fig. 10.91 Reflection of a plane figure in the line $y = -1$

Using $M_{y=c}$: $P(x, y) \rightarrow P'(x, 2c - y)$,
then $M_{y=-1}$: $P(x, y) \rightarrow P'(x, -2 - y)$,
since $2c = 2(-1) = -2$.

Also $M_{y=-1}$: $P(2, 1) \rightarrow P'(2, -2-1)$
$= P'(2, -3)$

$M_{y=-1}$: $Q(5, 3) \rightarrow Q'(5, -2-3)$
$= Q'(5, -5)$

and $M_{y=-1}$: $R(7, 1) \rightarrow R'(7, -2-1)$
$= R'(7, -3)$.

Hence the *coordinates* of P', Q' and R' are:
$P'(2, -3)$, $Q'(5, -5)$ and $R'(7, -3)$.

The *transformation* can be seen *illustrated* in Fig. 10.91.

This *problem* could also have been *solved* using a *graphical method.*

Inverse Reflection

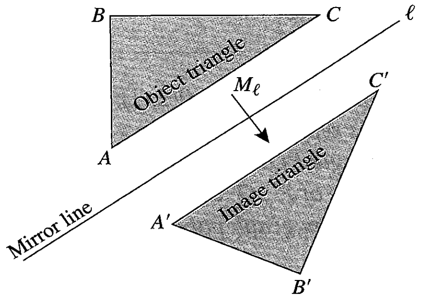


Fig. 10.92 Reflection

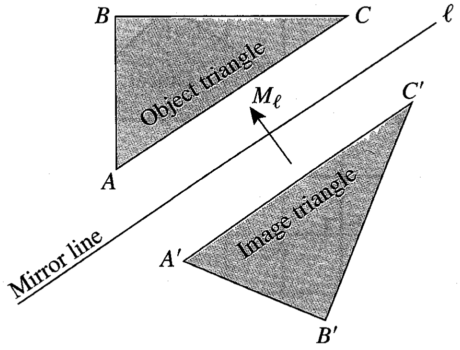


Fig. 10.93 Inverse reflection

In Figs. 10.92 and 10.93:

The *reflection* M_ℓ *maps triangle ABC onto triangle* $A'B'C'$ in the *line* ℓ.

The *inverse reflection* M_ℓ *maps triangle* $A'B'C'$ onto *triangle ABC* in the *same line* ℓ. That is, the *inverse reflection maps* the *image* of the *reflection onto* the *object* in the *same mirror line* ℓ.

Thus the *inverse of* M_ℓ: $\triangle ABC \rightarrow \triangle A'B'C'$ is
M_ℓ: $\triangle A'B'C' \rightarrow \triangle ABC$

Hence the *inverse of a reflection* is a *reflection* in the *same line* ℓ.

As a result, the *inverse reflection* is used to *find the object* that gives rise to an *image under a reflection.*

Example 11

Triangle ABC is mapped onto triangle $A'B'C'$ with vertices $A'(5, 6)$, $B'(6, 1)$ and $C'(1, 3)$ under a reflection in the line $y = -1.5$. What are the coordinates of the vertices of $\triangle ABC$?

Solution

The *coordinates* of the *vertices* of $\triangle ABC$ can be determined using a *graphical method.*

In Fig. 10.94, the *axis of reflection* $y = -1.5$ was drawn and then the *vertices A, B* and *C* were obtained using the property that, *image distance = object distance.*

From Fig. 10.94:

The *coordinates* of the *vertices* of *triangle ABC* are $A(5, -9)$, $B(6, -4)$ and $C(1, -6)$.

Alternative Method

Under *reflection* in the *line* $y = c$: $(x, y) \rightarrow (x, 2c - y)$.
So for *reflection* in the *line* $y = -1.5$:

$$(x, y) \rightarrow (x, -3-y),$$

since $2c = 2(-1.5) = -3$.

Hence under the *inverse reflection*:

$A'(5, 6) \rightarrow A(5, -3 - 6) = A(5, -9)$.
$B'(6, 1) \rightarrow B(6, -3 - 1) = B(6, -4)$.
$C'(1, 3) \rightarrow C(1, -3 - 3) = C(1, -6)$.

So the *coordinates* of the *vertices* of *triangle ABC* are:
$A(5, -9)$, $B(6, -4)$ and $C(1, -6)$.

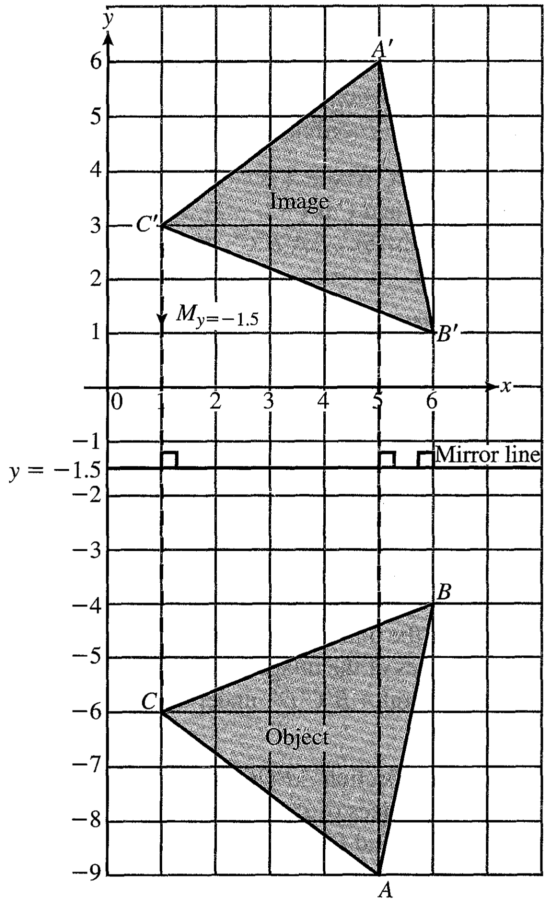


Fig. 10.94 Inverse reflection

Exercise 10e

1. Draw axes for x and y from -6 to 8. Draw $\triangle ABC$ with vertices $A(-5, -2)$, $B(-3, -4)$ and $C(-1, -2)$. Triangle $A'B'C'$ is the image of triangle ABC under a reflection in the x-axis. Draw triangle $A'B'C'$ and state the coordinates of its vertices.
2. Draw axes for x and y from -7 to 5. Draw $\triangle ABC$ with vertices $A(4, 2)$, $B(1, 2)$ and $C(1, 4)$. Triangle $A'B'C'$ is the image of triangle ABC under a reflection in the x-axis. Draw triangle $A'B'C'$ and state the coordinates of its vertices.
3. Draw axes for x and y from -5 to 7. Draw $\triangle PQR$ with vertices $P(-4, -2)$, $Q(-1, -2)$ and $R(-1, -4)$. $\triangle P'Q'R'$ is the image of $\triangle PQR$ under a reflection in the x-axis. Draw $\triangle P'Q'R'$ and state the coordinates of its vertices.
4. Draw axes for x and y from -8 to 8. Draw $\triangle PQR$ where P is $(-7, -3)$, Q is $(-5, -5)$, and R is $(-3, -2)$. $\triangle P'Q'R'$ is the image of $\triangle PQR$ under reflection in the x-axis. Draw $\triangle P'Q'R'$ and state the coordinates of its vertices.
5. A quadrilateral $ABCD$ is such that A is the point $(2, 1)$, B is the point $(6, 1)$, C is the point $(5, 3)$ and D is the point $(4, 3)$. State the coordinates of the image of $ABCD$, $A'B'C'D'$ after it is reflected in the x-axis.
6. Triangle ABC has vertices $A(-5, -2)$, $B(-3, -4)$ and $C(-1, -2)$. Determine the image of $\triangle ABC$ under a reflection in the y-axis.
7. ABC is a triangle with coordinates $(4, 2)$, $(1, 2)$ and $(1, 4)$, respectively. Determine the image of $\triangle ABC$ under a reflection in the y-axis.
8. PQR is a triangle with vertices $P(-4, -2)$, $Q(-1, -2)$ and $R(-1, -4)$. Find the vertices of the image of $\triangle PQR$ under a reflection in the y-axis.
9. Triangle PQR has coordinates $(-7, -3)$, $(-5, -5)$ and $(-3, -2)$, respectively. State the vertices of the image of $\triangle PQR$ under a reflection in the y-axis.
10. Quadrilateral $ABCD$ has vertices $A(2, 1)$, $B(6, 1)$, $C(5, 3)$ and $D(4, 3)$. State the coordinates of the image of $ABCD$, $A'B'C'D'$ after it is reflected in the y-axis.
11. The points $A(-5, -2)$, $B(-3, -4)$ and $C(-1, -2)$ are vertices of a $\triangle ABC$. The triangle is reflected in the line which passes through the points $O(0, 0)$ and $Q(2, 2)$ to produce $\triangle A'B'C'$.
    (a) State the coordinates of A', B' and C'.
    (b) State the single transformation that maps $\triangle ABC$ onto $\triangle A'B'C'$.
12. Triangle ABC has vertices $A(4, 2)$, $B(1, 2)$ and $C(1, 4)$. The triangle is reflected in the line which passes through the points $O(0, 0)$ and $P(3, 3)$ to produce $\triangle A'B'C'$.
    (a) State the coordinates of A', B' and C'.
    (b) State the single transformation that maps $\triangle ABC$ onto $\triangle A'B'C'$.
13. PQR is a triangle with coordinates $(-4, -2)$, $(-1, -2)$ and $(-1, -4)$, respectively. The triangle is reflected in the line which passes

through the points $K(1, 1)$ and $L(-1, -1)$ to produce $\Delta P'Q'R'$.

(a) Write down the coordinates of $A'B'$ and C'.

(b) State the single transformation that maps ΔPQR onto $\Delta P'Q'R'$.

14. PQR is a plane figure with vertices $P(-7, -3)$, $Q(-5, -5)$, and $R(-3, -2)$. Determine the image of the plane figure after reflection in the line $y = x$.

15. A quadrilateral $ABCD$ has vertices $A(2, 1)$, $B(6, 1)$, $C(5, 3)$ and $D(4, 3)$. State the coordinates of the image of quadrilateral $ABCD$, $A'B'C'D'$ after it is reflected in the line $y = x$.

16. The points $A(-5, -2)$, $B(-3, -4)$ and $C(-1, -2)$ are vertices of a ΔABC. The triangle is reflected in the line which passes through the points $P(2, -2)$ and $Q(-4, 4)$ to produce $\Delta A'B'C'$.

(a) State the coordinates of A', B' and C'.

(b) State the single transformation that maps ΔABC onto $\Delta A'B'C'$.

17. Triangle ABC has vertices $A(4, 2)$, $B(1, 2)$ and $C(1, 4)$. The triangle is reflected in the line $y = -x$. Determine the coordinates of the image of ΔABC, $A'B'C'$ after it is reflected in the line $y = -x$.

18. PQR is a triangle with coordinates $P(-4, -2)$, $Q(-1, -2)$ and $R(-1, -4)$. The triangle is reflected in the line which passes through the point $K(3, -3)$ and $L(-4, 4)$ to produce $\Delta P'Q'R'$.

(a) Write down the coordinates of P', Q' and R'.

(b) State the single transformation that maps ΔPQR onto $\Delta P'Q'R'$.

19. PQR is a triangle with vertices $P(-7, -3)$, $Q(-5, -5)$ and $R(-3, -2)$. The triangle is reflected in the line which passes through the points $A(1, -1)$ and $B(-3, 3)$.

(a) Write down the coordinates of P', Q' and R'.

(b) State the single transformation that maps ΔPQR onto $\Delta P'Q'R$.

20. Quadrilateral $ABCD$ has vertices $A(2, 1)$, $B(6, 1)$, $C(5, 3)$ and $D(4, 3)$. State the coordinates of the image of quadrilateral $ABCD$, quadrilateral $A'B'C'D'$, after it is reflected in the line $y = -x$.

21. Draw ΔABC with vertices $A(4, 2)$, $B(1, 2)$ and $C(1, 4)$ using a scale of 1 cm to represent 1 unit on each axis. $\Delta A'B'C'$ is the image of ΔABC under a reflection in the origin. Draw $\Delta A'B'C'$ and state the coordinates of its vertices.

22. Draw ΔABC where A is $(-5, -2)$, B is $(-3, -4)$, and C is $(-1, -2)$. Draw the image of ΔABC, $\Delta A'B'C'$, after reflection in the origin.

23. ΔPQR has vertices $P(-4, -2)$, $Q(-1, -2)$ and $R(-1, -4)$. Determine the coordinates of the image of ΔPQR, $\Delta P'Q'R'$, after reflection in the origin.

24. The coordinates of ΔPQR are $P(-7, -3)$, $Q(-5, -5)$ and $R(-3, -2)$. Draw the image of ΔPQR, $\Delta P'Q'R'$, after reflection in the origin.

25. A quadrilateral $ABCD$ is such that A is the point $(2, 1)$, B is the point $(6, 1)$, C is the point $(5, 3)$ and D is the point $(4, 3)$. Determine the coordinates of the image of quadrilateral $ABCD$, quadrilateral $A'B'C'D'$, under a reflection in the origin.

26. Under a reflection $\Delta PQR \rightarrow \Delta P'Q'R'$. Given that the vertices of ΔPQR are $P(0, 3)$, $Q(3.5, 4.5)$ and $R(5, 6)$, determine the vertices P', Q' and R' of $\Delta P'Q'R'$ under a reflection in the line $x = 2$.

27. Under a reflection $\Delta PQR \rightarrow \Delta P'Q'R'$. Given that the vertices of ΔPQR are $P(0, 3)$, $Q(3.5, 4.5)$ and $R(5, 6)$, determine the vertices P', Q' and R' of $\Delta P'Q'R'$ under a reflection in the line $x = -1$.

28. ΔABC has vertices $A(-4, -2)$, $B(-1, -2)$ and $C(-1, -4)$. State the vertices of the image of ΔABC, $\Delta A'B'C'$, under reflection in the line $x = 1$.

29. ΔABC has coordinates $(-7, -3)$, $(-5, -5)$ and $(-3, -2)$, respectively. Determine the vertices of the image of ΔABC, $\Delta A'B'C'$, under a reflection in the line $x = -3$.

30. ΔABC has vertices $A(4, 2)$, $B(1, 2)$ and $C(1, 4)$. Find the vertices of the image of ΔABC, $\Delta A'B'C'$, under a reflection in the line $x = -1$.

31. Draw ΔABC with vertices $A(-5, -2)$, $B(-3, -4)$ and $C(-1, -2)$. Determine the vertices of the image of ΔABC, $\Delta A'B'C'$, under a reflection in the line $x = -1$.

32. *PQRS* is a quadrilateral with vertices *P*(2, 1), *Q*(6, 1), *R*(5, 3) and *S*(4, 3). Under reflection in the line $x = -3.5$, the image of *PQRS* is $P'Q'R'S'$. Illustrate the transformation on graph paper.

33. Under reflection $\Delta PQR \rightarrow \Delta P'Q'R'$. Given that the vertices of ΔPQR are *P*(0, 3), *Q*(3.5, 4.5) and *R*(5, 6), determine the vertices P', Q' and R' of $\Delta P'Q'R'$ under a reflection in the line $y = 2$.

34. Under a reflection $\Delta PQR \rightarrow \Delta P'Q'R'$. Given that the vertices of ΔPQR are *P*(0, 3), *Q*(3.5, 4.5) and *R*(5, 6), determine the vertices P', Q' and R' of $\Delta P'Q'R'$ under a reflection in the line $y = -1$.

35. Draw ΔABC with vertices $A(-5, -2)$, $B(-3, -4)$, and $C(-1, -2)$. Find the vertices of the image of ΔABC, $\Delta A'B'C'$, under a reflection in the line $y = 2$.

36. Triangle *PQR* with vertices *P*(1, 2), *Q*(3, 5) and *R*(1, 4) is reflected in the line $y = -2$. The image of ΔPQR is $\Delta P'Q'R'$. Determine the coordinates of P', Q' and R'. Illustrate the transformation on graph paper.

37. *PQRS* is a quadrilateral with vertices *P*(2, 1), *Q*(6, 1), *R*(5, 3) and *S*(4, 3). Under reflection in the line $y = -3.5$, the image of *PQRS* is $P'Q'R'S'$. Find the coordinates of P', Q', R' and S'. Illustrate the transformation on graph paper.

38. Draw axes for *x* and *y* from -3 to 6, using 1 cm to represent 1 unit. *P* is the point $(-1, -2)$ and its image P' is the point (5, 3). Construct the mirror line so that the line PP' intersects it at *N*. State the coordinates of *N*.

39. ΔABC is mapped onto $\Delta A'B'C'$ with vertices $A'(-5, 6)$, $B'(-3, 3)$ and $C'(0, 1)$ under a reflection in the *x*-axis. State the coordinates of the vertices of ΔABC?

40. ΔABC is mapped onto $\Delta A'B'C'$ with vertices $A'(-3, -2)$, $B'(-4, -3)$ and $C'(-5, -1)$ under a reflection in the *y*-axis. Determine the coordinates of the vertices of ΔABC.

41. The points $A'(-3, 5)$, $B'(-4, 2)$ and $C'(-5, 1)$ are vertices of a $\Delta A'B'C'$ under a reflection in the line $y = x$. Find the coordinates of the vertices of ΔABC.

42. $A'B'C'$ is a triangle with coordinates $(-2, -5)$, $(-3, -4)$ and $(-6, -3)$, respectively, under a reflection in the line $y = -x$. State the coordinates of ΔABC.

43. The vertices of $\Delta P'Q'R'$ under a reflection in the origin are $P'(-3, 4)$, $Q'(-2, 5)$ and $R'(-1, 3)$. Determine the vertices of ΔPQR.

44. Quadrilateral *PQRS* is mapped onto quadrilateral $P'Q'R'S'$ with vertices $P'(-8, -4)$, $Q'(-10, -5)$, $R'(-11, -7)$ and $S'(-12, -9)$ under a reflection in the line $x = -2.5$. Determine the coordinates of the vertices of quadrilateral *PQRS*.

45. Quadrilateral *PQRS* is mapped onto quadrilateral $P'Q'R'S'$ with vertices $P(3, -5)$, $Q(5, -6)$, $R(6, -7)$ and $S(3, -8)$ under a reflection in the line $y = -2.5$. Determine the vertices of quadrilateral *PQRS*.

Rotations

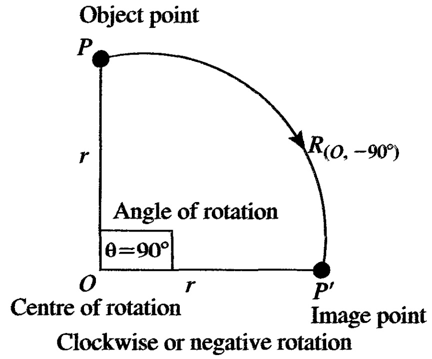
A *rotation* is a *transformation* in which *every point turns through* the *same angle about* the *same centre* in the *same direction*. In a *plane, rotation* is *about* a *single point* called the *centre of rotation*. The *centre of rotation* is the *only point* which *does not change its position* after the *rotation*. That is, it is the *one point* that is *invariant*. A *rotation* can either be *clockwise* or *anti-clockwise*.

$-\theta°$ means an *angle* of $\theta°$ *clockwise*, while $+\theta°$ means an *angle* of $\theta°$ *anti-clockwise*. For example: $-90°$ means an *angle* of 90° *clockwise*. $+90°$ means an *angle* of 90° *anti-clockwise*.

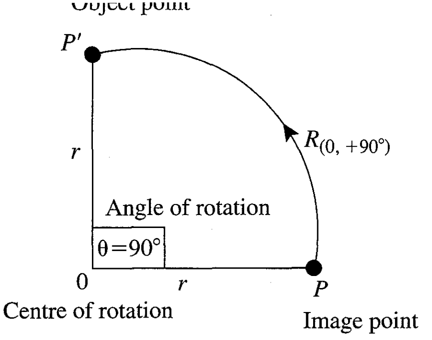
Also $-\theta° = +(360° - \theta°) \sim$ *anti-clockwise* and
$+\theta° = -(360° - \theta°) \sim$ *clockwise*.

For example: $-90° = +(360° - 90°) = +270°$.
$+90° = -(360° - 90°) = -270°$.

(a)



Clockwise or negative rotation



Anti-clockwise or positive rotation

Fig. 10.95 Rotations

Fig. 10.95 (a) shows the *movement* of an *object point P*, 90° to the *right about centre O*. This represents a *clockwise* or *negative rotation of P about centre O*, through 90°, or a *rotation of P* through −90° about *centre O*.

Fig. 10.95 (b) shows the *movement* of an *object point P*, 90° to the *left about centre O*. This represents an *anti-clockwise* or *positive rotation of P about centre O*, through 90°, or a *rotation of P* through +90° about *centre O*.

In Fig. 10.95 it can be *seen* that:

(i) *O is the centre of rotation*, the *centre of rotation* is the *point about* which *rotation takes place*.

(ii) $OP = OP' = r$, where r is the *radius* of a *circle centre O*. *P* and *P'* are *points* on the *circumference* of the *circle*. So a *rotation* is a *transformation* in which a *point changes its position* on the *circumference* of a *circle*.

(iii) *Rotation* takes place *through* an *angle* of $POP' = 90°$. 90° describes the *amount of rotation* that takes place. So the *angle* is referred to as the *angle of rotation*. The *angle through* which *rotation takes place* is called the *angle of rotation*. It describes the *amount of rotation* that takes place.

The *rotation* in Fig. 10.95 (a) can be *denoted* by:

$$R_{(0,\,-90^\circ)}: P \rightarrow P' \text{ or } R_{(0,\,-90^\circ)}(P) = P' \text{ or}$$

$$P \xrightarrow{R_{(0,\,-90^\circ)}} P',$$

where each of the *rotations* given above *means*:
P' is the *image of P* under a *clockwise rotation about* 0 *through* 90°, or *P is mapped onto P'* under a *clockwise rotation about* 0 *through* 90°.

The *rotation* in Fig. 10.95 (b) can be *denoted by*:

$$R_{(0,\,+90^\circ)}: P \rightarrow P' \text{ or } R_{(0,\,+90^\circ)}(P) = P' \text{ or}$$

$$P \xrightarrow{R_{(0,\,+90^\circ)}} P',$$

where each of the *rotations* given above *means*:
P' is the *image of P* under an *anti-clockwise rotation about* 0 *through* 90°, or *P is mapped onto P'* under an *anti-clockwise rotation about* 0 *through* 90°.

Properties of Rotation

Here we *investigate* the *properties of a figure* which are *invariant* under a *rotation*.

Fig. 10.96 on the next page, shows the *image P'Q'R'S'* of *trapezium PQRS* under an *anti-clockwise rotation* of 60° *about* 0. It also shows the *image P''Q''R''S''* of *trapezium P'Q'R'S'* under an *anti-clockwise rotation* of 30° *about* 0.

Hence *trapezium PQRS* is *mapped onto trapezium P''Q''R''S''* under an *anti-clockwise rotation* of 90° *about* 0.

Thus

$R_{(0,\,+60^\circ)}$: *Trapezium PQRS* → *Trapezium P'Q'R'S'*

and

$R_{(0,\,+30^\circ)}$: *Trapezium P'Q'R'S'* → *Trapezium P''Q''R''S''*.

Hence

$R_{(0,\,+90^\circ)}$: *Trapezium PQRS* → *Trapezium P''Q''R''S''*.

From Fig. 10.96 it can be *seen* that:

(i) $OP = OP' = OP'' = 5$ cm.
$OQ = OQ' = OQ'' = 10$ cm.
$OR = OR' = OR'' = 8.2$ cm.
$OS = OS' = OS'' = 5.4$ cm.

So, if the *object* is 5 cm from the *centre of rotation*, its *image* is also 5 cm from the *centre of rotation*.

(ii) $P\hat{O}P' = Q\hat{O}Q' = R\hat{O}R' = S\hat{O}S' = 60°$.
$P\hat{O}P'' = Q'\hat{O}Q'' = R'\hat{O}R'' = S'\hat{O}S'' = 30°$.
$P\hat{O}P'' = Q\hat{O}Q'' = R\hat{O}R'' = S\hat{O}S'' = 90°$.

Hence we can *conclude* that:

(i) *Under a rotation, the distance of an object and its image from the centre of rotation are equal.*

(ii) (a) *The angle of rotation is the angle formed by joining the object point, the centre of rotation and the corresponding image point.*

same value) *for all points under a given rotation about a common centre.*

From Fig. 10.96 it can be *seen* that:

$PQ = P'Q' = P''Q'' = 5$ cm.
$QR = Q'R' = Q''R'' = 2.8$ cm.
$RS = R'S' = R''S'' = 3$ cm.
$PS = P'S' = P''S'' = 2$ cm.

So *corresponding sides* are *equal* under the *rotations*.

Hence we can *conclude* that:

(i) *Rotations preserve the distance between points. Rotations preserve lengths.*

In Fig. 10.96 it can be *observed* that:

$\hat{P} = \hat{P}' = \hat{P}'' = 90°$, $\hat{Q} = \hat{Q}' = \hat{Q}'' = 45°$,
$\hat{R} = \hat{R}' = \hat{R}'' = 135°$ and $\hat{S} = \hat{S}' = \hat{S}'' = 90°$.

rotations.

Hence we can *conclude* that:

(ii) *Rotations preserve angles.*

From Fig. 10.96 it can be *seen* that:

$PQ \,//\, SR$, $P'Q' \,//\, S'R'$ and $P''Q'' \,//\, S''R''$.

So *parallel sides remain parallel* under the *rotations*.

Hence we can *conclude* that:

(iii) *Rotations preserve parallelism*

In Fig. 10.96 it can be *observed* that:

The *area of trapezium PQRS*
= The *area of trapezium P'Q'R'S'*
= The *area of trapezium P''Q''R''S''*
= 8 cm².

So *area remains* the *same* under the *rotations*.

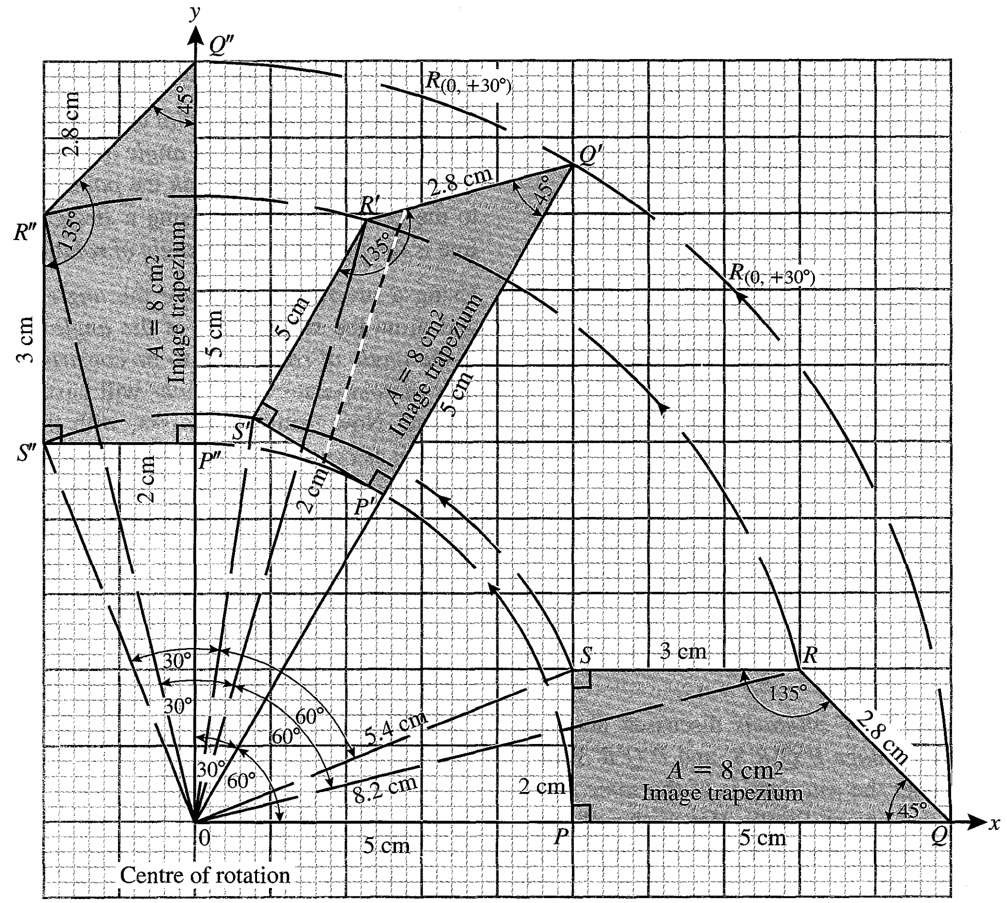


Fig. 10.96 Rotation of a plane figure about the origin

Hence we can *conclude* that:

(iv) *Rotations preserve area.*

From Fig. 10.96 it can be seen that:

$PQ:SR = 5:3$, $P'Q':S'R' = 5:3$

and $P''Q'':S''R'' = 5:3$.

$PS:QR = 1:1.4$, $P'S':Q'R' = 1:1.4$

and $P''S'':Q''R'' = 1:1.4$

So *ratios* are *unchanged* under the *rotations.*

Hence we can *conclude* that:

(v) *Rotations preserve ratios.*

In Fig. 10.96 it can be *observed* that:

The *order* of the *vertices* in the *original object* is *PQRS*. The *order* of the *vertices* in the *images* is $P'Q'R'S'$ and $P''Q''R''S''$. So the *order* of the *vertices remains* the *same.*

Hence we can *conclude* that:

(vi) *Rotations preserve the order of points.*

From Fig. 10.96 it can be *seen* that:

The *orientation* (or *sense*) of the *object* is *anti-clockwise.*

The *orientation* (or *sense*) of the *images* is *anti-clockwise.*

So the *orientation* (or *sense*) is *unchanged* under the *rotations.*

Hence we can *conclude* that:

(vii) *Rotations preserve the orientation (or sense) of a figure.*

We can *summarize* the *properties of a rotation* in a *table* as shown below.

Table 10.4

Invariant property	Length	Angle	Parallelism	Area	Ratio	Order of points	Orientation of sense
Invariant under rotation	Yes	Yes	Yes	Yes	Yes	Yes	Yes

From the *properties of rotations* discussed above, it follows that the *image* $P'Q'R'S'$ and $P''Q''R''S''$ of figure *PQRS* under the *rotations* is of the *same size* and *shape* as the *object PQRS*. Hence the *object PQRS* and the *images* $P'Q'R'S'$ and $P''Q''R''S''$ are said to be *congruent.*

Thus:

The *trapezium PQRS* ≡ The *trapezium* $P'Q'R'S'$

≡ The *trapezium* $P''Q''R''S''$

Hence *rotation* is a *congruency transformation.*

We can *summarize* the *properties of rotations* as follows:

(i) *All rotations have a centre of rotation and an angle of rotation.*

(ii) *The distance of an object and its image from the centre of rotation are equal.*

(iii) *All points in a plane under the same rotation move through the same angle in the same direction, except the centre of rotation which is the only fixed (or invariant) point.*

(iv) *Rotation is a congruency transformation.*

Image of a Point Under a Rotation

A *point* and its *image lie* on an *arc* (or on the *circumference*) of a *circle* drawn from the *point to its image* and *subtending* an *angle equal* to the *angle of rotation* at the *centre of rotation.* (i.e., the *centre of a circle*).

In *finding* the *image of a point* under a *rotation*, note the *point to be rotated*, the *angle of rotation* and the *centre of rotation.* Then *join* the *point* to be *rotated* to the *centre of rotation* using a *straight line.* This line is the *first arm* of the *angle of rotation.*

Using a *protractor*, *measure* the *angle of rotation* and draw the *second arm* of the *angle of rotation.* If the *angle of rotation* is to be *constructed*, then a *pair of compasses* and a *ruler* will have to be used instead. Now using *compasses*, with the *centre of rotation* as *centre*, and a *radius equal to the distance of the point from the centre of rotation*, draw an *arc* from the *point* to *meet* the *second arm* of the *angle of rotation.*

The *point* where the *arc* and the *second arm* of the *angle of rotation meet* is the *required image.*

It should be stated that the *centre of rotation* is *not necessarily* the *origin* $O(0, 0)$. The *centre of rotation* can be *any point* on the *Cartesian plane.*

Example 12

Find the image of the point P under $R_{(c,\,-105^\circ)}$, *if P is 3.5 cm from the centre of rotation, C. State the size of angle CPP'.*

Solution

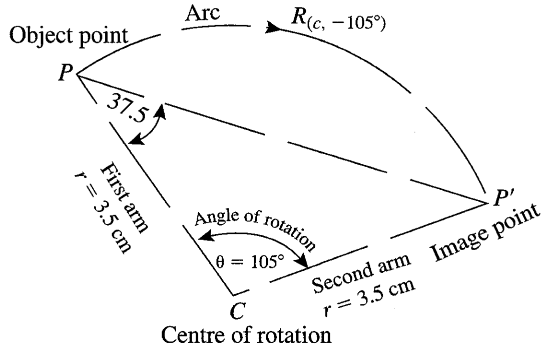


Fig. 10.97 Image of a point under a rotation

In Fig. 10.97, the *image* of the *point P*, P' under $R_{(c,\,-105^\circ)}$ and *radius* $r = 3.5$ cm was obtained from the *drawing*.

Note that: $CP = CP' = r = 3.5$ cm.

By *measurement*:

Angle $PCP' = 105^\circ$ *clockwise*.

The *size of angle* $CPP' = 37.5^\circ$

$\triangle PCP$ is isosceles, so $\angle CPP'$ can also be calculated.

The *image* of a *line segment* under a *rotation* is *found* by *first finding* the *images of the endpoints* of the *line segment*, then *joining* these *point images* by a *straight line*.

Image Under a Rotation about the Origin

In this section we will considered the rotation of a plane figure about the origin.

ANTI-CLOCKWISE ROTATION THROUGH 90°

From Fig. 10.98, it can be *seen* that under an *anti-clockwise rotation of* 90° about the *origin*:
$A(2, 0) \to A'(0, 2)$, $B(5, 0) \to B'(0, 5)$ and $C(7, 2) \to C'(-2, 7)$.

So in general, a *point* with *coordinates* (x, y) is *mapped onto* a *point* with *coordinates* $(-y, x)$ under an *anti-clockwise rotation of* 90° about the *origin*.

Hence $R_{(0,\,+90^\circ)}: (x, y) \to (-y, x)$.

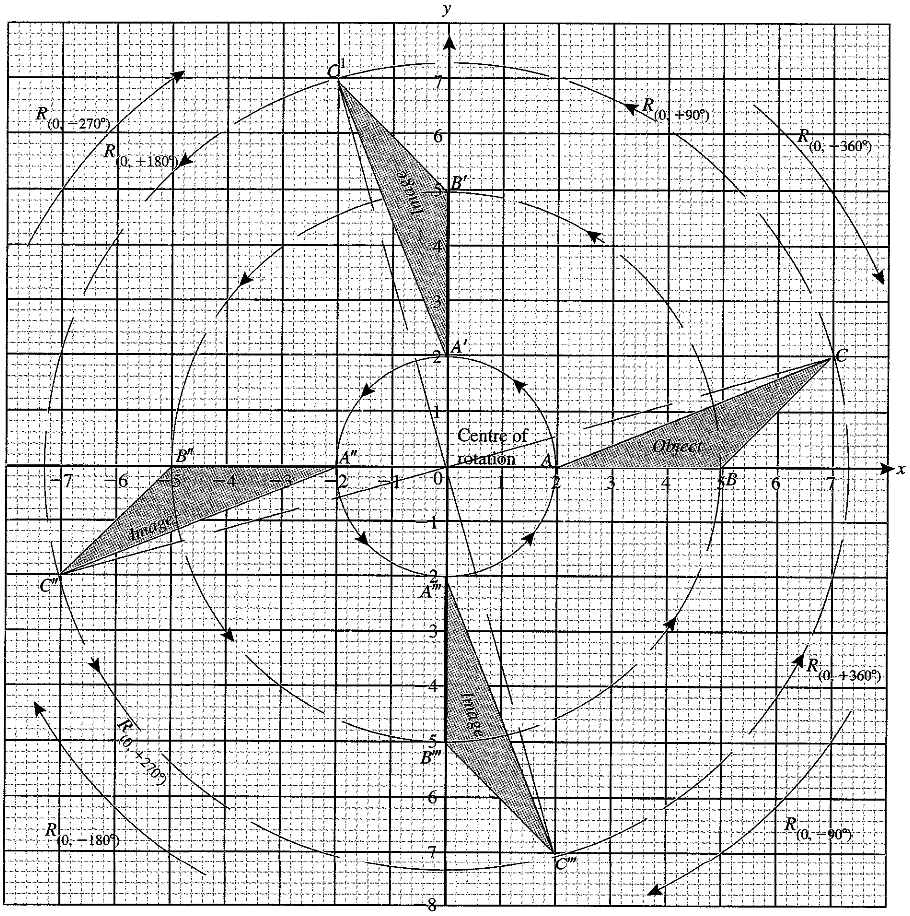


Fig. 10.98 Rotation of a plane figure about the origin

Example 13

ABC is a triangle with vertices (−3, −2), (−5, −3) and (−4, −7), respectively. Determine the image of ABC under a anti-clockwise rotation of 90° about the origin.

Solution

Now $R_{(0,\,+90°)}: (x, y) \rightarrow (-y, x)$.

So $R_{(0,\,+90°)}: A(-3, -2) \rightarrow A'(2, -3)$,

$R_{(0,\,+90°)}: B(-5, -3) \rightarrow B'(3, -5)$

and $R_{(0,\,+90°)}: C(-4, -7) \rightarrow C'(7, -4)$.

Hence the *image* of $\triangle ABC$ is $\triangle A'B'C'$ with *coordinates* $A'(2, -3)$, $B'(3, -5)$ and $C'(7, -4)$.

ANTI-CLOCKWISE ROTATION THROUGH 180°

From Fig. 10.98, it can be *seen* that under an *anti-clockwise rotation of* 180° about the *origin*:

$A(2, 0) \rightarrow A''(-2, 0)$, $B(5, 0)$, $\rightarrow B''(-5, 0)$ and $C(7, 2) \rightarrow C''(-7, -2)$.

So in general, a *point* with *coordinates* (x, y) is *mapped onto* a *point* with *coordinates* $(-x, -y)$ under an *anti-clockwise rotation of* 180° about the *origin*.

Hence $R_{(0,\,+180°)}: (x, y) \rightarrow (-x, -y)$.

Example 14

ABC is a plane figure with coordinates (−5, 3), (−2, 1) and (−1, 4), respectively. Determine the image of ABC under an anti-clockwise rotation of 180° about the origin.

Solution

Now $R_{(0,\,+180°)}: (x, y) \rightarrow (-x, -y)$.

So $R_{(0,\,+180°)}: A(-5, 3) \rightarrow A'(5, -3)$,

$R_{(0,\,+180°)}: B(-2, 1) \rightarrow B'(2, -1)$

and $R_{(0,\,+180°)}: C(-1, 4) \rightarrow C'(1, -4)$.

Hence the *image* of figure *ABC* is *figure A'B'C'* with *coordinates* $A'(5, -3)$, $B'(2, -1)$ and $C'(1, -4)$.

From Fig. 10.98, it can be *seen* that under an *anti-clockwise rotation of* 270° about the *origin*:

$A(2, 0) \rightarrow A'''(0, -2)$, $B(5, 0) \rightarrow B'''(0, -5)$ and $C(7, 2) \rightarrow C'''(2, -7)$.

So in general, a *point* with *coordinates* (x, y) is *mapped onto* a *point* with *coordinates* $(y, -x)$ under an *anti-clockwise rotation of* 270° about the *origin*.

Hence $R_{(0,\,+270°)}: (x, y) \rightarrow (y, -x)$.

Example 15

Determine the image of the line segment LM with end-points (3, −2) and (7, −4), respectively under an anti-clockwise rotation of 270° about the origin.

Solution

Now $R_{(0,\,+270°)}: (x, y) \rightarrow (y, -x)$.

So $R_{(0,\,+270°)}: L(3, -2) \rightarrow L'(-2, -3)$

and $R_{(0,\,+270°)}: M(7, -4) \rightarrow M'(-4, -7)$.

Hence the *image* of the line segment *LM* is the *line segment L'M'* with *coordinates* $L'(-2, -3)$ and $M'(-4, -7)$.

ANTI-CLOCKWISE ROTATION THROUGH 360°

From Fig. 10.98, it can be *seen* that under an *anti-clockwise rotation of* 360° about the *origin*:

$A(2, 0) \rightarrow A(2, 0)$, $B(5, 0) \rightarrow B(5, 0)$ and $C(7, 2) \rightarrow C(7, 2)$.

So in general, a *point* with *coordinates* (x, y) is *mapped onto* a *point* with *coordinates* (x, y) under an *anti-clockwise rotation of* 360° about the *origin*.

Hence $R_{(0,\,+360°)}: (x, y) \rightarrow (x, y)$.

Example 16

Determine the image of quadrilateral WXYZ with vertices W(2, 1), X(5, 1), Y(3, 5) and Z(0, 4) under an anti-clockwise rotation of 360° about the origin.

tion

Now $R_{(0,\,+360°)}$: $(x, y) \rightarrow (x, y)$.

So $R_{(0,\,+360°)}$: $W(2, 1) \rightarrow W'(2, 1)$.
$R_{(0,\,+360°)}$: $X(5, 1) \rightarrow X'(5, 1)$,
$R_{(0,\,+360°)}$: $Y(3, 5) \rightarrow Y'(3, 5)$
and $R_{(0,\,+360°)}$: $Z(0, 4) \rightarrow Z'(0, 4)$.

Hence the *image* of quadrilateral *WXYZ* is *quadrilateral W′X′Y′Z′* with the same *coordinates* as *WXYZ*.

CLOCKWISE ROTATIONS THROUGH 90°

From Fig. 10.98, it can be seen that a *clockwise rotation of* 90° about the *origin is equivalent* to an *anti-clockwise rotation of* 270° about the *origin.*

Hence $R_{(0,\,-90°)} = R_{(0,\,+270°)}$.

Thus $R_{(0,\,-90°)}$: $(x, y) \rightarrow (y, -x)$.

CLOCKWISE ROTATION THROUGH 180°

From Fig. 10.98, it can be *seen* that a *clockwise rotation of* 180° about the *origin* is *equivalent* to an *anti-clockwise rotation of* 180° about the *origin.*

Hence $R_{(0,\,-180°)} = R_{(0,\,+180°)}$.

Thus $R_{(0,\,-180°)}$: $(x, y) \rightarrow (-x, -y)$.

CLOCKWISE ROTATION THROUGH 270°

From Fig. 10.98, it can be *seen* that a *clockwise rotation of* 270° about the *origin* is *equivalent* to an *anti-clockwise rotation of* 90° about the *origin.*

Hence $R_{(0,\,-270°)} = R_{(0,\,+90°)}$.

Thus $R_{(0,\,-270°)}$: $(x, y) \rightarrow (-y, x)$.

CLOCKWISE ROTATION THROUGH 360°

From Fig. 10.98, it can be *seen* that a *clockwise rotation of* 360° about the *origin* is *equivalent* to an *anti-clockwise rotation of* 360° about the *origin.*

Hence $R_{(0,\,-360°)} = R_{(0,\,+360°)}$.

Thus $R_{(0,\,-360°)}$: $(x, y) \rightarrow (x, y)$.

Equivalent Rotations

From the *examples* above, it can be *seen* that:

(i) If a *rotation* is $+\theta°$ about a *point*, then the *equivalent rotation* is $-(360° - \theta°)$ about the *same point.*

(ii) If a *rotation* is $-\theta°$ about a *point*, then the *equivalent rotation* is $+(360° - \theta°)$ about the *same point.*

Inverse Rotations

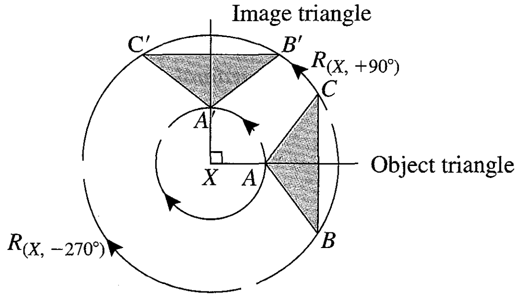


Fig. 10.99 Rotation

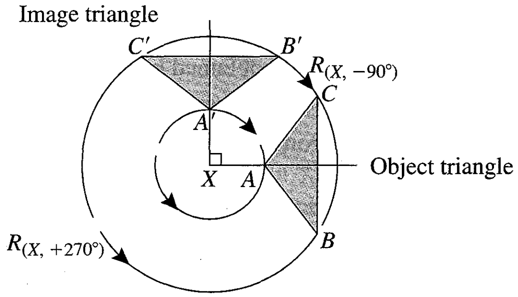


Fig. 10.100 Inverse rotation

From Figs. 10.99 and 10.100, it can be seen that:
$\triangle ABC$ is *mapped onto* $\triangle A'B'C'$ under an *anti-clockwise rotation* through 90° *about X* (i.e. $R_{(X,\,+90°)}$).

$\triangle A'B'C'$ is *mapped onto* $\triangle ABC$ under a *clockwise rotation* through 90° *about X* (i.e. $R_{(X,\,-90°)}$). This is the *inverse rotation of* $R_{(X,\,+90°)}$.

$\triangle ABC$ is also *mapped onto* $\triangle A'B'C'$ under a *clockwise rotation* through 270° *about X* (i.e. $R_{(X,\,-270°)}$).

$\triangle A'B'C'$ is also *mapped onto* $\triangle ABC$ under an *anti-clockwise rotation* through 270° about *X* (i.e. $R_{(X,\,+270°)}$). This is the *inverse rotation of* $R_{(X,\,-270°)}$.

Hence we can *conclude* that:

(i) The *inverse* of an *anti-clockwise rotation* is a *clockwise rotation* about the *same centre* and through the *same angle of rotation.*

That is, the *inverse of* $R_{(X,\,+\theta°)}$ *is* $R_{(X,\,-\theta°)}$.

(ii) The *inverse* of a *clockwise rotation* is an *anti-clockwise rotation* about the *same centre* and through the *same angle of rotation.*

That is, the *inverse of* $R_{(X, -\theta°)}$ *is* $R_{(X, +\theta°)}$.

The *inverse rotation* is used to *find the object* that gives rise to an *image under a rotation.*

Example 17

$\triangle A'B'C'$ with vertices $A'(4, 10)$, $B'(7, 11)$ and $C'(5, 7)$ is the image of $\triangle ABC$ under an anti-clockwise rotation of 70° about the point $X(2.5, 3)$. Determine the coordinates of $\triangle ABC$.

Solution

In Fig. 10.101, the image $\triangle A'B'C'$ was obtained using its *vertices. The object* $\triangle ABC$ was then obtained using the *inverse rotation*, that is, $R_{(X, -70°)}$, where the *centre of rotation X* has *coordinates* $X(2.5, 3)$.

The *rotation* under which $\triangle ABC \rightarrow A'B'C'$ is $R_{[(2.5, 3), +70°]}$.

Thus, the *object* $\triangle ABC$ is obtained by performing the *inverse rotation*, $R_{[(2.5, 3), -70°]}$ on $\triangle A'B'C'$.

Draw a straight line from X to C' using a ruler. Open the compasses from X to C' and draw an arc in a clockwise direction. Using a protractor and point X as centre, measure an angle of 70° clockwise, from XC'. Draw a straight line from X passing through the point indicating the 70° angle, to intersect the arc at C.

Repeat the process in order to obtain point B and point C, then complete the triangle ABC.

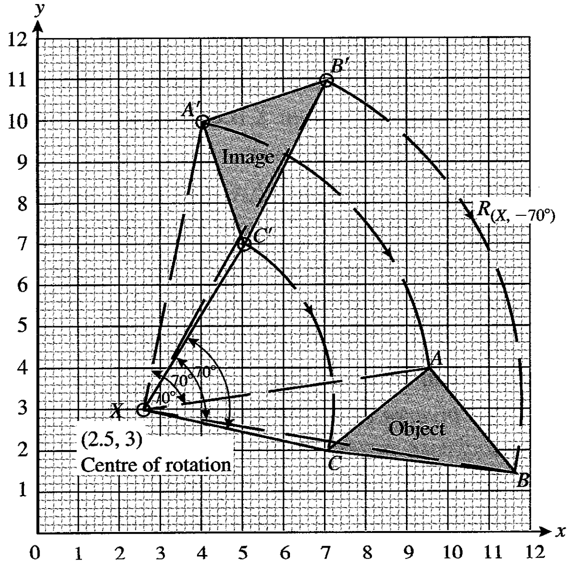


Fig. 10.101 Inverse rotation

Hence the *coordinates* of $\triangle ABC$ are: $A(9.5, 4)$, $B(11.5, 1.5)$ and $C(7, 2)$.

Finding the Centre of Rotation and the Angle of Rotation

The *centre of rotation* and the *angle of rotation* can be found by a *suitable construction.* This *method* involves *finding* the *centre of rotation* first and then *measuring off* the *angle of rotation.*

When the *object* under *rotation* is a *plane figure*, the *centre of rotation lies* on the *perpendicular bisector* of the *line segment joining* a *point on the object* and the *corresponding point on the image. Two perpendicular bisectors* are *sufficient* to find the *centre of a rotation.*

So in order to find the *centre of rotation* when the *object* under a *rotation* is a *plane figure* with *line segments* as *sides*:

(i) *Choose two suitable vertices* of the *object*, e.g. A and B.

(ii) *Identify* the *corresponding vertices* of the *image* onto which the *vertices of the object* are *mapped* e.g. A' and B'.

(iii) *Draw line segments* to *join each pair of object* and its *corresponding image*, e.g. A to A' and B to B'.

(iv) *Construct* the *perpendicular bisector of each line segment* to *intersect* at a *point*, e.g. the perpendicular bisectors of AA' and BB'. The *single point* where these *perpendicular bisectors intersect* is the *centre of rotation X.* The *coordinates* of the *centre of rotation* can then be *read off* from the *axes.*

(v) *Line segments* are then *drawn* from the *object vertex* and its *corresponding image vertex* to the *centre of rotation* to *complete an angle*, e.g. $A\hat{X}A'$ or $B\hat{X}B'$.

The *angle formed* is the *angle of rotation*, which can then be *measured* using a *protractor.*

The following diagram shows the construction for obtaining the centre of rotation and the angle of rotation.

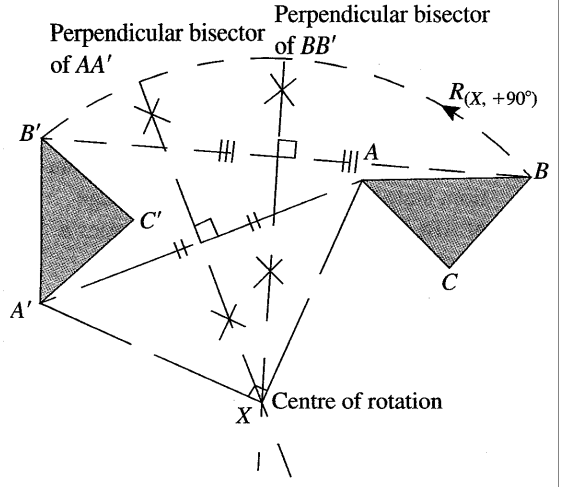


Fig. 10.102 Finding the centre of rotation

The *centre of rotation* was obtained by *constructing* the *perpendicular bisectors* of the *line segments* AA' and BB' to intersect at point X.

The *angle of rotation*, $A\hat{X}A' = 90°$ *anti-clockwise*.

Exercise 10f

1. Use suitable notations to represent each of the following statements:
   (a) P is mapped onto P' under a positive rotation of 225° about O.
   (b) Line segment $L'M'$ is the image of line segment LM under a rotation through $-90°$ about C.
   (c) Under a negative rotation through 33° about A, $\triangle ABC \rightarrow \triangle A'B'C'$.

2. R is a rotation defined as follows:
   $R_{(P, -21°)}$: $\triangle XYZ \rightarrow \triangle X'Y'Z'$.
   (a) Name the centre of rotation.
   (b) Name the object and the image under the rotation.
   (c) State the magnitude of rotation.

3. P' is the image of P under an anti-clockwise rotation through 35° about C.
   (a) Name the centre of rotation.
   (b) State a relationship that exists between PC and $P'C$.
   (c) Describe the shape of triangle PCP'.

4. Under a clockwise rotation of 60° about C, L is mapped onto L' with $CL = 5$ units.
   (i) State the amount of rotation.
   (ii) State the size of $L\hat{C}L'$.
   (iii) State the size of $L\hat{L'}C$.

5. The transformation R is such that $R_{(X, +\theta°)}$: $C \rightarrow C'$.
   If $C\hat{C'}X = 85°$, $C'\hat{C}X$ and θ.

6. Find the image of the point P under $R_{[C, -45°]}$, if P is 6 cm from the centre of rotation.
   State the size of angle CPP'.

7. Determine the image of the point A under $R_{(X, +130°)}$, if A is 7.5 cm from the centre of rotation.
   State the magnitude of angle $X\hat{A'}A$.

8. LM is any line segment of length 4 cm. Find its image under $R_{(C, -120°)}$, where C is any suitable point.
   State the size of the angle CLL'.

9. PQ is a line segment of length 5.5 cm. Determine its image under $R_{(X, +125°)}$, where X is any suitable point.
   State the magnitude of angle $X\hat{P'}P$.

10. Draw any triangle ABC. Find its image under $R_{(C, -75°)}$, where C is any convenient point. Measure and state the size of angle $C\hat{A}A'$.

11. Draw any triangle PQR. Find its image under $R_{(x, 94°)}$, where X is any convenient point. Measure and state the size of angle $X\hat{P'}P$.

12. ABC is a triangle with vertices (3, 2), (5, 4) and (4, 7), respectively. Find the image of $\triangle ABC$ under an anti-clockwise rotation of 90° about the origin.

13. $\triangle PQR$ has coordinates $P(-3, 4)$, $Q(-5, 3)$ and $R(-4, 5)$. Determine the image of $\triangle PQR$ under an anti-clockwise rotation of 90° about the origin.

14. KLM is a triangle with coordinates $(-3, -5)$, $(-4, -3)$ and $(-5, -6)$, respectively. Determine the image of triangle KLM under an anti-clockwise rotation of 180° about the origin.

15. ABC is a plane figure with vertices $(-5, 4)$, $(-3, 1)$ and $(-1, 6)$, respectively. Find the image of ABC under an anti-clockwise rotation of 180° about the origin.

16. $\triangle PQR$ has vertices $P(-3, -2)$, $Q(-5, -1)$ and $R(-4, -5)$. Find the image of $\triangle PQR$ under an anti-clockwise rotation of 270° about the origin.

17. Determine the image of the line segment LM with endpoints $(5, -3)$ and $(7, -5)$ respectively under an anti-clockwise rotation of 270° about the origin.

18. Find the image of quadrilateral $ABCD$ with vertices $A(-3, -2)$, $B(-5, -3)$, $C(-4, -4)$ and $D(0, -4)$ under an anti-clockwise rotation of 360° about the origin.

19. Determine the image of quadrilateral $WXYZ$ with coordinates $W(2, 1)$, $X(4, 1)$, $Y(3.5, 4)$ and $Z(5, 3)$ under an anti-clockwise rotation of 360° about the origin.

20. ABC is a triangle with vertices $(3, 2)$, $(5, 3)$ and $(4, 5)$ respectively. Find the image of $\triangle ABC$ under a clockwise rotation of 90° about the origin.

21. $\triangle PQR$ has coordinates $P(-3, 2)$, $Q(-4, 3)$ and $R(-5, 5)$. Determine the image of $\triangle PQR$ under a clockwise rotation of 90° about the origin.

22. KLM is a triangle with coordinates $(-3, -5)$, $(-4, -5)$ and $(-5, -6)$ respectively.

Determine the image of triangle KLM under a clockwise rotation of 180° about the origin.

23. ABC is a plane figure with vertices $(-5, 3)$, $(-3, 2)$ and $(-2, 5)$ respectively. Find the image of ABC under a clockwise rotation of 180° about the origin.

24. $\triangle PQR$ has vertices $P(-3, -1)$, $Q(-4, -2)$ and $R(-4, -5)$. Find the image of $\triangle PQR$ under a clockwise rotation of 270° about the origin.

25. Determine the image of the line segment LM with end-points $(4, -3)$ and $(6, -5)$ respectively under a clockwise rotation of 270° about the origin.

26. Find the image of quadrilateral $ABCD$ with vertices $A(3, -2)$, $B(4, -4)$, $C(5, -3)$ and $D(0, -3)$ under a clockwise rotation of 360° about the origin.

27. Determine the image of quadrilateral $WXYZ$ with coordinates $W(-5, -3.5)$, $X(-3, -3.5)$, $Y(-2, -1.5)$ and $Z(-4, -1.5)$ respectively under a clockwise rotation of 360° about the origin.

28. A quadrilateral $ABCD$ is such that A is the point $(2, 1)$, B is the point $(6, 1)$, C is the point $(5, 3)$ and D is the point $(4, 3)$.

(a) If the origin is the centre of rotation, state the coordinates of the image of $ABCD$, $A'B'C'D'$ after it undergoes an anti-clockwise rotation of 90°.

(b) If the point $(-1, -2)$ is the centre of rotation, state the coordinates of the image of $ABCD$. $A''B''C''D''$ after it undergoes a clockwise rotation of 90°.

29. Given the points $A(2, 1)$, $B(3, 5)$ and $C(0, 4)$ and that R is an anti-clockwise rotation of 180° about the origin and M is a reflection in the y-axis, draw accurate diagrams to show:

(a) the image $A'B'C'$ of ABC under R

(b) the image $A''B''C''$ of $A'B'C'$ under M.

30. A triangle ABC which is represented by the vertices $A(2, 3)$, $B(5, 3)$ and $C(5, 7)$ is reflected in the y-axis. The image, $\triangle A'B'C'$ of $\triangle ABC$ is then rotated through an anti-clockwise angle of 90° about the origin, and is mapped onto $\triangle A''B''C''$.

(a) Find:

  (i) the vertices of $\triangle A'B'C'$

  (ii) the vertices of $\triangle A''B''C''$.

(b) Plot on graph paper, $\triangle ABC$, $\triangle A'B'C'$ and $\triangle A''B''C''$.

31. Given that the points $A(0, 1)$, $B(3, 0)$ and $C(4, 5)$ undergo an anti-clockwise rotation of 90° about the origin, followed by a reflection in the x-axis, draw accurate diagrams to show:

(a) the image $A'B'C'$ of ABC under $R_{90°}$

(b) the image $A''B''C''$ of $A'B'C'$ under M_x.

32. The vertices of a triangle $A(3, -2)$, $B(7, -3)$ and $C(5, -5)$ are reflected in the line O_x and then rotated through an angle of 90° counter-clockwise about the origin. Determine the position of the images of A, B and C.

33. Given that $\mathbf{OA} = \begin{pmatrix}1\\2\end{pmatrix}$, $\mathbf{OB} = \begin{pmatrix}5\\8\end{pmatrix}$, $\mathbf{OC} = \begin{pmatrix}11\\8\end{pmatrix}$ and $\mathbf{OD} = \begin{pmatrix}7\\2\end{pmatrix}$, prove that $ABCD$ is a parallelogram. If $ABCD$ is rotated through 180° about the origin, find the position vectors of A', B', C' and D'.

34. (a) Given that the points $A(-4, 5)$, $B(-1, 3)$ and $C(-2, 7)$ under a reflection in the y-axis, followed by a translation represented by the matrix $T = \begin{pmatrix} 0 \\ -6 \end{pmatrix}$, draw accurate diagrams to show:

  (i) the image of $A'B'C'$ of ABC under the reflection M_y
  (ii) the image $A''B''C''$ of $A'B'C'$ under T.

(b) State the coordinates of the images A', B', C', A'', B'' and C''.

35. $\triangle A'B'C'$ with vertices $A'(-3, -2)$, $B'(-3, -5)$ and $C'(-7, -5)$ is the image of $\triangle ABC$ under an anti-clockwise rotation of 90° about the origin. Determine the vertices of $\triangle ABC$.

36. $P'Q'R'$ is a plane figure with coordinates $(-4, 3)$, $(-3, 1)$ and $(-1, 5)$, respectively. Find the coordinates of the plane figure PQR, if $P'Q'R'$ is the image of PQR under an anti-clockwise rotation of 180° about the origin.

37. The image of the line segment LM under an anti-clockwise rotation of 270° about the origin is the line segment $L'M'$ with endpoints $L'(-3, 5)$ and $M'(-7, 4)$. Determine the endpoints L and M.

38. $K'L'M'$ is a triangle with coordinates $(-2, -5)$, $(-4, -3)$, and $(-5, -7)$, respectively. $K'L'M'$ is the image of KLM under a clockwise rotation of 90° about the origin. Find the vertices of $\triangle KLM$.

39. $P'Q'R'S'$ is a quadrilateral with vertices $P'(-3, 1)$, $Q'(-5, 1)$, $R'(-4, 3)$ and $S'(-1, 6)$. $P'Q'R'S'$ is the image of $PQRS$ under a clockwise rotation of 180° about the origin. Determine the vertices of quadrilateral $PQRS$.

40. $\triangle P'Q'R'$ is the image of $\triangle PQR$ under a clockwise rotation of 270° about the origin. If $P'Q'R'$ has vertices $P'(3, 1)$, $Q'(5, 1)$ and $R'(4, 3)$, find the vertices of $\triangle PQR$.

41. The points $P(4, 3)$ and $Q(1, 1)$ are mapped onto $P'(-2, 1)$ and $Q'(1, 3)$ under a rotation about a centre C.

(a) Determine the coordinates of the centre of rotation C.

(b) State the magnitude of the angle of rotation.

42. The images of $L(1, 1)$ and $M(2, 5)$ under a single transformation P are $L'(-1, 1)$ and $M'(-5, 2)$, respectively.

(a) Describe geometrically the transformation P.

(b) Determine the equation of the straight line $L'M'$.

43. $\triangle ABC \to \triangle A'B'C'$ under a rotation such that $A(5, 3)$, $B(4, 3)$, $C(4, 1) \to A'(-2, -2)$, $B'(-1, -2)$, $C'(-1, 0)$.

(a) Determine the centre of rotation.

(b) State the angle of rotation.

44. $\triangle PQR \to \triangle P'Q'R'$ under a rotation such that $P(7, 6)$, $Q(5, 3)$, $R(4, 1) \to P'(3, -5)$, $Q'(0, -3)$, $R'(-2, -2)$.

(a) Determine the centre of rotation.

(b) State the angle of rotation.

45. The points $A(-5, 0)$ and $B(-9, 0)$ are mapped by a rotation with centre C onto the points $A'(3, 4)$ and $B'(3, 8)$.

(a) Plot the points A, B, A' and B' on graph paper.

(b) State the relation of B and B' to C.

(c) By a suitable construction, find the coordinates of C.

(d) Measure and state the magnitude of the angle of rotation to the nearest degree.

46. The points $A(4, 0)$ and $B(10, 0)$ are mapped by a rotation with centre C onto the points $A'(4, 8)$ and $B'(4, 14)$.

(a) Using a scale to represent 1 unit on both axes, plot the points A, B, A' and B'.

(b) State:

  (i) The relation of A and A' to C.
  (ii) The size of angle BMC, where M is the mid-point of BB'.

(c) By suitable construction find the coordinates of C. Measure and state the size of the angle of rotation to the nearest degree.

Enlargement

An *enlargement* is a *transformation* which *maps* a *shape onto a similar shape*, from a *centre of enlargement* using a *scale factor k*.

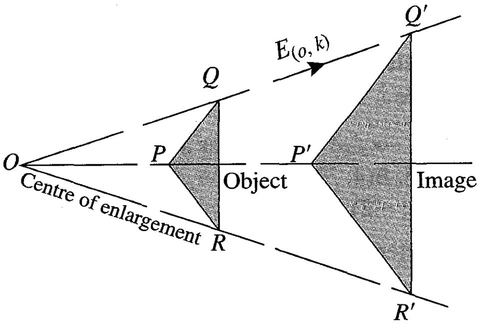


Fig. 10.103 Enlargement when k > 1

Fig. 10.103 indicates that in order to find the *image* $P'Q'R'$ of an *object PQR*, *lines* are drawn *from O*, the *centre of enlargement* through *P*, *Q* and *R*. Then *P' is on OP, Q' is on OQ* and *R' is on OR*. The *scale factor* of an *enlargement* is the *ratio* of *corresponding lengths* on the *image* to those on the *object*. That is,

$$k = \frac{P'Q'}{PQ} = \frac{Q'R'}{QR} = \frac{P'R'}{PR}$$

where $k =$ The *scale factor* (or *enlargement factor*). Alternatively, the *scale factor* of an *enlargement* is the *ratio* of *distances between the centre and the image* to *corresponding distances between the centre and the object*. That is,

$$k = \frac{OP'}{OP} = \frac{OQ'}{OQ} = \frac{OR'}{OR}$$

Thus $P'Q' = k \cdot PQ$, $Q'R' = k \cdot QR$ and $P'R' = k \cdot PR$.

That is, *under* an *enlargement*, the *length of a side* of the *image* is *equal* to the *scale factor* of the enlargement *times* the *length of the corresponding side* of the *object*.

Also $OP' = k \cdot OP$, $OQ' = k \cdot OQ$ and $OR' = k \cdot OR$.

That is, the *distance* of the *image point* from the *centre of enlargement* is *equal* to the *scale factor* of the enlargement *times* the *distance of the object point* from the *centre of enlargement*.
The *lines through O* used to *draw* the *image* are called *pattern lines*.

When the *scale factor k* is *greater than* 1, then the *image* is *larger than the object* as shown in Fig. 10.103. We say that the *image* is *magnified*.

When the *scale factor k* is a *fraction*, then the *image* is *smaller than the object*. We say that the *image is reduced* (or *diminished*). Fig. 10.104 below illustrates this fact.

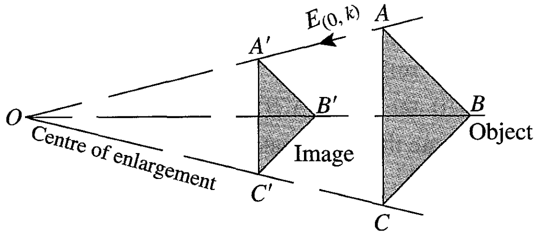


Fig. 10.104 Enlargement when k is a fraction

When the *scale factor k* is *negative*, then the *image* is also *rotated* 180° about the *centre O* (or the *image* is also *reflected* in the *centre O*). Fig. 10.105 illustrates this fact.

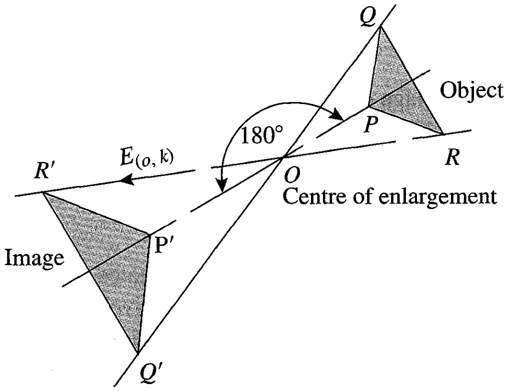


Fig. 10.105 Enlargement when k is negative

The *enlargements* above can be *denoted* by:

E_o: $\triangle ABC \to \triangle A'B'C'$ or
$E_o(\triangle ABC) = \triangle A'B'C'$ or $\triangle ABC \xrightarrow{E_o} \triangle A'B'C'$.

NOTE: The *scale factor k*, is the *linear scale factor*. The *area scale factor* is k^2.

Types of Enlargements

There are *four types of enlargements*.

ENLARGEMENT WITH SCALE FACTOR > 1

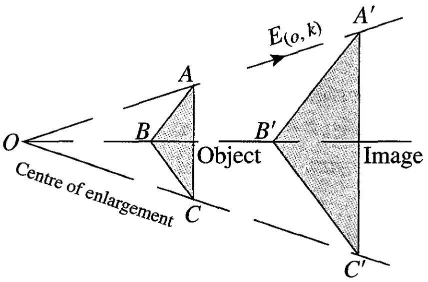


Fig. 10.106 Enlargement when k > 1

Fig. 10.106 illustrates an *enlargement* with a *scale factor greater than* 1.

When $k > 1$ then:

(i) The *image* is *larger* than the *object*.
That is, the *image is magnified*.

(ii) *Both* the *object* and the *image* are on the *same side* of the *centre of enlargement*.

ENLARGEMENT WITH SCALE FACTOR < −1

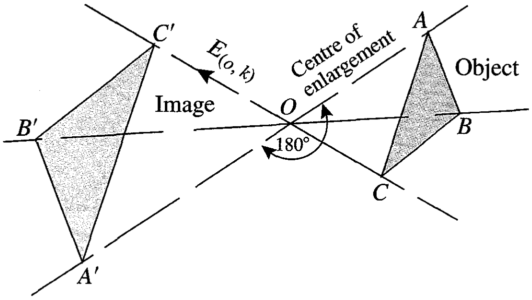


Fig. 10.107 Enlargement when k < −1

Fig. 10.107 illustrates an *enlargement* with a *scale factor less than* −1.

When $k < -1$ then:

(i) The *image* is *larger* than the *object*.
That is, the *image* is *magnified*.

(ii) The *object* and the *image* are on *opposite sides* of the *centre of enlargement*. That is, the *image* is also *rotated through* 180° about the *centre of enlargement*, we say that the *image is inverted*.

ENLARGEMENT WITH SCALE FACTOR BETWEEN 0 AND 1

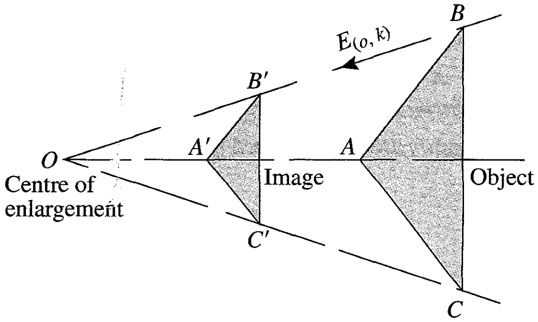


Fig. 10.108 Enlargement when 0 < k < 1

Fig. 10.108 illustrates an *enlargement* with a *scale factor between* 0 *and* 1.

When $0 < k < 1$ then:

(i) The *image* is *smaller* than the *object*.
That is, the *image is reduced* (or *diminished*).

(ii) *Both* the *object* and the *image* are on the *same side* of the *centre of enlargement*.

ENLARGEMENT WITH SCALE FACTOR BETWEEN −1 AND 0

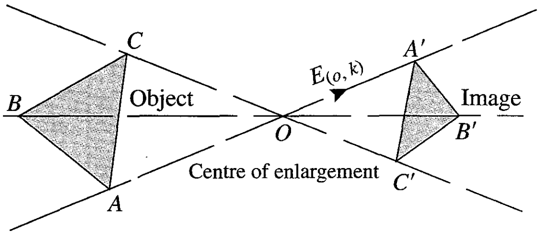


Fig. 10.109 Enlargement when −1 < k < 0

Fig. 10.109 illustrates an *enlargement* with a *scale factor between* −1 *and* 0.

When $-1 < k < 0$ then:

(i) The *image* is *smaller* than the *object*.
That is, the *image is reduced* (or *diminished*).

(ii) The *object* and the *image* are on *opposite sides* of the *centre of enlargement*. That is, the *image* is also *rotated through* 180° about the *centre of enlargement*, we say that the *image is inverted*.

NOTE: *Mathematically*, the term '*Enlargement*' refers to the *transformation* of an *object onto* an *image* which is *similar* to it. This *image* may be *larger* or *smaller* than the *object*.

(i) When the *modulus* of the *scale factor k*, $|k|$ is *numerically greater than* 1, $|k| > 1$, then the *image is larger* than the *object*. That is, the *image is magnified*.

(ii) When the *modulus* of the *scale factor k*, $|k|$ is *numerically less than* 1, $|k| < 1$, then the *image is smaller* than the *object*. That is, the *image is reduced* (or *diminished*).

However in *both cases*, *mathematically* we refer to the *transformations* as *enlargements*.

Enlargements are also referred to as *size transformations*.

Properties of Enlargements

Here we *investigate* the *properties* of a *figure* which are *invariant* under an *enlargement*.

Fig. 10.110(a), shows the *image A′B′C′D′ of trapezium ABCD* under an *enlargement* with the *origin O* as the *centre of enlargement* and *scale factor* $k = 3$. That is, $k > 1$.

Thus:

$E_{(0,\,3)}$: *Trapezium ABCD* → *Trapezium A′B′C′D′*.

Fig. 10.110(b), shows the *image A″B″C″D″ of trapezium ABCD* under an *enlargement* with the *origin O* as the *centre of enlargement* and *scale factor* $k = -3$. That is, $k < -1$. Thus:

$E_{(0,\,-3)}$: *Trapezium ABCD* → *Trapezium A″B″C″D″*.

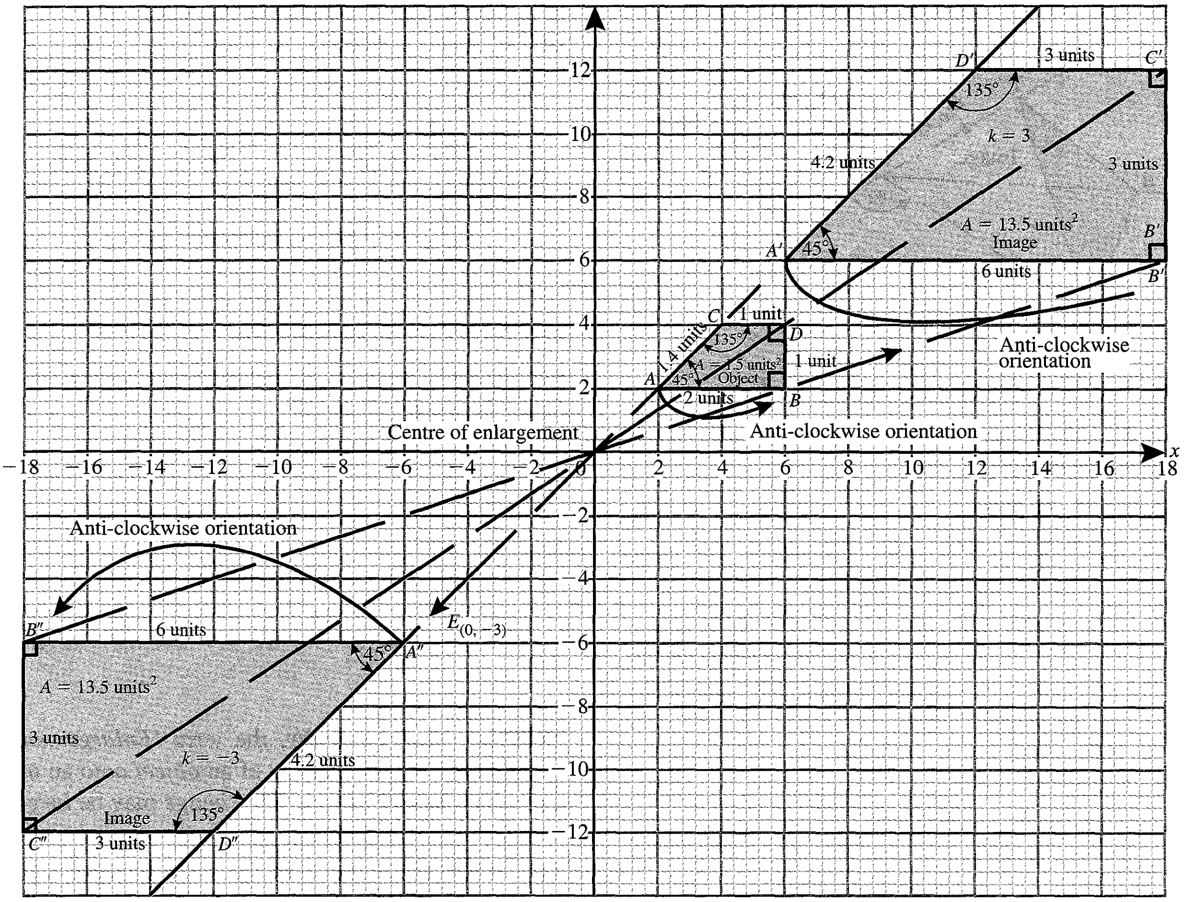
Fig. 10.111(a), shows the *image A′B′C′D′ of trapezium ABCD* under an *enlargement* with the *point X*(4, 2) as the *centre of enlargement* and *scale factor* $K = \frac{1}{2}$. That is $0 < k < 1$. Thus:

$E_{(X,\,\frac{1}{2})}$: *Trapezium ABCD* → *Trapezium A′B′C′D′*.

Fig. 10.111(b), shows the *image A″B″C″D″ of trapezium ABCD* under an *enlargement* with the *point X*(4, 2) as the *centre of enlargement* and *scale factor k* $k = -\frac{1}{2}$. That is $-1 < k < 0$. Thus:

$E_{(X,\,-\frac{1}{2})}$: *Trapezium ABCD* → *Trapezium A″B″C″D″*.

(a) Enlargement with scale factor > 1.



(b) Enlargement with scale factor < −1.

Fig. 10.110 Origin is the center of enlargement

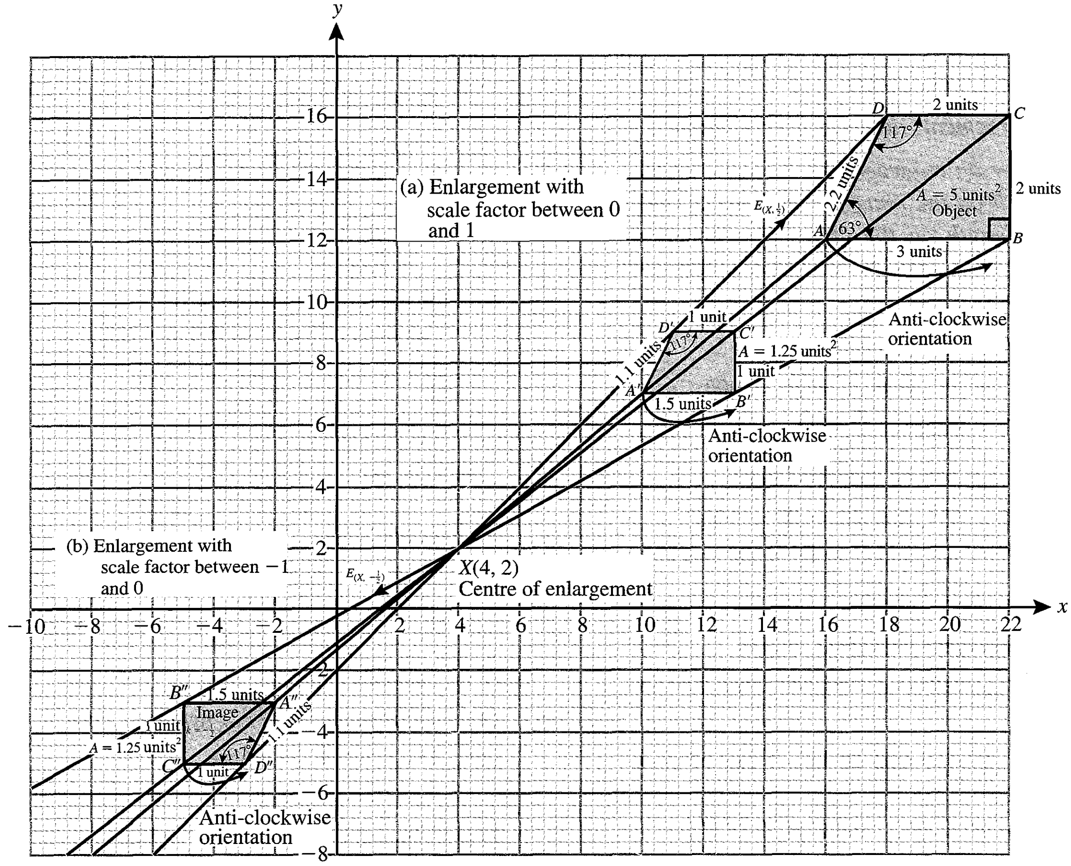


Fig. 10.111 Origin is not the center of enlargement

From Figs. 10.110 and 10.111 it can be *seen* that:

O, A and *A′ lie on the same straight line*. So too are the *points O, A* and *A″*; *X, A* and *A′*; *X, A* and *A″* et cetera.

Hence we can *conclude* that:

(i) *Under an enlargement, the image of a point, the point itself and the centre of enlargement are collinear.*

From Fig. 10.110, it can be *seen* that:

$$\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC} = \frac{OD'}{OD} = \frac{3}{1} = 3 = k$$

$$\text{and } -\frac{OA''}{OA} = -\frac{OB''}{OB} = -\frac{OC''}{OC} = -\frac{OD''}{OD}$$

$$= -\frac{3}{1} = -3 = k.$$

(The *negative sign* implies that the *image* is *inverted.*)

From Fig. 10.111 it can be *seen* that:

$$\frac{XA'}{XA} = \frac{XB'}{XB} = \frac{XC'}{XC} = \frac{XD'}{XD} = \frac{1}{2} = k$$

$$\text{and } -\frac{XA''}{XA} = -\frac{XB''}{XB} = -\frac{XC''}{XC} = -\frac{XD''}{XD}$$

$$= -\frac{1}{2} = k.$$

(The *negative sign* implies that the *image* is *inverted.*)

Hence we can *conclude* that:

(ii) *Under an enlargement, the ratio of the length of the line segment joining points on the image to the centre of enlargement, and the length of the line segment joining the corresponding points on the object to the centre of enlargement is a constant called the scale factor (or enlargement factor).*

From Fig. 10.111 it can be *seen* that:

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC} = \frac{3}{1} = 3 = k$$

$$\text{and } -\frac{A''B''}{AB} = -\frac{B''C''}{BC} = -\frac{A''C''}{AC} = -\frac{3}{1} = k.$$

(The *negative sign* implies that the *image* is *inverted.*)

From Fig. 10.111, it can be *seen* that:

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC} = \frac{1}{2} = k$$

$$\text{and } -\frac{A''B''}{AB} = -\frac{B''C''}{BC} = -\frac{A''C''}{AC} = -\frac{1}{2} = k.$$

(The *negative sign* implies that the *image* is *inverted.*)

Hence we can *conclude* that:

(i) (a) *Enlargements do not preserve lengths.*

(b) *Under an enlargement, the ratio of the length on the image to the corresponding length on the object is a constant called the scale factor (or enlargement factor).*

In Fig. 10.110 it can be *observed* that:

$\hat{A} = \hat{A}' = \hat{A}'' = 45°$, $\hat{B} = \hat{B}' = \hat{B}'' = 90°$,
$\hat{C} = \hat{C}' = \hat{C}'' = 90°$, and $\hat{D} = \hat{D}' = \hat{D}'' = 135°$.

In Fig. 10.111 it can be *observed* that:

$\hat{A} = \hat{A}' = \hat{A}'' = 63°$, $\hat{B} = \hat{B}' = \hat{B}'' = 90°$,
$\hat{C} = \hat{C}' = \hat{C}'' = 90°$, and $\hat{D} = \hat{D}' = \hat{D}'' = 117°$.

Hence we can *conclude* that:

(ii) (a) *Enlargement preserves angles.*

(b) *Enlargement is a similarity transformation.*

From Figs. 10.110 and 10.111 it can be *seen* that:

$AB \,//\, DC$, $A'B' \,//\, D'C'$ and $A''B'' \,//\, D''C''$.

Hence we can *conclude* that:

(iii) *Enlargements preserve parallelism.*

In Fig. 10.110 it can be *observed* that:

The *area of trapezium ABCD*,

$$\begin{aligned} A &= \frac{1}{2}(a+b)h \\ &= \frac{1}{2}(1+2) \times 1 \text{ cm}^2 \\ &= \frac{1}{2}(3) \text{ cm}^2 \\ &= 1.5 \text{ cm}^2 \end{aligned}$$

And the *area of trapezium A′B′C′D′*

= The *area of trapezium A″B″C″D″*

$$\begin{aligned} &= \frac{1}{2}(3+6) \times 3 \text{ cm}^2 \\ &= \frac{1}{2}(9) \times 3 \text{ cm}^2 \\ &= 4.5 \times 3 \text{ cm}^2 \\ &= 13.5 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{Also } 1.5 \text{ cm}^2 \times k^2 &= 1.5 \text{ cm}^2 \times 3^2 \\ &= 1.5 \text{ cm}^2 \times 9 \\ &= 13.5 \text{ cm}^2 \end{aligned}$$

In Fig. 10.111 it can be *observed* that:

The *area of trapezium ABCD,*

$$\begin{aligned} A &= \frac{1}{2}(a+b)h \\ &= \frac{1}{2}(2+3) \times 2 \text{ cm}^2 \\ &= \frac{1}{2}(5) \times 2 \text{ cm}^2 \\ &= 5 \text{ cm}^2 \end{aligned}$$

And the *area of trapezium A′B′C′D′*

= The area of *trapezium A″B″C″D″*

$$\begin{aligned} &= \frac{1}{2}(1+1.5) \times 1 \text{ cm}^2 \\ &= \frac{1}{2}(2.5) \text{ cm}^2 \\ &= 1.25 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{Also } 5 \text{ cm}^2 \times k^2 &= 5 \text{ cm}^2 \times \left(\frac{1}{2}\right)^2 \\ &= 5 \text{ cm}^2 \times \frac{1}{4} \\ &= 1.25 \text{ cm}^2 \end{aligned}$$

Hence we can *conclude* that:

(iv) (a) *Enlargements do not preserve area.*

(b) *The area of the image*
= $k^2 \cdot$ *The area of the object*

From Fig. 10.111 it can be *seen* that:

$AB{:}DC = 2{:}1$, $A'B'{:}D'C' = 2{:}1$
and $A''B''{:}D''C'' = 2{:}1$;
$BC{:}AD = 1{:}14$, $B'C'{:}A'D' = 1{:}1.4$
and $B''C''{:}A''D'' = 1{:}1.4$

From Fig. 10.111 it can be seen that:

$AB{:}DC = 3{:}2$, $A'B'{:}D'C' = 3{:}2$
and $A''B''{:}D''C'' = 3{:}2$;
$BC{:}AD = 1{:}1.1$, $B'C'{:}A'D' = 1{:}1.1$
and $B''C''{:}A''D'' = 1{:}1.1$

Hence we can *conclude* that:

(v) *Enlargements preserve ratios.*

In Figs. 10.110 and 10.111 it can be *observed* that:

The *order* of the *vertices* in the *original object* is *ABCD*. The *order* of the *vertices* in the *images* is *A′B′C′D′* and *A″B″C″D″*. So the *order* of the *vertices remains* the *same.*

Hence we can *conclude* that:

(vi) *Enlargements preserve the order of points.*

From Figs. 10.110 and 10.111 it can be *observed* that:

The *orientation* (or *sense*) of the *object* is *anti-clockwise*.

The *orientation* (or *sense*) of the *images* is *anti-clockwise*.

So the *orientation* (or *sense*) is *unchanged* under the *enlargements*.

Hence we can *conclude* that:

(vii) *Enlargements preserve the orientation (or sense) of a figure.*

We can *summarize* the *properties of an enlargement* in a *table* as shown below.

Table 10.5

Invariant property	Length	Angle	Parallelism	Area	Ratio	Order of points	Orientation or sense
Invariant under enlargement	No	Yes	Yes	No	Yes	Yes	Yes

We can further *summarize* the *properties of enlargements* as follows:

(i) *Under an enlargement, the image of a point, the point itself and the centre of enlargement are collinear.*

(ii) *The ratio of the length of a line segment joining an image point to the centre of enlargement, and the length of a line segment joining the corresponding object point to the centre of enlargement is a constant called the scale factor (or enlargement factor).*

(iii) *The ratio of a length on the image to the corresponding length on the object is a constant called the scale factor (or enlargement factor).*

(iv) *The area of the image*
= $k^2 \cdot$ The area of the object.

(v) *Enlargement is a similarity transformation.*

Example 18

ABC is a triangle with AB = 2 cm. Under an enlargement, the image of triangle ABC is triangle A′B′C′.

(a) Calculate the scale factor of the enlargement if
$A'B' = 5$ cm.

(b) Hence, determine the length of B′C′ if
$BC = 3$ cm.

Solution

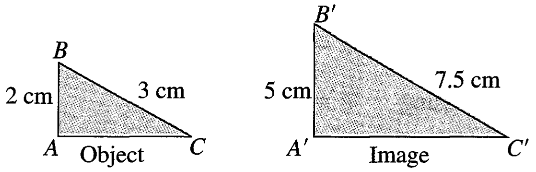


Fig. 10.112 Similar triangles

(a) Given that $AB = 2$ cm
and $A'B' = 5$ cm,

then the *scale factor*, $k = \dfrac{A'B'}{AB} = \dfrac{5\ \cancel{cm}}{2\ \cancel{cm}} = 2.5$

Hence the *scale factor* of the *enlargement* is 2.5.

Note that the *scale factor* has *no units*.

(b) Given that $BC = 3$ cm
and $k = 2.5$,
then $B'C' = k \cdot BC = 2.5 \times 3 \text{ cm} = 7.5$ cm

Hence the *length* of $B'C'$ is 7.5 cm.

Example 19

ABC is a triangle with AB = 3 cm and BC = 7 cm. Under an enlargement, $\triangle ABC \rightarrow \triangle A'B'C'$ with $A'B' = 4.5$ cm and $A'C' = 7.5$ cm. If $\triangle A'B'C'$ is inverted calculate:

(a) the scale factor of the enlargement

(b) the length of the side AC of $\triangle ABC$

(c) the length of the side B′C′ of $\triangle A'B'C'$.

Solution

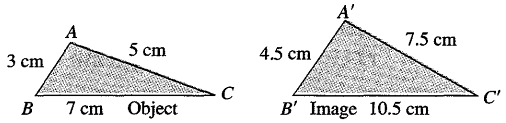


Fig. 10.113 Similar triangles

(a) Since the *image* $\triangle A'B'C'$ is *inverted*, then the *scale factor is negative*.

Given that $AB = 3$ cm
and $A'B' = 4.5$ cm,
then the *scale factor*,

$$k = -\frac{A'B'}{AB} = -\frac{4.5\ \cancel{cm}}{3\ \cancel{cm}} = -1.5$$

Hence the *scale factor* of the *enlargement* is −1.5.

(b) Given that $A'C' = 7.5$ cm

and $k = -1.5$,

then using $k = -\dfrac{A'C'}{AC}$

we have $-1.5 = -\dfrac{7.5\text{ cm}}{AC}$

So $AC = \dfrac{-7.5\text{ cm}}{-1.5} = 5$ cm

Hence the *length* of the side AC of $\triangle ABC$ is 5 cm.

(c) Given that $BC = 7$ cm

and $k = -1.5$,

then using $k = -\dfrac{B'C'}{BC}$

we have $-1.5 = -\dfrac{B'C'}{7\text{ cm}}$

So $B'C' = -1.5 \times (-7)$ cm $= 10.5$ cm

Hence the *length* of the side $B'C'$ of $\triangle A'B'C'$ is 10.5 cm.

Example 20

O is the centre of an enlargement with scale factor 2.

(a) If OA is 2.5 cm, calculate the distance of A', the image of A from the enlargement centre.

(b) If OB' is 7 cm, calculate the distance of B from the enlargement centre.

Solution

(a) Given that $k = 2$

and $OA = 2.5$ cm,

then $OA' = k \cdot OA = 2 \times 2.5$ cm $= 5$ cm

Hence the *distance of A' from the enlargement centre* is 5 cm.

(b) Given that $k = 2$

and $OB' = 7$ cm,

then using $k = \dfrac{OB'}{OB}$

We have $2 = \dfrac{7\text{ cm}}{OB}$

So $OB = \dfrac{7\text{ cm}}{2} = 3.5$ cm

Hence the *distance of B from the enlargement centre* is 3.5 cm.

Example 21

Under an enlargement, the image of the point P is 5 cm from P which is 4 cm from the centre of enlargement. Calculate the scale factor of the enlargement, if P and its image are on the same side of the centre of enlargement.

Solution

Let the centre of enlargement be O.

Then $OP = 4$ cm.

And $OP' = OP + PP'$

$= (4 + 5)$ cm $= 9$ cm

So the *scale factor*, $k = \dfrac{OP'}{OP} = \dfrac{9\,\cancel{\text{cm}}}{4\,\cancel{\text{cm}}} = 2.25$

Hence the *scale factor* of the *enlargement* is 2.25.

Example 22

An enlargement with centre O and scale factor $-1\frac{2}{3}$ *maps* $\triangle PQR$ *onto* $\triangle P'Q'R'$*. If P' is 5 cm from O, calculate how far P is from O.*

Solution

Given that $k = -1\frac{2}{3} = -\dfrac{5}{3}$

and $OP' = 5$ cm,

then using $k = -\dfrac{OP'}{OP}$

we have $-\dfrac{5}{3} = -\dfrac{5\text{ cm}}{OP}$

So $OP = \dfrac{5\text{ cm}}{-\frac{5}{3}} = 5\text{ cm} \times \dfrac{3}{5} = 3$ cm

Hence OP is 3 cm *from O.*

Example 23

Under an enlargement, the area of an object is 4 cm² and the area of its image is 36 cm². Calculate the numerical value of the scale factor of the enlargement.

Solution

Given that the area of the image $= 36\text{ cm}^2$
and the area of the object $= 4\text{ cm}^2$
The *area scale factor*, k^2

$$= \frac{\textit{The area of the image}}{\textit{The area of the object}}$$

$$= \frac{36}{4}$$

$$= 9$$

So the *linear scale factor*, $k = \pm\sqrt{9} = \pm 3$.
Hence the *numerical value* of the *scale factor* of the enlargement is 3.

Image Under an Enlargement

A *point* and its *image lie on a straight line passing through* the *centre of enlargement.*

If the *scale factor* of the *enlargement* is *positive*, then the *object* and its *image* are *both* on the *same side* of the *centre of enlargement.*

If the *scale factor* of the *enlargement* is *negative*, then the *object* and its *image* are on *opposite sides* of the *centre of enlargement.*

In finding the image of a plane figure bounded by straight lines under an enlargement:

(i) Note the *scale factor* of the *enlargement.* If it is *positive*, then the *image* and the *object* are on the *same side* of the *centre of enlargement.* If it is *negative*, then the *image* and the *object* are on *opposite sides* of the *centre of enlargement.*

(ii) *Select suitable points* on the *object*. The *most suitable points* are the *vertices.*

(iii) *Draw a straight line* of *suitable length* through *each point* (or *vertex*) and the *centre of enlargement.*

(iv) *Mark off* the *corresponding point* of the *image* on the *straight line*, noting that $OA' = k \cdot OA$, $OB' = k \cdot OB$ et cetera.

(v) *Join* the *image points* obtained. Then the *resulting figure* is the *required image.*

Example 24

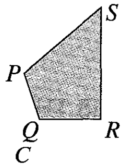


Fig. 10.114 Quadrilateral

In the diagram above, C is the centre of enlargement with scale factor $-1\frac{1}{2}$ *and quadrilateral PQRS is the object. Find the image.*

Solution

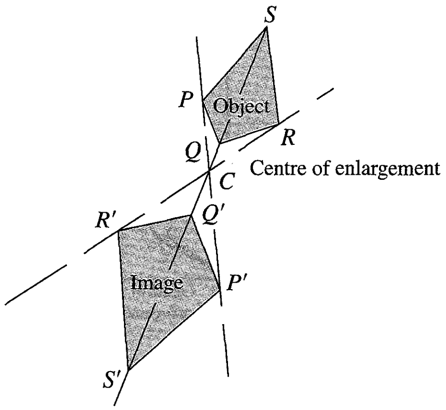


Fig. 10.115 Image of a plane figure under an enlargement

Since the *scale factor* is *negative*, then the *image* is on the *opposite side* of the *centre of enlargement. Line segments* of suitable lengths through *P* and *C*, *Q* and *C*, *R* and *C*, and *S* and *C* are *drawn.* The *distances* of *P*, *Q*, *R* and *S from C* are *measured.* The distances are found to be as follows:

$CP = 1$ cm, $CQ = 0.4$ cm, $CR = 1$ cm and $CS = 2$ cm.

Since $CP' = k \cdot CP$
then $CP' = 1.5 \times 1\text{ cm} = 1.5\text{ cm}$,
$CQ' = 1.5 \times 0.4\text{ cm} = 0.6\text{ cm}$,
$CR' = 1.5 \times 1\text{ cm} = 1.5\text{ cm}$
and $CS' = 1.5 \times 2\text{ cm} = 3\text{ cm}$.

The *point* P', Q', R' and S' of stated *lengths* are then *marked* off on the *opposite side on the lines drawn* using a *pair of compasses*. The *points* P', Q', R' and S' which were obtained are then *joined.* The resulting figure, *quadrilateral* $P'Q'R'S'$ is the required *image.*

Enlargement on the Cartesian Plane

WHEN THE ORIGIN IS THE CENTRE OF ENLARGEMENT

If an *object* (or *pre-image*) $P(x, y)$ is *magnified* (or *reduced*) by a *scale factor* of k and the *origin* is the *centre* of *enlargement*, then it is *mapped onto* $P'(kx, ky)$.

Thus, $E_{(0,\, k)}: P_{(x,\, y)} \to P'_{(kx,\, ky)}$.

If $k > 1$, then the *image* is *magnified* and it is on the *same side* as the *object*.

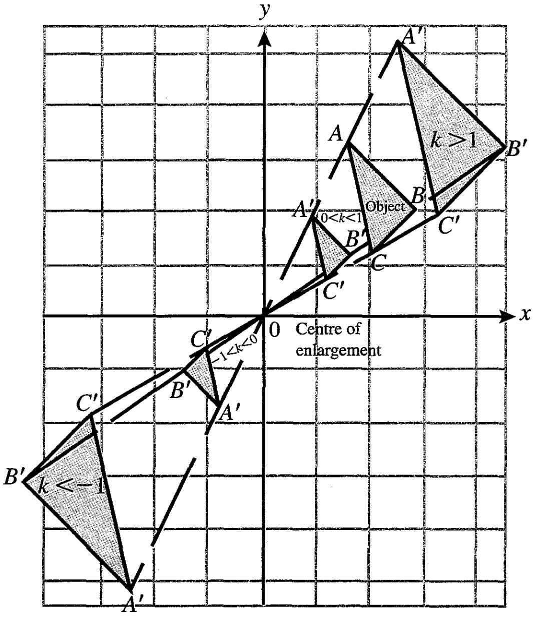


Fig. 10.116 Enlargement

If $k < -1$, then the *image* is *magnified*, *inverted* and on the *opposite side* of the *centre of enlargement*.

If $0 < k < 1$, then the *image* is *reduced* and it is on the *same side* as the *object*.

If $-1 < k < 0$, then the *image* is *reduced*, *inverted* and on the *opposite side* of the *centre of enlargement*.

Example 25

ABC is a triangle with vertices $(-1, -1)$, $(-4, -1)$ and $(-5, -3)$, respectively. Locate the image of triangle ABC under an enlargement:

(a) $E_{(0,\, 2)}$ *(b)* $E_{(0,\, \frac{1}{2})}$

(c) $E_{(0,\, -2)}$ *(d)* $E_{(0,\, \frac{3}{4})}$

Solution

(a) Since $E(0, k): P(x, y) \to P'(kx, ky)$,

then $E_{(0,\, 2)}: P(x, y) \to P'(2x, 2y)$.

So $E_{(0,\, 2)}: A(-1, -1) \to A'(2 \times [-1], 2 \times [-1])$,

$= A'(-2, -2)$.

$E_{(0,\, 2)}: B(-4, -1) \to B'(2 \times [-4], 2 \times [-1])$

$= B'(-8, -2)$.

and $E_{(0,\, 2)}: C(-5, -3) \to C'(2 \times [-5], 2 \times [-3])$

$= C'(-10, -6)$.

(b) Since $E_{(0,\, k)}: P(x, y) \to P'(kx, ky)$,

then $E_{(0,\, \frac{1}{2})}: P(x, y) \to P'(\frac{1}{2}x, \frac{1}{2}y)$.

So $E_{(0,\, \frac{1}{2})}: A(-1, -1)$

$\to A'(\frac{1}{2} \times [-1], \frac{1}{2} \times [-1]) = A'(-\frac{1}{2}, -\frac{1}{2})$,

$E_{(0,\, \frac{1}{2})}: B(-4, -1)$

$\to B'(\frac{1}{2} \times [-4], \frac{1}{2} \times [-1]) = B'(-2, -\frac{1}{2})$

and $E_{(0,\, \frac{1}{2})}: C(-5, -3)$

$\to C'(\frac{1}{2} \times [-5], \frac{1}{2} \times [-3]) = C'(-2\frac{1}{2}, -1\frac{1}{2})$.

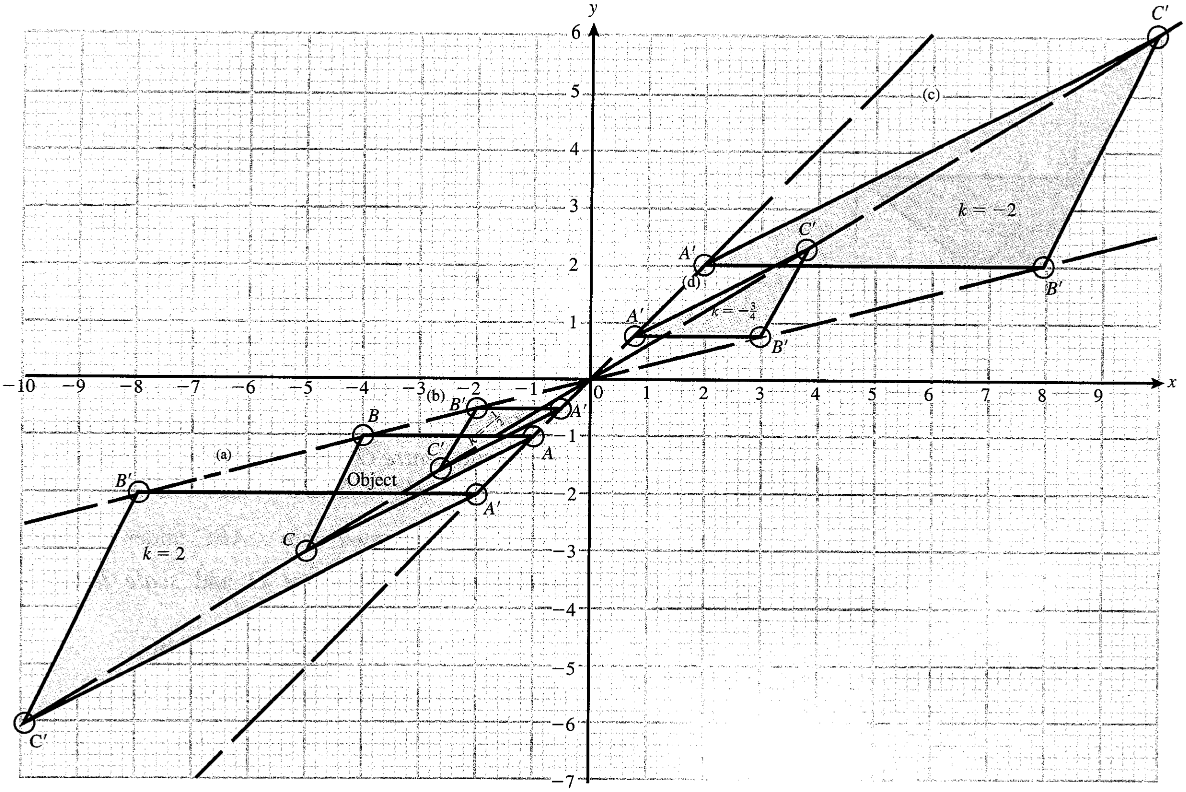


Fig. 10.117 Enlargement of a plane figure with the origin as centre

(c) Since $E_{(0,\,k)}$: $P(x, y) \rightarrow P'(kx, ky)$,

then $E_{(0,\,-2)}$: $P(x, y) \rightarrow P'(-2x, -2y)$.

So $E_{(0,\,-2)}$: $A(-1, -1)$

$\rightarrow A'(-2 \times [-1], -2 \times [-1]) = A'(2, 2)$,

$E_{(0,\,-2)}$: $B(-4, -1)$

$\rightarrow B'(-2 \times [-4], -2 \times [-1]) = B'(8, 2)$

and $E_{(0,\,-2)}$: $C(-5, -3)$

$\rightarrow C'(-2 \times [-5], -2 \times [-3]) = C'(10, 6)$.

(d) Since $E_{(0,\,k)}$: $P(x, y) \rightarrow P'(kx, ky)$,

then $E_{(0,\,-\frac{3}{4})}$: $P(x, y) \rightarrow P'(-\frac{3}{4}x, -\frac{3}{4}y)$.

So $E_{(0,\,-\frac{3}{4})}$: $A(-1, -1)$

$\rightarrow A'(-\frac{3}{4} \times [-1], -\frac{3}{4} \times [-1]) = A'(\frac{3}{4}, \frac{3}{4})$,

$E_{(0,\,-\frac{3}{4})}$: $B(-4, -1)$

$\rightarrow B'(-\frac{3}{4} \times [-4], -\frac{3}{4} \times [-1]) = B'(3, \frac{3}{4})$

and $E_{(0,\,-\frac{3}{4})}$: $C(-5, -3)$

$\rightarrow C'(-\frac{3}{4} \times [-5], -\frac{3}{4} \times [-3]) = C'(3\frac{3}{4}, 2\frac{1}{4})$.

In Fig. 10.117, the graphs of the four different enlargements can be seen.

WHEN THE ORIGIN IS NOT THE CENTRE OF ENLARGEMENT

When the *centre of enlargement* is *not* the *origin*, then we use the fact that $XA' = k \cdot XA$, $XB' = k \cdot XB$, $XC' = k \cdot XC$ etc, where X is the *centre of enlargement*.

Example 26

Triangle LMN with vertices L(−7, 8), M(−3, 8) and N(−8, 4) is mapped onto triangle L'M'N' by an enlargement with center X(−3, 4) and scale factor

(a) $\frac{1}{2}$ *(b)* $-\frac{1}{2}$

Solution

Locate the coordinates of L', M' and N'.

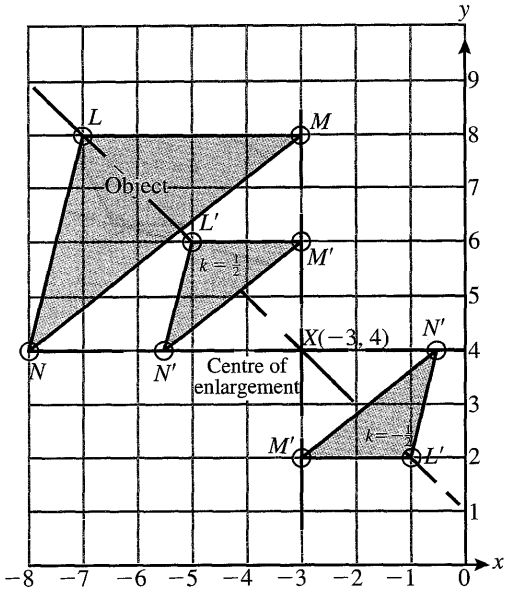


Fig. 10.118 Enlargement of a plane figure when the origin is not the centre

As can be *seen* from Fig. 10.118.

(a) Under an *enlargement* with centre $X(-3, 4)$ and *scale factor* $\frac{1}{2}$, the *coordinates* of $L'M'N'$ are: $L'(-5, 6)$, $M'(-3, 6)$, and $N'(5\frac{1}{2}, 4)$.

(b) Under an *enlargement* with *center* $X(-3, 4)$ and *scale factor* $-\frac{1}{2}$, the *coordinates* of $L'M'N'$ are: $L'(-1, 2)$, $M'(-3, 2)$ and $N'(-\frac{1}{2}, 4)$.

Inverse Enlargements

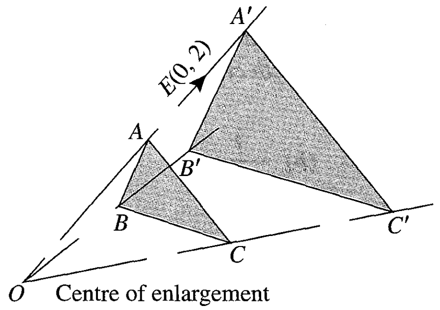


Fig. 10.119 Enlargement E(0, 2)

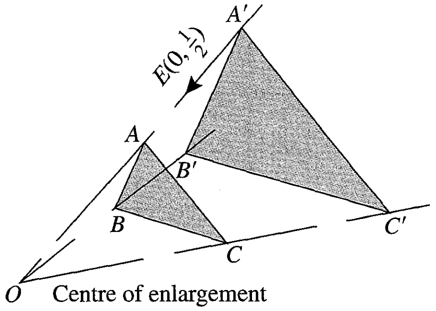


Fig. 10.120 Inverse enlargement $E_{(0,\frac{1}{2})}$

In Figs. 10.119 and 10.120:

$\triangle ABC$ is *mapped* onto $\triangle A'B'C'$ under an *enlargement* with *centre O* and *scale factor* 2. That is

$$E_{(0,\,2)}\colon \triangle ABC \to \triangle A'B'C'.$$

$\triangle A'B'C'$ is *mapped* onto $\triangle ABC$ under an *inverse enlargement* with *centre O* and *scale factor* $\frac{1}{2}$. That is

$$E_{(0,\frac{1}{2})}\colon \triangle A'B'C' \to \triangle ABC.$$

So $E_{(0,\,2)}$is the *inverse enlargement* of $E_{(0,\frac{1}{2})}$.

And $E_{(0,\frac{1}{2})}$ is the *inverse enlargement* of $E_{(0,\,2)}$.

Hence we can *conclude* that:

The *inverse of an enlargement* with a given *scale factor k* is an *enlargement* with the *same centre* and *scale factor* $\frac{1}{k}$.

Thus the *inverse of* $E_{(0,\,k)}$ is $E_{(0,\frac{1}{k})}$.

And the *inverse of* $E_{(0,\frac{1}{k})}$ is $E_{(0,\,k)}$.

Under the *enlargement*, $\triangle ABC \to \triangle A'B'C'$.

Under the *inverse enlargement*, $\triangle A'B'C' \to \triangle ABC$.

Hence, the *inverse enlargement* maps the *image* of the *enlargement* onto the *object.*

As a result, the *inverse enlargement* is used to find the *object* that gives rise to an *image under an enlargement.*

Example 27

Under an enlargement with the origin as centre and scale factor 3, $\triangle ABC \to \triangle A'B'C'$ *with vertices* $A'(-3, 6)$, $B'(-6, 9)$ *and* $C'(0, 3)$. *Find the vertices of* $\triangle ABC$.

Solution

Since $E(0, 3)$: $\triangle ABC \rightarrow \triangle A'B'C'$,

then $E(0, \frac{1}{3})$: $\triangle A'B'C' \rightarrow \triangle ABC$.

So $E(0, \frac{1}{3})$: $A'(-3, 6)$

$\rightarrow A(\frac{1}{3} \times -3, \frac{1}{3} \times 6) = A(-1, 2)$,

$E(0, \frac{1}{3})$: $B'(-6, 9)$

$\rightarrow B(\frac{1}{3} \times -6, \frac{1}{3} \times 9) = B(-2, 3)$

and $E(0, \frac{1}{3})$: $C'(0, 3)$

$\rightarrow C(\frac{1}{3} \times 0, \frac{1}{3} \times 3) = C(0, 1)$.

Exercise 10g

1. E is an enlargement defined as follows: $E(0, -2) : LP \rightarrow L'P'$.

(a) What is the location of the centre of enlargement of E?

(b) State the scale factor of E.

2. Use suitable notations to represent each of the following statements:

(a) PQ is mapped onto $P'Q'$ by an enlargement of scale factor of 3.5 and centre 0.

(b) $\triangle A'B'C'$ is the image of $\triangle ABC$ under the enlargement $E(x, -\frac{3}{4})$.

(c) The enlargement with centre C and scale factor $\frac{1}{4}$ maps quadrilateral $PQRS$ onto quadrilateral $P'Q'R'S'$.

3. ABC is a triangle with $AB = 3$ cm. Under an enlargement, the image of $\triangle ABC$ is $\triangle A'B'C'$.

(a) Calculate the scale factor of the enlargement if $A'B' = 6$ cm.

(b) Hence, determine the length of $B'C'$ if $BC = 4.5$ cm.

4. PQR is a triangle with $PR = 3.5$ cm. Given that $E(\triangle PQR) = \triangle P'Q'R'$.

(a) Calculate the scale factor of the enlargement if $P'R' = 10.5$ cm.

(b) Hence, determine the length of $Q'R'$ if $QR = 4.5$ cm.

5. ABC is a triangle with $AC = 3$ cm. Under an enlargement, the image of $\triangle ABC$ *is* $\triangle A'B'C'$.

(a) Calculate the scale factor of the enlargement if $A'C' = 7.5$ cm.

(b) Hence, determine the length of AB if $A'B' = 10$ cm.

6. PQR is a triangle with $PQ = 3.2$ cm. Given that $E : \triangle PQR \rightarrow \triangle P'Q'R'$.

(a) Calculate the scale factor of the enlargement if $P'Q' = 5.6$ cm.

(b) Hence, determine the length of QR if $Q'R' = 4.2$ cm.

7. ABC is triangle with $AB = 3$ cm and $BC = 5$ cm. Under an enlargement, $\triangle ABC \rightarrow \triangle A'B'C'$ with $A'B' = 10.5$ cm and $A'C' = 24.5$ cm. If $\triangle A'B'C'$ is inverted, calculate:

(a) the scale factor of the enlargement

(b) the length of the side AC of $\triangle ABC$

(c) the length of the side $B'C'$ of $\triangle A'B'C'$.

8. PQR is a triangle with $PR = 3.2$ cm and $QR = 4.6$ cm. Given that $E(\triangle PQR) = \triangle P'Q'R'$, $P'R' = 1.6$ cm and $P'Q' = 2.7$ cm. If $\triangle PQR$ is inverted, determine:

(a) the scale factor of the enlargement

(b) the length of the side PQ of $\triangle PQR$

(c) the length of the side $Q'R'$ of $\triangle P'Q'R'$.

9. O is the centre of an enlargement with scale factor 2.

(a) If OA is 3.5 cm, calculate the distance of A', the image of A from the enlargement centre.

(b) If OB' is 9 cm, find the distance of B from the enlargement centre.

10. O is the centre of an enlargement with scale factor -2.5.

(a) If OP is 3 cm, calculate the distance of P', the image of P from the enlargement centre.

(b) If OQ' is 12.5, calculate the distance of Q from the enlargement centre.

11. Under an enlargement, the image of the point P is 6 cm from P which is 2 cm from the centre of enlargement. Calculate the scale factor of the enlargement, if P and its image are on the same side of the centre of enlargement.

12. Under an enlargement, the image of the point A is 4.5 cm from A which is 2.5 cm from the centre of enlargement. Calculate the scale factor of the enlargement, if A and its image are on the same side of the centre of enlargement.

13. An enlargement with centre O and scale factor $-1\frac{1}{2}$ maps $\triangle PQR$ onto $\triangle P'Q'R'$. If P' is 6 cm from O, find how far P is from O.

14. An enlargement with centre O and scale factor $-\frac{3}{4}$ maps $\triangle ABC$ onto $\triangle A'B'C'$. If A' is 5.1 cm from O, find how far A is from O.

15.

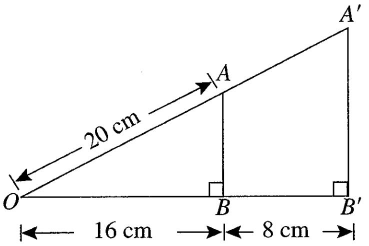


Fig. 10.121 Similar triangles

The figure above shows an enlargement, centre O, of triangle OAB. $OB = 16$ cm, $BB' = 8$ cm and $OA = 20$ cm.

(a) Write down the scale factor of the enlargement.

(b) Calculate the lengths of AB, $A'B'$ and AA'.

16.

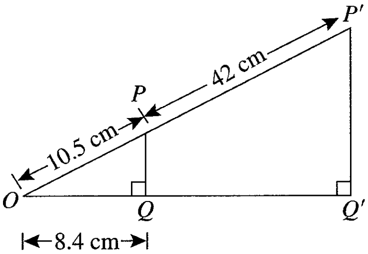


Fig. 10.122 Similar triangles

The figure above shows an enlargement, centre O, of $\triangle OPQ$. $OP = 10.5$ cm, $PP' = 42$ cm and $OQ = 8.4$ cm.

(a) Write down the scale factor of the enlargement.

(b) Calculate the lengths of PQ, $P'Q'$ and QQ'.

17.

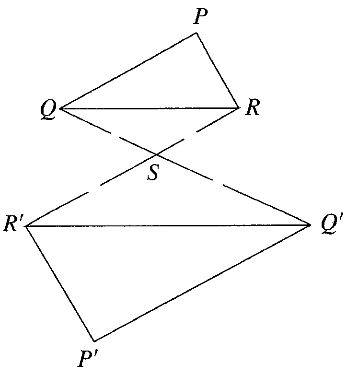


Fig. 10.123 Similar triangles

In the diagram above, $\triangle P'Q'R'$ is the image of $\triangle PQR$ under an enlargement with centre S, and scale factor $-1\frac{1}{2}$.

(a) If $QR = 9$ cm, what value is the length of $Q'R'$?

(b) If $SR = 5$ cm, what value is the length of $S'R'$?

(c) If $PR = 6$ cm, what value is the length of $P'R'$?

18.

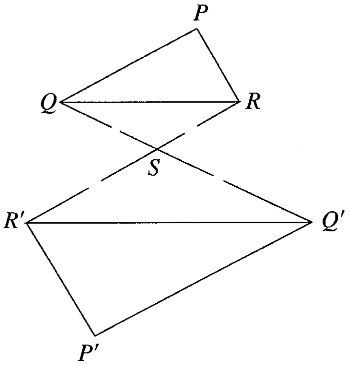


Fig. 10.124 Similar triangles

In the diagram above, $\triangle P'Q'R'$ is the image of $\triangle PQR$ under an enlargement with center S, and scale factor $-1\frac{3}{4}$.

(a) If $QR = 5$ cm, what value is the length of $Q'R'$?

(b) If $SR = 3$ cm, what value is the length of $S'R'$?

(c) If $PR = 4$ cm, what value is the length of $P'R'$?

19. Under an enlargement, the area of an object is 4 cm^2 and the area of its image is 64 cm^2.

Calculate:

(a) the area scale factor of the enlargement

(b) the numerical value of the scale factor of the enlargement.

20. Under an enlargement, the area of an object is 25 cm^2 and the area of the image is 625 cm^2. Determine:

(a) the area scale factor of the enlargement

(b) the numerical value of the scale factor of the enlargement.

21. A rectangle of dimensions 6 and 15 cm is enlarged to a rectangle of dimensions 18 and 45 cm.

(a) What is the scale factor of the enlargement?

(b) Calculate the area of each rectangle.

(c) Determine the area scale factor.

22. A rectangle of dimensions 5 and 12 cm is enlarged to a rectangle of dimensions 12.5 and 30 cm.

(a) What is the scale factor of the enlargement?

(b) Find the area of each rectangle.

(c) Determine the area scale factor.

23. Draw the image of any triangle *ABC* under an enlargement of scale factor 2, if the vertices of *A*, *B* and *C* are 5, 7 and 4 cm, respectively from the centre of enlargement.

24. Draw the image of any triangle *PQR* under an enlargement of scale factor -2, if the vertices of *P*, *Q* and *R* are 4.5, 2.3 and 3.7 cm, respectively from the centre of enlargement.

25. Draw the image of any quadrilateral *ABCD* under an enlargement of scale factor $1\frac{1}{2}$, if the vertices of *A*, *B*, *C* and *D* are 3, 5, 6 and 2 cm, respectively from the centre of enlargement.

26. Draw the image of any quadrilateral *PQRS* of scale factor $-1\frac{1}{2}$, if the vertices of *P*, *Q*, *R* and *S* are 9, 6.4, 5.8 and 4.2 cm, respectively from the center of enlargement.

27. *ABC* is a triangle with coordinates (1, 2), (3, 4) and (1.5, 4.5), respectively. Find the image of triangle *ABC* under an enlargement with the origin as centre and scale factor 2.

28. *PQR* is a triangle with vertices $P(-1, 2)$, $Q(-3, 4)$ and $R(-4.5, 6)$. Determine the image of triangle *PQR* under an enlargement with the origin as centre and scale factor 3.

29. *ABCD* is a quadrilateral with coordinates $(-1, -2)$, $(-4, -1)$, $(-5, -3)$ and $(-2, -3)$, respectively. Find the image of quadrilateral *ABCD* under an enlargement with the origin as centre and scale factor -2.

30. *KLM* is a triangle with vertices $K(3, -2)$, $L(5, -4)$ and $M(4, -5)$. Determine the image of triangle *KLM* under an enlargement with the origin as centre and scale factor $\frac{1}{2}$.

31. *PQRS* is a quadrilateral with coordinates $(-1, -2)$, $(-3, -2)$ $(-4, -5)$ and $(-2, -6)$, respectively. Find the image of quadrilateral *PQRS* under an enlargement with the origin as centre and scale factor $-\frac{1}{4}$.

32. The vertices of a quadrilateral are $A(-1, 3)$, $B(2, 4)$, $C(3, -2)$ and $D(-3, -3)$.

(a) $E_{(0,\,3)}: ABCD \rightarrow A'B'C'D'$. Find the vertices of quadrilateral $A'B'C'D'$.

(b) $E_{(0,\,\frac{1}{2})}: ABCD \rightarrow A^*B^*C^*D^*$. Determine the vertices of quadrilateral $A^*B^*C^*D^*$.

33. A quadrilateral *ABCD* has vertices $A(2, 1)$, $B(6, 1)$, $C(5, 3)$ and $D(4, 3)$.

(a) *E* is an enlargement of magnitude 10 and $AB = 12$ cm. If E(quadrilateral $ABCD$) = quadrilateral $A'B'C'D'$, find $A'B'$.

(b) State the coordinates of the image of $ABCD$, $A'B'C'D'$ if it undergoes a size transformation of magnitude $\frac{1}{2}$ with centre at the origin.

34. Using a scale of 1 cm 1 unit on each axis, draw on graph paper triangle *ABC* whose vertices are $A(2, 1)$, $B(5, 2)$ and $C(3, 4)$. $\triangle A'B'C'$ is the image of $\triangle ABC$ under an enlargement with centre (0, 0) and scale factor $k = 2$. Draw $\triangle A'B'C'$ and state the coordinates of its vertices.

35. Using a scale of 1 cm to 1 unit on each axis, draw on graph triangle *ABC* whose vertices are $A(3, 1)$, $B(6, 2)$ and $C(4, 5)$.

$\triangle A'B'C'$ is the image of $\triangle ABC$ under an enlargement, centre (0, 0) and scale factor $k = 1.5$. Draw $\triangle A'B'C'$ and state the coordinates of its vertices.

36 Triangle *ABC* with coordinates (2, 1), (3, 3) and (5, 1), respectively is mapped onto triangle $A'B'C'$ with the point *A* as the centre of enlargement and a scale factor of 2. Find the coordinates of A', B' and C'.

37. Triangle PQR with vertices $P(2, 0)$, $Q(4, 3)$ and $R(0, 1)$ is mapped onto triangle $P'Q'R'$ with the point R as the centre of enlargement and a scale factor of 2. Determine the coordinates of P', Q' and R'.

38. Triangle KLM with vertices $K(-3, 1)$, $L(-1, 1)$ and $M(-1, 2)$ is mapped onto triangle $K'L'M'$ with the point L as the centre of enlargement and a scale factor of 3. Find the vertices K', L' and M'.

39. Triangle LMN with vertices $L(-5, 6)$, $M(-3, 6)$ and $N(-5.5, 4)$ is mapped onto triangle $L'M'N'$ by an enlargement with centre $X(-3, 4)$ and scale factor 2. Determine the coordinates of L', M' and N'.

40. Triangle LMN with vertices $L(-1, 2)$, $M(-3, 2)$ and $N(-\frac{1}{2}, 4)$ is mapped onto triangle $L'M'N'$ by an enlargement with centre $X(-3, 4)$ and scale factor -2. Determine the coordinates of L', M' and N'.

41. $\triangle ABC$ with vertices $A(-3, -3)$, $B(-12, -3)$ and $C(-15, -9)$ is mapped onto $\triangle A'B'C'$ by an enlargement with centre $X(-1, -1)$ and scale factor $\frac{1}{2}$. Determine the vertices A', B' and C'.

42. Under an enlargement with the origin as centre and scale factor 2, $\triangle ABC \to \triangle A'B'C'$ with vertices $A'(1, 2)$, $B'(3, 4)$ and $C'(5, 2)$. Find the vertices of $\triangle ABC$.

43. Under an enlargement with the origin as centre and scale factor 3, $\triangle ABC \to \triangle A'B'C'$ with vertices $A'(-2, -3)$, $B'(-3, -6)$ and $C'(-6, -9)$. Determine the vertices of $\triangle ABC$.

44. $E_{(0,\,2.5)}$: $\triangle PQR \to \triangle P'Q'R'$. $\triangle P'Q'R'$ has vertices of $P'(-5, 1)$, $Q'(-5, 2)$ and $R'(-1, 3)$. Find the vertices of $\triangle PQR$.

45. $E_{(0,\,\frac{1}{3})}$: $\triangle KLM \to \triangle K'L'M'$. $\triangle K'L'M'$ has vertices $K'(-1, -2)$, $L'(-3, -4)$ and $M'(-2, -3)$. Determine the vertices of $\triangle KLM$.

46. $E_{(0,\,-2)}$: $\triangle ABC \to \triangle A'B'C'$. $\triangle A'B'C'$ has vertices $A'(2, -1)$, $B'(4, -3)$ and $C'(6, -4)$. Find the vertices of $\triangle ABC$.

47. $E_{(0,\,-\frac{1}{2})}$: $\triangle PQR \to \triangle P'Q'R'$. $\triangle P'Q'R'$ has vertices $P'(1, -1)$, $Q'(2, -3)$ and $R'(5, -4)$. Determine the vertices of $\triangle PQR$.

C.X.C. Past Paper Questions

The following supplementary questions were taken from C.X.C. Past Papers.

Exercise 10h

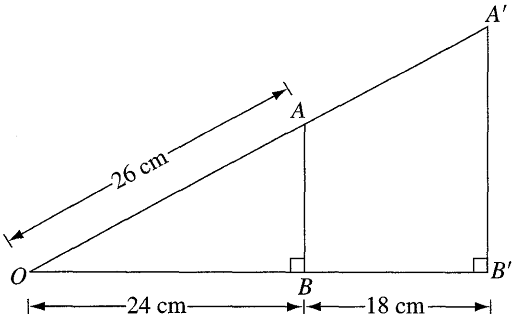
1. An isosceles triangle PQR has $PQ = PR =$ 5 cm and angle $QPR = 150°$. The triangle is reflected in the side PQ.
   (a) Draw and label the figure with its image PQR'.
   (b) Find the length of RR'. Explain your working.

   Question 3(ii). C.X.C. (Basic). June 1981.

2. (a) Construct a triangle ABC such that $AB = 6$ cm, $BC = 4.5$ cm and $CA = 3$ cm.
   (b) Construct the image of triangle ABC formed by reflecting triangle ABC in side AB. Label the image of C as C'.
   (c) Measure and state the length of CC'.
   (d) If AB intersects CC' at N write down two statements about NC'.
   (e) State the type of polygon formed by the composite figure ABC and its image.

   Question 9. C.X.C. (Basic). June 1983.

3.



   (i) The figure above, which is not drawn to scale, shows an enlargement, centre O, of triangle OAB. $OB = 24$ cm, $BB' = 18$ cm and $OA = 26$ cm.
   (a) Write down the scale factor of the enlargement.
   (b) Calculate the lengths of AB and $A'B'$.

   Question 10(i). C.X.C. (Basic). June 1984.

4. The points $A(2, 2)$, $B(5, 2)$ and $C(2, 6)$ are vertices of a triangle ABC. The triangle ABC is reflected in the y-axis to produce triangle $A'B'C'$. The original triangle ABC is also reflected in the line which passes through the points $O(0, 0)$ and $P(-3, 3)$ to produce triangle $A''B''C''$.

(a) Using a scale of 1 cm to represent 1 unit, draw on the graph paper ABC, $A'B'C'$ and $A''B''C''$.

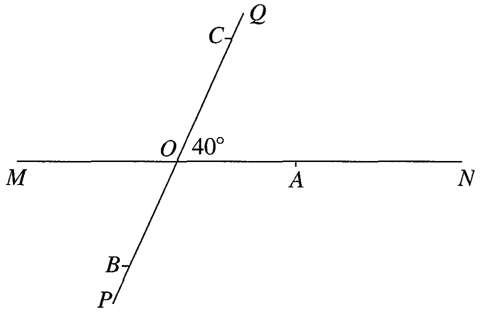
(b) State the single transformation that will map triangle $A'B'C'$ onto triangle $A''B''C''$.

(c) Calculate the length of AA''.

(d) Determine the size of angle $A'AA''$.

Question 7. C.X.C. (Basic). June 1986.

5.



The diagram above (not drawn to scale) shows two line segments PQ and MN intersecting at O. The points A, B and C are equidistant from O. Angle $AOC = 40°$.

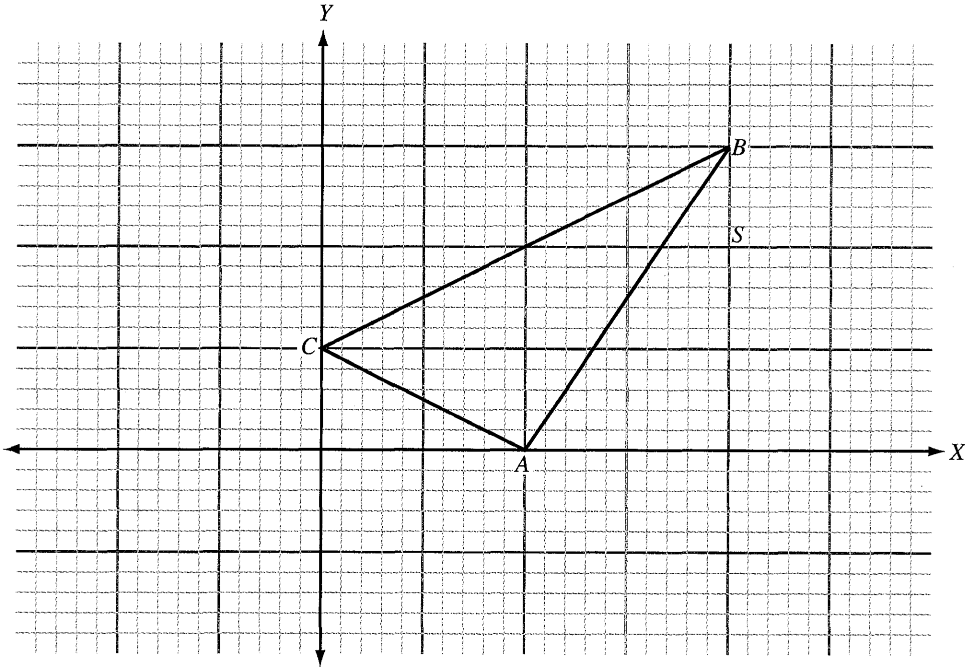
(a) A is mapped onto C by means of a reflection in the line l (not shown). Determine the acute angle which l makes with ON. Give a reason for your answer.

(b) A is mapped onto B by means of a reflection in the line k (not shown). Calculate the size of the acute angle which k makes with ON.

(c) Describe fully a transformation which maps the line k onto the line l.

Question 6. C.X.C. (Basic). June 1987.

6.



The coordinates of the vertices of triangle ABC above are $A(2, 0)$, $B(4, 3)$ and $C(0, 1)$, respectively.

(a) Draw the triangle ABC on graph paper, and using a scale factor of 2 and the origin as centre, construct an enlargement $A'B'C'$ of triangle ABC. State the co-ordinates of the points $A'B'C'$.

(b) After a transformation P, the original triangle ABC is mapped onto triangle $A''B''C''$, where $A''(0, -2)$, $B''(3, -4)$ and $C''(1, 0)$ are the vertices of the image. Describe fully the transformation P.

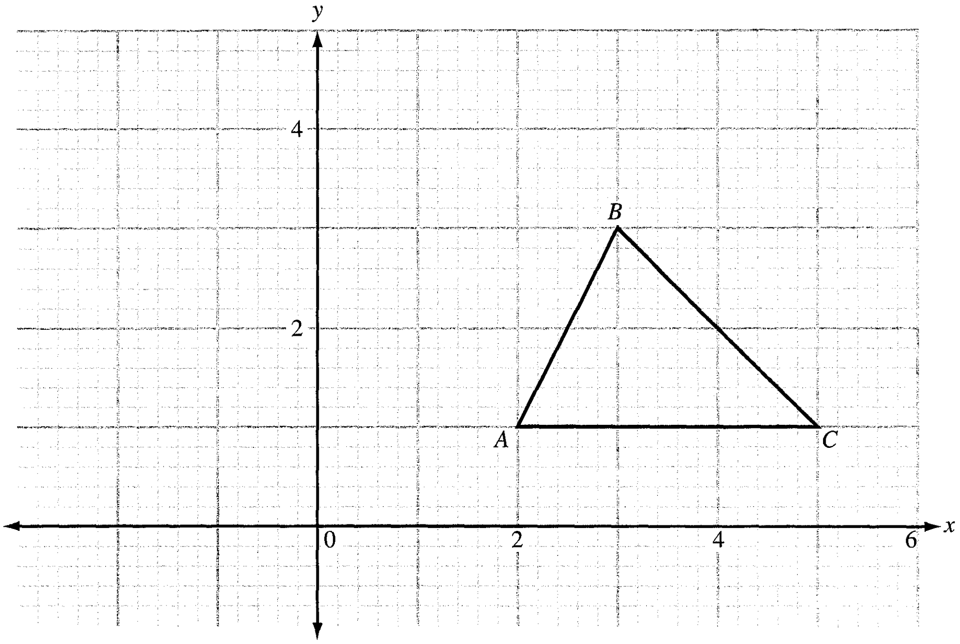
(c) The original triangle ABC is reflected in the y-axis to form triangle $A'''B'''C'''$. State the co-ordinates of the image A'''B'''C'''.

(d) State the relationship between the area of

- (i) triangle ABC and triangle $A'B'C'$
- (ii) triangle ABC and triangle $A''B''C''$
- (iii) triangle ABC and triangle $A'''B'''C'''$.

Question 8. C.X.C. (Basic). June 1988.

7.



(a) (i) State co-ordinates of the points A, B and C in the triangle in the graph above.

(ii) *Using the same scales as in the diagram above*, draw on graph paper, triangle ABC and its image, triangle $A'B'C'$, after a reflection in the x-axis.

(b) (i) On your graph draw also triangle $A''B''C''$, the enlargement of triangle ABC, using a scale factor of 2 and the point A as the centre of enlargement.

(ii) Calculate the area of triangle ABC and, hence determine the area of triangle $A''B''C''$.

Question 3. C.X.C. (Basic). June 1991.

8. A man walks due north, from a point P for a distance of 12 km, to a point X. He then walks 5 km due east to a point Q.

(a) Draw a diagram to represent the man's journey, showing by means of an arrow, the north direction. Determine

- (i) the distance PQ
- (ii) the bearing of Q from P.

(b) A point $R(3, 4)$ is translated to a point S, by the vector $\begin{pmatrix}2\\-5\end{pmatrix}$.

- (i) Determine the co-ordinates of the point S.
- (ii) If the point R is reflected in the line $y = x$, determine the co-ordinates of R', the image of R.

Question 8. C.X.C. (Basic). June 1993.

CHAPTER 11

Geometry: Trigonometry

This Chapter will teach you about

- the notation for a triangle
- the sine, cosine, tangent of an angle and complementary angles
- how to find an unknown angle using 3-figure tables and a scientific calculator
- how to find the length of an unknown side
- the angle of elevation and depression
- bearings

Introduction

In this chapter we will be dealing with suitable *right-angled triangles.*

Trigonometry is a *branch of Mathematics* concerned, at its *simplest level*, with the *measurement of triangles*. *Trigonometrical ratios* such as *sine*, *cosine*, and *tangent* are used to *calculate unknown angles* or *lengths*.

Notation for a Triangle

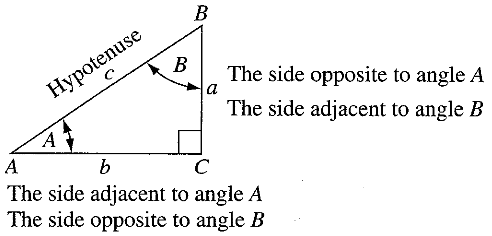


Fig. 11.1 Right-angled triangle

In any *right-angled triangle*:

(i) The *side* that is *opposite* to the *right-angle* is called the *hypotenuse*.

(ii) The *side* that is *opposite* to the *angle being considered* is called the *opposite side.*

(iii) The *third side* which forms the *angle being considered* with the *hypotenuse* is called the *adjacent side*.

In Fig. 11.1 it can be *seen* that:

(i) The *hypotenuse* is the *side AB*.
The word *hypotenuse* can be abbreviated to *hyp.*

(ii) (a) The *side opposite* to *angle A* is *BC*.
(b) The *side opposite* to *angle B* is *AC*.
The word *opposite* can be abbreviated to *opp.*

(iii) (a) The *side adjacent* to *angle A* is *AC*.
(b) The *side adjacent* to *angle B* is *BC*.
The word *adjacent* can be abbreviated to *adj.*

Hence we can *conclude* that the *opposite side* and the *adjacent side depend* on the *angle* being *considered.* However the *hypotenuse* is *fixed.*

Sine of an Angle

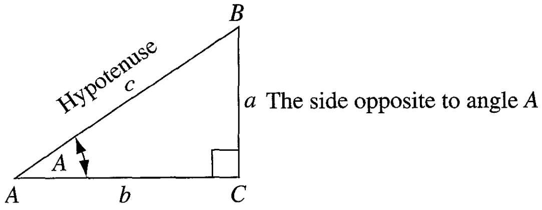


Fig. 11.2 Right-angled triangle

The *sine of an angle* is defined as the *ratio* of the *side opposite to the angle* to the *hypotenuse*. Thus:

$$\textit{The sine of an angle} = \frac{\textit{The side opposite to the angle}}{\textit{The hypotenuse}}$$

The word *sine* can be abbreviated to *sin*.

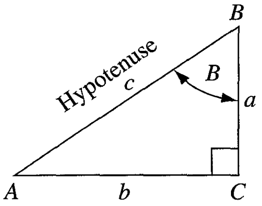
From Fig. 11.2:

$$\textit{The sine of angle A} = \frac{\textit{The side opposite to angle A}}{\textit{The hypotenuse}}$$

or $\quad sin\hat{A} = \dfrac{Opp}{Hyp}$

or $\quad sin\hat{A} = \dfrac{BC}{AB}$

or $\quad sin\hat{A} = \frac{a}{c}$



The side opposite to angle *B*

Fig. 11.3 Right-angled triangle

From Fig. 11.3:

$$\textit{The sine of angle B} = \frac{\textit{The side opposite to angle B}}{\textit{The hypotenuse}}$$

or $\quad sin\hat{B} = \dfrac{Opp}{Hyp}$

or $\quad sin\hat{B} = \dfrac{AC}{AB}$

or $\quad sin\hat{B} = \frac{b}{c}$

Sin θ

The *sine of an angle* θ can be obtained from the '*Natural Sines*' *table* in *3-figure tables* or from a *scientific calculator.*

USING 3-FIGURE TABLES

Natural Sines

Table 11.1 Extract from a table of 'Natural Sines'

Degrees	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	0.000	002	003	005	007	009	010	012	014	016
1	.017	019	021	023	024	026	028	030	031	033
2	.035	037	038	040	042	044	045	047	049	051
3	**.052**	054	056	058	**059**	061	063	065	066	**068**
4	.070	071	073	075	077	078	080	082	084	085
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 11.1 is an *extract* from a *table* of '*Natural Sines*'. It shows the *sines of angles* from 0.0 *degrees to* 4.9 *degrees.* The *first column* indicates *whole number* of *degrees* from 0 *to* 4 and the *top row* indicates *decimal parts of a degree* from 0.0 *to* 0.9. The *sine of an angle* is that *value corresponding* to a *particular whole number* of *degrees* and a *particular decimal parts of a degree.* For example:

(a) The *sine of* 3.0 *degrees* = *sin* 3.0° = 0.052

(b) The *sine of* 3.4 *degrees* = *sin* 3.4° = 0.059

(c) The *sine of* 3.9 *degrees* = *sin* 3.9° = 0.068

USING A SCIENTIFIC CALCULATOR

Table 11.2 Using a scientific calculator

	Input	*Display*
(a)	sin 3.0 =	sin 3.0 0.052 335 956
(b)	sin 3.4 =	sin 3.4 0.059 306 373
(c)	sin 3.9 =	sin 3.9 0.068 015 29

Table 11.2 illustrates how to *find* the *sine of an angle* using a *scientific calculator*. The sine function key is first pressed (input), then the magnitude of the angle is input, after which the equals sign is pressed (input). On the *display* we will then see the *value corresponding* to the *sine* of that *particular angle*. We must bear in mind that a *calculator* is far *more accurate* than 3-*figure tables* and that we should *state* our *answer correct* to *three significant figures*.

Thus:

(*a*) $sin 3.0° = 0.052|335956 = 0.052$ (*correct* to 3 *s.f.*).

(*b*) $sin 3.4° = 0.059|306373 = 0.059$ (*correct* to 3 *s.f.*).

(*c*) $sin 3.9° = 0.068|01529 = 0.068$ (*correct* to 3 *s.f.*).

Example 1

Use 3-figure tables or a scientific calculator to find:

(a) sin 15.7° *(b) sin 46.3°*

(c) sin 78.5° *(d) sin 83.9°*

Solution

Using 3-*figure tables*:

(a) $sin 15.7° = 0.271$

(b) $sin 46.3° = 0.723$

(c) $sin 78.5° = 0.980$

(d) $sin 83.9° = 0.994$

Using *a scientific calculator*:

(a) $sin 15.7° = 0.270|600446 = 0.271$ (*correct* to 3 *s.f.*).

(b) $sin 46.3° = 0.722|967145 = 0.723$ (*correct* to 3 *s.f.*).

(c) $sin 78.5° = 0.979|924704 = 0.980$ (*correct* to 3 *s.f.*).

(d) $sin 83.9° = 0.994|337944 = 0.994$ (*correct* to 3 *s.f.*).

It should be noted that $0 \leqslant sin\theta \leqslant 1$, if $0° \leqslant \theta \leqslant 90°$ (i.e. if θ is an *acute angle*).

The *smallest value* of $sin\theta$ is 0 and it *occurs when* $\theta = 0°$, i.e. $sin 0° = 0$.

The *greatest value* of $sin\theta$ is 1 and it *occurs when* $\theta = 90°$, i.e. $sin 90° = 1$.

$Sin^{-1}x$ or $Arcsin\, x$

The *angle whose sine* is of a *given value* can be obtained from the '*Natural Sines*' *table* in 3-*figure tables* or from a *scientific calculator*.

USING 3-FIGURE TABLES

Table 11.3 is an *extract* from a *table* of '*Natural Sines*'. It shows the *sines of angles* from 60.0 *degrees to* 64.9 *degrees*. The *first column* indicates *whole number of degrees* from 60 *to* 64 and the *top row* indicates *decimal parts of a degree* from 0.0 *to* 0.9.

Natural Sines

Table 11.3 Extract from a table of 'Natural Sines'

Degrees	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
60	**0.866**	867	868	869	869	**870**	871	872	**873**	874
61	.875	875	876	877	878	879	880	880	881	882
62	.883	884	885	885	886	887	888	889	889	890
63	.891	892	893	893	894	895	896	896	897	898
64	.899	900	900	901	902	903	903	904	905	906
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

The *angle whose sine* is a *given value* can be obtained from the *'Natural Sines' table*. For example:

(a) The *angle whose sine is* $0.866 = sin^{-1} 0.866 = arcsin 0.866 = 60.0°$

(b) The *angle whose sine is* $0.870 = sin^{-1} 0.870 = arcsin 0.870 = 60.5°$

(c) The *angle whose sine is* $0.873 = sin^{-1} 0.873 = arcsin 0.873 = 60.8°$

It can be *seen* that sin^{-1} (or *arcsin*) means *'The angle whose sine is'*.

USING A SCIENTIFIC CALCULATOR

Table 11.4 Using a scientific calculator

	Input	*Display*
(a)	inv sin 0.866 =	$\sin^{-1}0.866$ 59.997 089 07
(b)	inv sin 0.870 =	$\sin^{-1}0.870$ 60.458 639 5
(c)	inv sin 0.873 =	$\sin^{-1}0.873$ 60.809 152 8

Table 11.4 illustrates how to *find* the *angle whose sine* is of a *given value* using a *scientific calculator*. The *INV key* (*or* 2*nd FUN key*) is first *pressed*, after which the *sine function key is pressed* (*input*), the value is then input, after which the equals sign is pressed. On the *display* we will then see the *angle corresponding* to the *sine* of that *particular value.* We again must bear in mind that a *calculator* is far *more accurate* than 3-*figure tables* and then we should *state* our *answer correct* to 3 *significant figures.*

Thus:

(a) $sin^{-1}0.866 = \text{arcsin } 0.866 = 59.9|97\,089\,07^\circ$
$= 60.0^\circ$(*correct* to 3 *s.f.*).

(b) $sin^{-1}0.870 = \text{arcsin } 0.870 = 60.4|58\,639\,5^\circ$
$= 60.5^\circ$(*correct* to 3 *s.f.*).

(c) $sin^{-1}0.873 = \text{arcsin } 0.873 = 60.8|09\;152\,8^\circ$
$= 60.8^\circ$(*correct* to 3 *s.f.*).

Example 2

Use 3-figure tables or a scientific calculator to find:

(a) $sin^{-1}0.429$ *(b)* $sin^{-1}0.635$

(c) $arcsin\,0.844$ *(d)* $arcsin\,0.905$

Solution

Using 3-*figure tables*:

(a) $sin^{-1}0.429 = 25.4^\circ$

(b) $sin^{-1}0.635 = 39.4^\circ$

(c) $arcsin\,0.844 = 57.6^\circ$

(d) $arcsin\,0.905 = 64.8^\circ$

Using *a scientific calculator*:

(a) $sin^{-1}0.429 = 25.4|04\,114\,36^\circ$
$= 25.4^\circ$ (*correct* to 3 *s.f.*).

(b) $sin^{-1}0.635 = 39.4|19\,984\,47^\circ$
$= 39.4^\circ$ (*correct* to 3 *s.f.*).

(c) $arcsin\,0.844 = 57.5|64\,951\,93^\circ$
$= 57.6^\circ$ (*correct* to 3 *s.f.*).

(d) $arcsin\,0.905 = 64.8|23\,283\,06^\circ$
$= 64.8^\circ$ (*correct* to 3 *s.f.*).

Example 3

Use 3-figure table or a scientific calculator to determine the size of each of the following unknown angles, given that:

(a) $sin\,x = 0.091$ *(b)* $sin\,y = 0.368$

(c) $sin\,\theta = 0.555$ *(d)* $sin\,\phi = 0.913$

Solution

Using 3-*figure tables*:

(a) Given that $\sin x = 0.091$
then $x = sin^{-1}0.091 = 5.2^\circ$

(b) Given that $\sin y = 0.368$
then $y = sin^{-1}0.368 = 21.6^\circ$

(c) Given that $\sin\theta = 0.555$
then $\theta = sin^{-1}0.555 = 33.7^\circ$

(d) Given that $\sin\phi = 0.913$
then $\phi = sin^{-1}0.913 = 65.9^\circ$

Using *a scientific calculator*:

(a) Given that $\sin x = 0.091$
then $x = arcsin\,0.091$
$= 5.22|1\,138\,957^\circ$
$= 5.2^\circ$ (*correct* to 1 *d.p.*)

(b) Given that $\sin y = 0.368$
then $y = arcsin\,0.368$
$= 21.5|92\,324\,87^\circ$
$= 21.6^\circ$ (*correct* to 3 *s.f.*)

(c) Given that $\sin\theta = 0.555$
then $\theta = arcsin\,0.555$
$= 33.7|10\,714\,78^\circ$
$= 33.7$ (*correct* to 3 *s.f.*)

(d) Given that $\sin\phi = 0.913$
then $\phi = arcsin\,0.913$
$= 65.9|23\,278\,36^\circ$
$= 65.9^\circ$ (*correct* to 3 *s.f.*)

Calculating an Unknown Side

The *sine of an angle* can be used to *calculate* either the *opposite side* to a *given angle* or the *hypotenuse*.

Example 4

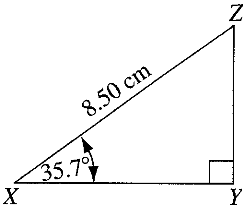


Fig.11.4 Right-angled triangle

In the diagram above, △XYZ is right-angled at Y with $\hat{X} = 35.7°$ and XZ = 8.50 cm.

Calculate the length of the side YZ correct to three significant figures.

Solution

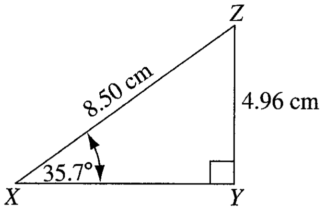


Fig. 11.4 Right-angled triangle

Considering the *right-angled* △*XYZ*:

$$\sin 35.7° = \frac{Opp}{Hyp} = \frac{YZ}{XZ} = \frac{YZ}{8.50 \text{ cm}}$$

So $$YZ = 8.50 \text{ cm} \times \sin 35.7°$$
$$= 8.50 \text{ cm} \times 0.584$$
$$= 4.96 \text{ cm}$$ (*correct* to 3 *s.f.*)

Hence the *length* of the side *YZ* is 4.96 cm.

Example 5

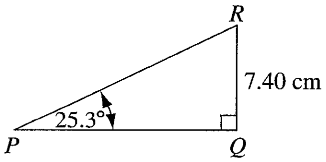


Fig. 11.5 Right-angled triangle

In the diagram above, △PQR is right-angled at Q with $\hat{P} = 25.3°$ and QR = 7.40 cm.

Determine the length of the side PR correct to three significant figures.

Solution

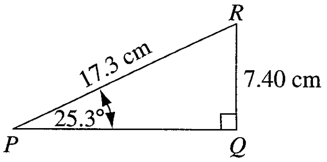


Fig. 11.5 Right-angled triangle

Considering the *right-angled* △*PQR*:

$$\sin 25.3° = \frac{Opp}{Hyp} = \frac{QR}{PR} = \frac{7.40 \text{ cm}}{PR}$$

So $$PR = \frac{7.40 \text{ cm}}{\sin 25.3}$$
$$= \frac{7.40 \text{ cm}}{0.427}$$
$$= 17.3 \text{ cm}$$ (*correct* to 3 *s.f.*).

Hence the *length* of the side *PR* is 17.3 cm.

Calculating an Unknown Angle

The *sine of an angle* can be used to *calculate* an *angle*, given the *hypotenuse* and the *length of the side opposite to the angle*.

Example 6

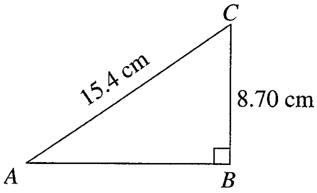


Fig. 11.6 Right-angled triangle

In the diagram above, △ABC is right-angled at B with AC = 15.4 cm and BC = 8.70 cm.

Calculate the size of angle BAC correct to three significant figures.

Solution

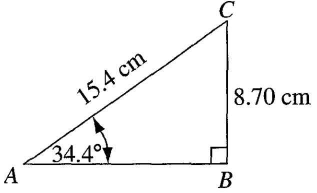


Fig. 11.6 Right-angled triangle

Considering the *right-angled* $\triangle ABC$:

$$\sin B\hat{A}C = \frac{Opp}{Hyp} = \frac{BC}{AC} = \frac{8.70\,\cancel{cm}}{15.4\,\cancel{cm}} = 0.565$$

So $\quad B\hat{A}C = \sin^{-1} 0.565$

$\qquad = 34.4°$ (*correct* to 3 *s.f.*)

Hence the size of *angle BAC* is 34.4°.

Exercise 11a

1. Use 3-figure tables or a scientific calculator to find:

(a) $\sin 8°$ (b) $\sin 15°$
(c) $\sin 39°$ (d) $\sin 48°$

2. Use 3-figure tables or a scientific calculator to determine:

(a) $\sin 53°$ (b) $\sin 61°$
(c) $\sin 74°$ (d) $\sin 86°$

3. Use 3-figure tables or a scientific calculator to find:

(a) $\sin 9.3°$ (b) $\sin 17.4°$
(c) $\sin 25.1°$ (d) $\sin 34.7°$

4. Use 3-figure tables or a scientific calculator to evaluate:

(a) $\sin 48.5°$ (b) $\sin 53.2°$
(c) $\sin 64.9°$ (d) $\sin 75.8°$

5. Use 3-figure tables or a scientific calculator to find:

(a) $\sin 67.9°$ (b) $\sin 79.5°$
(c) $\sin 87.3°$ (d) $\sin 89.9°$

6. Use 3-figure tables or a scientific calculator to find:

(a) $\sin^{-1} 0.087$ (b) $\sin^{-1} 0.326$
(c) $\sin^{-1} 0.469$ (d) $\sin^{-1} 0.515$

7. Use 3-figure tables or a scientific calculator to determine:

(a) $\sin^{-1} 0.656$ (b) $\sin^{-1} 0.799$
(c) $\sin^{-1} 0.927$ (d) $\sin^{-1} 0.951$

8. Use 3-figure tables or a scientific calculator to determine:

(a) $\arcsin 0.078$ (b) $\arcsin 0.239$
(c) $\arcsin 0.489$ (d) $\arcsin 0.550$

9. Use 3-figure tables or a scientific calculator to evaluate:

(a) $\arcsin 0.617$ (b) $\arcsin 0.764$
(c) $\arcsin 0.816$ (d) $\arcsin 0.850$

10. Use 3-figure tables or a scientific calculator to find:

(a) $\sin^{-1} 0.897$ (b) $\sin^{-1} 0.964$
(c) $\arcsin 0.966$ (d) $\arcsin 0.980$

11. Use 3-figure tables or a scientific calculator to find the size of each of the following unknown angles, given that:

(a) $\sin x = 0.156$ (b) $\sin y = 0.309$
(b) $\sin \theta = 0.423$ (d) $\sin \emptyset = 0.559$

12. Use 3-figure tables or a scientific calculator to determine the size of each of the following unknown angles, given that:

(a) $\sin a = 0.602$ (b) $\sin b = 0.669$
(c) $\sin \alpha = 0.809$ (d) $\sin \beta = 0.883$

13. Use 3-figure tables or a scientific calculator to determine the magnitude of each of the following unknown angles, given that:

(a) $\sin A\hat{B}C = 0.151$ (b) $\sin X\hat{Y}Z = 0.235$
(c) $\sin P\hat{Q}R = 0.261$ (d) $\sin S\hat{T}U = 0.299$

14. Use 3-figure tables or a scientific calculator to evaluate the magnitude of each of the following unknown angles, given that:

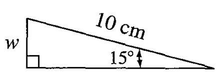
(a) $\sin C\hat{A}B = 0.710$ (b) $\sin B\hat{A}C = 0.741$
(c) $\sin P\hat{Q}R = 0.830$ (d) $\sin L\hat{M}N = 0.865$

15. Use 3-figure tables or a scientific calculator to find the size of each of the following unknown angles, given that:

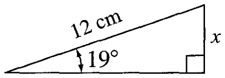
(a) $\sin x = 0.909$ (b) $\sin y = 0.922$
(c) $\sin \theta = 0.964$ (d) $\sin \emptyset = 0.984$

16. Calculate the length of the marked side in each of the following right-angled triangles:

(a)



(b)



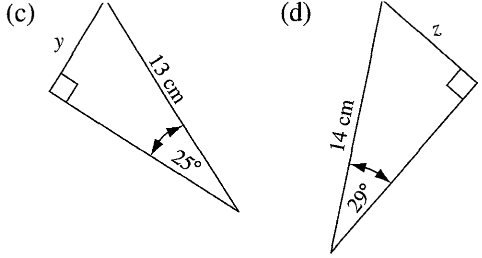


Fig. 11.7 Right-angled triangles

17. Determine the length of the marked side in each of the following right-angled triangles:

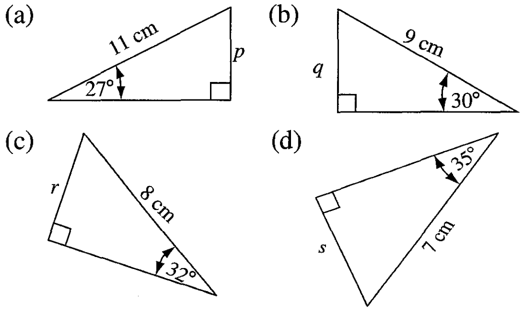


Fig. 11.8 Right-angled triangles

18. Evaluate the length of the marked side in each of the following right-angled triangles:

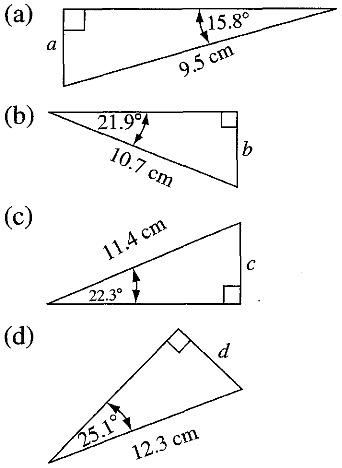
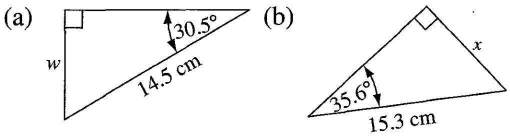


Fig. 11.9 Right-angled triangles

19. Evaluate the length of the marked side in each of the following right-angled triangles:



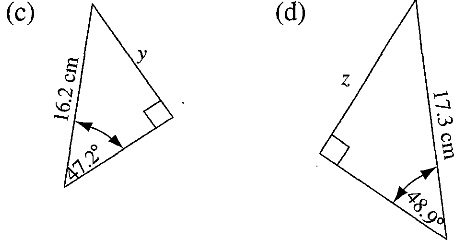


Fig. 11.10 Right-angled triangles

20. Calculate the length of the marked side in each of the following right-angled triangles:

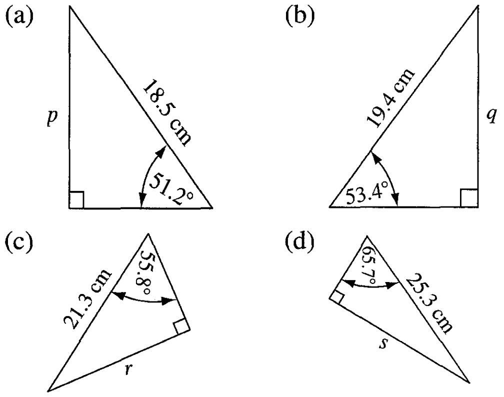


Fig. 11.11 Right-angled triangles

21. Calculate the length of the marked side in each of the following right-angled triangles:

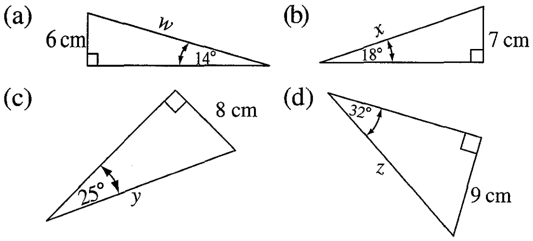


Fig. 11.12 Right-angled triangles

22. Determine the length of the marked side in each of the following right-angled triangles:

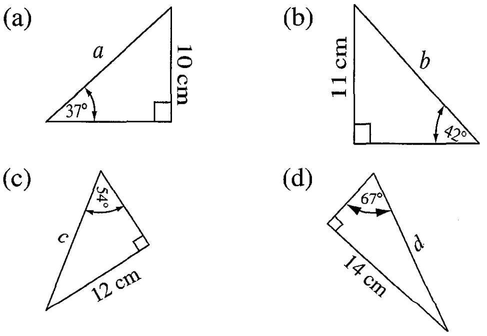


Fig. 11.13 Right-angled triangles

23. Determine the length of the marked side in each of the following right-angled triangles:

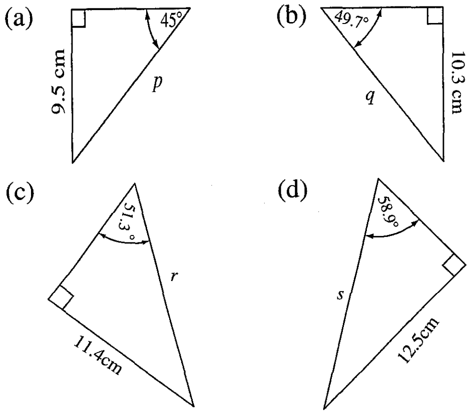


Fig. 11.14 Right-angled triangles

24. Evaluate the length of the marked side in each of the following right-angled triangles:

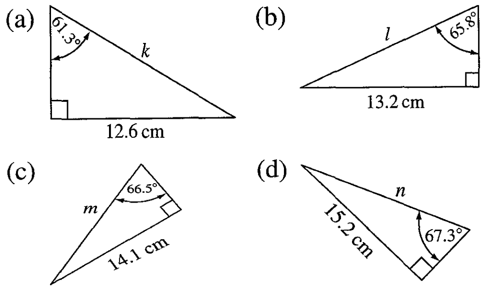
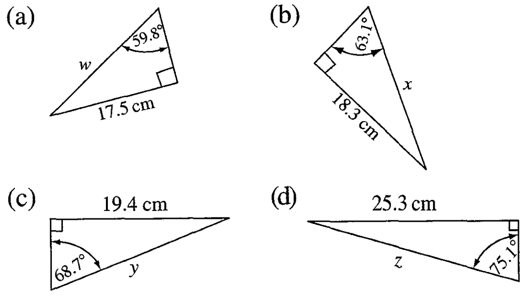
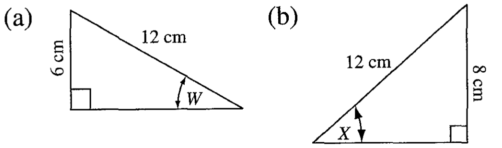


Fig. 11.15 Right-angled triangles

25. Calculate the length of the marked side in each of the following right-angled triangles:



26. Determine the size of the marked angle in each of the following right-angled triangles:



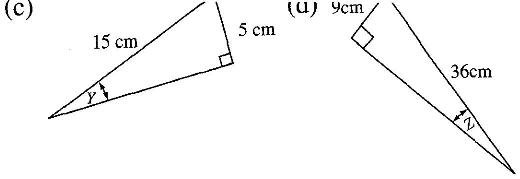


Fig. 11.16 Right-angled triangles

Fig. 11.17 Right-angled triangles

27. Determine the size of the marked angle in each of the following right-angled triangles:

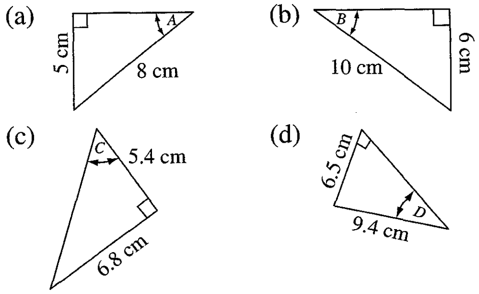


Fig. 11.18 Right-angled triangles

28. Calculate the magnitude of the marked angle in each of the following right-angled triangles:

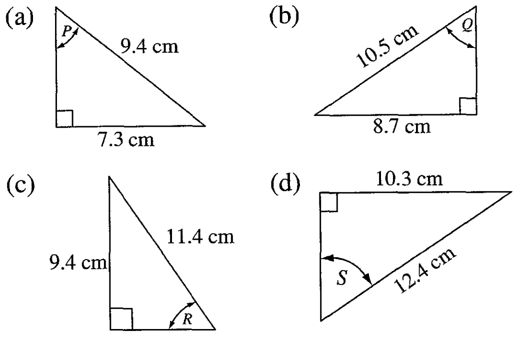


Fig. 11.19 Right-angled triangles

29. Evaluate the magnitude of the marked angle in each of the following right-angled triangles:

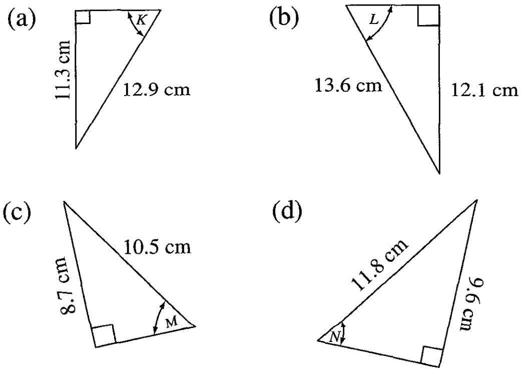


Fig. 11.20 Right-angled triangles

30. Calculate the size of the marked angle in each of the following right-angled triangles:

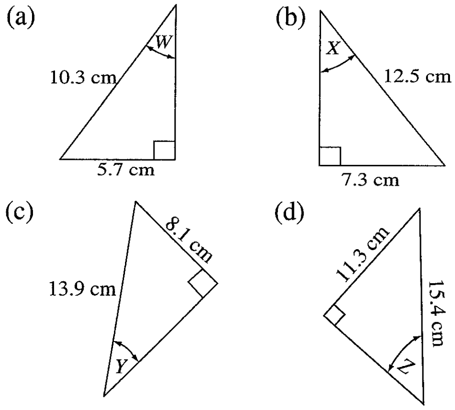
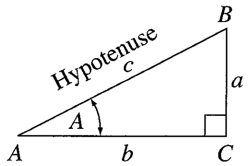


Fig. 11.21 Right-angled triangles

Cosine of an Angle



The side adjacent to angle *A*

Fig. 11.22 Right-angled triangle

The *cosine of an angle* is defined as the *ratio* of the *side adjacent to the angle* to the *hypotenuse*. Thus:

$$\text{The cosine of an angle} = \frac{\text{The side adjacent to the angle}}{\text{The hypotenuse}}$$

The word *cosine* can be abbreviated to *cos*.

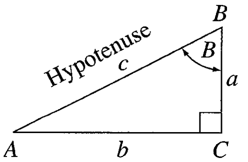
From Fig. 11.22:

$$\text{The cosine of angle } A = \frac{\text{The side adjacent to angle } A}{\text{The hypotenuse}}$$

or $\cos\hat{A} = \frac{Adj}{Hyp}$

or $\cos\hat{A} = \frac{AC}{AB}$

or $\cos\hat{A} = \frac{b}{c}$



The side adjacent to angle *B*

Fig. 11.23 Right-angled triangle

From Fig. 11.23:

$$\text{The cosine of angle } B = \frac{\text{The side adjacent to angle } B}{\text{The hypotenuse}}$$

or $\cos\hat{B} = \frac{Adj}{Hyp}$

or $\cos\hat{B} = \frac{BC}{AB}$

or $\cos\hat{B} = \frac{a}{c}$

Cos θ

The *cosine of an angle* θ can be obtained from the *'Natural Cosines' table* in 3-*figure table* or from a *scientific calculator.*

USING 3-FIGURE TABLES

Natural Cosines

Table 11.5 Extract from a table of 'Natural Cosines'

Degrees	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
20	0.940	939	938	938	937	937	936	935	935	934
21	.934	933	932	932	931	930	930	929	928	928
22	.927	927	926	925	925	924	923	923	922	921
23	.921	**920**	919	918	918	**917**	916	916	**915**	914
24	.914	913	912	911	911	910	909	909	908	907
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 11.5 is an *extract* from a *table* of *'Natural Cosines'*. It shows the *cosines of angles from* 20.0 *degrees to* 24.9 *degrees.* The *first column* indicates *whole number of degrees* from 20 to 24 and the *top row* indicates *decimal parts of a degree* from 0.0 to 0.9. The *cosine of an angle* is that *value corresponding* to a *particular whole number of degrees* and a *particular decimal parts of a degree.*

For example:

(a) The *cosine of* 23.1 *degrees* = $\cos 23.1° = 0.920$

(b) The *cosine of* 23.5 *degrees* = $\cos 23.5° = 0.917$

(c) The *cosine of* 23.8 *degrees* = $\cos 23.8° = 0.915$

USING A SCIENTIFIC CALCULATOR

Table 11.6 Using a scientific calculator

	Input	*Display*
(a)	cos 23.1 =	cos 23.1 0.919 821 497
(b)	cos 23.5 =	cos 23.5 0.917 060 074
(c)	cos 23.8 =	cos 23.8 0.914 959 667

Table 11.6 illustrates how to *find* the *cosine of an angle* using a *scientific calculator*. The *cosine function key* is *first pressed* (*input*), *then the magnitude of the angle* is *input, after* which the *equals sign* is *pressed* (*input*). On the *display* we will then see the *value corresponding* to the *cosine* of that *particular angle.* We must bear in mind that a calculator is far *more accurate* than 3-*figure tables* and that we should *state* our *answer correct* to *three significant figures.*

Thus:

(a) $cos\, 23.1° = 0.919\,|821\,497$
$= 0.920$ (*correct* to 3 *s.f.*).

(b) $cos\, 23.5° = 0.917\,|060\,074$
$= 0.917$ (*correct* to 3 *s.f.*).

(c) $cos\, 23.8° = 0.914\,|959\,667$
$= 0.915$ (*correct* to 3 *s.f.*).

Example 7

Use 3-figure tables or a scientific calculator to find:

(a) cos 9.5° *(b) cos 28.7°*

(c) cos 67.3° *(d) cos 78.1°*

Solution

Using 3-*figure tables*:

(a) $cos\, 9.5° = 0.986$

(b) $cos\, 28.7° = 0.877$

(c) $cos\, 67.3° = 0.386$

(d) $cos\, 78.1° = 0.206$

Using a *scientific calculator*:

(a) $cos\, 9.5° = 0.986|285\,601 = 0.986$
(*correct* to 3 *s.f.*).

(b) $cos\, 28.7° = 0.877|146\,163 = 0.877$
(*correct* to 3 *s.f.*).

(c) $cos\, 67.3° = 0.385|906\,042 = 0.386$
(*correct* to 3 *s.f.*).

(d) $cos\, 78.1° = 0.206|204\,185 = 0.206$
(*correct* to 3 *s.f.*).

It should be noted that $0 \leqslant cos\, \theta \leqslant 1$, if $0° \leqslant \theta \leqslant 90°$ (i.e. if θ is an *acute angle*).
The *smallest value* of $cos\, \theta$, is 0 and it *occurs when* $\theta = 90°$, i.e. $cos\, 90° = 0$.
The *greatest value* of $cos\, \theta$ is 1 and it *occurs when* $\theta = 0°$, i.e. $cos\, 0° = 1$.

Cos^{-1} x or Arccos x

The *angle whose cosine* is of a *given value* can be obtained from the '*Natural Cosines*' *table* in 3-*figure tables* or from a *scientific calculator.*

USING 3-FIGURE TABLES

Natural Cosines

Tables 11.7 Extract from a table of a 'Natural Cosines'

Degrees	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
85	0.087	085	084	082	080	078	077	075	073	071
86	.070	068	066	065	063	061	059	058	056	054
87	**.052**	051	049	047	045	044	**042**	040	038	**037**
88	.035	033	031	030	028	026	024	023	021	019
89	.017	016	014	012	010	009	007	005	003	002
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 11.7 is an *extract* from a *table of 'Natural Cosines'*. It shows the *cosines of angles* from 85.0 *degrees to* 89.9 *degrees.* The *first column* indicates *whole number of degrees* from 85 *to* 89 and the *top row* indicates *decimal parts of a degree* from 0.0 *to* 0.9.

The *angle whose cosine* is a *given value* can be obtained from the '*Natural Cosines*' *table.* For example:

(a) The *angle whose cosine is* $0.052 = cos^{-1} 0.052$
$= arccos\, 0.052 = 87.0°$.

(b) The *angle whose cosine is* 0.042 = $cos^{-1}0.42$ = *arccos* 0.042 = 87.6°.

(c) The *angle whose cosine is* 0.037 = $cos^{-1}0.037$ = *arccos* 0.037 = 87.9°.

It can be *seen* that cos^{-1} (or *arccos*) means *'The angle whose cosine is'*.

USING A SCIENTIFIC CALCULATOR

Table 11.8 Using a scientific calculator

	Input	*Display*
(a)	inv cos 0.052 =	$\cos^{-1}0.052$ 87.019 275 12
(b)	inv cos 0.042 =	$\cos^{-1}0.042$ 87.592 869 21
(c)	inv cos 0.037 =	$\cos^{-1}0.052$ 87.879 572 16

Table 11.8 above illustrates how to *find* the *angle whose cosine* is of a *given value* using a *scientific calculator.* The *INV key* (or *2nd FUN key*) is first *pressed*, after which the *cosine function key is pressed* (*input*), the value is then input, after which the equals sign is pressed. On the *display* we will then see the angle *corresponding* to the *cosine of that particular value.* We again must bear in mind that a *calculator* is far *more accurate* then 3-*figure tables* and that we should *state* our *answer correct to three significant figures.*

Thus:

(a) $cos^{-1}0.052$ = arccos 0.052
= 87.0|19 275 12°
= 87.0° (*correct* to 3 *s.f.*).

(b) $cos^{-1}0.042$ = arccos 0.042
= 87.5|92 869 21°
= 87.6° (*correct* to 3 *s.f.*).

(c) $cos^{-1}0.037$ = arccos 0.037
= 87.8|79 572 16°
= 87.9° (*correct* to 3 *s.f.*).

Example 8

Use 3-figure tables or a scientific calculator to find:

(a) $cos^{-1}0.250$ (b) $cos^{-1}0.497$
(c) arccos 0.636 (d) arccos 0.936

Solution

Using *3-figure tables*:

(a) $cos^{-1}0.250$ = 75.5°
(b) $cos^{-1}0.497$ = 60.2°
(c) *arccos* 0.636 = 50.5°
(d) *arccos* 0.936 = 20.6°

Using a *scientific calculator*:

(a) $cos^{-1}0.250$ = 75.5|22 487 81°
= 75.5° (*correct* to 3 *s.f.*).

(b) $cos^{-1}0.497$ = 60.1|98 280 72°
= 60.2° (*correct* to 3 *s.f.*).

(c) *arccos* 0.636 = 50.5|05 807 88°
= 50.5° (*correct* to 3 *s.f.*).

(d) *arccos* 0.936 = 20.6|09 692 94°
= 20.6° (*correct* to 3 *s.f.*).

Example 9

Use 3-figure tables or a scientific calculator to determine the size of each of the following unknown angles, given that:

(a) $\cos x = 0.917$ (b) $\cos y = 0.871$
(c) $\cos \theta = 0.768$ (d) $\cos \varnothing = 0.274$

Solution

Using 3-*figure tables*:

(a) Given that $\cos x = 0.917$
then $x = cos^{-1}0.917 = 23.5°$

(b) Given that $\cos y = 0.871$
then $y = cos^{-1}0.871 = 29.4°$

(c) Given that $\cos \theta = 0.768$
then $\theta = cos^{-1}0.768 = 39.8°$

(d) Given that $\cos \varnothing = 0.274$
then $\varnothing = cos^{-1}0.274 = 74.1°$

Using a *scientific calculator*:

(a) Given that $\cos x = 0.917$
then x = *arccos* 0.917
= 23.5|08 630 52°
= 23.5° (*correct* to 3 *s.f.*).

(b) Given that $\cos y = 0.871$

then $y = \textit{arccos}\ 0.871$

$= 29.4|24\,945\,31°$

$= 29.4°$ (*correct* to 3 *s.f.*).

(c) Given that $\cos \theta = 0.768$

then $\theta = \textit{arccos}\ 0.768$

$= 39.8|25\,371\,26°$

$= 39.8°$ (*correct* to 3 *s.f.*).

(d) Given that $\cos ø = 0.274$

then $ø = \textit{arccos}\ 0.274$

$= 74.0|97\,570\,44°$

$= 74.1°$ (*correct* to 3 *s.f.*).

Calculating an Unknown Side

The *cosine of an angle* can be used to *calculate* either the *adjacent side to a given angle* or the *hypotenuse*.

Example 10

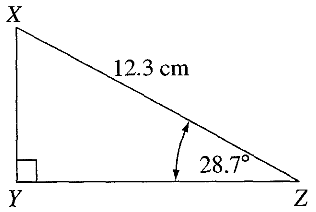


Fig. 11.24 Right-angled triangle

In the diagram above, △XYZ is right-angled at Y with angle XZY = 28.7° and XZ = 12.3 cm. Calculate the length of the side YZ correct to three significant figures.

Solution

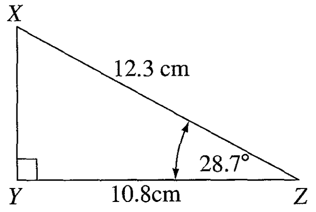


Fig. 11.24 Right-angled triangle

Considering the *right-angled* △XYZ.

$$\cos 28.7° = \frac{Adj}{Hyp} = \frac{YZ}{XZ} = \frac{YZ}{12.3 \text{ cm}}$$

So $YZ = 12.3 \text{ cm} \times \cos 28.7°$

$= 12.3 \text{ cm} \times 0.877$

$= 10.8$ cm (*correct* to 3 *s.f.*).

Hence the *length* of the side *YZ* is 10.8 cm.

Example 11

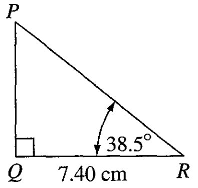


Fig. 11.25 Right-angled triangle

In the diagram above, △PQR is right-angled at Q with angle PRQ = 38.5° and QR = 7.40 cm. Determine the length of the side PR correct to three significant figures.

Solution

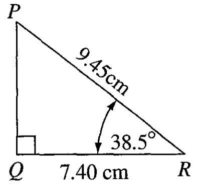


Fig. 11.25 Right-angled triangle

Considering the *right-angled* △*PQR*:

$$\textit{cos}\, 38.5° = \frac{Adj}{Hyp} = \frac{QR}{PR} = \frac{7.40 \text{ cm}}{PR}$$

So $PR = \dfrac{7.40 \text{ cm}}{\cos 38.5°}$

$= \dfrac{7.40 \text{ cm}}{0.783}$

$= 9.45$ cm (*correct* to 3 *s.f.*).

Hence the *length* of *PR* is 9.45 cm.

Calculating an Unknown Angle

The *cosine of an angle* can be used to calculate an *angle* given the *hypotenuse and the length of the side adjacent to the angle.*

Example 12

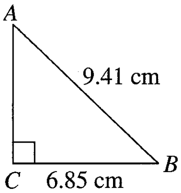


Fig. 11.26 Right-angled triangle

In the diagram above, $\triangle ABC$ is right-angled at C with $AB = 9.41$ cm and $BC = 6.85$ cm.

Calculate the size of angle ABC correct to three significant figures.

Solution

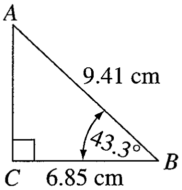


Fig. 11.26 Right-angled triangle

Considering the *right-angled* $\triangle ABC$:

$$\cos A\hat{B}C = \frac{Adj}{Hyp} = \frac{BC}{AB} = \frac{6.85 \cancel{cm}}{9.41 \cancel{cm}} = 0.728$$

So $A\hat{B}C = \cos^{-1} 0.728$

$= 43.3°$ (*correct* to 3 *s.f.*)

Hence the size of *angle ABC* is 43.3°.

Exercise 11b

1. Use 3-figure tables or a scientific calculator to find:

(a) $\cos 5°$ (b) $\cos 18°$
(c) $\cos 21°$ (d) $\cos 29°$

2. Use 3-figure tables or a scientific calculator to determine:

(a) $\cos 47°$ (b) $\cos 53°$
(c) $\cos 63°$ (d) $\cos 75°$

3. Use 3-figure tables or a scientific calculator to determine:

(a) $\cos 7.1°$ (b) $\cos 9.5°$
(c) $\cos 27.9°$ (d) $\cos 39.3°$

4. Use 3-figure tables or a scientific calculator to evaluate:

(a) $\cos 45.1°$ (b) $\cos 53.4°$
(c) $\cos 67.3°$ (d) $\cos 75.8°$

5. Use 3-figure tables or a scientific calculator to find:

(a) $\cos 76.3°$ (b) $\cos 81.2°$
(c) $\cos 85.4°$ (d) $\cos 89.7°$

6. Use 3-figure tables or a scientific calculator to find:

(a) $\cos^{-1} 1.000$ (b) $\cos^{-1} 0.927$
(c) $\cos^{-1} 0.857$ (d) $\cos^{-1} 0.731$

7. Use 3-figure tables or a scientific calculator to determine:

(a) $\cos^{-1} 0.602$ (b) $\cos^{-1} 0.469$
(c) $\cos^{-1} 0.259$ (d) $\cos^{-1} 0.017$

8. Use 3-figure tables or a scientific calculator to determine:

(a) $\arccos 0.853$ (b) $\arccos 0.791$
(c) $\arccos 0.759$ (d) $\arccos 0.702$

9. Use 3-figure tables or a scientific calculator to evaluate:

(a) $\arccos 0.639$ (b) $\arccos 0.546$
(c) $\arccos 0.466$ (d) $\arccos 0.413$

10. Use 3-figure tables or a scientific calculator to find:

(a) $\cos^{-1} 0.272$ (b) $\cos^{-1} 0.196$
(c) $\arccos 0.115$ (d) $\arccos 0.065$

11. Use 3-figure tables or a scientific calculator to find the size of each of the following unknown angles, given that:

(a) $\cos x = 0.899$ (b) $\cos y = 0.829$
(c) $\cos \theta = 0.755$ (d) $\cos \phi = 0.616$

12. Use 3-figure tables or a scientific calculator to find the size of each of the following unknown angles, given that:

(a) $\cos a = 0.454$ (b) $\cos b = 0.259$
(c) $\cos \alpha = 0.052$ (d) $\cos \beta = 0.035$

13. Use 3-figure tables or a scientific calculator to determine the magnitude of each of the following unknown angles, given that:

(a) $\cos A\hat{B}C = 0.939$ (b) $\cos X\hat{Y}Z = 0.884$

(c) $\cos P\hat{Q}R = 0.837$ (d) $\cos S\hat{T}U = 0.759$

14. Use 3-figure tables or a scientific calculator to evaluate the magnitude of each of the following unknown angles, given that:

(a) $\cos C\hat{A}B = 0.655$ (b) $\cos B\hat{A}C = 0.584$

(c) $\cos P\hat{Q}R = 0.549$ (d) $\cos L\hat{M}N = 0.446$

15. Use 3-figure tables or a scientific calculator to find the size of each of the following unknown angles, given that:

(a) $\cos x = 0.322$ (b) $\cos y = 0.201$
(c) $\cos \theta = 0.158$ (d) $\cos \phi = 0.005$

16. Calculate the length of the marked side in each of the following right-angled triangles:

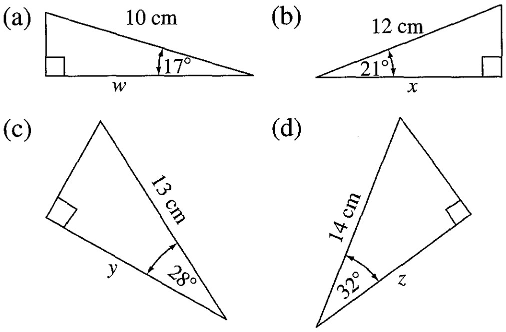


Fig. 11.27 Right-angled triangles

17. Determine the length of the marked side in each of the following right-angled triangles:

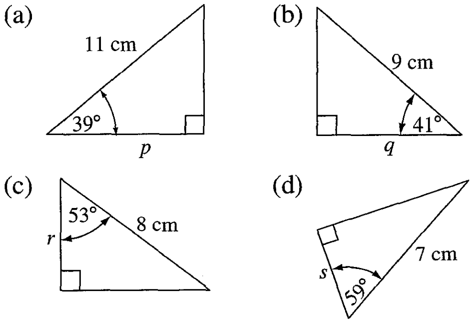


Fig. 11.28 Right-angled triangles

18. Evaluate the length of the marked side in each of the following right-angled triangles:

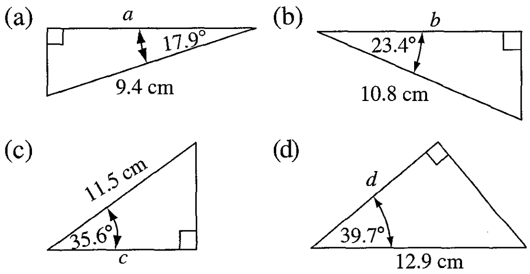


Fig. 11.29 Right-angled triangles

19. Evaluate the length of the marked side in each of the following right-angled triangles:

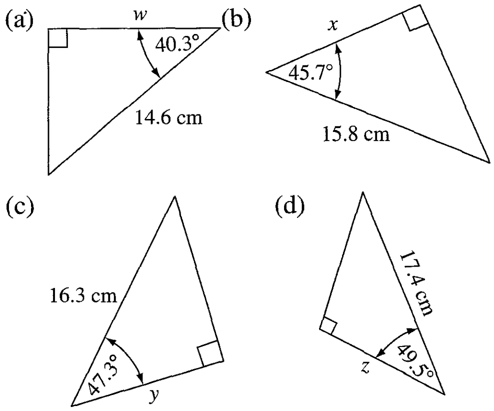


Fig. 11.30 Right-angled triangles

20. Calculate the length of the marked side in each of the following right-angled triangles:

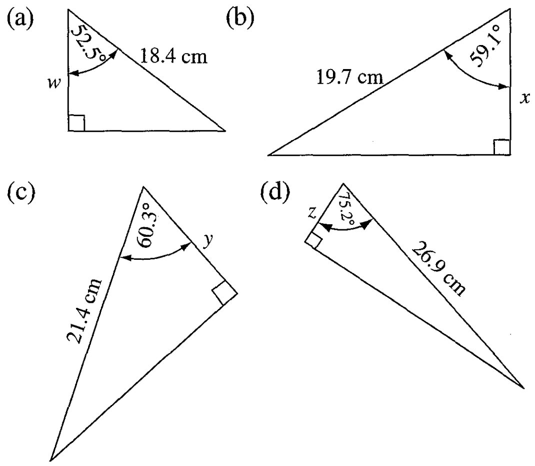


Fig. 11.31 Right-angled triangles

21. Calculate the length of the marked side in each of the following right-angled triangles:

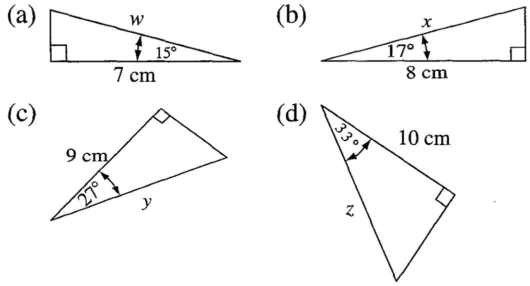


Fig. 11.32 Right-angled triangles

22. Determine the length of the marked side in each of the following right-angled triangles:

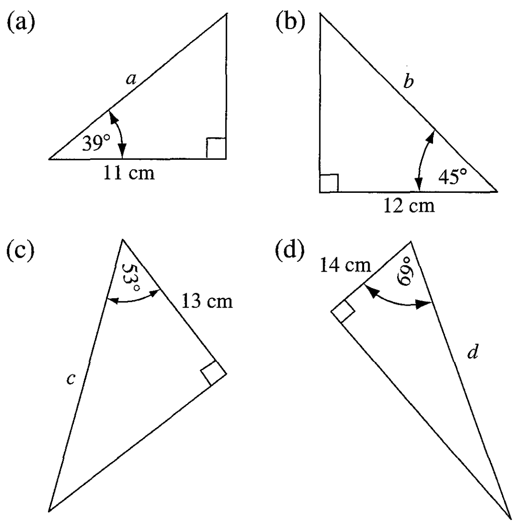


Fig. 11.33 Right-angled triangles

23. Determine the length of the marked side in each of the following right-angled triangles:

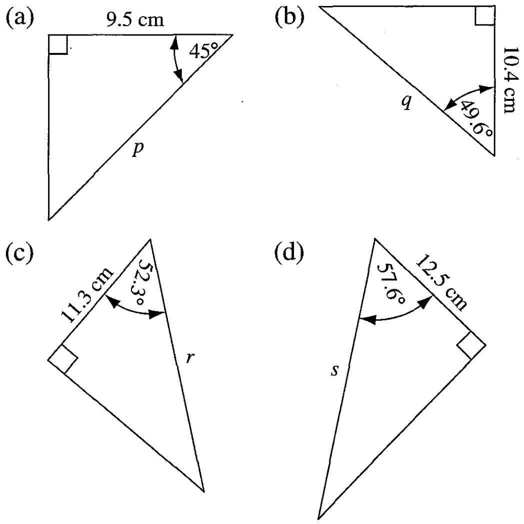
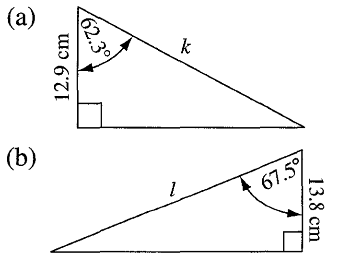


Fig. 11.34 Right-angled triangles

24. Evaluate the length of the marked side in each of the following right-angled triangles:



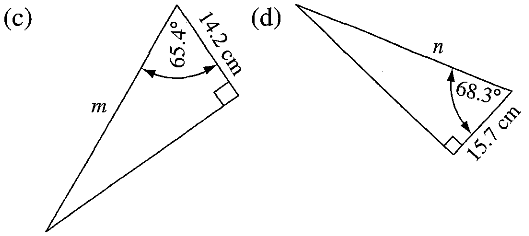


Fig. 11.35 Right-angled triangles

25. Calculate the length of the marked side in each of the following right-angled triangles:

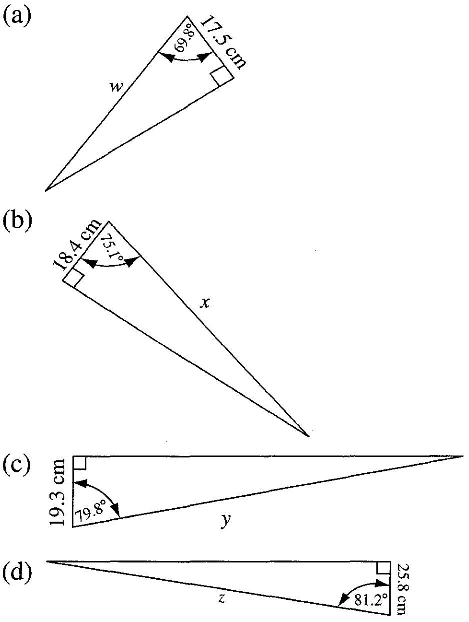


Fig. 11.36 Right-angled triangles

26. Determine the size of the marked angle in each of the following right-angled triangles:

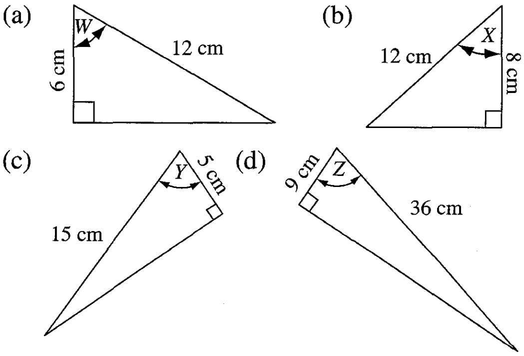


Fig. 11.37 Right-angled triangles

27. Determine the size of the marked angle in each of the following right-angled triangles:

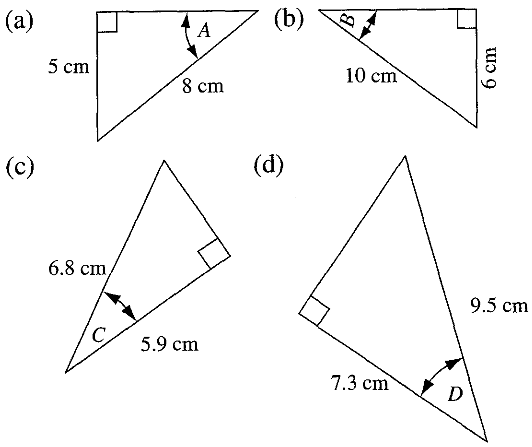


Fig. 11.38 Right-angled triangles

28. Calculate the magnitude of the marked angle in each of the following right-angled triangles:

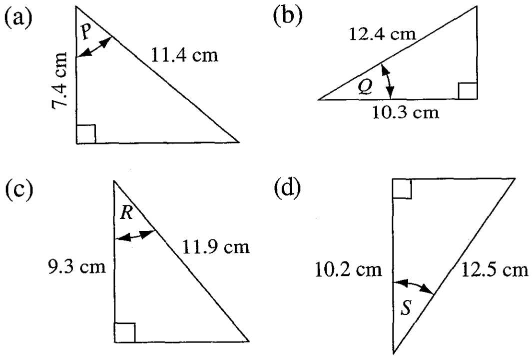


Fig. 11.39 Right-angled triangles

29. Evaluate the magnitude of the marked angle in each of the following right-angled triangles:

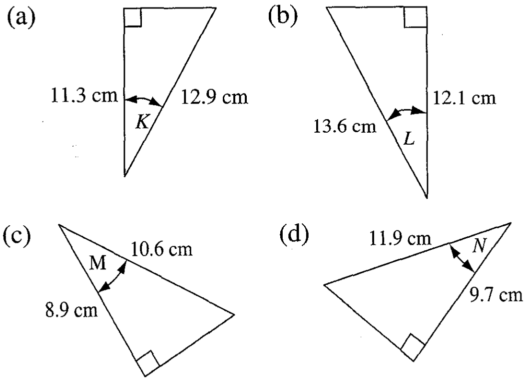


Fig. 11.40 Right-angled triangles

30. Calculate the size of the marked angle in each of the following right-angled triangles:

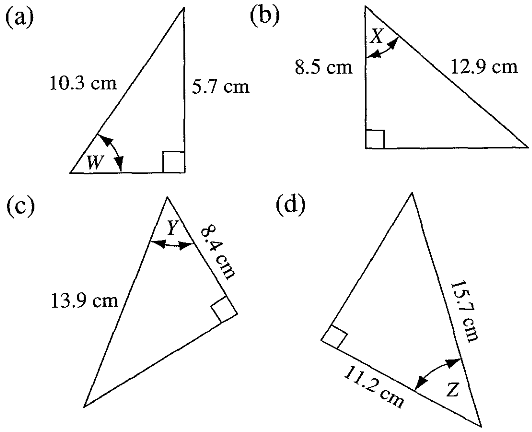


Fig. 11.41 Right-angled triangles

Tangent of an Angle

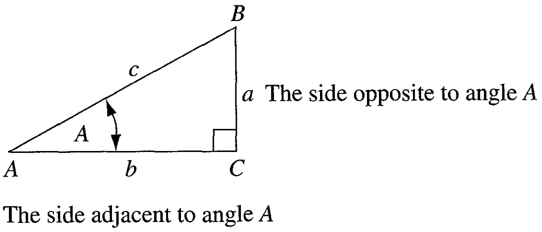


Fig. 11.42 Right-angled triangle

The *tangent of an angle* is defined as the *ratio* of the *side opposite to the angle* to the *side adjacent to the angle*. Thus:

$$\text{The tangent of an angle} = \frac{\text{The side opposite to the angle}}{\text{The side adjacent to the angle}}$$

The word *tangent* can be abbreviated to *tan*.

From Fig. 11.42:

$$\text{The tangent of angle } A = \frac{\text{The side opposite to angle } A}{\text{The side adjacent to angle } A}$$

or $$tan\hat{A} = \frac{Opp}{Adj}$$

or $$tan\hat{A} = \frac{AB}{AC}$$

or $$tan\hat{A} = \frac{a}{b}$$

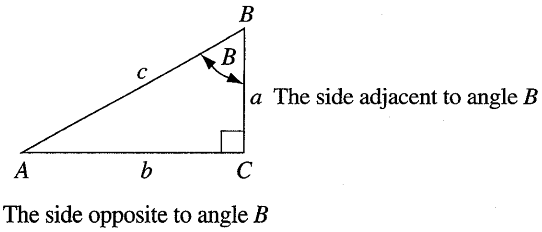


Fig. 11.43 Right-angled triangle

$$\text{The tangent of angle } \hat{B} = \frac{\text{The side opposite to angle B}}{\text{The side adjacent to angle B}}$$

or $$\tan \hat{B} = \frac{Opp}{Adj}$$

or $$\tan \hat{B} = \frac{AC}{BC}$$

or $$\tan \hat{B} = \frac{b}{a}$$

Tan θ

The *tangent of an angle* θ can be obtained from the '*Natural Tangents*' table in 3-*figure tables* or from a *scientific calculator.*

USING 3-FIGURE TABLES

Natural Tangents

Table 11.9 Extract from a table of 'Natural Tangents'

Degrees	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
60	1.732	739	746	753	760	767	775	782	789	797
61	.804	811	819	827	834	842	849	857	865	873
62	.881	889	897	905	913	921	929	937	946	954
63	1.963	**971**	980	988	997	**2.006**	2.014	2.023	2.032	2.041
64	2.050	059	069	078	087	097	106	116	125	135
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 11.9 is an *extract* from a *table of 'Natural Tangents'*. It shows the *tangents of angles* from 60.0 *degrees* to 64.9 *degrees.* The *first column* indicates *whole number of degrees* from 60 to 64 and the *top row* indicates *decimal parts of a degree* from 0.0 to 0.9. The *tangent of an angle* is that *value corresponding* to a *particular whole number of degrees* and a *particular decimal parts of a degree.*

For example:

(a) The *tangent of* 63.1 *degrees* = *tan* 63.1° = 1.971

(b) The *tangent of* 63.5 *degrees* = *tan* 63.2° = 2.006

(c) The *tangent of* 63.8 *degrees* = *tan* 63.8° = 2.032

USING A SCIENTIFIC CALCULATOR

Table 11.10 Using a scientific calculator

	Input	*Display*
(a)	tan 63.1 =	tan 63.1 1.971 107 679
(b)	tan 63.5 =	tan 63.5 2.005 689 708
(c)	tan 63.8 =	tan 63.8 2.032 268 347

Table 11.10 above illustrates how to *find* the *tangent of an angle* using a *scientific calculator.* The *tangent function key* is *first pressed* (*input*), then the *magnitude* of the *angle* is *input, after* which the *equals sign* is *pressed* (*input*). On the *display* we will then see the *value corresponding* to the *tangent* of that *particular angle.* We must bear in mind that a *calculator* is far *more accurate* than 3-*figure tables* and that we should *state* our *answer correct to three significant figures* (and sometimes *four significant figures*).

Thus:

(a) $\tan 63.1° = 1.971|107\,679 = 1.971$ (*correct* to 4 *s.f.*).

(b) $\tan 63.5° = 2.005|689\,708 = 2.006$ (*correct* to 4 *s.f.*).

(c) $\tan 63.8° = 2.032|268\,347 = 2.032$ (*correct* to 4 *s.f.*).

Example 13

Use 3-figure tables or a scientific calculator to find:

(a) tan 24.9° *(b) tan 49.6°*

(c) tan 64.8° *(d) tan 89.3°*

Solution

Using 3-*figure tables*:

(a) *tan* 24.9° = 0.464

(b) *tan* 49.6° = 1.175

(c) *tan* 64.8° = 2.125

(d) *tan* 89.3° = 81.85

Using a *scientific calculator*:

(a) tan 24.9° = 0.464|1845 = 0.464 (*correct* to 3 *s.f.*).

(b) tan 49.6° = 1.174|996 = 1.175 (*correct* to 4 *s.f.*).

(c) tan 64.8° = 2.125|1082 = 2.125 (*correct* to 4 *s.f.*).

(d) tan 89.3° = 81.84|7041 = 81.85 (*correct* to 4 *s.f.*).

It should be noted that $0 \leq \tan\theta \leq \propto$, if $0° \leq \theta \leq 90°$ (i.e. if θ is an *acute angle*). The *smallest value* of *tan* θ is 0 and it *occurs when* θ = 0°. i.e. *tan* 0° = 0. *As* θ *approaches* 90°, then *tan* θ *increases rapidly.*

Tan^{-1} x or Arctan x

The *angle whose tangent* is of a *given value* can be obtained from the *'Natural Tangents' table* in 3-*figure tables* or from a *scientific calculator.*

USING 3-FIGURE TABLES

Natural Tangents

Table 11.11 Extract from a table of 'Natural Tangents'

Degrees	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
20	0.364	366	368	370	372	374	376	378	380	382
21	.384	386	388	390	392	394	396	398	400	402
22	.404	406	408	410	412	414	416	418	420	422
23	**.424**	427	429	**431**	433	435	437	**439**	441	443
24	.445	447	449	452	454	456	458	460	462	464
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 11.11 is an *extract* from a *table of 'Natural Tangents'*. It shows the *tangents of angles* from 20.0 *degrees to* 24.9 *degrees.* The *first column* indicates *whole number of degrees* from 20 *to* 24 *and the top row* indicates *decimal parts of a degree from* 0.0 *to* 0.9.

The *angle whose tangent* is a *given value* can be obtained from the *'Natural Tangents' table.*

For example:

(a) The *angle whose tangent is* 0.424 = *tan*$^{-1}$0.424 = *arctan* 0.424 = 23.0°

(b) The *angle whose tangent is* 0.431 = *tan*$^{-1}$0.431 = *arctan* 0.431 = 23.3°

(c) The *angle whose tangent is* 0.439 = *tan*$^{-1}$0.439 = *arctan* 0.439 = 23.7°

It can be *seen* that *tan*$^{-1}$ (or *arctan*) means *'The angle whose tangent is'.*

USING A SCIENTIFIC CALCULATOR

Table 11.12 Using a scientific calculator

	Input	*Display*
(a)	inv tan 0.424 =	tan^{-1}0.424 22.9769445
(b)	inv tan 0.431 =	tan^{-1}0.431 23.31604221
(c)	inv tan 0.439 =	tan^{-1}0.439 23.70147431

Table 11.12 illustrates how to *find* the *angle whose tangent* is of a *given value* using a *scientific calculator.* The *INV key* (or 2*nd FUN key*) is first *pressed, after which the tangent function key is pressed (input), the value is then input, after which the equals sign is pressed.* On the *display* we will then see the *angle corresponding* to the *tangent of that particular value.* We must bear in mind that a *calculator* is far *more accurate* than 3-*figure tables* and that we should *state* our *answer correct* to *three significant figures.*

Thus:

(a) tan^{-1}0.424 = arctan 0.424 = 22.9|769445° = 23.0° (*correct* to 3 *s.f.*).

(b) tan^{-1}0.431 = arctan 0.431 = 23.3|1604221° = 23.3° (*correct* to 3 *s.f.*).

(c) tan^{-1}0.439 = arctan 0.439 = 23.7|0147431° = 23.7° (*correct* to 3 *s.f.*).

Example 14

Use 3-figure tables or a scientific calculator to find:

(a) tan^{-1}*0.169*	*(b) tan*$^{-1}$*1.678*
(c) arctan 3.923	*(d) arctan 13.62*

Solution

Using 3-*figure tables*:

(a) $\tan^{-1} 0.169 = 9.6^\circ$

(b) $\tan^{-1} 1.678 = 59.2^\circ$

(c) $\arctan 3.923 = 75.7^\circ$

(d) $\arctan 13.62 = 85.8^\circ$

Using a *scientific calculator*:

(a) $\tan^{-1} 0.169 = 9.59|234969^\circ$
$= 9.59^\circ$ (*correct* to 3 *s.f.*).
$= 9.6^\circ$ (*correct* to 1 *d.p.*).

(b) $\tan^{-1} 1.678 = 59.2|0727514^\circ$
$= 59.2^\circ$ (*correct* to 3 *s.f.*).

(c) $\arctan 3.923 = 75.6|9945365^\circ$
$= 75.7^\circ$ (*correct* to 3 *s.f.*).

(d) $\arctan 13.62 = 85.8|0079616^\circ$
$= 85.8^\circ$ (*correct* to 3 *s.f.*).

Example 15

Use 3-figure tables or a scientific calculator to determine the size of each of the following unknown angles, given that:

(a) $\tan x = 0.348$ *(b)* $\tan y = 1.209$

(c) $\tan \theta = 2.747$ *(d)* $\tan \phi = 7.300$

Solution

Using 3-*figure tables*:

(a) Given that $\tan x = 0.348$
then $x = \tan^{-1} 0.348 = 19.2^\circ$

(b) Given that $\tan y = 1.209$
then $y = \tan^{-1} 1.209 = 50.4^\circ$

(c) Given that $\tan \theta = 2.747$
then $\theta = \tan^{-1} 2.747 = 70.0^\circ$

(d) Given that $\tan \phi = 7.300$
then $\phi = \tan^{-1} 7.300 = 82.2^\circ$

Using a *scientific calculator*:

(a) Given that $\tan x = 0.348$
then $x = \arctan 0.348$
$= 19.1|878966^\circ$
$= 19.2^\circ$ (*correct* to 3 *s.f.*).

(b) Given that $\tan y = 1.209$
then $y = \arctan 1.209$
$= 50.4|0483355^\circ$
$= 50.4^\circ$ (*correct* to 3 *s.f.*).

(c) Given that $\tan \theta = 2.747$
then $\theta = \arctan 2.747$
$= 69.9|9679968^\circ$
$= 70.0^\circ$ (*correct* to 3 *s.f.*).

(d) Given that $\tan \phi = 7.300$
then $\phi = \arctan 7.300$
$= 82.1|9981212^\circ$
$= 82.2^\circ$ (*correct* to 3 *s.f.*).

Calculating an Unknown Side

The *tangent of an angle* can be used to *calculate* either the *opposite side* or the *adjacent side* to a *given angle.*

Example 16

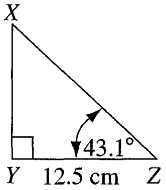


Fig. 11.44 Right-angled triangle

In the diagram above, $\triangle XYZ$ *is right-angled at Y with angle* $XZY = 43.1^\circ$ *and* $YZ = 12.5$ *cm. Calculate the length of the side XY correct to three significant figures.*

Solution

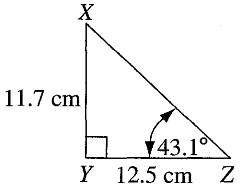


Fig. 11.44 Right-angled triangle

Considering the *right-angled* $\triangle XYZ$:

$$\tan 43.1^\circ = \frac{Opp}{Adj} = \frac{XZ}{YZ} = \frac{XY}{12.5 \text{ cm}}$$

So $XY = 12.5 \text{ cm} \times \tan 43.1^\circ$
$= 12.5 \text{ cm} \times 0.936$
$= 11.7$ cm (*correct* to 3 *s.f.*)

Hence the *length* of the side *XY* is 11.7 cm.

Example 17

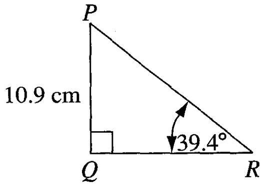


Fig. 11.45 Right-angled triangle

In the diagram above, $\triangle PQR$ is right-angled at Q with angle $R = 39.4°$ and $PQ = 10.9$ cm. Determine the length of the side QR correct to three significant figures.

Solution

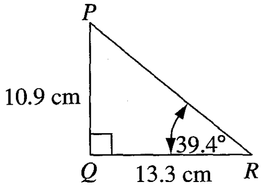


Fig. 11.45 Right-angled triangle

Considering the *right-angled* $\triangle PQR$:

$$tan\ 39.4° = \frac{Opp}{Adj} = \frac{PQ}{QR} = \frac{10.9 \text{ cm}}{QR}$$

So
$$QR = \frac{10.9 \text{ cm}}{\tan 39.4°}$$
$$= \frac{10.9 \text{ cm}}{0.821}$$
$$= 13.3 \text{ cm } (\textit{correct} \text{ to } 3\ \textit{s.f.})$$

Hence the *length* of the side QR is 13.3 cm.

Calculating an Unknown Angle

The *tangent of an angle* can be used to *calculate* an *angle*, *given* the *opposite side* and the *adjacent side* to the *angle*.

Example 18

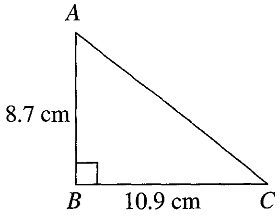


Fig. 11.46 Right-angled triangle

In the diagram above, $\triangle ABC$ is right-angled at B with $AB = 8.7$ cm and $BC = 10.9$ cm. Calculate the size of angle ACB correct to three significant figures.

Solution

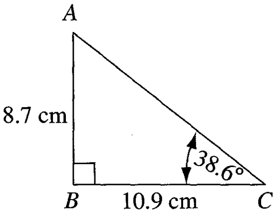


Fig. 11.46 Right-angled triangle

Considering the *right-angled* $\triangle ABC$:

$$tan\ A\hat{C}B = \frac{Opp}{Adj} = \frac{AB}{BC} = \frac{8.7 \cancel{cm}}{10.9 \cancel{cm}} = 0.798$$

So $A\hat{C}B = \tan^{-1} 0.798 = 38.6°$

(*correct* to 3 *s.f.*)

Hence the *size* of *angle ACB* is 38.6°.

Exercise 11c

1. Use 3-figure tables or scientific calculator to evaluate:
(a) tan 8° (b) tan 15°
(c) tan 23° (d) tan 28°

2. Use 3-figure tables or a scientific calculator to determine:
(a) tan 45° (b) tan 52°
(c) tan 61° (d) tan 74°

3. Use 3-figure tables or a scientific calculator to determine:
(a) tan 8.1° (b) tan 9.8°
(c) tan 25.7° (d) tan 35.6°

4. Use 3-figure tables or a scientific calculator to evaluate:

(a) $\tan 45.3°$ (b) $\tan 52.4°$
(c) $\tan 65.2$ (d) $\tan 76.8°$

5. Use 3-figure tables or a scientific calculator to find:

(a) $\tan 79.2°$ (b) $\tan 80.1°$
(c) $\tan 87.3°$ (d) $\tan 89.1°$

6. Use 3-figure tables or a scientific calculator to find:

(a) $\tan^{-1} 0.158$ (b) $\tan^{-1} 0.194$
(c) $\tan^{-1} 0.287$ (d) $\tan^{-1} 0.384$

7. Use 3-figure tables or a scientific calculator to determine:

(a) $\tan^{-1} 0.625$ (b) $\tan^{-1} 0.810$
(c) $\tan^{-1} 1.428$ (d) $\tan^{-1} 2.145$

8. Use 3-figure tables or a scientific calculator to evaluate:

(a) arctan 0.368 (b) arctan 0.543
(c) arctan 0.613 (d) arctan 0.827

9. Use 3-figure tables or a scientific calculator to evaluate:

(a) arctan 0.851 (b) arctan 1.183
(c) arctan 1.303 (d) arctan 1.718

10. Use 3-figure tables or a scientific calculator to find:

(a) $\tan^{-1} 2.059$ (b) $\tan^{-1} 2.633$
(c) arctan 3.096 (d) arctan 5.614

11. Use 3-figure tables or a scientific calculator to find the size of each of the following unknown angles, given that:

(a) $\tan x = 0.035$ (b) $\tan y = 0.287$
(b) $\tan \theta = 0.510$ (d) $\tan \varnothing = 0.727$

12. Use 3-figure tables or a scientific calculator to determine the size of each of the following unknown angles, given that:

(a) $\tan a = 0.869$ (b) $\tan b = 1.376$
(c) $\tan \alpha = 2.050$ (d) $\tan \beta = 3.271$

13. Use 3-figure tables or a scientific calculator to estimate the magnitude of each of the following unknown angles, given that:

(a) $\tan A\hat{B}C = 0.149$ (b) $\tan X\hat{Y}Z = 0.317$
(c) $\tan P\hat{Q}R = 0.396$ (d) $\tan S\hat{T}U = 0.613$

14. Use 3-figure tables or a scientific calculator to evaluate the magnitude of each of the following unknown angles, given that:

(a) $\tan C\hat{A}B = 1.179$ (b) $\tan B\hat{A}C = 1.271$
(b) $\tan P\hat{Q}R = 1.505$ (d) $\tan L\hat{M}N = 1.980$

15. Use 3-figure tables or a scientific calculator to find the size of each of the following unknown angles, given that:

(a) $\tan x = 2.300$ (b) $\tan y = 3.398$
(c) $\tan \theta = 3.923$ (d) $\tan \varnothing = 8.264$

16. Find the length of the marked side in each of the following right-angled triangles:

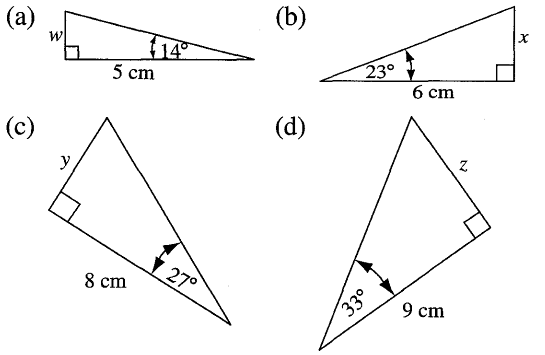


Fig. 11.47 Right-angled triangles

17. Determine the length of the marked side in each of the following right-angled triangles:

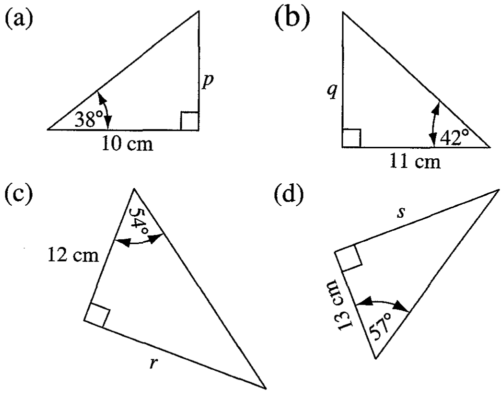


Fig. 11.48 Right-angled triangles

18. Calculate the length of the marked side in each of the following right-angled triangles:

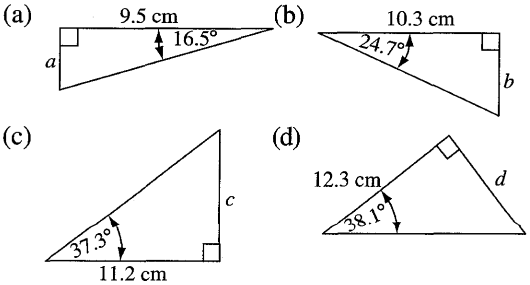


Fig. 11.49 Right-angled triangles

19. Evaluate the length of the marked side in each of the following right-angled triangles:

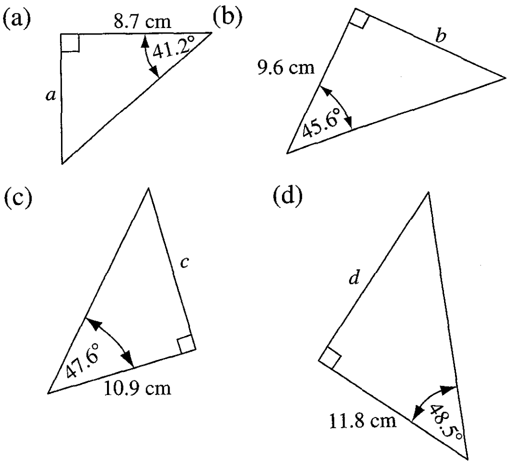


Fig. 11.50 Right-angled triangles

20. Determine the length of the marked side in each of the following right-angled triangles:

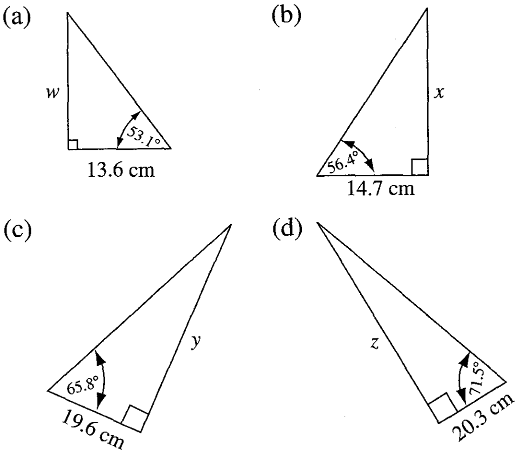


Fig. 11.51 Right-angled triangles

21. Calculate the length of the marked side in each of the following right-angled triangles:

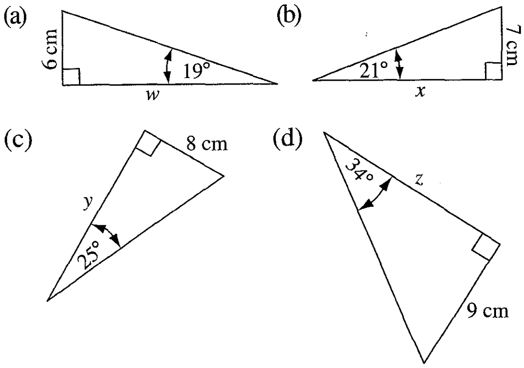


Fig. 11.52 Right-angled triangles

22. Determine the length of the marked side in each of the following right-angled triangles:

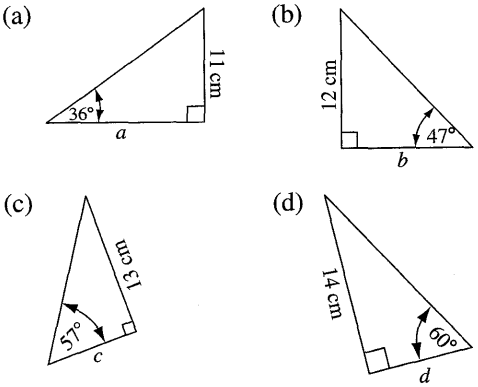


Fig. 11.53 Right-angled triangles

23. Calculate the length of the marked side in each of the following right-angled triangles:

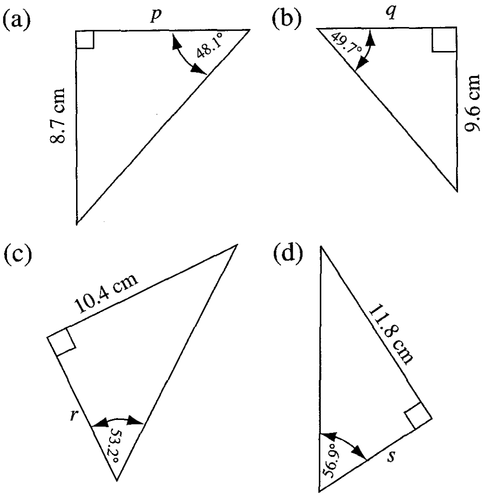


Fig. 11.54 Right-angled triangles

24. Evaluate the length of the marked side in each of the following right-angled triangles:

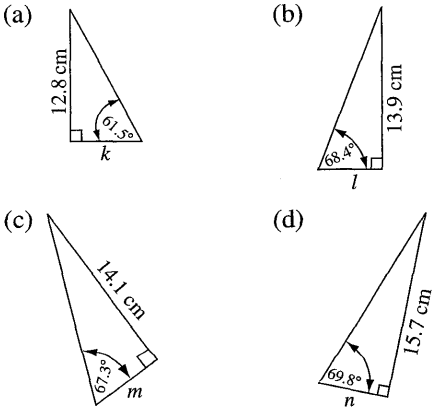


Fig. 11.55 Right-angled triangles

25. Calculate the length of the marked side in each of the following right-angled triangles:

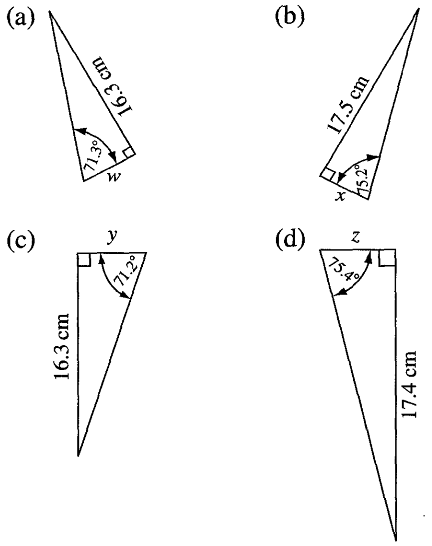


Fig. 11.56 Right-angled triangles

26. Calculate the size of the marked angle in each of the following right-angled triangles:

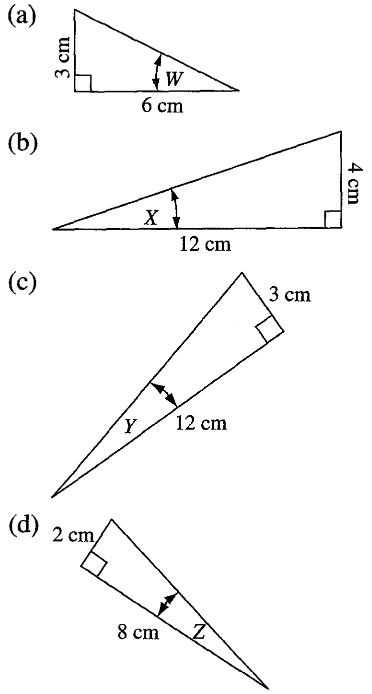
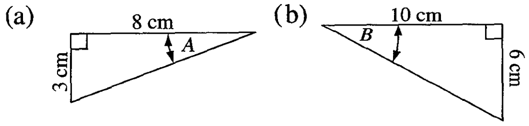


Fig. 11.57 Right-angled triangles

27. Determine the size of the marked angle in each of the following right-angled triangles:



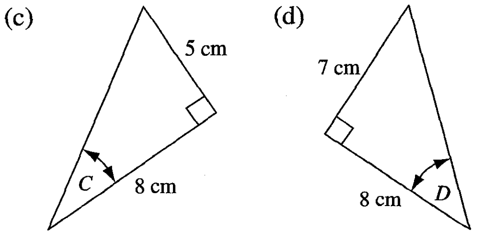


Fig. 11.58 Right-angled triangles

28. Calculate the magnitude of the marked angle in each of the following right-angled triangles:

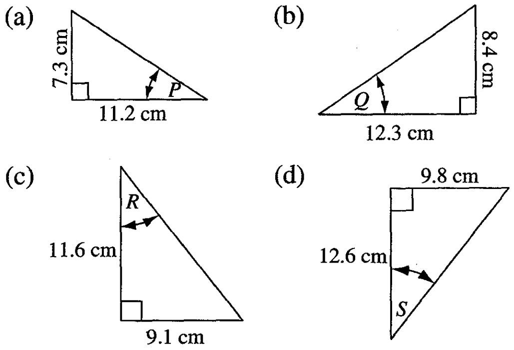


Fig. 11.59 Right-angled triangles

29. Evaluate the magnitude of the marked angle in each of the following right-angled triangles:

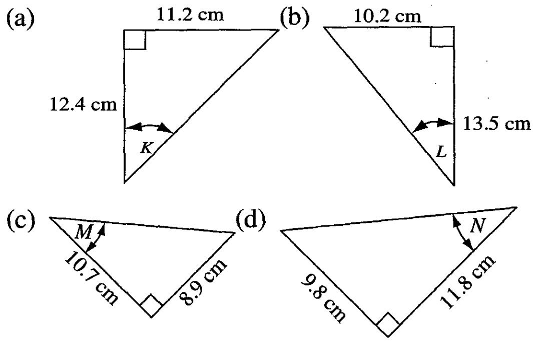
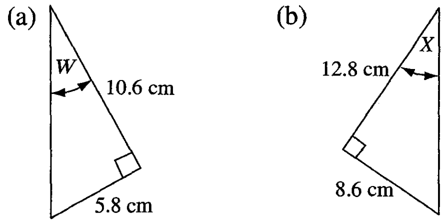


Fig. 11.60 Right-angled triangles

30. Calculate the size of the marked angle in each of the following right-angled triangles:



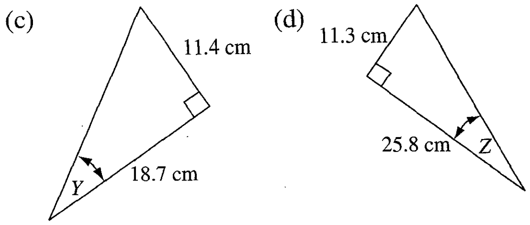


Fig. 11.61 Right-angled triangles

Mixed Problems

The *trigonometric ratios*: *sine, cosine* and *tangent*, can be used to *solve* the *problems* given below.

Exercise 11d

1. Calculate the length of the side AB in the following triangle:

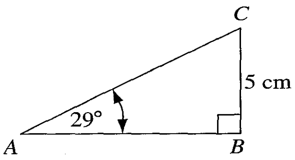


Fig. 11.62 Right-angled triangle

2. Determine the length of the side AC in the following triangle:

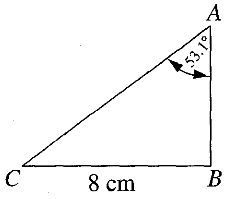


Fig. 11.63 Right-angled triangle

3. Evaluate the length of the side AB in the following triangle:

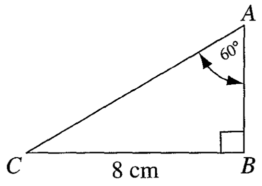


Fig. 11.64 Right-angled triangle

4. Calculate the size of the angle BAC in the following triangle:

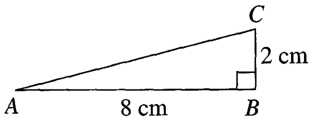


Fig. 11.65 Right-angled triangle

5. Determine the size of angle BAC in the following triangle:

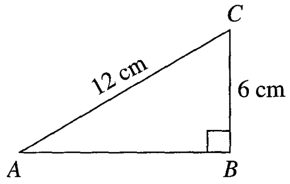


Fig. 11.66 Right-angled triangle

6. Calculate the magnitude of angle BAC in the following triangle:

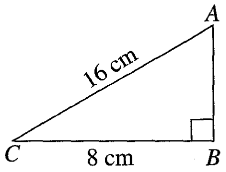


Fig. 11.67 Right-angled triangle

7.

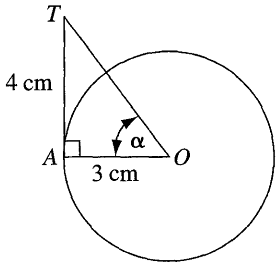


Fig.11.68 Triangle and circle

(a) Calculate the magnitude of angle α in the diagram above.

(b) Determine the length of OT.

8.

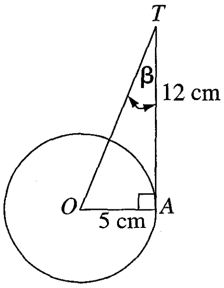


Fig. 11.69 Triangle and circle

(a) Calculate the size of angle β in the diagram.

(b) Determine the length of OT.

9.

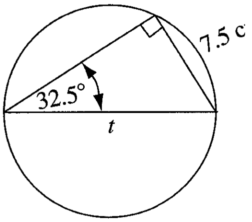


Fig. 11.70 Triangle and circle

Calculate the length of t.

10.

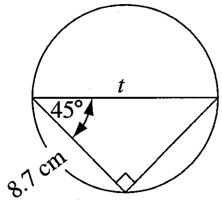


Fig. 11.71 Triangle and circle

Calculate the length of t.

11.

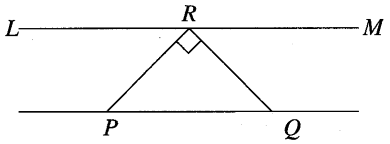


Fig. 11.72 Triangle

In the diagram above PQ and LM represent parallel edges of an east–west river bank. Angle $PRQ = 90°$.

Given that $PR = RQ = 5$ m, calculate

(a) the size of the angle RPQ

(b) the width of the river

(c) the distance PQ.

12.

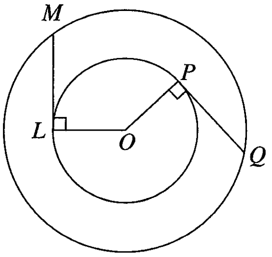


Fig. 11.73 Concentric circles

The diagram above shows two circles with their centres at O. The radius of the smaller circle is 6 cm, $\angle MLO = 90°$, $\angle OPQ = 90°$ and $ML = 8$ cm.

(a) Determine the length of QP, stating reasons.

(b) Calculate the radius of the large circle.

(c) Write down the values of tan $\angle OML$, sin $\angle OML$ and cos $\angle OML$.

13.

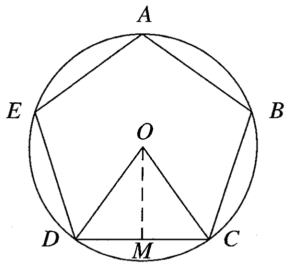


Fig. 11.74 Inscribed pentagon

$ABCDE$ is a regular pentagon inscribed in a circle centre O, radius 15 cm, as shown in the diagram above. M is the mid-point of DC.

(a) Calculate the size of angle DOC (in degrees).

(b) Calculate the length of DM.

(c) Hence, determine the perimeter of the pentagon.

14.

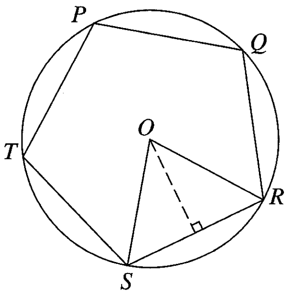


Fig. 11.75 Inscribed pentagon

$PQRST$ is a regular pentagon inscribed in a circle centre O, radius 17 cm, as shown in the diagram above. M is the mid-point of RS.

(a) Calculate the magnitude of angle ROS (in degrees).

(b) Calculate the length of RS.

(c) Hence, find the perimeter of the pentagon.

Complementary Angles

Complementary angles are angles whose *sum is* 90°. Normally we look at *two angles* whose *sum is* 90°.

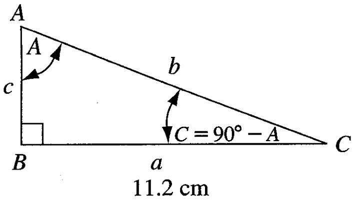


Fig. 11.76 Right-angled triangle

In the *right-angled* $\triangle ABC$ above:

$$\hat{A} + \hat{C} = 90° \text{ (complementary } \angle s\text{)}$$

So $\quad \hat{C} = 90° - A.$

Considering the *right-angled* $\triangle ABC$ above:

$$sin\,\hat{A} = \frac{AB}{AC} = \frac{a}{b}$$

and $\quad cos\,\hat{C} = cos\,(90° - A) = \frac{BC}{AC} = \frac{a}{b}$

Hence $\quad sin\,\hat{A} = cos\,(90° - \text{A}).$

Now $\quad cos\,\hat{A} = \frac{AB}{AC} = \frac{c}{b}$

and $\quad sin\,\hat{C} = sin\,(90° - A) = \frac{AB}{AC} = \frac{c}{b}$

Hence $\quad cos\,\hat{A} = sin\,(90° - A).$

From the *results* above we can *conclude* that:

(i) The *sine of an angle* is *equal* to the *cosine* of its *complementary angle.*

(ii) The *cosine of the an angle* is *equal* to the *sine* of its *complementary angle.*

Example 19

(a) Given that sin 25° = 0.423, without using tables, determine the value of cos 65°.

(b) Given that cos 25° = 0.906, without using tables, determine the value of sin 65°.

Solution

(a) Given that sin 25° = 0.423

then $cos(90° - 25°) = cos65° = sin25° = 0.423$

(*complementary* $\angle s$)

So the *value* of cos 65° is 0.423.

(b) Given that cos 25° = 0.906

then $sin\,(90° - 25°) = sin65° = cos25° = 0.906$

(*complementary* $\angle s$)

So the *value* of sin 65° is 0.906.

Exercise 11e

1. Given that sin 27° = 0.454, without using tables, determine the value of cos 63°.

2. Given that sin 59° = 0.857, without using tables, determine the value of cos 31°.

3. Given that sin 83° = 0.993, determine the value of cos 7° without using tables.

4. Given that sin 35.7° = 0.584, determine the value of cos 54.3° without using tables.

5. If sin 48.3° = 0.747, determine the value of cos 41.7° without using tables.

6. If sin 61.4° = 0.878, determine the value of cos 28.6° without using tables.

7. Given that cos 32° = 848, without using tables, determine the value of sin 58°.

8. Given that cos 48° = 0.669, without using tables, determine the value of sin 42°.

9. Given that cos 67° = 0.391, determine the value of the sin 23° without using tables.

10. Given that cos 59.3° = 0.511, determine the value of sin 30.7° without using tables.

11. If cos 68.5° = 0.367, determine the value of sin 21.5° without using tables.

12. If cos 71.8° = 0.312, determine the value of sin 18.2 without using tables.

Angles of Elevation and Depression

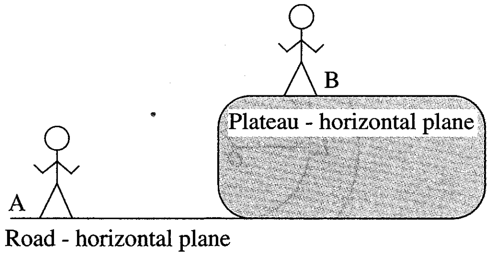


Fig. 11.77 Horizontal planes

Fig 11.77 above shows a *Person A standing* on a *horizontal plane* (e.g. a *road*) and a *Person B standing* on another *horizontal plane* (e.g. a *plateau*).

(a)

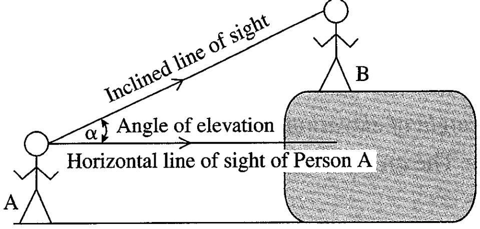


Fig. 11.78 Angle of elevation

If *Person A* looks along a *horizontal line of sight*, then he would *not be able to see person B.*

If however his *line of sight is inclined* at the *correct angle* and he *looks upwards*, then he would be able to *see Person B*. The *angle* formed by *Person A's horizontal line of sight* and his *inclined line of sight* is called the *angle of elevation.*

(b)

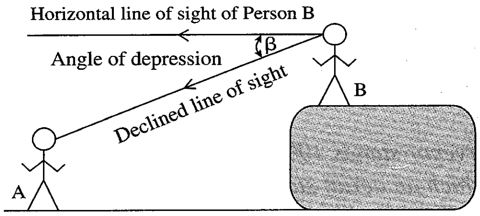


Fig. 11.79 Angle of depression

If *Person B* looks along a *horizontal line of sight*, then he would *not be able to see Person A.*

If however his *line of sight is declined* at the *correct angle* and he *looks downwards* then he would be able *to see Person A*. The *angle* formed by *Person B's horizontal line of sight* and his *declined line of sight* is called the *angle of depression.*

(c)

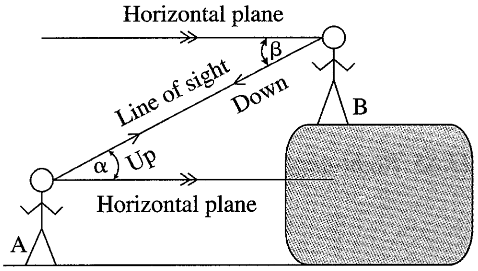


Fig. 11.80 Angles of elevation depression

The *two horizontal lines of sight* can be considered to be on *two horizontal planes.*

From Fig. 11.80 above, it can be *seen* that.

The angle of elevation = The angle of depression
i.e. $\alpha = \beta$ *(alternate angles)*

This fact is very useful in the *solution of practical problems.*

From what was discussed above we can *conclude* that:

(i)
- The *angle of elevation* of an *object* for an *observer viewing the object from below*, is the *angle formed* by the *line joining* the *object* and the *observer*, and the *horizontal plane.*
- Or, the *angle of elevation* is the *angle measured upwards from the horizontal to an object.*
- Or, the *angle of elevation* is the angle between the line of sight of a person looking upwards and the horizontal.

For example:

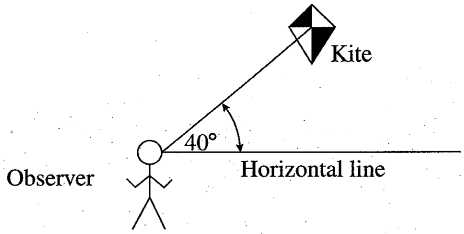


Fig. 11.81 Angle of elevation

The *angle of elevation* of the *kite* from the *observer is* 40°.

(ii)
- The *angle of depression* of an *object* for an *observer viewing the object from above*, is the *angle formed* by the *line joining* the *object* and the *observer*, and the *horizontal plane.*
- Or, the *angle of depression* is the *angle measured downwards from the horizontal to an object.*
- Or, the *angle of depression* is the *angle between the line of sight of a person looking downwards and the horizontal.*

For example:

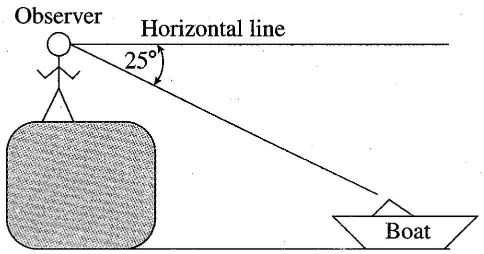


Fig. 11.82 Angle of depression

The *angle of depression* of the *boat* from the *observer* is 25°.

It should also be noted that *both* the *angle of elevation* and the *angle of depression* are *acute angles.* Hence *angles of elevation* and *depression* are always *less than* 90°.

Example 20

The angle of elevation of the top of a vertical tree from a man standing on level ground 25 m from the base of the tree is 38.5°. Calculate the height of the tree correct to the nearest metre.

Solution

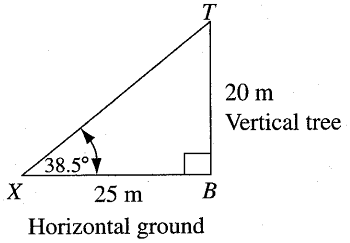


Fig. 11.83 Right-angled triangle

Considering the *right-angled* $\triangle TBX$:

$$tan\ 38.5° = \frac{TB}{BX} = \frac{TB}{25\text{ m}}$$

So $\qquad TB = 25\text{ m} \times \tan 38.5°$

$\qquad\qquad = 25\text{ m} \times 0.795$

$\qquad\qquad = 19.875\text{ m}$

i.e. $\qquad TB = 20\text{ m}$ (*correct* to the *nearest metre*).

Hence the *height* of the tree is 20 m.

Example 21

A girl 1.0 m in height, standing on top of a vertical building 45.0 m high sees a car some distance away when the angle of depression is 55°. What distance is the car from the base of the building?

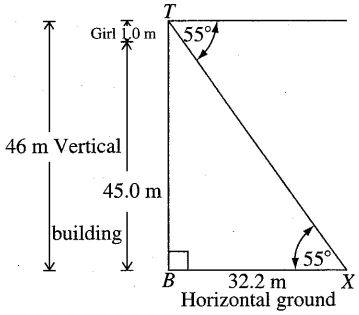


Fig. 11.84 Right-angled triangle

Solution

The *height* of the girl *from the ground*,

$TB = (1.0 + 45.0)\text{ m}$

$\quad = 46\text{ m}.$

The *angle of elevation*, $T\hat{X}B$

$\quad$ = The *angle of depression*

$\quad = 55°.$

Considering the *right-angled* $\triangle TBX$:

$$tan 55° = \frac{TB}{BX} = \frac{46\text{ m}}{BX}$$

So $\qquad BX = \dfrac{46\text{ m}}{\tan 55°}$

$\qquad\qquad = \dfrac{46\text{ m}}{1.428}$

$\qquad\qquad = 32.2\text{ m}$ (*correct* to 3 *s.f.*)

Hence the *distance* of the car from the *base of the building* is 32.2 m.

Example 22

A surveyor stands 100 m from the base of a tower on which an aerial stands. He measures the angle of elevation to the top and bottom of the aerial as 55° and 51° respectively. Calculate the height of the aerial.

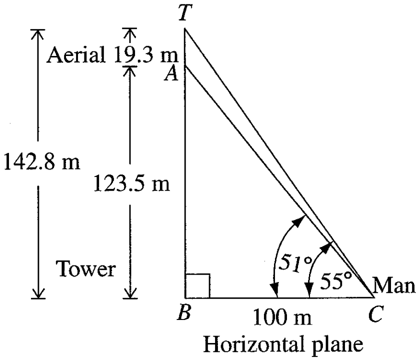


Fig. 11.85 Right-angled triangle

Solution

Considering the *right-angled* $\triangle TBC$:

$$tan 55° = \frac{TB}{BC} = \frac{TB}{100\text{ m}}$$

So $\qquad TB = 100\text{ m} \times \tan 55°$

$\qquad\qquad = 100\text{ m} \times 1.428$

$\therefore \qquad TB = 142.8\text{ m}$

Considering the *right-angled* $\triangle ABC$:

$$tan\,51° = \frac{AB}{BC} = \frac{AB}{100 \text{ m}}$$

So $\quad AB = 100 \text{ m} \times \tan 51°$

$\quad = 100 \text{ m} \times 1.235$

$\therefore \quad AB = 123.5 \text{ m}$

Now $\quad TA = TB - AB$

$\quad = (142.8 - 123.5) \text{ m}$

$\therefore \quad TA = 19.3 \text{ m}$

Hence the *height* of the aerial is 19.3 m.

Example 23

From a coastal lookout point P, 100 m above the sea, a sailor sights two boats A and B in the same direction. The angles of depression of the two boats are 15° and 23° respectively. Calculate the distance between the two boats.

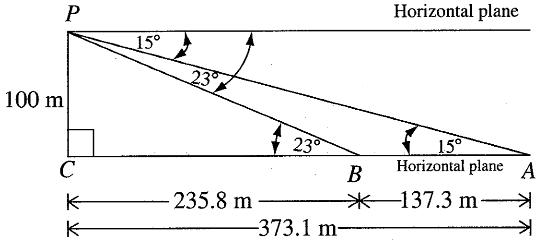


Fig. 11.86 Right-angled triangle

Solution

Considering the *right-angled* $\triangle PAC$:

$$tan\,15° = \frac{PC}{AC} = \frac{100 \text{ m}}{AC}$$

So $\quad AC = \frac{100 \text{ m}}{\tan 15°} = \frac{100 \text{ m}}{0.268}$

$\quad = 373.1 \text{ m}$

(*correct* to 1 *d.p.*)

Considering the *right-angled* $\triangle PBC$:

$$tan\,23° = \frac{PC}{BC} = \frac{100 \text{ m}}{BC}$$

So $\quad BC = \frac{100 \text{ m}}{\tan 23°} = \frac{100 \text{ m}}{0.424}$

$\quad = 235.8 \text{ m}$

(*correct* to 1 *d.p.*)

Now $\quad AB = AC - BC$

$\quad = (373.1 - 235.8) \text{ m}$

$\therefore \quad AB = 137.3 \text{ m}$

Hence the *distance* between the two boats is 137.3 m.

Exercise 11f

1. A sailor sights the top of a cliff at an angle of elevation of 12°. He knows that the height of the cliff is about 90 m above sea level. Calculate his distance from the base of the cliff to the nearest metre.

2.

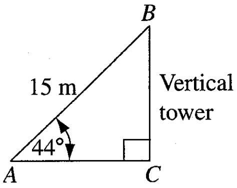


Fig. 11.87 Right-angled triangle

The diagram above shows a vertical tower BC situated on level ground AC. Given that $AB = 15$ m and the angle of elevation $BAC = 44°$, calculate correct to 1 decimal place:

(a) the height of the tower, BC

(b) the distance of A from the base of the tower, AC.

3. From a point P on the ground which is 100 m from the foot of a church tower, the angle of elevation of the top of the tower is 50°. Calculate the height of the tower.

4. From a point, the angle of elevation of the top of a tower is 26°. If the tower is 30 m away from the point on the same horizontal level, what value is the height of the tower?

5. A girl 1.2 m in height is 25 m away from a tower 18 m high. What value is the angle of elevation of the top of the tower from her eyes?

6. A woman 1.7 m in height observes the angle of elevation of a tree to be 24°. If she is standing 15 m from the tree, determine the height of the tree.

7. From a point P on ground level which is 100 m from the foot of a church tower, the angle of elevation of the top of the tower is 35°. Use a scale of 1 cm to 10 m to make a scale drawing. Use your drawing to calculate the height of the tower.

8. An instrument in an aircraft flying at a height of 400 m measures the angle of depression of the beginning of the runway as 25°. Calculate the horizontal distance of the aircraft from the runway.

9. A man 1.5 m in height standing on top of a vertical building 42 m high, sees a truck some distance away, at an angle of depression of 53.5°. At what distance is the truck from the base of the building?

10. From a coastal lookout point P, 100 m above sea-level, a sailor sights a boat at an angle of depression of 27°. Calculate the horizontal distance of the boat from the sailor.

11. A woman standing 20 m away from a tower observes the angles of elevation to the top and bottom of a flag-staff standing on the tower as 73° and 70° respectively. Calculate the height of the flag-staff.

12. A surveyor stands 100 m from the base of a tower on which an aerial stands. He measures the angles of elevation to the top and bottom of the aerial as 52° and 49° respectively. Determine the height of the aerial.

13. A surveyor stands 100 m from the base of a tower on which an aerial stands. He measures the angles of elevation to the top and bottom of the aerial as 56° and 49°. Calculate the height of the aerial.

14. A surveyor stands 100 m from the base of a tower on which aerial stands. He measures the angles of elevation to the top and bottom of the aerial as 63° and 58° respectively. Determine the height of the aerial.

15. From a coastal lookout point A, 100 m above the sea, a sailor sights two boats B and C in the same direction. The angles of depression of the two boats are 12° and 26° respectively. Calculate the distance between the two boats.

16. A man standing on top of a cliff 90 m high is in line with two buoys whose angles of depression are 15° and 19°. Calculate the distance between the buoys.

17. A woman of height 1.4 m standing on top of a building 34.6 m high views a tree some distance away. She observes that the angle of depression of the bottom of the tree is 35°, and the angle of depression of the top of the tree is 29°.

Assume that the building and the tree stand on level ground.

(a) Calculate the distance of the woman from the top of the tree measured along her line of sight.

(b) Determine the height of the tree.

18. A man of height 1.5 m standing on top of a building of height 48.5 m views another building across the square. He observes that the angle of depression of the bottom of the building is 40° and the angle of depression of the top of the building is 25°. Both buildings stand on the same level ground.

(a) Calculate the distance of the man from the base of the building across the square measured along his line of sight.

(b) Calculate the height of the building.

Bearings

The *four cardinal directions* are *north, south, east* and *west.* The *directions mid-way* between these directions are also used, that is, *north-east* (*NE*), *south-east* (*SE*), *south-west* (*SW*), and *north-west* (*NW*). These facts are illustrated in Fig. 11.88 below.

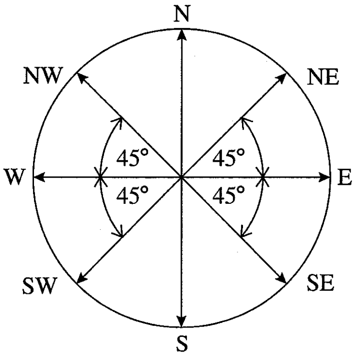


Fig. 11.88 Cardinal directions

The *position* of an *object relative* to *another object* is called its *bearing*. The *bearing* of an *object* is the *angle measured* in a *clockwise direction from north* to the *object. Bearings* are always *written* using *three digits*. Thus: *north* is 000°, *east* is 090°, *south* is 180° and *west* is 270°. Also *NE* is 045°, *SE* is 135°, *SW* is 225° and *NW* is 315°. These facts are illustrated in Fig. 11.89 below.

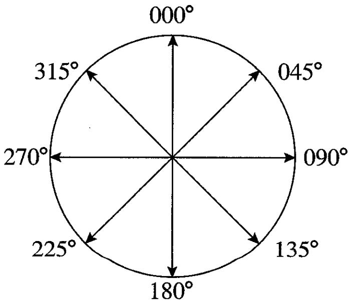


Fig. 11.89 Bearings

For example:

(a)

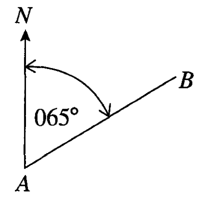


Fig. 11.90 Bearing

In Fig. 11.90, the *bearing of B from A is* 065°.

(b)

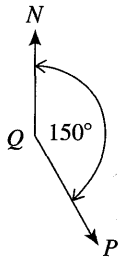


Fig. 11.91 Bearing

In Fig. 11.91, the *bearing of P from Q is* 150°.

(c)

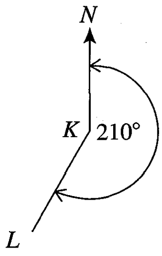


Fig. 11.92 Bearings

In Fig. 11.92, the *bearing of L from K is* 210°.

(d)

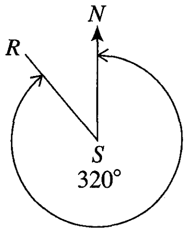


Fig. 11.93 Bearing

In Fig. 11.93, the *bearing of R from S is* 320°.

Example 24

(a) The bearing of a point A from a point B is 075°. State the bearing of B from A.

(b) The bearing of a point Q from a point P is 325°. What value is the bearing of P from Q?

Solution

(a)

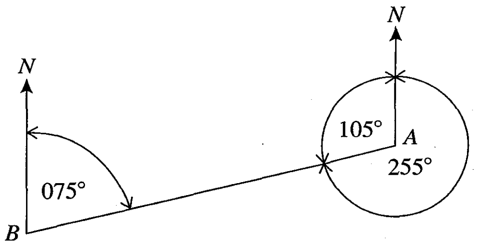


Fig. 11.94 Bearings

Since *NB* and *NA* are *parallel*,

then $N\hat{A}B = 180^\circ - 75^\circ = 105^\circ$ (*supplementary ∠s*)

So the *bearing of B from A* $= 360^\circ - 105^\circ$

$= 255^\circ$ (*∠s at a point*)

Hence the *bearing of B from A* is 255°.

(b)

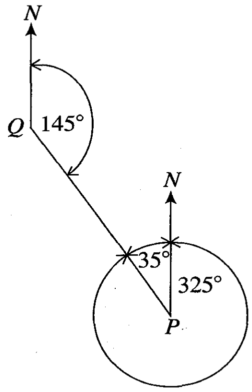


Fig. 11.95 Bearings

Now $NPQ = 360^\circ - 325^\circ = 35^\circ$ (*∠s at a point*), since *NQ* and *NP* are *parallel*.

So the *bearing of P from Q*

$= 180^\circ - 35^\circ$ (*supplementary ∠s*)

$= 145^\circ$

Hence the *bearing of P from Q* is 145°.

Alternative Method

(a)

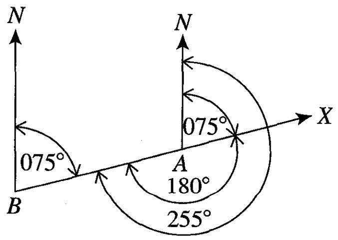


Fig. 11.96 Bearings

Now $N\hat{A}X = N\hat{B}A = 75°$ (*corres.* $\angle s$).

So the *bearing of B from A* = 75° + 180° = 255°.

(b)

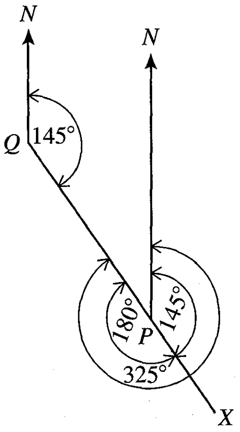


Fig. 11.97 Bearings

Now $N\hat{P}X = 325° - 180 = 145°$.

$N\hat{Q}P = N\hat{P}X = 145°$ (*corres.* $\angle s$).

So the *bearing of P from Q* is 145°.

Example 25

(a) By drawing a diagram, find the distance travelled north and the distance travelled east by a plane flying on a bearing of 48° for 80 km.

(b) By drawing a diagram, determine the distance travelled south and the distance travelled west by a car driving on a bearing of 230° for 65 km.

Scale: 1 cm represents 10 km

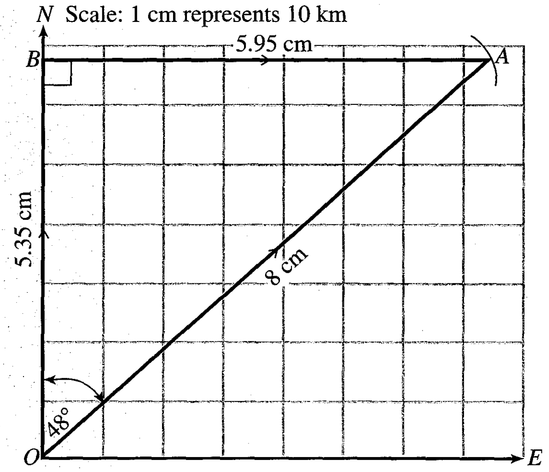


Fig. 11.98 Right-angled triangle

Solution

(a) If 1 cm represents 10 km,

then 8 cm will represent 80 km.

From fig. 11.98 by *measurement*:

The *distance travelled north* by the plane

= 5.35 × 10 km

= 53.5 km

The *distance travelled east* by the plane

= 5.95 × 10 km

= 59.5 km

Scale: 1 cm represents 10 km

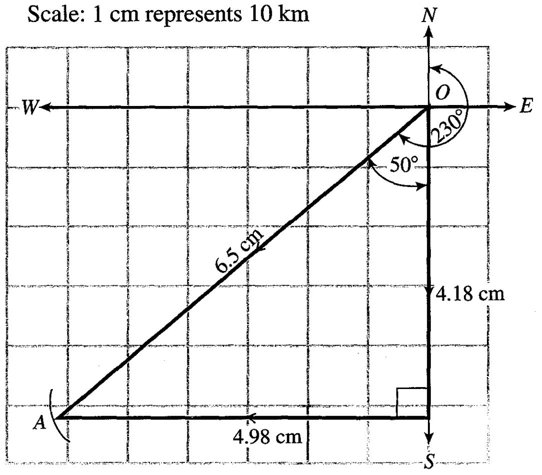


Fig. 11.99 Right-angled triangle

(b) If 1 cm represents 10 km,

then 6.5 cm will represent 65 km.

From Fig. 11.99 by *measurement*:

The *distance travelled south* by the car = 4.18 × 10 km

= 41.8 km

The *distance travelled west* by the car = 4.98 × 10 km

= 49.8 km

Example 26

(a) Calculate the distance travelled south and the distance travelled east by a ship sailing on a bearing of 150° for 90 km.

(b) Determine the distance travelled north and the distance travelled west by a yacht sailing on a bearing of 310° for 75 km.

Solution

(a)

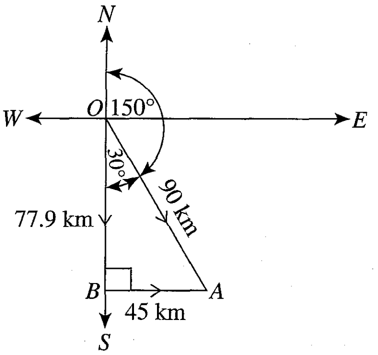


Fig. 11.100 Right-angled triangle

Considering the *right-angled* $\triangle OAB$:

$$A\hat{O}B = 180° - 150° = 30° \text{ (}\angle\text{s on a st. line)}$$

So $\quad \cos 30° = \frac{OB}{OA} = \frac{OB}{90 \text{ km}}$

i.e. $\quad OB = 90 \text{ km} \times \cos 30°$

$= 90 \text{ km} \times 0.866$

$= 77.94 \text{ km}$

$= 77.9 \text{ km}$ (*correct* to 3 *s.f.*)

Hence the *distance travelled south* by the ship is 77.9 km.

Now $\quad \sin 30° = \frac{AB}{OA} = \frac{AB}{90 \text{ km}}$

So $\quad AB = 90 \text{ km} \times \sin 30°$

$= 90 \text{ km} \times 0.5$

$= 45 \text{ km}$

Hence the *distance travelled east* by the ship is 45 km.

(b)

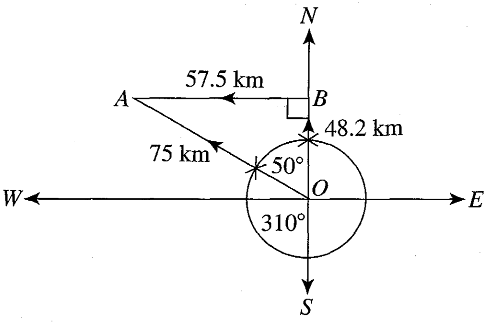


Fig. 11.101 Right-angled triangle

Considering the *right-angled* $\triangle OAB$:

$$A\hat{O}B = 360° - 310° = 50° \text{ (}\angle\text{s at a point)}$$

So $\quad \cos 50° = \frac{OB}{OA} = \frac{OB}{75 \text{ km}}$

i.e. $\quad OB = 75 \text{ km} \times \cos 50°$

$= 75 \text{ km} \times 0.643$

$= 48.225 \text{ km}$

$= 48.2 \text{ km}$ (*correct to* 3 *s.f.*)

Hence the *distance travelled north* by the yacht is 48.2 km.

Now $\quad \sin 50° = \frac{AB}{OA} = \frac{AB}{75 \text{ km}}$

So $\quad AB = 75 \text{ km} \times \sin 50°$

$= 75 \text{ km} \times 0.766$

$= 57.45 \text{ km}$

$= 57.5 \text{ km}$ (*correct* to 3 *s.f.*)

Hence the *distance travelled west* by the yacht is 57.5 km.

Exercise 11g

1. Draw a rough sketch to illustrate each of the following bearings. Mark the angle in your sketch.
   (a) From a point *P*, the bearing of a point *Q* is 30°.
   (b) From a place *A*, the bearing of a place *B* is 140°.
   (c) The bearing of a point *K* from a point *L* is 250°.
   (d) The bearing of a place *M* from a place *N* is 330°.
2. Draw a rough sketch to illustrate each of the following bearings. Mark the angle in your sketch.
   (a) From a ship, *P*, the bearing of a yacht, *Q*, is 50°.
   (b) From a point, *H*, the bearing of a mosque, *C*, is 220°.
   (c) From an aircraft, *A*, the bearing of an airport, *L*, is 150°.
   (d) The bearing of a flagpole, *F*, from a tent, *T*, is 150°.
3. Draw a rough sketch to illustrate each of the following bearings. Mark the angle in your sketch.
   (a) The bearing of a ship *S* from a lighthouse *L* is 65°.
   (b) The bearing of a boat *B* from a harbour *H* is 175°.
   (c) The bearing of a plane *P* from an airport *A* is 315°.
   (d) From a building *B*, the bearing of an aerial *A* is 235°.

4. The bearing of a point A from a point B is 65°. State the bearing of B from A.

5. The bearing of a point P from a point Q is 70°. Determine the bearing of Q from P.

6. The bearing of a point K from a point L is 84°. Calculate the bearing of L from K.

7. The bearing of a point A from a point of B is 135°. State the bearing of B from A.

8. The bearing of a point P from a point Q is 155°. Determine the bearing of Q from P.

9. The bearing of a point K from a point L is 164°. Calculate the bearing of L from K.

10. The bearing of ship S from a yacht Y is 220°. State the bearing of the yacht Y from the ship S.

11. The bearing of a boat B from a harbour H is 250°. Calculate the bearing of the harbour H from the boat B.

12. The bearing of a place X from a place Y is 265°. Calculate the bearing of the place Y from the place X.

13. The bearing of an airport A from a plane P is 310°. State the bearing of the plane P from the airport A.

14. The bearing of submarine S from a port P is 325°. Evaluate the bearing of the port P from the submarine S.

15. The bearing of ship S from a harbour H is 339°. Calculate the bearing of the harbour H from the ship S.

16. By drawing a diagram, determine the distance travelled north and the distance travelled east by a plane flying on a bearing of 50° for 100 km.

17. By drawing a diagram, determine the distance travelled south and the distance travelled east by ship sailing on a bearing of 140° for 90 km.

18. By drawing a diagram, determine the distance travelled south and the distance travelled west by a car being driven on a bearing of 220° for 85 km.

19. By drawing a diagram, determine the distance travelled north and the distance travelled west by a yacht sailing on a bearing of 300° for 65 km.

20. Calculate the distance travelled north and the distance travelled east by a plane flying on a bearing of 45° for 165 km.

21. Evaluate the distance travelled south and the distance travelled east by a ship sailing on a bearing of 158° for 95 km.

22. Calculate the distance travelled south and the distance travelled west by a car driving on a bearing of 225° for 100 km.

23. Determine the distance travelled north and the distance travelled west by a yacht sailing on a bearing of 325° for 87 km.

24. By drawing a diagram, determine the bearing on which a ship sails from port if it finishes 40 km east and 20 km south.

25. By drawing a diagram, determine the bearing on which a plane files from an airport if it finishes 35 km west and 75 km south.

26. By drawing a diagram, determine the bearing on which a yacht sails from harbour if it finishes 40 km west and 50 km north.

27. By drawing a diagram, determine the bearing on which a car drives from a park if it finishes 39 km east and 52 km north.

28. Calculate the bearing on which a plane flies from an airport if it finishes 75 km east and 30 km north.

29. Determine the bearing on which a ship sails from port if it finishes 65 km west and 20 km north.

30. Calculate the bearing on which a yacht sails from harbour if it finishes 48 km west and 100 km south.

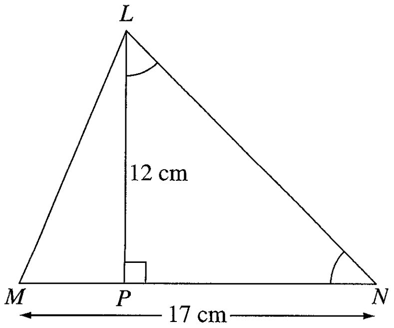
31. Determine the bearing on which a car drives from a park if it finishes 30 km east and 75 km south.

32. From a point P, the bearing of a tree, T, is 60°. From a second point Q, which is 200 m due east of P, the bearing of the tree is 330°. Use a scale of 1 cm to 20 m to make a scale diagram and determine the distance of the tree from P.

C.X.C. Past Paper Questions

The following supplementary questions were taken from C.X.C. Past Papers.

1.



In triangle *LMN* above (not drawn to scale)

MN = 17 cm. *LP* is perpendicular to *MN*.

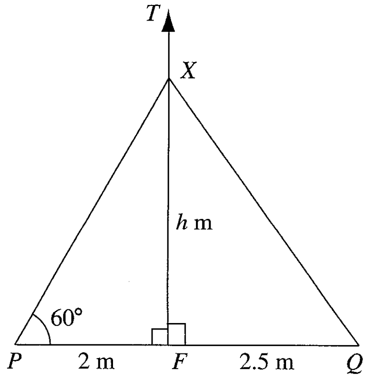
LP = 12 cm and angle *PLN* = angle *PNL*.

(i) State the length of *PN*. Give a reason for your answer.

(ii) Calculate the length of *LM*.

Question 7 (b). C.X.C. (Basic). June 1987.

2.



A flagpole is put up at a point *F*. Wires to hold the pole vertically in place are attached at *X* on the pole *TF*. One wire is fixed at a point *P* on the grounds that *PF* = 2 m. The angle of elevation of *X* from *P* is 60°. Another wire is to be fixed to the ground at *Q* where *QF* = 2.5 m.

(a) Calculate the height of *X* above *F*. Give your answer correct to two significant figures.

(b) An extra 40 cm is needed to attach the wire at each of the points *X* and *Q*. Calculate the total length of wire to be used for the support at *Q*.

Question 9. C.X.C. (basic). June 1989.

CXC

MODEL

EXAMINATIONS –

BASIC PROFICIENCY

C.X.C. MODEL EXAMINATION 1
MATHEMATICS
Paper 1 – Basic Proficiency
90 minutes
Answer ALL the questions

1. 39.98 × 0.5 is approximately equal to
(A) 0.2 (B) 2.0 (C) 20.0 (D) 200

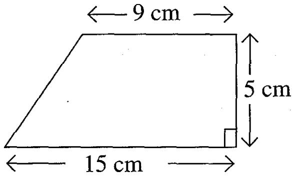
2. $(6.7)^2 - (0.3)^2 =$
(A) 44.8 (B) 7.0 (C) 49.0 (D) 40.96

3. $105.00 is divided among 3 friends in the ratio 3:5:7. How much is the largest share?
(A) $21.00 (B) $49.00 (C) $35.00 (D) $63.00

4. A square has the same area as a parallelogram with altitude 6 cm and base 24 cm. What is the length of the side of the square?
(A) 72 cm (B) 48 cm (C) 24 cm (D) 12 cm

5. If TT $6.30 ≈ US $1.00. How much approximately in TT would one get for US $5.50?
(A) $550.00 (B) $34.65
(C) $63.00 (D) $67.30

6.



The area of the trapezium above (not drawn to scale) is
(A) 120 cm² (B) 135 cm²
(C) 60 cm² (D) 29 cm²

7. If $P = \{5, 6, 7, 8\}$, $Q = \{2, 4, 6, 8\}$ and $R = \{2, 3, 7, 9\}$, then $P \cap Q \cap R =$
(A) $\{9\}$ (B) $\{\ \}$ (C) $\{6, 8\}$ (D) $\{5\}$

8. If $5n$ is an odd number, which of the following is an even number?
(A) $5n - 2$ (B) $5n + 2$
(C) $5n + 7n$ (D) $5n - 1$

9. If a car travels 180 kilometres on 18 litres of petrol, then on a full 40 litre – tank it should travel
(A) 198 km (B) 360 km
(C) 400 km (D) 720 km

10. 8 075 in standard form is
(A) 8.075×10^{-3} (B) 8.075×10^{-2}
(C) 8.075×10^{3} (D) 8.075×10^{2}

11. If an article is sold for $90.00, a profit of 25% is made. The cost price of the article is
(A) $80 (B) $72 (C) $115 (D) $65

12. $\sqrt{490}$ is approximately =
(A) 2.2×10 (B) 2.5×10
(C) 7.0×10 (D) 14.5×10^{-1}

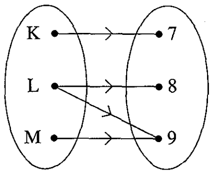
13. The coordinates of the image of the point $P(2, -5)$ under the translation
$T = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ are
(A) $(5, -9)$ (B) $(-5, 9)$
(C) $(1, -1)$ (D) $(6, -20)$

14. Each of the letters of the word 'JEREMIAH' are written on separate pieces of paper. The pieces of paper are then placed in a box. What is the probability of drawing a letter '*E*'?
(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{1}{7}$

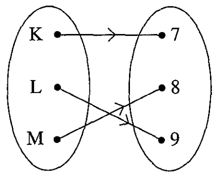
15. The sum of $5 400 was borrowed from a bank at the rate of 10% per annum for 3 years. The simple interest payable is
(A) $1 800 (B) $540 (C) $1 620 (D) $486

16. Which of the following arrow diagrams represents a one-to-one relation?

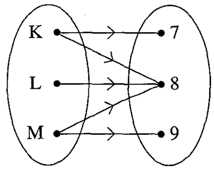
(A)



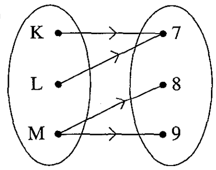
(B)



(C)



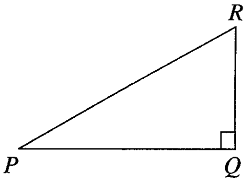
(D)



17. \$900 invested at simple interest for 2 years earns \$99. The rate of interest per annum is
(A) 11% (B) 5.5% (C) 10% (D) 9%

18. The mean of eight numbers is 14.5. The number 10 is added to the set. The new mean is
(A) 12.5 (B) 14.0 (C) 4.5 (D) 16.5

19.

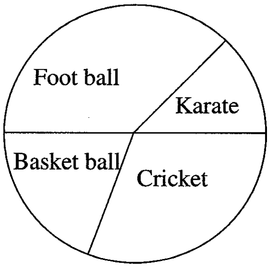


In the triangle PQR above, the ratio expressing $\tan \hat{P} =$
(A) $\frac{PQ}{RQ}$ (B) $\frac{PR}{PQ}$ (C) $\frac{RQ}{PQ}$ (D) $\frac{RQ}{PR}$

20. $4x - 3(x + 5) =$
(A) $7x - 15$ (B) $x + 15$
(C) $-7x + 15$ (D) $x - 15$

21. If $p * q \equiv 3p + q$, then $1 * 3 =$
(A) 10 (B) 9 (C) 4 (D) 6

22.

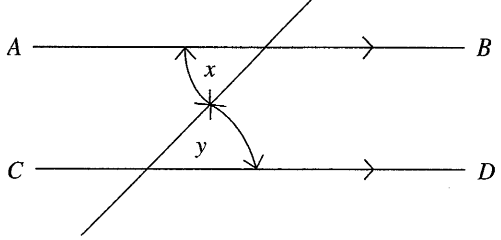


The pie chart above shows how 200 students spend their physical education period. The number of children who spend it playing karate is approximately equal to
(A) 25 (B) 50 (C) 75 (D) 60

23. The point $P(2, 3)$ is reflected in the line $y = x$. What are the co-ordinates of the image P'?
(A) (3, 2) (B) (−3, −2)
(C) (−2, −3) (D) (−2, 3)

24. If $45 - 2x = 2x - 3$, then $x =$
(A) 7 (B) 24 (C) 12 (D) 0

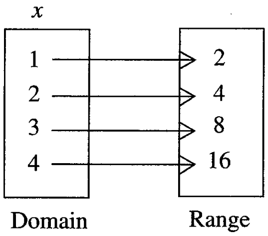
25.



In the figure above, AB and CD are parallel. Which of the following statements best describes the relation between x and y?
(A) $x - y = 0$ (B) $x \neq y$
(C) $x = 2y$ (D) $x > y$

26. If $\tan A° = \frac{3}{4}$, and $\hat{A}$ is acute, then the value of $\sin A° =$
(A) $\frac{3}{5}$ (B) $\frac{4}{5}$ (C) $\frac{4}{3}$ (D) $\frac{3}{7}$

27.



The diagram above represents the mapping
(A) $x \rightarrow 2x$ (B) $x \rightarrow x + 2$
(C) $x \rightarrow x^2$ (D) $x \rightarrow 2^x$

28. $5^2 - 3^2 =$
(A) 2 (B) 8 (C) −2 (D) 16

29. Which of the following best describes a quadrilateral with its opposite sides parallel and equal?
(A) kite (B) rhombus
(C) rectangle (D) trapezium

30. $503_7 =$
(A) $5 \times 7^3 + 3$ (B) $5 \times 7^2 + 3$
(C) $5 \times 7^3 + 3 \times 7^2$ (D) $5 \times 7^2 + 3 \times 7$

31. 345_6 can be written in base ten as
(A) 12 (B) 60 (C) 240 (D) 137

32. Which of the following tables of values shows that there is a direct variation between x and y?

(A)

x	y
0	0
1	2
2	8
3	12

(B)

x	y
0	0
1	5
2	20
3	40

(C)

x	y
0	0
1	3
2	6
3	9

(D)

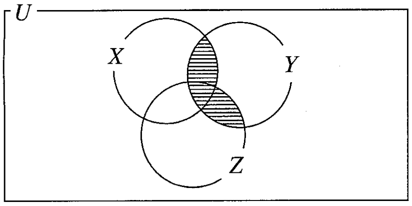
x	y
0	0
1	2
2	4
3	8

33. $2a^2 + (3a)^2 =$
(A) $5a^2$ (B) $2a^2$ (C) $11a^2$ (D) $8a^2$

34. The curved surface area of a closed cylinder of height h cm and radius r cm is
(A) $2\pi r(r + h)$ (B) $2\pi h$
(C) $\pi r^2 h$ (D) $2\pi rh$

35. If a sphere has radius r cm and surface area A cm^2, then $A =$
(A) $\frac{3}{4}\pi r^3$ cm^2 (B) $4\pi r^2$ cm^2
(C) $2\pi r$ cm^2 (D) πr^2 cm^2

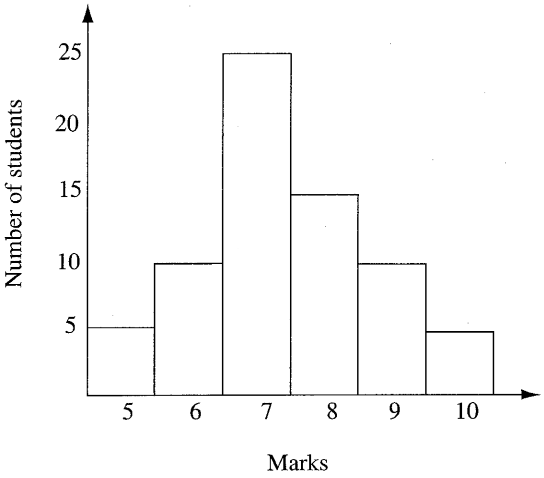
36.



In the Venn diagram above, the shaded portion represents
(A) $X \cap Y \cap Z$ (B) $Y \cap (X \cup Z)$
(C) $X \cup Y \cup Z$ (D) $(X \cap Y) \cup Z$

37. The median of the numbers 1, 1, 3, 3, 5, 6, 6, 6, 7, 8 is
(A) 6 (B) 11 (C) 5 (D) 5.5

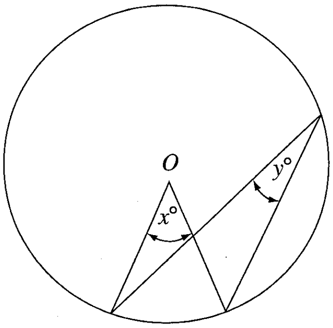
Items 38–39 refer to the following histogram which shows the marks received by a group of students in an examination.



38. The total number of students that were examined was
(A) 25 (B) 50 (C) 70 (D) 100

39. The modal mark was
(A) 6 (B) 7 (C) 8 (D) 9

40.



In the diagram above, if O is the centre of the circle, then $x =$
(A) y (B) $2y$ (C) $\frac{1}{2}y$ (D) $180° - y$

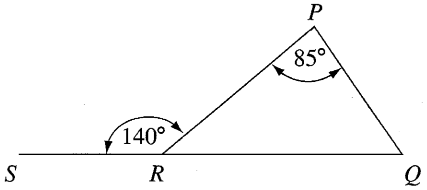
41. If $x = \frac{4}{7}$ and $y = \frac{2}{5}$, then $x + \frac{1}{y} =$
(A) $\frac{8}{35}$ (B) $\frac{10}{7}$ (C) $\frac{35}{8}$ (D) $\frac{7}{10}$

42. A straight line passing through the point (2, 5) and with gradient -3 can be represented by the particular equation
(A) $y = -3x + 11$ (B) $y = -3x - 11$
(C) $y = -3x + 1$ (D) $y = -3x - 1$

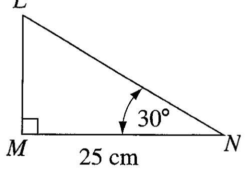
43. $a(x + y) - b(x + y) =$
(A) $a - b(x + y)$ (B) $ax - by$
(C) $(ax - by)^2$ (D) $(a - b)(x + y)$

44. The gradient of the straight line $3x + 2y = 5$ is
(A) $\frac{5}{2}$ (B) $\frac{2}{3}$ (C) $-\frac{3}{2}$ (D) $-\frac{2}{5}$

45.



In the triangle PQR above, angle $PRS = 140°$ and angle $RPQ = 85°$. The angle $PQR =$
(A) 40° (B) 60° (C) 55° (D) 95°



The triangle LMN above is right-angled at M. $L\hat{N}M = 30°$ and $MN = 25$ cm. The length of LM, in cm, is

(A) 25 sin 30° (B) 25 tan 30°

(C) $\dfrac{25}{\tan 30°}$ (D) $\dfrac{25}{\sin 30°}$

47. The heights in cm of ten girls are 135, 139, 147, 154, 136, 135, 140, 152, 153, 141. The range is

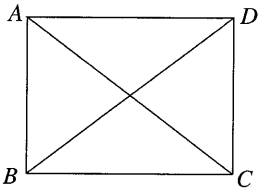
(A) 19 cm (B) 25 cm

(C) 135 cm (D) 154 cm

48. A man bought a car for \$60 800. After one year, it was worth \$51 680. Its depreciation, as a percentage of the selling price was

(A) 5 (B) 10 (C) 15 (D) 20

49.



In the rectangle above (not drawn to scale), the lengths of the diagonals are 10 cm and $AD = BC = 8$ cm. The altitude of the triangle, in cm, is

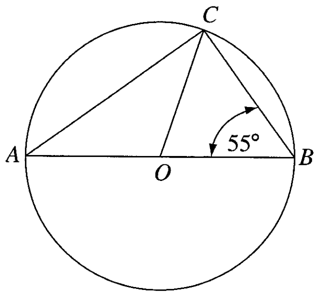
(A) 2 (B) 6 (C) 64 (D) 80

50. The height of a tower is 90 m. The tower is represented on a scale drawing by a length of 3 cm. The scale used was

(A) 1:3 000 (B) 1:30 000

(C) 1:300 000 (D) 1:3 000 000

51.



In the figure above, AOB is a diameter of a circle centre O. If angle $CBO = 55°$, what is the size of angle CAB?

(A) 15° (B) 90° (C) 45° (D) 35°

52. If $v = u + at$, then t is equal to

(A) $\dfrac{v - u}{a}$ (B) $\dfrac{u - v}{a}$

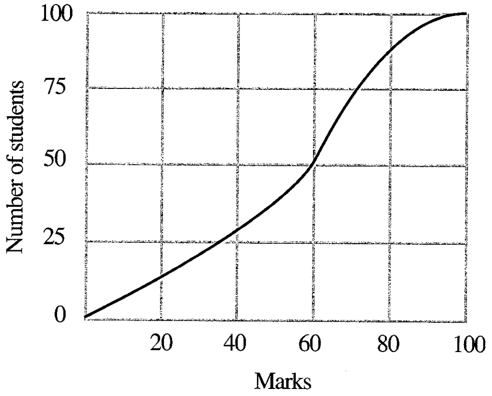
(C) $\dfrac{vu}{a}$ (D) $vu - a$

53. $x^3y \div xy^2 =$

(A) $\dfrac{x}{y}$ (B) $\dfrac{x^2}{y}$

(C) $\dfrac{x}{y^2}$ (D) x^2y^{-2}

54.



The diagram above shows the cumulative frequency curve for marks obtained by 100 students in a Mathematics examination at school. The estimated median mark is

(A) 50 (B) 60

(C) 65 (D) 100

55. The number of possible subsets that can be obtained from the set {p, q, r} is

(A) 6 (B) 8

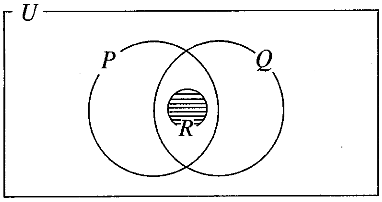
(C) 12 (D) 15

56. If $12\frac{1}{2}$% value-added tax is paid on articles purchased, then the ratio of the price of an article inclusive of value-added tax to the price of the article exclusive of the value-added tax is

(A) $\dfrac{1}{9}$ (B) $\dfrac{7}{8}$

(C) $\dfrac{8}{9}$ (D) $\dfrac{9}{8}$

57.



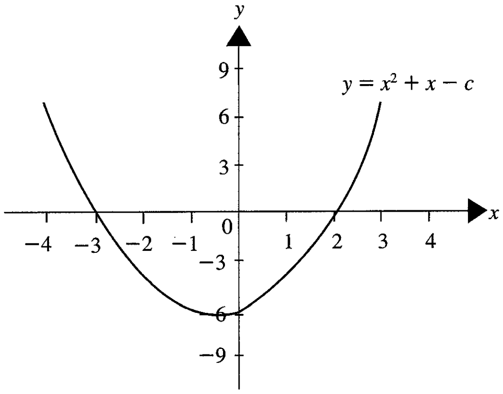
In the Venn diagram above, the shaded region represents

(A) $P \cup Q \cup R$ (B) $P \cap Q \cup R'$
(C) $P \cap Q \cap R$ (D) $(P \cap Q)' \cup R$

58. After a 5% increase a teacher's salary was \$1 260. The teacher's salary if a 10% increase was given instead is

(A) \$1 320 (B) \$1 512
(C) \$1 458.95 (D) \$1 400

59.



In the given parabola above (not drawn to scale), the value of c is

(A) −3 (B) 2 (C) −6 (D) 1

60. If $312_n = 54_{10}$, then $n =$

(A) 3 (B) 4 (C) 5 (D) 6

C.X.C. MODEL EXAMINATION 2
MATHEMATICS
Paper 1 – Basic Proficiency
90 minutes
Answer ALL the questions

1. $\frac{1}{5}$ expressed as a percentage =
(A) 5% (B) 10% (C) 20% (D) 25%

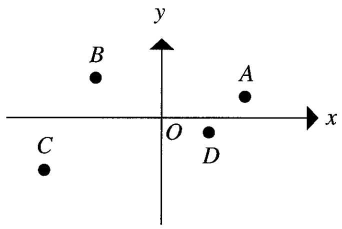
2. The decimal equivalent of $\frac{3}{8}$ =
(A) 0.375 (B) 0.875 (C) 0.625 (D) 0.3

3. 49×101 is equivalent to
(A) $(49 \times 100) + 1$
(B) $(49 \times 100) + (49 \times 1)$
(C) $(49 \times 100) - (49 \times 1)$
(D) $(49 \times 100)(49 \times 1)$

4. If x is an even number, which of the following is also even?
(A) $x + 1$ (B) $x - 1$ (C) $2x + 1$ (D) $x + 2$

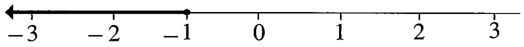
5. Raymond is 32 years of age and Yuri is 8 years of age. The ratio of Raymond's age to Yuri's age is
(A) 16:1 (B) 1:4 (C) 4:1 (D) 1:8

6.



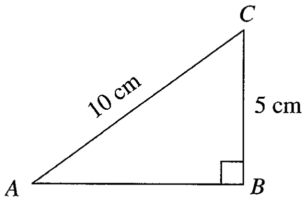
In the figure above, the point for which the x-coordinate is negative and the y-coordinate is positive is
(A) A (B) B (C) C (D) D

7. A plane is heading on a bearing 55°, and changes course in a clockwise direction to 125°. The angle through which the plane turns is
(A) 70° (B) 180° (C) 235° (D) 90°



8. Which of the inequalities in x best describes the number line above?
(A) $\{x: x > 0\}$ (B) $\{x: x < 0\}$
(C) $\{x: x < -1\}$ (D) $\{x: x > -1\}$

9.



In the right-angled triangle ABC above, angle A is
(A) 50° (B) 60° (C) 45° (D) 30°

10. If a circle has a radius r cm, diameter d cm, and area A cm^2, then A =
(A) πd^2 cm^2 (B) πr^2 cm^2
(C) $2\pi d$ cm^2 (D) $2\pi r$ cm^2

11. 0.032 75 in standard form correct to 3 significant figures is
(A) 3.27×10^{-2} (B) 3.28×10^{2}
(C) 3.28×10^{-2} (D) 3.28×10^{2}

12. $(-1)^2 + (-2)^3 =$
(A) 7 (B) −7 (C) 10 (D) −3

13. The marked price of a refrigerator was $2 350. A worker bought the refrigerator on terms by paying $1 000 down and $99 for 15 months. How much would have been saved if the refrigerator had been bought cash?
(A) $135 (B) $1 500 (C) $1 485 (D) $485

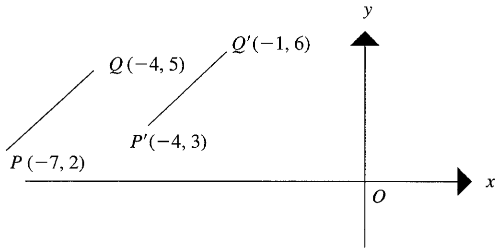
14. A square has the same area as a rectangle with sides of lengths 4 centimetres and 25 centimetres. What is the length of a side of the square?
(A) 100 cm (B) 10 cm (C) 5 cm (D) 58 cm

15. If a rectangle with perimeter 46 centimetres has sides of lengths 8 centimetres and $3x$ centimetres, then x =
(A) 15 cm (B) 23 cm (C) 5 cm (D) 5.75 cm

16. The set of numbers greater than -5 but not more than 3 can be written as
(A) $\{x: -5 \leqslant x < 3\}$ (B) $\{x: -5 < x < 3\}$
(C) $\{x: -5 \leqslant x \leqslant 3\}$ (D) $\{x: -5 < x \leqslant 3\}$

17. Which of the following plane figures best describes a polygon with all its interior angles equal?
(A) equilateral triangle (B) pentagon
(C) convex polygon (D) re-entrant polygon

18.



The translation in which PQ is mapped onto $P'Q'$ can be represented by the matrix
(A) $\begin{pmatrix}-5\\2\end{pmatrix}$ (B) $\begin{pmatrix}5\\-2\end{pmatrix}$ (C) $\begin{pmatrix}3\\1\end{pmatrix}$ (D) $\begin{pmatrix}-3\\-1\end{pmatrix}$

19. If $P = \{\text{factors of } 12\}$ and $Q = \{\text{factors of } 16\}$, then $P \cap Q =$
(A) $\{12, 16\}$ (B) $\{1, 2, 4\}$
(C) $\{4, 8\}$ (D) $\{\ \cdot\ \}$

20. The square root of 209 lies between
(A) 13 and 14 (B) 14 and 15
(C) 15 and 16 (D) 16 and 17

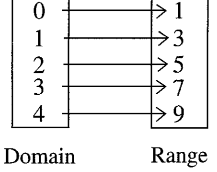
21. A shopkeeper bought articles at \$4.00 a dozen and sold them at \$4.50 a dozen. Her percentage profit was
(A) 20% (B) 25% (C) 12.5% (D) 50%

22. If $p = 2^3 \times 3^2$, then $p^5 =$
(A) $2^{15} \times 3^2$ (B) $2^3 \times 3^{10}$
(C) $2^{15} \times 3^{10}$ (D) $2^{30} \times 3^{30}$

23. If $x \rightarrow x^2 - 1$, then $f(-2) =$
(A) 3 (B) -3 (C) 5 (D) -5

24. $\dfrac{3m}{5r} + \dfrac{4n}{7s} =$
(A) $\dfrac{3m + 4n}{5r + 7s}$ (B) $\dfrac{21ms + 20nr}{35rs}$
(C) $\dfrac{3mn}{21rs}$ (D) $\dfrac{12mn}{21rs}$

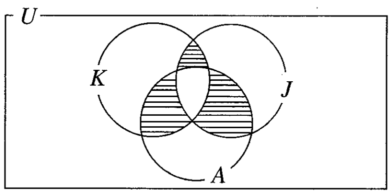
25.



The diagram given represents the mapping
(A) $x \rightarrow 2x - 1$ (B) $x \rightarrow 2x - 3$
(C) $x \rightarrow 2x + 3$ (D) $x \rightarrow 2x + 1$

26. $6x - 3(x - 4) =$
(A) $-3x - 12$ (B) $-3x + 12$
(C) $3x - 12$ (D) $3x + 12$

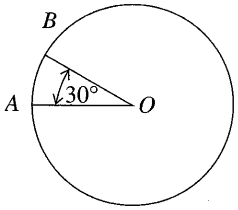
27.



If $K = \{\text{students who play Karate}\}$
$J = \{\text{students who play Judo}\}$
and $A = \{\text{students who play Aikido}\}$,
then the shaded regions in the Venn diagram represent
(A) {students who play one sports only}
(B) {students who play two sports only}
(C) {students who play three sports only}
(D) {students who play all three sports}

28. $5^x \times 5^y =$
(A) 25^{xy} (B) 25^{x+y} (C) 5^{xy} (D) 5^{x+y}

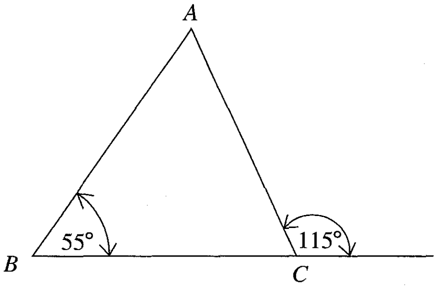
29.



In the circle above, the radius is 28 cm. The length of the arc AB in centimetres is
(A) $\dfrac{44}{3}$ (B) $\dfrac{56}{3}$ (C) $\dfrac{136}{3}$ (D) $\dfrac{616}{3}$

30. The volume of a cuboid whose edges are 4 cm, 5 cm and 6 cm is
(A) 240 cm³ (B) 120 cm³
(C) 360 cm³ (D) 480 cm³

31.



In the triangle ABC above, the exterior angle is 115° and angle $ABC = 55°$.
Angle BAC =
(A) 125° (B) 65° (C) 10° (D) 60°

32. If $12\frac{1}{2}\%$ of a sum of money is given as \$60, then the total sum of money is
(A) \$960 (B) \$480 (C) \$750 (D) \$1 000

33. The circumference of a circle of radius 14 metres =
(A) 44 m (B) 88 m (C) 56 m (D) 70 m

34. In a container, there are 5 red balls, 4 yellow balls and 3 green balls. If one of the balls is chosen at random, what is the probability that it is not green?
(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) 1

35. The simple interest on \$700 invested for 5 years at 6.5% per annum is
(A) $\frac{\$700 \times 6.5 \times 5}{100}$ (B) $\frac{\$700 \times 5}{5 \times 100}$
(C) $\frac{\$700 \times 5}{5 \times 100}$ (D) $\frac{\$100 \times 5}{6.5 \times 600}$

36. The cost price of an article is \$95. In selling the article a shopkeeper made a profit of 20%. The selling price was
(A) \$115 (B) \$114 (C) \$79 (D) \$128

37. What is the least number of sweets that can be shared equally among 9, 12 or 15 girls?
(A) 15 (B) 108 (C) 135 (D) 180

38. By selling a shirt for \$72, a businesswoman made a profit of \$12. The profit as a percentage of the cost price is
(A) 16.7% (B) 20%
(C) 60% (D) 5%

39. A discount of \$2.50 is given on a dress. If a girl paid \$135.00 for two dresses that had the same sale price, what was the original cost per dress?
(A) \$70.00 (B) \$65.00
(C) \$67.50 (D) \$132.50

40. If x and y are both integers, then $2(x - y)^2$ means
(A) two times the difference of their squares
(B) four times their difference
(C) the square root of two times their difference
(D) two times the square of their difference

41. The marked price of a pair of pants is \$100. If VAT of 15% is payable, then the price paid by the customer is
(A) \$100.15 (B) \$85.00
(C) \$195 (D) \$115.00

42. A ship was sailing on a bearing of 180°. In which direction was it travelling?
(A) North (B) South (C) East (D) West

43. If $x = -8$ and $y = -16$, then $x - y$ is
(A) −8 (B) −24 (C) 8 (D) 24

44. If $0.05x = 20$, then x =
(A) 400 (B) 40 (C) 4 (D) 0.4

45. 32_{10} can be written in base two as
(A) 1 000 000 (B) 100 000
(C) 10 000 (D) 1000

46. In the numeral 312_7, the digit 1 represents
(A) 21 (B) 14 (C) 7 (D) 1

47. The sum of two interior angles of a convex pentagon is 225°. If the remaining angles are equal, what amount does each measure?
(A) 112.5° (B) 90° (C) 100° (D) 105°

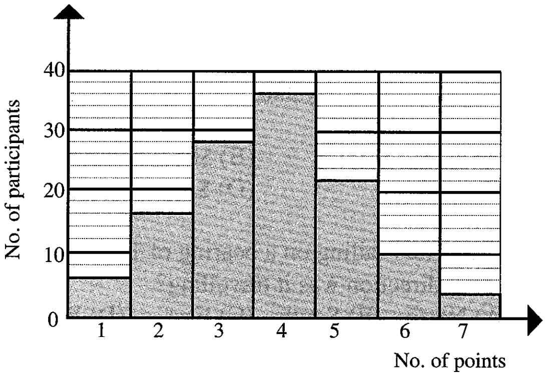
48. The gradient of the straight line passing through the points (1, −3) and (2, 5) is
(A) 1 (B) 8 (C) −8 (D) 2

49. A vehicle travels with a speed of 72 kilometres per hour. Its speed in metres per second is
(A) 15 (B) 20 (C) 25 (D) 30

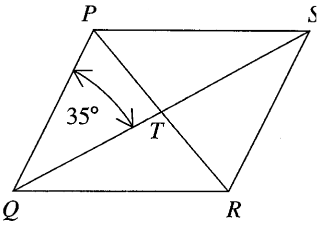
50. In a bank, the interest rate on investments increased from $6\frac{1}{2}\%$ per annum to 7% per annum. The increase in annual interest on a fixed deposit of \$3 000 is
(A) \$15 (B) \$210 (C) \$195 (D) \$405

51. The mean of 11 numbers is 9. One of the numbers 19, is deleted. The mean of the remaining numbers is
(A) 7 (B) 8 (C) 8.5 (D) 6.1

52. The histogram following shows the number of points scored by participants in a shooting competition. The number of participants is
(A) 7 (B) 100 (C) 120 (D) 280



53. When rotated through 180° about the origin, the image of the point (5, 1) is
(A) (−1, −5) (B) (5, −1)
(C) (−5, 1) (D) (−5, −1)



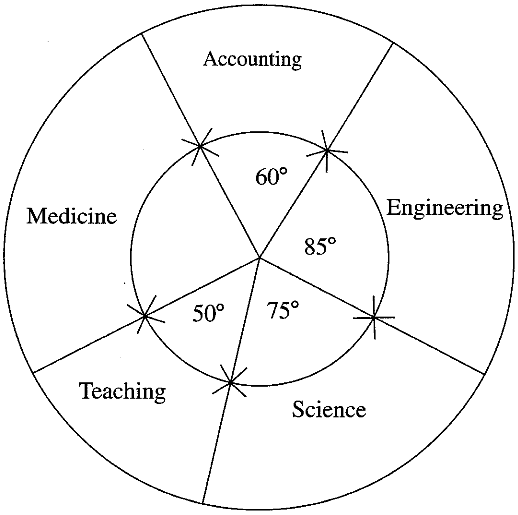
54. In the rhombus above, if $\angle PQT = 35°$, then $\angle QPT =$
(A) 72.5° (B) 55° (C) 90° (D) 35°

55. In a certain hotel, guests are served 125 g of meat per meal. What mass of meat in kilograms is required for 6 meals to feed 1 500 guests?
(A) 1 125 (B) 187 500
(C) 9 000 (D) 750

56. The set of fractions $\left\{\frac{3}{4}, \frac{2}{5}, \frac{1}{3}, \frac{5}{8}\right\}$ written in decending order of magnitude is
(A) $\left\{\frac{1}{3}, \frac{2}{5}, \frac{5}{8}, \frac{3}{4}\right\}$ (B) $\left\{\frac{3}{4}, \frac{2}{5}, \frac{5}{8}, \frac{1}{3}\right\}$
(C) $\left\{\frac{5}{8}, \frac{3}{4}, \frac{2}{5}, \frac{1}{3}\right\}$ (D) $\left\{\frac{3}{4}, \frac{5}{8}, \frac{2}{5}, \frac{1}{3}\right\}$

57. The modal value is most suitable for use by
(A) Engineers (B) Scientists
(C) Business people (D) Teachers

Items 58 − 60 refer to the pie chart below which shows the favourite fields of further study of 1 080 students at a particular school.



58. How many preferred Medicine?
(A) 90 (B) 180 (C) 270 (D) 360

59. How many preferred Law?
(A) 225 (B) 150 (C) 100 (D) 0

60. If a student is chosen at random, the probability that he/she prefers teaching is
(A) $\frac{5}{36}$ (B) $\frac{1}{6}$ (C) $\frac{1}{5}$ (D) $\frac{5}{66}$

C.X.C. MODEL EXAMINATION 3
MATHEMATICS
Paper 1 – Basic Proficiency
90 minutes
Answer ALL the questions

1. 7 853 to the nearest hundred =
(A) 7 800 (B) 7 900 (C) 7 950 (D) 8 000

2. The total simple interest paid on \$500 borrowed for 2 years at a rate of 10% per annum is
(A) \$100 (B) \$80 (C) \$60 (D) \$40

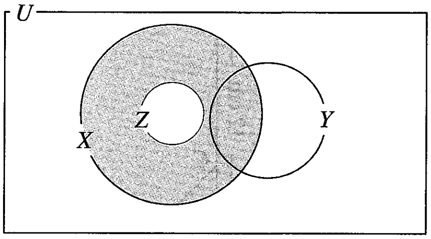
3. In selling an article, a shopkeeper made a profit of 25% on his cost price of \$80. The selling price was
(A) \$130 (B) \$98 (C) \$100 (D) \$125

4. If x is an odd number, which of the following is also odd?
(A) $x - 1$ (B) $x + 1$ (C) $2x$ (D) $2x - 1$

5. $15.2 \div 0.01 =$
(A) 1.52 (B) 15.2 (C) 152 (D) 1520

6. Oliver is x years older than Savi. If Savi is 27 years of age, then Olivers's age in years is
(A) $x - 27$ (B) $x + 27$ (C) $27x$ (D) $27 - x$

7.

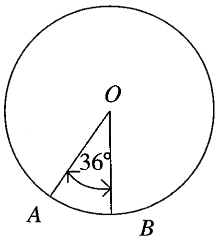


In the Venn diagram above, the shaded region represents the set
(A) $X \cap Y \cap R$ (B) Y'
(C) $X \cap Z'$ (D) $X' \cup Z$

8. By the distributive law, $28 \times 15 + 7 \times 15 =$
(A) 28×30 (B) 15×30
(C) $35 + 15$ (D) 35×15

9. 0.003 58 expressed in standard form =
(A) 3.58×10^{-2} (B) 3.58×10^{-3}
(C) 3.58×10^{2} (D) 3.58×10^{3}

10.



The circumference of the circle, centre O, is 25 cm. Therefore the length of the arc AB is
(A) $\frac{1}{10} \times 25$ cm (B) $\frac{1}{36} \times 25$ cm
(C) 36×25 cm (D) $\frac{1}{5} \times 25$ cm

11. If $P = \{2, 3, 5, 7\}$, $Q = \{3, 6, 7\}$ and $R = \{2, 4, 5\}$, then $P \cup Q \cup R =$
(A) { } (B) {3, 6, 7}
(C) { 4 } (D) {2, 3, 4, 5, 6, 7}

12. An aeroplane leaves Airport A at 18:00 h and arrives at Airport B at 18:45 h, travelling at an average speed of 752 kilometres per hour. The distance from Airport A to Airport B is
(A) 892 km (B) 1 002.7 km
(C) 564 km (D) 655 km

13. The cost of insuring articles valued at \$200 is \$3.00. So the cost of insuring articles valued at \$120 is
(A) \$1.50 (B) \$1.80 (C) \$2.00 (D) \$2.20

14.

157	167	182	145
155	139	153	149

(in cm)

The median of the eight heights in the table above is
(A) 155.5 cm (B) 159 cm
(C) 160 cm (D) 154 cm

15. The median of 11 numbers is 9. The number 15 is deleted. The mean of the remaining numbers is
(A) 5 (B) 8.4 (C) 17 (D) 9.2

16. If $\frac{5x-3}{2} = 6$, then $x =$
(A) 5 (B) 3 (C) 17 (D) 25

17. If $\sqrt{9.7} = 3.11$, then $\sqrt{9.7 \times 10^4} =$
(A) 3.11×10 (B) 3.11×10^2
(C) 3.11×10^3 (D) 3.11×10^4

18. A ship was sailing on a bearing of 270°. In what direction was it sailing?
(A) North (B) South (C) East (D) West

19. The area of square is 144 cm^2. What is its perimeter?
(A) 48 cm (B) 96 cm (C) 72 cm (D) 216 cm

20. $\frac{2}{3x} - \frac{1}{2y} =$
(A) $\frac{4y-3x}{6xy}$ (B) $\frac{1}{6xy}$
(C) $\frac{1}{3x+2y}$ (D) $\frac{3x-2y}{6}$

21. How many litres of soft drink could a container of 4000 cm^3 hold?
(A) 2 (B) 4 (C) 6 (D) 8

22. $\frac{5p}{3q} + \frac{2r}{7s} =$
(A) $\frac{5p+2r}{3q+7s}$ (B) $\frac{5p+2r}{21qs}$
(C) $\frac{10pr}{21qs}$ (D) $\frac{35ps+6qr}{21qs}$

23. The equation of a straight line is $y = mx + c$, where m represents the
(A) x-intercept (B) y-intercept
(C) gradient (D) horizontal distance

24. In a quadrilateral $ABCD$, angle $A = x°$, angle $B = 2x°$ and angle $C = 3x°$. What is the magnitude of angle D?
(A) $(360 - 6x)°$ (B) $\frac{360°}{6x}$
(C) $(180 - 6x)°$ (D) $360° \times 6x°$

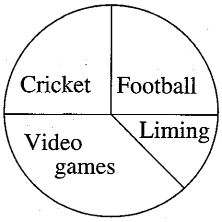
25. If \$560 is divided into two parts in the ratio 3:4, the larger part is
(A) \$300 (B) \$280 (C) \$240 (D) \$320

26. 36549 written correct to 3 significant figures is
(A) 37000 (B) 36000 (C) 36500 (D) 36600

27. The cube root of 125 is
(A) 5 (B) 25 (C) 12.5 (D) 15

28. A merchant bought sweets at \$3.00 a dozen and sold them at \$2.50 a dozen. His percentage loss was
(A) 50% (B) 19.5% (C) 16.7% (D) 21%

29.

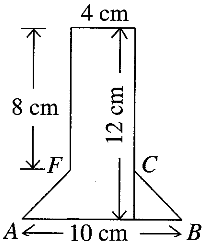


The pie-chart above shows how 100 students spent their time at home during a specified period. The number of students who spent it playing video games is approximately
(A) 25 (B) 50 (C) 38 (D) 75

30. $8x - 5(x + 3) =$
(A) $3x - 15$ (B) $3x + 15$
(C) $13x - 3$ (D) $-13x + 3$

31. Given that, $x = 0$, $5y + 4x = 1$ and $10y + px = 2$, then $\hat{P} =$
(A) −6 (B) 4 (C) 6 (D) 8

32.



The area of the plane figure $ABCDEF$ above is
(A) 34 cm^2 (B) 60 cm^2
(C) 75 cm^2 (D) 37 cm^2

33. If 10 books which costs \$25.00 each are sold for \$3.00, then the profit, expressed as a percentage of the cost is
(A) 15% (B) 20% (C) 25% (D) 30%

34. A polygon with all its sides equal is called a
(A) pentagon (B) convex polygon
(C) re-entrant polygon (D) regular polygon

35. In a sale a Nintendo set marked \$1 200 was sold for \$960. What percent discount was given?
(A) 240 (B) 10 (C) 15 (D) 20

36. A businesswoman sells an article at a profit of 20%. What was the cost price of the article if she sells it for \$4.80?
(A) \$3.00 (B) \$4.00 (C) \$4.60 (D) \$0.80

37. The solution set of $-7 < x - 6 < 5$ will include the elements
(A) $-7, 6, 8$ (B) $2, 6, 15$
(C) $2, 6, 9$ (D) $-1, 6, 9$

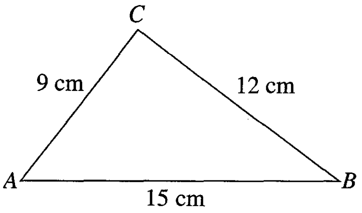
38. y is directly proportional to x. When $x = 15$ then $y = 12$. What is the value of y when $x = 30$?
(A) 45 (B) 27 (C) 24 (D) 90

39. Which of the following sets represents the relation $f: x \to 2x + 3$?
(A) $\{(0, 3), (1, 6), (2, 9), (3, 12)\}$
(B) $\{(0, 3), (1, 5), (2, 6), (3, 8)\}$
(C) $\{(0, 3), (1, 4), (2, 5), (3, 6)\}$
(D) $\{(0, 3), (1, 5), (2, 7), (3, 9)\}$

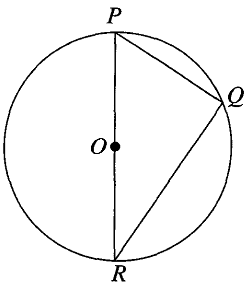
40. 316_7 written in base 10 =
(A) 70 (B) 160 (C) 60 (D) 87

41. Which of the following is rational?
(A) $\sqrt{15}$ (B) $\sqrt{\frac{25}{36}}$ (C) $\frac{5\pi}{6}$ (D) $\frac{3}{\sqrt{5}}$

42. The best estimate of 0.0925×25.98 is
(A) 0.024 (B) 0.24 (C) 2.4 (D) 24.0



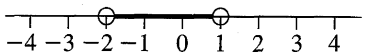
43. What is the area of the triangle above?
(A) 54 cm² (B) 67.5 cm²
(C) 90 cm² (D) 810 cm²

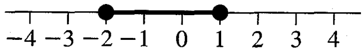


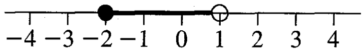
44. In the figure above, O is the centre of the circle. The triangle PQR is circumscribed by the circle. If $PQ = 9$ cm and $RQ = 12$ cm, then $QO =$
(A) 6.5 cm (B) 7.5 cm
(C) 8.5 cm (D) 10.5 cm

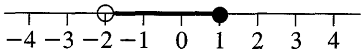
45. Which of the following are exactly divisible by $x + 2$?
I. $x + 4$ II. $-x - 2$
III. $x^2 + 6$ IV. $x^2 - 4$
(A) I and II (B) II and III
(C) III and IV (D) II and IV

46. Which of the following line graphs represents $\{x: -2 \leqslant x < 1\}$?

(A)


(B)


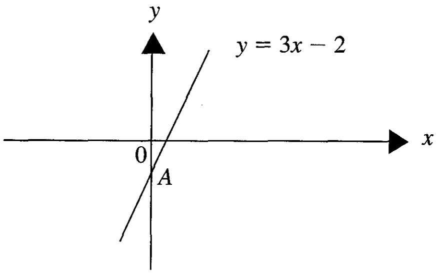
(C)


(D)


47. The lengths of the sides of a parallelogram are $2x$ and $5x$ cm. If the perimeter is 140 cm, then $2x$ =
(A) 10 (B) 20 (C) 30 (D) 40

48. One year after a car was bought, its value dropped by 10% to \$8100. The original cost of the car was
(A) \$8 110 (B) \$8 700 (C) \$8 910 (D) \$9 000

49.



In the figure above, A is the point with coordinates
(A) $(3, -2)$ (B) $(-2, 0)$
(C) $(0, -2)$ (D) $(-2, 3)$

50. Which of the following sets of values could not be lengths of the sides of a right-angled triangle?
(A) 3, 4, 5 (B) 5, 12, 13
(C) 6, 8, 10 (D) 3, 4, 7

51. A square is given an enlargement of scale factor 2. What is the ratio of the area of the image to the area of the original square?
(A) 2:1 (B) 4:1 (C) 1:2 (D) 1:4

52. Which of the following sets represents the function $f: x \rightarrow x^2 + 5$?
(A) $\{(0, 5), (1, 6), (2, 9), (3, 14)\}$
(B) $\{(0, 5), (1, 7), (2, 9), (3, 11)\}$
(C) $\{(0, 5), (1, 6), (2, 7), (3, 8)\}$
(D) $\{(0, 5), (1, 4), (2, 3), (3, 2)\}$

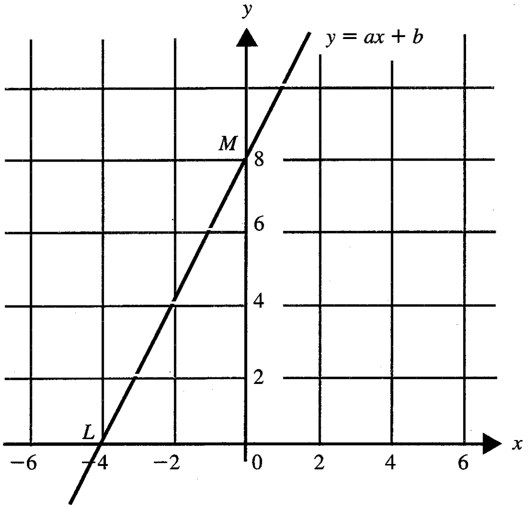
53. On Monday a shopkeeper increased his stock by 20%, but by Sunday his sales had reduced his stock by 20%. How does his stock on Monday compare with his stock on Sunday?
(A) It is greater (B) It is the same
(C) It is less (D) They are equal

54. What are the coordinates of the image of $(2, -3)$ under the displacement $\begin{pmatrix}3\\-2\end{pmatrix}$?
(A) $(4, 6)$ (B) $(-4, -6)$
(C) $(5, -5)$ (D) $(-5, 5)$

55. If $p \square q = p^2 - 3q$, then $5 \square 2 =$
(A) -11 (B) 19 (C) 21 (D) 25

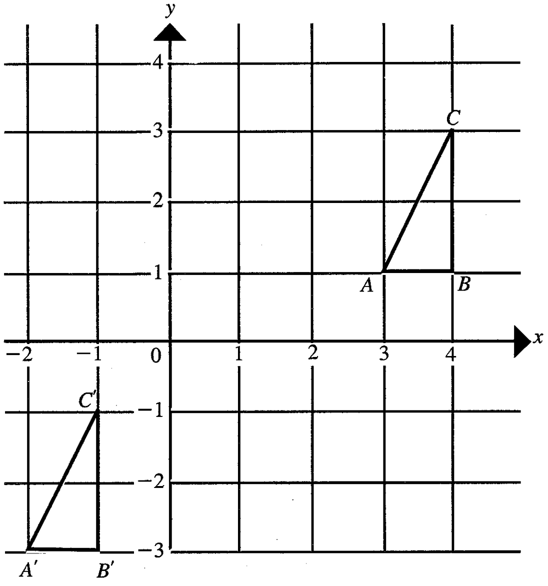
56. If the point $P(3, 5)$ is rotated anti-clockwise through 270°, what are the coordinates of the image of P?
(A) $(-3, -5)$ (B) $(-3, 5)$
(C) $(-5, 3)$ (D) $(5, -3)$

57.

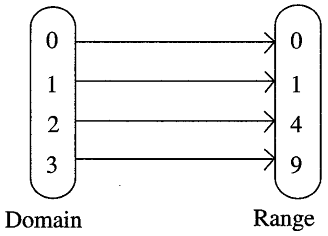


The equation of the straight line LM in the graph above is
(A) $y = 2x - 8$ (B) $y = -2x + 8$
(C) $y = -2x - 8$ (D) $y = 2x + 8$

58.



What single transformation maps triangle ABC onto triangle $A'B'C'$ in the figure above?
(A) Reflection (B) Rotation
(C) Translation (D) Shear



59. The diagram above represents the mapping
(A) $x \rightarrow 2x$ (B) $x \rightarrow 2x + 2$
(C) $x \rightarrow 2^x$ (D) $x \rightarrow x^2$

60. The number of subsets of a set of n elements is equal to
(A) $n^2 + 1$ (B) 2^n
(C) $2^n + 1$ (D) n^2

C.X.C. MODEL EXAMINATION 4
MATHEMATICS
Paper 1 – Basic Proficiency
90 minutes
Answer ALL the questions

1. If $5x = 9$, then $x =$

(A) $\frac{1}{5} \times 9$ (B) $\frac{1}{9} \times 5$

(C) 5×9 (D) $9 - 5$

2. 23.96×0.25 is approximately

(A) 0.06 (B) 0.6 (C) 6 (D) 60

3. If 1 300 out of 6 500 members voted in an association's elections, then the per cent which voted is?

(A) 5% (B) 10% (C) 15% (D) 20%

4. If x is an odd number, which of the following is even?

(A) $x - 1$ (B) $x - 2$ (C) $x + 2$ (D) $2x + 1$

5. The simple interest on \$900 for 5 years at 7% is per annum is

(A) $\frac{\$900 \times 7 \times 5}{100}$ (B) $\frac{\$900 \times 7}{5 \times 100}$

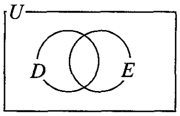
(C) $\frac{\$900 \times 5}{7 \times 100}$ (D) $\frac{\$100 \times 7 \times 5}{900}$

6. What is least number of sweets which can be shared equally among either 9, 12 or 18 children?

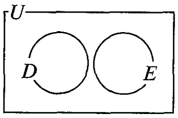
(A) 24 (B) 36 (C) 48 (D) 60

7. If $U = \{\text{real numbers}\}$,
$D = \{\text{odd numbers}\}$
and $E = \{\text{even numbers}\}$.
Which of the Venn diagrams below illustrates the statement. "No odd numbers are even numbers"

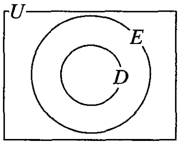
(A)



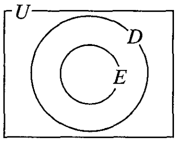
(B)



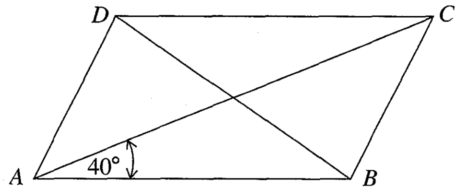
(C)



(D)



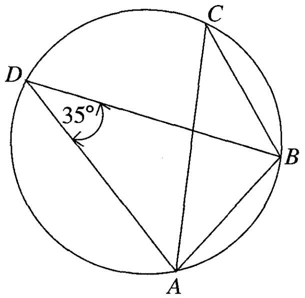
8.



In the parallelogram $ABCD$ above, if $\angle BAC = 40°$, then $\angle ACD =$

(A) 20° (B) 40° (C) 60° (D) 80°

9.



In the diagram above, $A\hat{D}B = 35°$, so $A\hat{C}B =$

(A) 35° (B) 70° (C) 55° (D) 17.5°

10. In a triangle ABC, angle $A = x°$ and angle $B = 3x°$. What is the magnitude of angle C?

(A) $(180 - 4x)°$ (B) 45°

(C) 90° (D) $\left(\frac{180}{4x}\right)^{\circ}$

11. The perimeter of a square is 60 cm. What is its area in cm^2?

(A) 240 cm^2 (B) 96 cm^2

(C) 30 cm^2 (D) 225 cm^2

12. If \$360 is divided into two parts in the ratio 3:5 the smaller part is

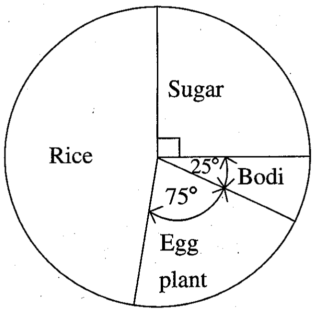
(A) \$260 (B) \$135 (C) \$225 (D) \$315

13. The missing term in the series

$12,\ 11\frac{1}{3},\ 10\frac{2}{3},\ \square,\ 9\frac{1}{3}$ is

(A) 10 (B) $10\frac{1}{3}$ (C) 9 (D) $9\frac{1}{5}$

14.



The pie-chart above shows the allotment of land to various crops. If sugar was cultivated on 18 hectares, what area of land was allotted to rice?

(A) 34 hectares (B) 17 hectares
(C) 1.8 hectares (D) 28 hectares

15. $3p(p + 4q) - q(2p - 3q) \equiv$

(A) $3p^2 - 10pq + 3q^2$ (B) $3p^2 - 10pq - 3q^2$
(C) $3p^2 + 10pq - 3q^2$ (D) $3p^2 + 10pq + 3q^2$

16. If $P = \{2, 3, 5, 7\}$, $Q = \{2, 5, 8\}$ and $R = \{2, 4, 6, 8\}$, then $P \cap Q \cap R =$

(A) $\{2, 8\}$ (B) $\{2\}$ (C) $\{7\}$ (D) $\{\ \}$

17. If the interest rate on investments at a particular bank increased from 6% per annum to $7\frac{1}{2}\%$ per annum, then the difference in annual interest on a fixed deposit of $3 000 would be

(A) $225 (B) $45 (C) $180 (D) $1.50

18. The marked price on a sewing machine was $1540. A seamstress bought the sewing machine on hire purchase by making a down payment of $300 and paying $114 monthly for one year.
Calculate the amount she would have saved if the sewing machine had been bought for cash.

(A) $172 (B) $128 (C) $414 (D) $186

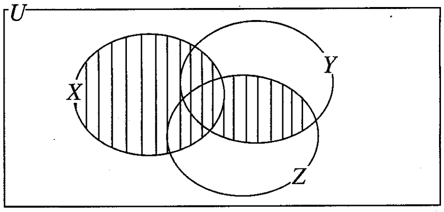
19. $(-p)^2 \times (-q)^3 =$

(A) $-6pq$ (B) $6pq$ (C) p^2q^3 (D) $-p^2q^3$

20. $\frac{3}{4x} + \frac{2}{4x} =$

(A) $\frac{12x + 8x}{4x}$ (B) $\frac{5}{16x}$
(C) $\frac{5}{4x}$ (D) $\frac{5}{8x}$

21.



In the Venn diagram above, the shaded region represents the set

(A) $(X \cap Z) \cup Y$ (B) $X \cup (Y \cap Z)$
(C) $(X \cap Y) \cup Z$ (D) $X \cup (Y \cap Z)$

22. A man gave his wife 25% of his wage, banked 35%, and the remainder he used to purchase groceries for the home. What amount did he spend on groceries if his salary was $540?

(A) $135 (B) $189 (C) $216 (D) $324

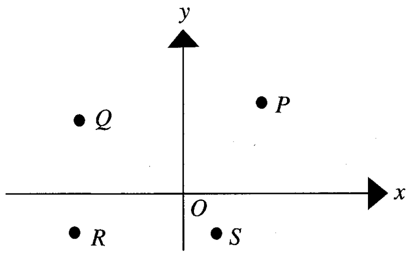
23. Six calculators costing $45 each are sold for $337.50. The profit expressed as a percentage of the cost price is

(A) 15% (B) 20% (C) 25% (D) 30%

24. The lengths of the sides of a parallelogram are $2x$ and $5x$ cm. Given that the perimeter is 42 cm, then $3x =$

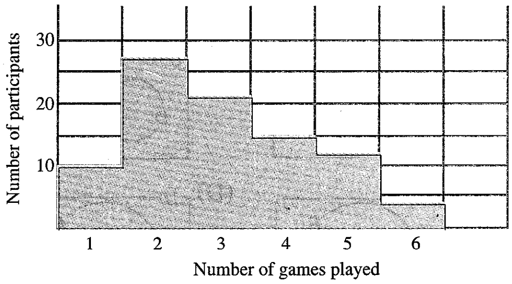
(A) 6 cm (B) 3 cm (C) 14 cm (D) 9 cm

25.



In the figure above, for which point are both the x-coordinate and the y-coordinate negative?

(A) P (B) Q (C) R (D) S



26. The histogram above shows the number of participants who played a given number of games. The total number of participants is
(A) 6 (B) 50 (C) 100 (D) 90

27. The number 26 725 written correct to four significant figures is
(A) 26 720 (B) 2672 (C) 2673 (D) 26 730

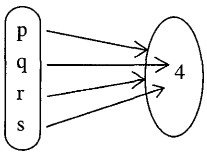
28. Two fair dice are tossed. What is the probability of scoring 10?
(A) $\frac{1}{36}$ (B) $\frac{1}{18}$ (C) $\frac{1}{12}$ (D) $\frac{1}{6}$

29. 0.003 768 expressed in scientific notation =
(A) 3.768×10^3 (B) 3.768×10^2
(C) 3.768×10^{-3} (D) 3.768×10^{-2}

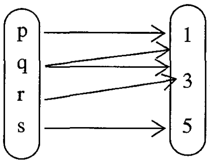
30. y is directly proportional to x. When $x = 15$, then $y = 12$. What is the value of y when $x = 45$
(A) 30 (B) 36 (C) 40 (D) 42

31. Which of the following is not a function?

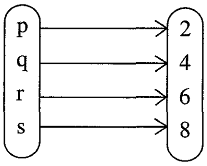
(A)



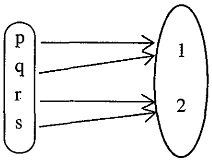
(B)



(C)



(D)



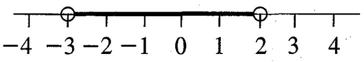
32. If $\frac{x}{y} = \frac{1}{2}$, which of the following is true?
(A) $x - 2 = y - 1$ (B) $x - y = 2$
(C) $y = 2x$ (D) $x = 2y$

33. If $x = \{1, 3, 5\}$, then the number of subsets which can be formed =
(A) 3 (B) 6 (C) 8 (D) 9

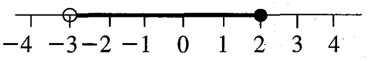
34. A rectangular plot of land is 800 m long and 600 m wide. The area of the plot of land in hectares is
(A) 4800 (B) 48 (C) 480 (D) 4.8

35. Which of the following line graphs represents the set $\{x: -3 \leqslant x < 2\}$?

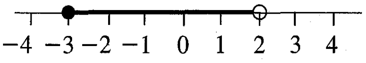
(A)



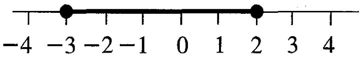
(B)



(C)



(D)



36. If the equation of a straight line is $y = mx + c$, then c represents the
(A) gradient of the line
(B) x-intercept
(C) y-intercept
(D) horizontal distance

37. Using set builder notation, the set $\{-3, -2, -1, 0, 1, 2, 3\}$ can be represented as
(A) $\{x: -3 \leqslant x \leqslant 3, x \in N\}$
(B) $\{x: -3 \leqslant x \leqslant 3, x \in Z\}$
(C) $\{x: -3 < x < 3, x \in N\}$
(D) $\{x: -3 < x < 3, x \in Z\}$

38. The fractions $\frac{3}{8}, \frac{1}{3}, \frac{1}{5}, \frac{1}{2}$, written in descending order of magnitude =
(A) $\frac{1}{5}, \frac{1}{3}, \frac{3}{8}, \frac{1}{2}$ (B) $\frac{1}{2}, \frac{3}{8}, \frac{1}{5}, \frac{1}{3}$
(C) $\frac{1}{2}, \frac{3}{8}, \frac{1}{3}, \frac{1}{5}$ (D) $\frac{3}{8}, \frac{1}{5}, \frac{1}{3}, \frac{1}{2}$

39. If x and y are two integers, then $3(x + y)^2$ means
(A) three times the square of their sum
(B) six times their sum
(C) the square of three times their sum
(D) three times the sum of their squares

40. The total surface area of a closed cylindrical container of radius r and height h is
(A) $\pi r^2(h + r)$ (B) $2\pi rh$
(C) $\pi r^2 h$ (D) $2\pi r(r + h)$

41. Under the translation $T = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, the image of $(2, -5)$ is
(A) $(5, -7)$ (B) $(6, -10)$
(C) $(1, -3)$ (D) $(-1, 7)$

42. The circumference of a circle with a radius of 14 cm is
(A) 44 cm (B) 88 cm (C) 140 cm (D) 616 cm

43. Which of the following sets represents the function $f: x \rightarrow x^2 - 1$?
(A) $\{(0, 1), (1, 2), (2, 4), (3, 9)\}$
(B) $\{(0, -1), (1, 1), (2, 3), (3, 5)\}$
(C) $\{(0, -1), (1, 0), (2, 3), (3\ 8)\}$
(D) $\{(0, 3), (1, 4), (2, 5), (3, 6)\}$

44. The number property used in writing $\left(\frac{3}{19} + \frac{4}{9}\right) + \frac{23}{29}$ as $\frac{3}{19} + \left(\frac{4}{9} + \frac{23}{29}\right)$ is the
(A) identity law (B) distributive law
(C) commutative law (D) associative law

45. In which of the following transformations are shape, area and symmetry invariant?
I. Shear II. Displacement
III. Stretch IV. Glide reflection
(A) I, II and III only (B) II, III and IV only
(C) I and III only (D) II and IV only

46. If (x', y') is the image of $(4, 7)$ after a reflection in the line $x = 1$, then (x', y') is
(A) $(-3, 7)$ (B) $(-2, 7)$ (C) $(-5, 7)$ (D) $(2, 7)$

47. 2 520 expressed as a product of prime numbers is
(A) $2^3 \times 3^2 \times 5 \times 7$ (B) $2^3 \times 3^2 \times 5 \times 7$
(C) $2^4 \times 3 \times 7$ (D) $2^3 \times 5 \times 11$

48. The arithmetic mean of the set of numbers 9, 11, 15, 17, 18, 25, 38 is
(A) 16 (B) 17 (C) 18 (D) 19

49. The semi-interquartile range of the set of numbers 11, 15, 38, 17, 9, 18, 25 is
(A) 21 (B) 14 (C) 7 (D) 3.5

50.

x	1	2	3	4	5	6	7	8
f	5	3	6	7	2	1	3	4

What is the value of the mode of the frequency distribution above?
(A) 7.00 (B) 4.00 (C) 8.00 (D) 4.53

51. In an urn there are 5 red balls and 3 blue balls. If one is selected at random what is the probability that it is blue?
(A) $\frac{3}{8}$ (B) $\frac{5}{8}$ (C) $\frac{8}{3}$ (D) $\frac{8}{5}$

52. An article marked to give 120% profit originally cost \$60. What is the marked prize?
(A) \$120 (B) \$132 (C) \$72 (D) \$180

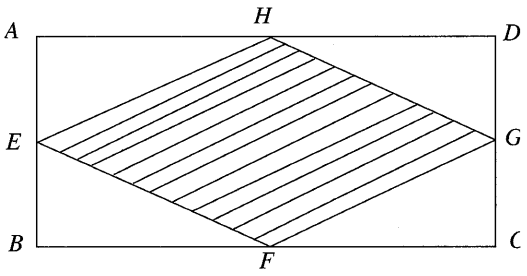
53. If $x = 2$ and $xy = 6$, then $(x - y)^2 - (x^2 - y^2) =$
(A) -9 (B) 12 (C) -5 (D) 6

54. y varies inversely as x^2. When $y = 16$ then $x = 9$. What is the value of x when $y = 9$?
(A) 4 (B) 25 (C) 12 (D) 16

55. An importer pays customs duty of 30% on compact disks valued at \$1 350 each. He then sells them to make a profit of \$565. What is selling price of such a compact disk.
(A) \$1 915 (B) \$2 320
(C) \$2 489.50 (D) \$1 945

56. After 15% of a woman's salary had been deducted for tax she received \$1 326. The amount of money she paid in tax was
(A) \$2 020 (B) \$1 341 (C) \$234 (D) \$329

57.



The rectangle $ABCD$ above has sides of length 12 cm and 4 cm. If E, F, G and H are mid-points of their respective sides, then the shaded area is
(A) 24 cm² (B) 48 cm² (C) 6 cm² (D) 12 cm²

58. In a triangle ABC, if $\hat{A} = x°$, $\hat{B} = 2x°$ and $\hat{C} = 3x°$, then $\hat{B} =$
(A) 30° (B) 60° (C) 90° (D) 45°

59. An anti-clockwise rotation of 90° about the origin maps the point $K(3, 6)$ onto K'. What are the coordinates of K'
(A) $(-3, -6)$ (B) $(-6, -3)$
(C) $(-6, 3)$ (D) $(3, -6)$

60. The prices of all articles in a store were marked up by 25%. During a sale a 25% discount on marked prices was given. How did the final selling prices compare with the prices before the mark up?
(A) The prices were the same.
(B) The final selling price was less than the original price.
(C) The final selling price was greater than the original price.
(D) No decisions can be made.

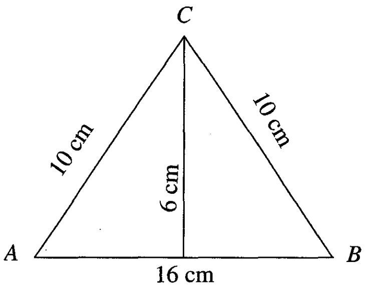
C.X.C. MODEL EXAMINATION 5
MATHEMATICS
Paper 1 – Basic Proficiency
90 minutes
Answer ALL the questions

1. 2.5 expressed as a percentage of 1 is
(A) 25% (B) 0.25% (C) 250% (D) 2 500%

2. If 1 200 out of 4 800 persons ran in a marathon, the per cent that ran is
(A) 25% (B) 20% (C) 15% (D) 10%

3. If x is an even number, which of the following is odd?
(A) $2x$ (B) $2x - 1$ (C) $x - 2$ (D) $x + 2$

4.



The perimeter of the triangle ABC above is
(A) 42 cm (B) 32 cm (C) 80 cm (d) 36 cm

5. The least number of mangoes which can be shared equally among 5, 10 or 15 orphans is
(A) 15 (B) 30 (C) 45 (D) 60

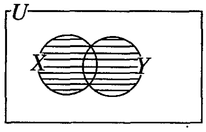
6. State which of the following sets is well defined?
(A) {Boys in the cricket team}
(B) {Boys good at running}
(C) {Tall boys at Presentation College}
(D) {Intelligent boys playing chess}

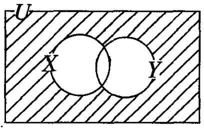
7. $\sqrt{10^2 - 8^2} =$
(A) 2 (B) 4 (C) 6 (D) 8

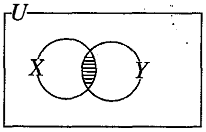
8. Using the distributive law, $37 \times 18 + 18 \times 5 =$
(A) 42×18 (B) $42 + 18$
(C) 42×36 (D) $42 + 36$

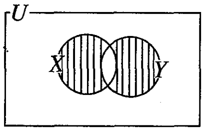
9. If (x', y') is the image of (3, 5) after reflection in the x-axis, then (x', y') is
(A) (−3, 5) (B) (−3, −5)
(C) (4, 5) (D) (3, −5)

10. In which of the Venn diagrams below does the shaded region represent $(X \cup Y)'$?

(A)


(B)


(C)


(D)


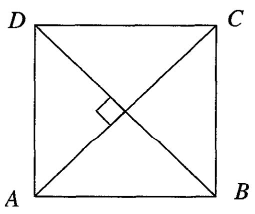
11. 423_6 may be written in base 10 as
(A) 159 (B) 9 (C) 47 (D) 99

12. $(-5)^2 + (-2)^3 =$
(A) −7 (B) 17 (C) −17 (D) −25

13. Eight books which cost $15 each are sold for $90. The loss per cent is
(A) 30% (B) 50% (C) 20% (D) 25%

14. If $\sqrt{9.5} = 3.08$, then $\sqrt{9.5 \times 10^2} =$
(A) 1.54×10 (B) 1.54×10^2
(C) 3.08×10 (D) 3.08×10^2

15. 0.000451, expressed in scientific notation =
(A) 4.51×10^{-4} (B) 4.51×10^{-3}
(C) 4.51×10^4 (D) 4.51×10^3



16. In the square $ABCD$ above, $\angle BDC =$
(A) 90° (B) 45° (C) 30° (D) 60°

17. $14.5 \times 28.6 = 414.7$, Hence $1.45 \times 2.86 =$
(A) 41.47 (B) 4.147
(C) 0.4147 (D) 0.041 47

18. $\frac{2}{3x} - \frac{1}{2y} =$
(A) $\frac{1}{6xy}$ (B) $\frac{1}{3x - 2y}$
(C) $\frac{4y - 3x}{6xy}$ (D) $\frac{7}{6xy}$

19. Under a reflection in the y-axis, the image of the point (3, 5) is
(A) (3, −5) (B) (−3, −5)
(C) (4, −5) (D) (−3, 5)

20. The total surface area of a cube with sides of length 5 cm is
(A) 125 cm² (B) 150 cm²
(C) 25 cm² (D) 200 cm²

21. Which of the following quadrilaterals is not a subset of the set of all parallelograms?
(A) squares (B) rectangles
(C) rhombuses (D) kites

22. If $\frac{P}{Q} = \frac{2}{3}$, which of the following is true?
(A) $3p = 2q$ (B) $2p = 3q$
(C) $p - 2 = q - 3$ (D) $p + 2 = q + 3$

23. The cost of insuring a parcel of mass 10 kg is \$22.50. What is the cost of insuring a parcel of mass 6 kg?
(A) \$11.50 (B) \$15.50 (C) \$13.50 (D) \$14.50

24. y is directly proportional to x. When $x = 13$, then $y = 16$. When $x = 52$, then $y =$
(A) 29 (B) 64 (C) 47 (D) 55

25. Robert is x years younger than Rita. If Rita is 25 years of age, then Robert's age in years is
(A) $\frac{25x}{2}$ (B) $25 - x$ (C) $x - 25$ (D) $x + 25$

26. If $A = \{\text{factors of } 24\}$
$B = \{\text{even numbers less than } 12\}$
and $C = \{\text{prime numbers less than } 12\}$,
then $A \cap B \cap C =$
(A) {2} (B) {12, 24}
(C) { } (D) {1}

27. If $\frac{x}{y} = 0$, then $(x, y) =$
(A) (2, 0) (B) (0, −2)
(C) (3, 3) (D) (−2, −2)

28. If $x = 3$, $y = 4$ and $z = -2$, then $5x - 2y - z =$
(A) 5 (B) 9 (C) 10 (D) 25

29. The lengths of the sides of a triangle are x, $2x$ and $3x$ cm. If the perimeter of the triangle is 24 cm, then $2x =$
(A) 4 cm (B) 7 cm (C) 8 cm (D) 12 cm

30. After a depreciation of 10%, the value of a personal computer was \$8100. What was its book value at the beginning of the year?
(A) \$9 290 (B) \$8 290 (C) \$8 910 (D) \$9 000

31. If $y = x^3$ and $z = y^2$, then $z =$
(A) y^9 (B) y^6 (C) x^9 (D) x^6

9	6	2	4
7	5	1	8

32. The median of the eight scores in the table above is
(A) 3.5 (B) 5.5 (C) 6 (D) 5

Score	1	2	3	4	5	6	7	8	9	10
Frequency	2	1	4	6	3	2	1	4	5	2

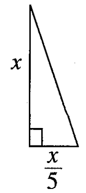
33. The table above shows the scores obtained by 30 students in a test. What was the modal score?
(A) 4 (B) 5.5 (C) 6 (D) 2

34. Which of the following numbers is a prime number?
(A) 154 (B) 155 (C) 156 (D) 157

35. $2^0 =$
(A) 1 (B) 2 (C) 0 (D) $\frac{0}{2}$

36. If the universal set U is the set of all counting numbers less than 10, and $P' = \{2, 3, 5, 7\}$, then $P =$
(A) {1, 4, 6, 8, 9} (B) {4, 6, 8}
(C) {1, 3, 5, 7, 9} (D) {4, 6}

37.



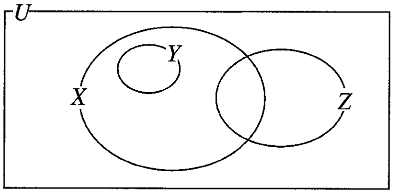
What is the least number of triangles such as the one shown above, which may be placed together to form a perfect square?
(A) 10 (B) 20 (C) 8 (D) 5

38. After 25% of a man's salary had been deducted for tax he receives \$285. The tax paid was
(A) \$47.50 (B) \$95.00 (C) \$57.00 (D) \$85.00

39. If $x * y$ means $3x - 2y$, then $4 * 1 =$
(A) 12 (B) −5 (C) 15 (D) 10

40. The probability than an event will not happen is $\frac{9}{13}$. The probability that the event will happen is
(A) 1 (B) 0 (C) $\frac{4}{13}$ (D) $\frac{9}{13}$

41.



Which of the following statements are true of the relationship among the three sets X, Y and Z, illustrated in the Venn diagram above?
I. $X \subset Y$ II. $Y \subset X$
III. $X \cap Z = \{\ \}$. IV. $Y \cap Z = \{\ \}$.
(A) I and III only (B) II and IV only
(C) I, II and III only (D) I, II, III and IV

42. In a boxing match of Holyfield vs. Tyson, the probability that Holyfield wins is $\frac{7}{13}$. The probability that Tyson wins is $\frac{5}{13}$. The probability of a draw is
(A) $\frac{1}{13}$ (B) $\frac{2}{13}$ (C) $\frac{5}{13}$ (D) $\frac{7}{13}$

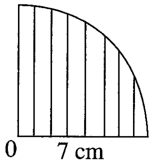
43. In a bank, the exchange rate of US \$1.00 is TT \$6.26. The value of US \$5.50 is
(A) TT \$34.43 (B) TT \$62.60
(C) TT \$19.76 (D) TT \$10.76

44. What is the total simple interest paid on \$5 000 borrowed for 3 years at a rate of 10% per annum?
(A) \$600 (B) \$900 (C) \$1 500 (D) \$3 000

45. A car moves with a maximum speed of 180 km/h. Its maximum speed in metres per second is
(A) 50 (B) 100 (C) 75 (D) 150

46. A man borrowed \$2 000 at $9\frac{1}{2}$% per annum simple interest on October 31, 2001. On November 1, 2001 the rate per cent per annum increased to 10% per annum. What amount of money will the man save annually by having taken his loan before the increase?
(A) \$10 (B) \$100 (C) \$190 (D) \$200

47.



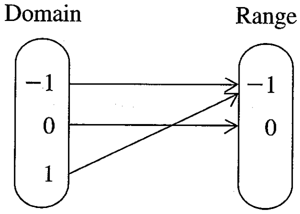
The area of the quadrant shown above is
(A) 44 cm (B) 154 cm²
(C) $\frac{22}{7}$ cm² (D) $\frac{77}{2}$ cm²

48. The government rates of any property is 2% in the dollar. A property has a rateable value of \$125 000. Determine the rates payable per annum.
(A) \$125 (B) \$1 250 (C) \$2 500 (D) \$12 500

49. An entrepreneur gained 30% by selling a shirt for \$117. The cost price to the entrepreneur was
(A) \$81.90 (B) \$87 (C) \$100 (D) \$90

50. Solve the equation $y = x - \frac{1}{2}$ when $x = 3\frac{1}{4}$
(A) $\frac{13}{4}$ (B) $2\frac{1}{2}$ (C) $3\frac{3}{4}$ (D) $2\frac{3}{4}$

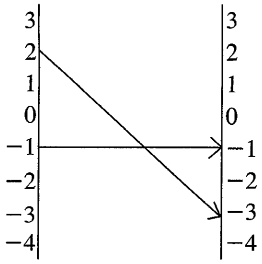
51.



Which of the following defines the function in the mapping diagram above?
(A) $f: x \to x$ (B) $f: x \to -x$
(C) $f: x \to x^2$ (D) $f: x \to -x^2$

52. If $f(x) = 2x$, then $f(x + 1) =$
(A) $2x + 1$ (B) $4x + 1$ (C) $2x + 2$ (D) $4x + 2$

53.



The diagram given shows two members of a linear relation. (x, y) is a member of this relation. The relation is

(A) $x \rightarrow -2x + 1$ (B) $x \rightarrow 2x + 1$
(C) $x \rightarrow -2x + 2$ (D) $x \rightarrow 2x + 2$

54. The volume of a cuboid of length l cm, breadth b cm and height h cm is

(A) $3\ lbh$ cm³ (B) $(lbh)^3$ cm³
(C) lbh cm³ (D) $\frac{lbh}{3}$ cm³

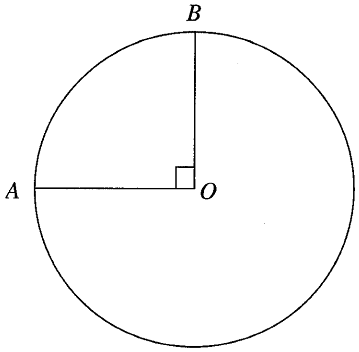
55.

A	60	90
V	150	*w*

If V varies directly as A, then the value of w in the table above is

(A) 200 (B) 225
(C) 250 (D) 275

56.

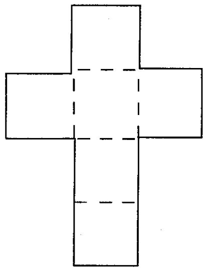


In the circle above, the circumference is 30 cm. The length of the major arc AB in centimetres is

(A) $\frac{3}{4} \times 30$ (B) $\frac{1}{4} \times 30$

(C) $\frac{3}{8} \times 30$ (D) $\frac{1}{8} \times 30$

57.



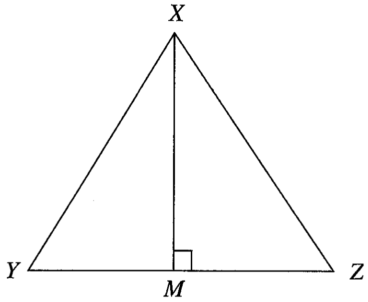
The figure above consists of six equal squares and has an area of 54 m². Its perimeter, in metres, is

(A) 42 (B) 27 (C) 126 (D) 324

58. The statistical average that makes use of all the data in its calculations is called the

(A) mode (B) mean
(C) median (D) lower quartile

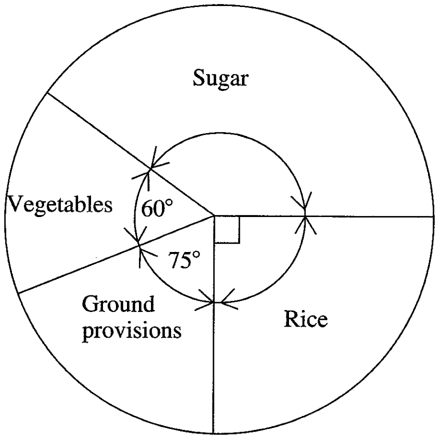
59.



In the equilateral triangle above, $XY = 9$ cm. M is the mid-point of YZ. Angle YXM is

(A) 60° (B) 45° (C) 30° (D) 15°

60.



60. The diagram above shows the portions of land allotted to various crops. If rice was cultivated on 4.5 hectares, what area of land was given to sugar?

(A) 6 hectares (B) 6.25 hectares
(C) 6.5 hectares (D) 6.75 hectares

C.X.C. MODEL EXAMINATION 6
MATHEMATICS
Paper 1 – Basic Proficiency
90 minutes
Answer ALL the questions

1. $7x - 3y + 2x - y =$

(A) $4x + y$ (B) $9x - 2y$
(C) $6x - y$ (D) $9x - 4y$

2. $\frac{5}{8}$ is equivalent to

(A) 0.375 (B) 0.625 (C) 0.875 (D) 0.75

3. The simple interest on \$600 for 3 years at 5% per annum is

(A) $\frac{\$600 \times 5}{100 \times 3}$ (B) $\frac{\$600 \times 3}{100 \times 5}$
(C) $\$6 \times 5 \times 3$ (D) $\frac{\$5 \times 3}{600}$

4. If $P = \{2, 3, 5, 7, 9\}$, $Q = \{2, 3, 6, 7\}$ and $R = \{2, 3, 5\}$, then $P \cap Q \cap R =$

(A) $\{2, 3, 5, 6, 7, 9\}$ (B) $\{\ \}$
(C) $\{6, 7, 9\}$ (D) $\{2, 3\}$

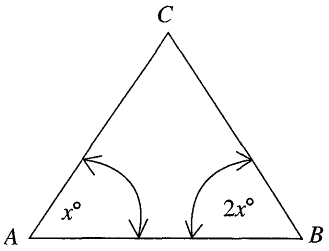
5. The square root of 841 is

(A) 29 (B) 33 (C) 52 (D) 61

6. A discount of 20% is offered on articles at a sale. The sale price of an article marked \$560 is

(A) \$540 (B) \$504 (C) \$672 (D) \$448

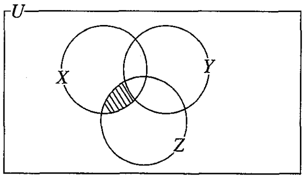
7.



In the triangle ABC above, angle $A = x°$ and angle $B = 2x°$, so angle $C =$

(A) 60° (B) $\left(\frac{180}{3x}\right)^{\circ}$
(C) $(180 - 3x)°$ (D) 45°

8.



In the Venn diagram given, the shaded region represents the set

(A) $(X \cap Z) \cap Y'$ (B) $(X \cup Z)' \cap Y$
(C) $(X \cup Z) \cap Y'$ (D) $(X \cap Y)' \cup Y$

9. If $P = \frac{1}{0.5}$, then $P =$

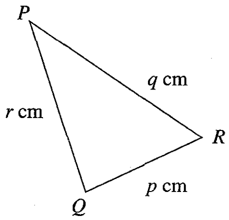
(A) 0.02 (B) 0.2 (C) 2 (D) 20

10. If x, y and z are integers, which of the following will always be an integer?

(A) $\frac{x + y}{5z}$ (B) $\frac{x + 1}{y}$ (C) $x(y + z)$ (D) $x - \frac{yz}{3}$

11. $\frac{x}{5} + \frac{5}{x} =$

(A) $\frac{x + 5}{5x}$ (B) $\frac{x^2 + 25}{5x}$ (C) $\frac{5(x + 5)}{x}$ (D) $\frac{x + 5}{5 + x}$



12. The area of $\triangle PQR$ above in cm² is

(A) $\frac{1}{2}pr \sin R$ (B) $rq \sin P$
(C) $\frac{1}{2}pqr$ (D) $\frac{1}{2}pq \sin R$

13. How many litres of sweet drink would a container of 2 000 cm³ be able to hold?

(A) 2 (B) 20 (C) 200 (D) 2 000

14. If $a = 1$ and $b = -2$, then $\frac{a - 2b}{ab} =$

(A) $-\frac{3}{2}$ (B) $\frac{5}{3}$ (C) $\frac{3}{4}$ (D) $-\frac{5}{2}$

15. If $L = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $M = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, then $L + M =$

(A) $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ (B) $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ (C) $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ (D) $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

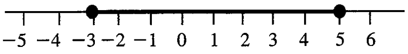
16. A refrigerator which cost $1 860 can be purchased by making a deposit of $402 and monthly payments of $81 each. How many months are required for the purchaser to become the owner of the refrigerator?

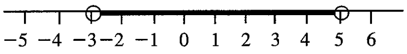
(A) 12 (B) 18 (C) 24 (D) 30

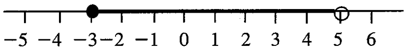
17. A man leaves his workplace at 15:45 h and reaches home at 02:30 h the following morning. How many hours did he take to reach home?

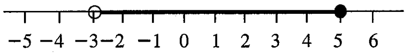
(A) $18\frac{1}{4}$ (B) $13\frac{1}{4}$ (C) $10\frac{3}{4}$ (D) $15\frac{3}{4}$

18. Which of the following line graphs represents $\{x: -3 < x < 5\}$.

(A)


(B)


(C)


(D)


19. $100 is divided among three friends in the ratio 2:3:5. The largest share is

(A) $80.00 (B) $60.00 (C) $50.00 (D) $75.00

20. If $\cos A° = \frac{15}{17}$, and angle A is acute, then the value of $\tan A° =$

(A) $\frac{17}{15}$ (B) $\frac{15}{8}$ (C) $\frac{8}{15}$ (D) $\frac{8}{17}$

21. The statement "A number p decreased by three times another number q is equal to r", can be expressed algebraically as

(A) $r = 3q - p$ (B) $3q + r = -p$

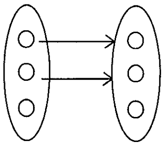
(C) $q = r - \frac{1}{3}p$ (D) $r = p - 3q$

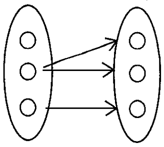
22. $x^3 = y$ means that

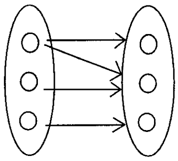
(A) $\log x = 3 \log y$ (B) x is a third of y

(C) x is the cube root of y (D) x is three times y

23. Which of the following mapping diagrams illustrates a function?

(A)


(B)


(C)


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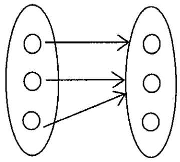
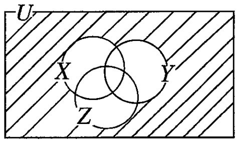


24.



In the Venn diagram above, the shaded region represents the set

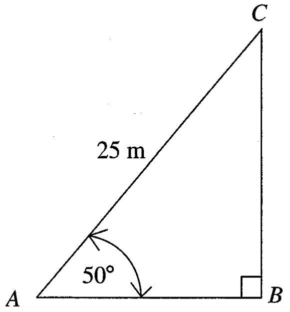
(A) $(X \cup Y \cup Z)'$ (B) $(X \cap Y \cap Z)'$

(C) $(X \cup Y)' \cup Z$ (D) $X \cap (Y \cup Z)'$

25. The simple interest on $480 after 2 years is $72. The rate per cent per annum is

(A) 7 (B) 7.5 (C) 8 (D) 8.5

26.

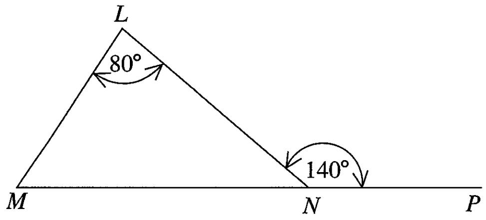


In the triangle ABC above, the length of AB, in m, is

(A) $25 \sin 50°$ (B) $\frac{25}{\sin 50°}$

(C) $\frac{25}{\cos 50°}$ (D) $25 \cos 50°$

27.



In the figure above, angle $LNP = 140°$ and angle $MLN = 80°$. The angle $LMN =$

(A) 30° (B) 60° (C) 50° (D) 40°

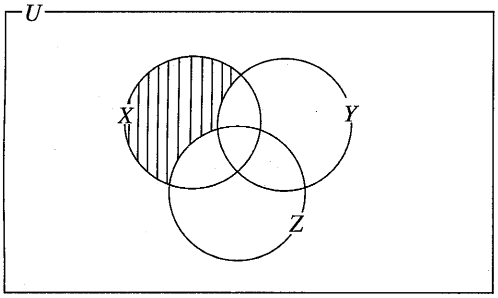
28. In the number 132_5 the digit 3 represents

(A) 40 (B) 21 (C) 15 (D) 8

29. Which of the following numbers is a prime number?

(A) 147 (B) 128
(C) 150 (D) 151

30.



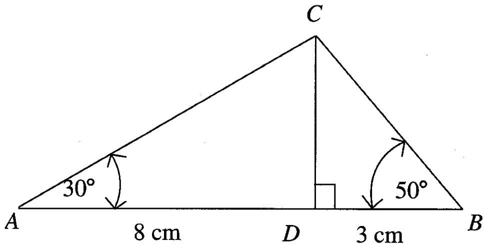
In the Venn diagram the shaded portion represents the set

(A) $(Y \cup Z)' \cap X$ (B) $X \cap Y'$
(C) $X \cap Z'$ (D) $(Y \cap Z) \cup X'$

31. The mean of six numbers is 47. If one of the numbers 52 is removed, then the mean of the remaining five numbers is

(A) 51 (B) 35
(C) 39 (D) 46

32.



In the figure above, angle $CAD = 30°$, angle $CBD = 50°$, $AD = 8$ cm and $BD = 3$ cm. The height of triangle ABC, $CD =$

(A) 3 cm × tan 50° (B) $\dfrac{8\text{ cm}}{\tan 30°}$

(C) 3 cm × sin 50° (D) $\dfrac{8\text{ cm}}{\sin 30°}$

33. y is directly proportional to x^2. When $x = 5$, then $y = 25$. What is the value of y when $x = 3$?

(A) 9 (B) 27
(C) 15 (D) 23

34. Given that $1500 \times 13 = 19\,500$, then $1499 \times 13 =$

(A) $19\,500 - 1000$ (B) $19\,500 - 1$
(C) $19\,500 - 1500$ (D) $19\,500 - 13$

35. $\sqrt{\dfrac{400}{25}} =$

(A) 16 (B) 4

(C) $\dfrac{1}{4}$ (D) $\dfrac{1}{16}$

36. If $p = 2$ and $pq = 8$, then $q =$

(A) −4 (B) −36
(C) 4 (D) 36

37. The length of a rectangle is measured as 8.0 cm, correct to the nearest cm. The minimum and maximum possible values of the length of the rectangle can be expressed as

(A) (8.0 ± 1) cm (B) (8.0 ± 1.5) cm
(C) (8.0 ± 0) cm (D) (8.0 ± 0.5) cm

38. The volume of a cube is 64 cm^3. The area of one of its plane faces is

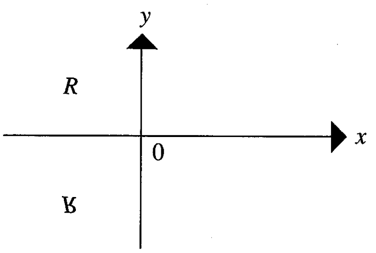
(A) 64 cm^2 (B) 16 cm^2
(C) 24 cm^2 (D) 4 cm^2

39. If $\cos A° = \dfrac{4}{5}$, and angle A is acute, then the value of $\tan A°$ is

(A) $\dfrac{4}{3}$ (B) $\dfrac{5}{4}$

(C) $\dfrac{3}{5}$ (D) $\dfrac{3}{4}$

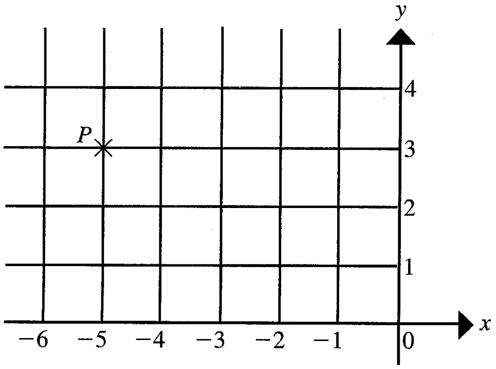
40.



In the diagram above the letter 'R' and an image of it are shown. The image may be obtained by a

(A) reflection in the line $x = 0$
(B) reflection in the line $y = 0$
(C) translation parallel to the y-axis
(D) rotation through 180° about the origin, 0

41.



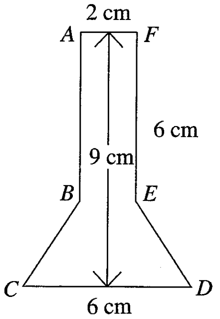
In the diagram above, the image of the point $P(-5, 3)$ under the translation $\begin{pmatrix}3\\-1\end{pmatrix}$ is

(A) $(-3, 1)$ (B) $(5, -3)$
(C) $(-2, 2)$ (D) $(8, 4)$

42. If $2x + 3y = 1$ and $ax + 6y = 2$, and $x \neq 0$, then $a =$

(A) 3 (B) 6 (C) 2 (D) 4

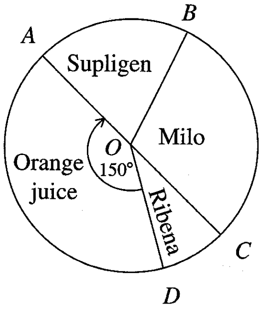
43.



In the figure above, the area enclosed by *ABCDEF*, in cm^2, is

(A) 12 (B) 24 (C) 36 (D) 40

44.



The pie chart given shows the preference in quarter–litre packed drinks of a group of students. *AC* is the diameter. If 26 students prefer Ribena, then the total number of students is

(A) 312 (B) 260
(C) 360 (D) 285

45. A merchant bought eggs at \$4.50 a dozen and sold them at \$6.00 a dozen. His percentage profit was

(A) $33\frac{1}{3}\%$ (B) 40% (C) 25% (D) 1.5%

46. The heights in centimetres of 10 students are 150, 155, 152, 170, 160, 153, 156, 165, 155, 158. The modal height is

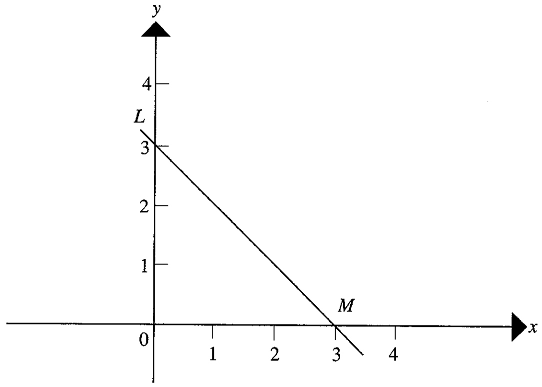
(A) 150 (B) 155 (C) 165 (D) 170

47. In which of the following are the diagonals perpendicular to each other?

I. Trapezium II. Parallelogram
III. Rhombus IV. Square

(A) I and II only (B) II and III only
(C) II and IV only (D) III and IV only

48.

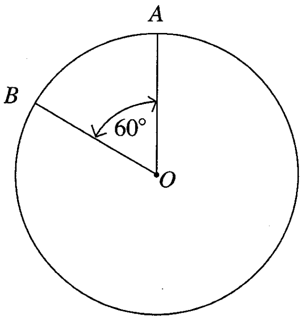


The equation of the *straight* line *LM* above is

(A) $y = \frac{1}{2}x + 3$ (C) $y = -x + 3$

(B) $y = -\frac{1}{2}x - 3$ (D) $y = x + 3$

49.

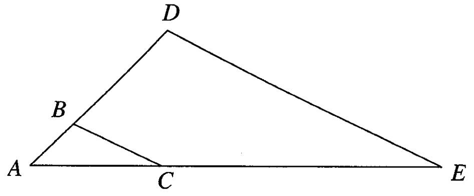


In the above circle the circumference is 43 cm. The length of the minor arc *AB* in centimetres is

(A) 6×43 (B) $\frac{1}{6} \times 43$

(C) $\frac{1}{9} \times 43$ (D) $\frac{1}{3} \times 43$

50.



In the diagram above $\triangle ADE$ is an enlargement of $\triangle ABC$ such that $\dfrac{AB}{BD} = \dfrac{AC}{CE} = \dfrac{1}{2}$

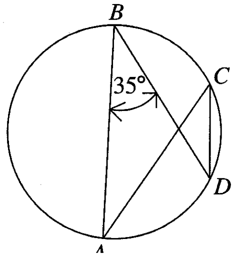
If the area of $\triangle ABC = 20\ \text{cm}^2$, then the area of $\triangle ADE$ in cm^2 is

(A) 40 (B) 60 (C) 80 (D) 180

51. Given that the point P is $(2, -2)$, then the length of the straight line OP is

(A) 2 (B) -2 (C) $2\sqrt{2}$ (D) $-2\sqrt{2}$

52.



In the diagram above angle $ABD = 35°$, so angle $ACD =$

(A) 55° (B) 35° (C) 75° (D) 65°

Items 53–55 refer to the table below which shows the distribution of the ages of 40 children in a class taking music.

Age	11	12	13	14	15
No. of children	3	9	12	14	2

53. What is the probability that a child chosen at random is at least 14 years of age?

(A) $\dfrac{14}{17}$ (B) $\dfrac{2}{5}$ (C) $\dfrac{3}{5}$ (D) $\dfrac{11}{16}$

54. The modal age of the distribution is

(A) 12 (B) 13 (C) 14 (D) 15

55. The median age of the distribution is

(A) 12 (B) 13 (C) 14 (D) 15

56. The width of a rectangular block of cheese is w centimetres. If its thickness is two-fifths its width and its length is seven times its thickness, then its volume, in cubic centimetres is

(A) $\dfrac{28}{25}w^3$ (B) $\dfrac{25}{28}w^3$ (C) $\dfrac{28}{25}w^2$ (D) $\dfrac{25}{28}w^2$

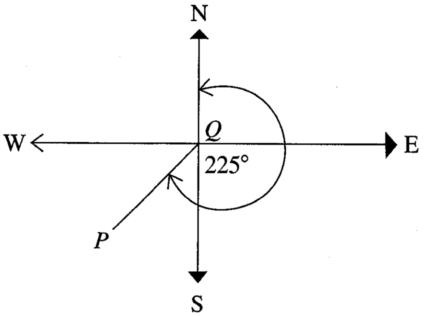
57. Robert is nine times as old as Lawrence who is x years old. Five years from now their ages will respectively be

I. $9x + 5$ II. $\dfrac{9}{x} + 5$

III. $x + 5$ IV. $\dfrac{x}{5}$

(A) I and II (B) I and III
(C) II and III (D) II and IV

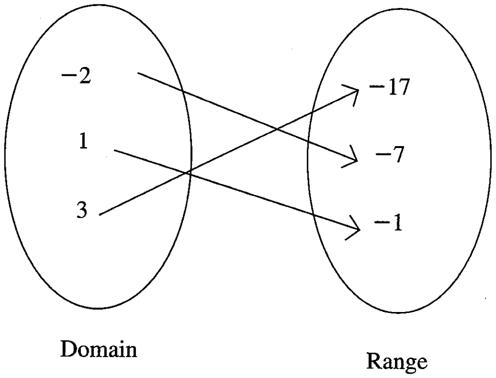
58.



In the diagram above, the bearing of Q from P is

(A) 45° (B) 60°
(C) 225° (D) 135°

59.



The mapping diagram shows three members of a linear relation. (x, y) is a member of this relation. The relation is

(A) $y + 1 = 2x^2$ (B) $y - 1 = 2x^2$
(C) $y - 2x^2 = 1$ (D) $y + 2x^2 = 1$

60. x^{-3} written with a positive index is

(A) $\dfrac{1}{x^3}$ (B) $\dfrac{x}{3}$

(C) $\dfrac{3}{x}$ (D) $\sqrt{x^3}$

C.X.C. MODEL EXAMINATION 1
MATHEMATICS
Paper 2 – Basic Proficiency
2 hours 40 minutes
Answer ALL the questions

1. You must not use slide rules, tables or calculators to work your answers to this question. All steps and calculations must be clearly shown to earn credit for your solutions.

(a) Calculate the exact value of

$$\frac{0.426 \times 0.03}{0.018}$$ (4 marks)

(b) (i) A cyclist left Turkeyen Campus at 16:35 h and arrived in a Place X, 5 min late at 17:15 h. If the place X is 8 km from Turkeyen Campus, calculate the average speed of the cyclist on the journey from Turkeyen Campus to the Place X.

(ii) Calculate the average speed of a second cyclist who left Turkeyen Campus at 16:35 h but reaches the Place X by the same route on time.

(6 marks)

2. A man earns $2 500 per month and his wife earns $1 200 per month. They have five children.
National Insurance of 4% on all earnings must be paid before taxes are deducted. Allowances and taxes are calculated on their combined salaries. Tax free allowances and tax rates are as follows:

Tax Free Allowances	*Rates on Taxable Income*
$2 500 per annum for each adult	10% on first $2 000
$700 per annum per child	20% on next $2 000
Earned income relief	30% on next $2 000
10% of husband's salary	40% on the remainder
Non-taxable income 30% of wife's salary	

Calculate

(a) the total taxable annual income

(b) the total tax paid by the employees for the whole year.

(9 marks)

3. (a) Using ruler and compasses only, construct a triangle PQR with $\angle P = 60°$, $\angle Q = 45°$ and $PQ = 9$ cm. Determine by measurement the length of RQ in centimetres.

(5 marks)

(b) An isosceles triangle LMN has $LM = LN = 7$ cm and angle $MLN = 140°$. The triangle is reflected in the side LM. Draw and label the figure with its image LMN^1. Determine the length of NN^1. Explain your working.

(6 marks)

4. (a) Calculate the number which must be added to both the numerator and the denominator of the fraction $\frac{3}{5}$ so that the resulting fraction is equal to $\frac{3}{4}$.

(5 marks)

(b) An arrangement of points from the set of whole numbers {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} is plotted on graph paper.

(i) List the points whose coordinates sum is 12.

(ii) Calculate the probability that if a point is chosen at random, its coordinates sum is 12.

(5 marks)

5. A map is drawn to a scale of 1:200000.

(a) Calculate the length on the map which represents a distance of 135 m on the ground.

(b) A rectangualr recreation field is shown on the map. Its dimensions on the map can be measured only to the nearest $\frac{1}{10}$ mm. The dimensions on the map are measured as 7.4 mm and 5.8 mm. Calculate the least possible area of the recreation field (on the ground).

(10 marks)

6. Copy and complete the table below for the function $y = 2^{x+1}$.

x	-3	-2	-1	0	1
y		$\frac{1}{2}$			

Using a scale of 4 cm to represent 1 unit on each axis, on graph paper plot the graph of $y = 2^{x+1}$ for $-3 \leqslant x \leqslant 1$.

Using the same axes and scales draw the line $y = x + 3$. From your graphs write down the values of x for which $2^{x+1} = x + 3$.

(10 marks)

7. The result of a survey of the income earned per hour for 2 samples of 50 families is given in the table below:

Income per hour in dollars	*Sample A No. of families*	*Sample B No. of families*
1–5	0	10
6–10	7	4
11–15	10	3
16–20	18	16
21–25	11	13
26–30	4	4

(a) using the same scales and axes, plot two frequency polygons so as to compare the two distributions given in the table. (Use graph paper).

(b) Using the data given, state which of the families, A or B, had a more equitable distribution of income. State the reasons for your choice.

(13 marks)

8. (a) (i) A stove can be bought on hire-purchase by making a deposit of \$750 and 15 monthly instalments of \$185 each. Calculate the hire-purchase price of the stove.

(ii) The marked price of the stove is \$2 475. This includes a sales tax of 12.5%. Calculate the price of the stove if no sales tax is included.

(8 marks)

(iii) Calculate the difference between the hire-purchase price and the sale price.

(b) The table below is an extract from a ready reckoner showing the cost of a number of articles at 21 cents per article:

5	1.05	15	3.15	45	9.45	95	19.95	220	46.20
6	1.26	16	3.36	46	9.66	96	20.16	230	48.30
7	1.47	17	3.57	47	9.87	97	20.37	240	50.40
8	1.68	18	3.78	48	10.08	98	20.58	250	52.50
9	1.89	19	3.99	49	10.29	99	20.79	260	54.60
10	2.10	20	4.20	50	10.50	100	21.00	270	56.70

Ready-reckoner

Use the table above to state the cost of

(i) 15 sweets at 21 cents per sweet
(ii) 55 sweets at 21 cents per sweet
(iii) 155 sweets at 21 cents per sweet
(iv) 3 255 sweets at 21 cents per sweet.

To gain full marks intermediate values must be shown.

(6 marks)

9. In the determination of the surface tension of water by the torsion balance method, the formula

$$\delta = \frac{mg}{2(l + b)}$$

is used.

Calculate correct to three significant figures, the value of δ in the formula, where $m = 1.12 \times 10^{-3}$, $g = 9.81$, $l = 7.5 \times 10^{-2}$ and $b = 1.10 \times 10^{-3}$.

(13 marks)

C.X.C. MODEL EXAMINATION 2
MATHEMATICS
Paper 2 – Basic Proficiency
2 hours 40 minutes
Answer ALL the questions

1. You must not use slide rules, tables or calculators to work your answers to this question. All steps and calculations must be clearly shown to earn credit for your solutions.

(a) Calculate the exact value of .

$$\frac{13.51 - (2.37 + 1.41)}{0.5 \times 1.4}$$ (5 marks)

(b) In December Ms. Kahn's telephone bill was calculated on the following information:

Long distance calls to	*Duration of calls in minutes*	*Fixed charge for 3 minutes or less*	*Additional charge per minute*
Xanadu	15	$42.75	$15.00
Paris	17	$17.25	$6.00
Moscow	20	$28.50	$10.00

Monthly rental for telephone = $22.50
Rebate received on rental for
3 weeks when the telephone
was not working = $18.00

Calculate Ms. Kahn's actual telephone bill for December. (5 marks)

2. (a) A salesman is paid a salary of $2 300 per month and a commission of 10% on all sales above $9 000.

Calculate:
- (i) the salesman's gross salary if his sales for a particular month was $18 500
- (ii) the value of his sales for December, when his gross salary was $4 960.

(b) If his sales for January was $7 485, state his gross salary. (8 marks)

3. The cash price of a refrigerator is $2 945.00.

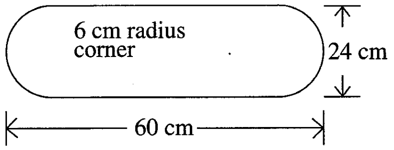
(a) It can be bought on hire purchase if a deposit of $810 is first paid. Simple interest at 12% per annum for 2 years is then added to the outstanding balance. The amount must then be paid off in equal monthly instalments over the two-year period.

Calculate:
- (i) the value of each monthly instalment
- (ii) the total amount paid for the refrigerator.

(b) The refrigerator can also be bought by borrowing the cash price from a bank at 12% simple interest and the principal and interest must be repaid at the end of 2 years. Calculate the amount to be paid by this arrangement.

(c) Which arrangement is better and by how much? (10marks)

4.



Base of aquarium

The figure above (not drawn to scale) is the base of an aquarium of altitude 40 cm. Each corner consist of a radius of 6 cm.

Calculate:
- (a) the area of the figure
- (b) the volume of the aquarium.

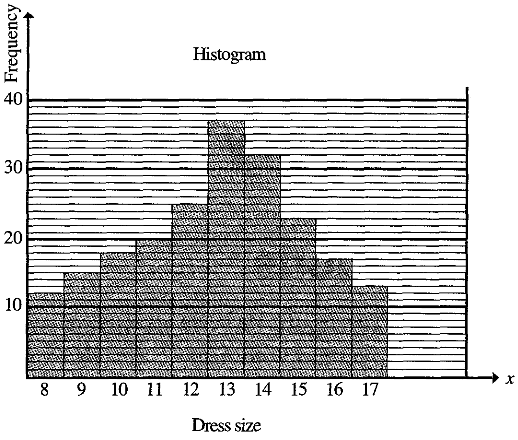
$\left[\text{Take } \pi \text{ to be } \frac{22}{7}\right]$ (10 marks)

5. A saleswoman sold 25 Mathematics books and 10 English books for a total of $775. If she had sold 10 Mathematics books and 40 English books, she would have got $75 more.

Calculate the price of EACH type of book.

(9 marks)

6.



The histogram above shows the frequency of dress sizes for a random sample of 212 dresses sold by a boutique just before a sale.

(a) Draw up a frequency table to represent this information.

(b) Determine the mean size, median size and mode size of this sample.

(c) The boutique manageress wishes to replenish her stock. Which of the three measures of statistical average should she use to determine what size to order in the largest quantity? Give a reason for your choice.

(d) Calculate the probability that a dress chosen at random from this sample of 212 dresses is a size 14. (16 marks)

7. (a) Determine the range of values of p for which $5(p + 1) \leq 35 - 2(p - 1)$ (6 marks)

(b)

Value of $1 at certain rates of simple interest	*No. of years*	*Rate of interest*				
		10%	*11%*	*12%*	*13%*	*14%*
	6	1.600	1.660	1.720	1.780	1.840
	7	1.700	1.770	1.840	1.840	1.980
	8	1.800	1.880	1.960	2.040	2.120
	9	1.900	1.990	2.080	2.170	2.260

Simple interest table

A teacher borrowed $25 900 for 7 years at 11% simple interest. Using the table above, calculate the interest he had to pay on the loan. (6 marks)

8. A model car is tested over a course of fixed length. In each trial it is kept as near as physically possible to a fixed speed and timed. The results obtained are given in the table below, where

s = the observed speed in km per hour
t = the time in minutes.

s	10	15	25	35	45	55	65	75	85	95
t	15.6	10.4	6.2	4.5	3.5	2.8	2.4	2.1	1.8	1.6

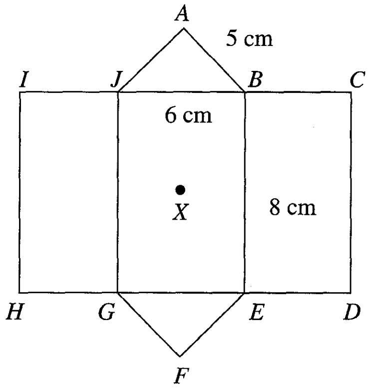
(a) Using the points above, plot a graph of s against t (the independent variable), using 2 cm to represent 10 km/h on the s-axis and 1 cm to represent 1 min on the t-axis.

(b) Draw a smooth curve through the points.

(c) From your graph determine

(i) the time when the speed is 50 km/h^{-1}
(ii) the length of the fixed course using your estimate of t when s = 50 km/h^{-1}. (11 marks)

9.



Net of a prism

(a) *ABCDEFGHIJ* is a sketch of the net of a prism on a rectangular base *BEGJ* of dimensions 6 cm and 8 cm with slant edges of length 5 cm.

(i) Draw an accurate full size diagram of the net.
(ii) Measure and write down the lengths of *AF* and *EJ*.

(b) The net is assembled as a prism with top edge *AF* and base *BEGJ*.

(i) By calculation, determine *XY* the altitude of the prism.
Where *X* is the centre of the base *BEGJ* and *Y* is the mid-point of the top edge *AF*.
(ii) Sketch the section *JXY* of the prism indicating clearly on your sketch the lengths of *JX* and *JY*.
(iii) Hence, calculate the angle *XJY*.

(14 marks)

C.X.C. MODEL EXAMINATION 3
MATHEMATICS
Paper 2 – Basic Proficiency
2 hours 40 minutes
Answer ALL the questions

1. You must not use slide rules, tables or calculators to work your answers to this question. All steps and calculations must be clearly shown to earn credit for your solutions.

Calculate the exact value of

$$\frac{(0.5)^2 + 1.4}{0.75} \qquad \text{(5 marks)}$$

2. Solve the equation:

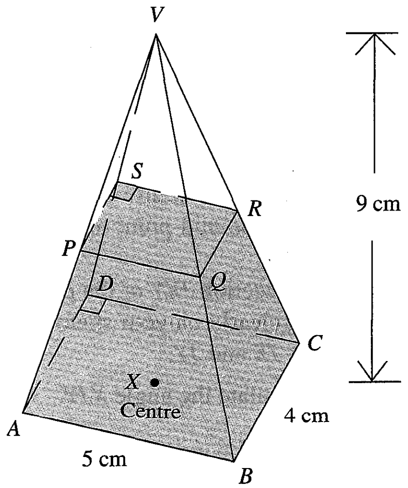
$5x - 3(2x - 7) = 11x + 3$ (5 marks)

3. (a) A sum of money was shared among three friends Aileen, Shirley and Janet in the ratio 8:13:19, respectively. If Janet received $822 more than Shirley, calculate the amount each receives.

(b) Draw a pie chart to represent the shares received by the three friends. (10 marks)

4. (a) A bus is scheduled to depart at 13:25 h for a journey which is scheduled to take $2\frac{3}{4}$ h. The bus actually departed at 14:05 h and arrived at its destination 25 min later than scheduled. What length of time did this journey take?

(3 marks)



Pyramid

(b) (i) The figure *VABCD* above represents a pyramid with $AB = 5$ cm, $BC = 4$ cm, and $VX = 9$ cm. X is the centre of the rectangular base *ABCD*, and $VP = VQ = VR = VS = \frac{1}{3}$ (the length of a slant edge).

Calculate the volume of the pyramid *VABCD*.

(ii) The pyramid *VPQRS* is cut along *PQRS* and removed from the pyramid *VABCD*. Calculate the volume of the solid *ABCDPQRS* which remains.

(7 marks)

5. (a) The Rate of exchange at a bank are as follows:

US $1.00 = TT $5.73
CAN $1.00 = TT $2.95

(i) An American tourist changed the US $800.00 to Trinidad and Tobago currency. Calculate the amount of money she received.

(ii) The tourist spent TT $3404 and changed the remaining amount to Canadian currency. Calculate the amount of money she received.

(9 marks)

Note: TT means Trinidad and Tobago.
US means United States.
CAN means Canada.

(b) An insurance company estimates that the book value of a car depreciates by 10% each year if it does not have an accident. Mr. Deygoo's car is valued by the insurance company at $48 000 on February 14, 2001. Calculate the value of the same car on February 14, 2004, assuring that the car was accident free. (6 marks)

6. Mr. Albert used 175 m^3 of domestic gas for the first half of 2001. In 2001, gas rates for domestic users for a half year were as follows:

\$0.75 per cubic metre for the first 60 m^3. \$0.90 per cubic metre for amounts in excess of 60 m^3. 10% discount on bills paid before July 14, 2001.

Calculate the amount Mr. Albert paid for the half year

(a) assuming that the bill was paid on July 21, 2001

(b) assuming that the bill was paid on July 7, 2001. (8 marks)

7. On Sunday, Mrs. Agnes bought 10 chickens and 4 ducks for \$440. On Friday, Mr. Samuel bought 4 chickens and 3 ducks for \$246. The price was the same for each type of bird on both days. Using *c* dollars to represent the cost of a chicken and *d* to represent the cost of a duck,

(a) write down TWO equations in *c* and *d* to represent the purchase on Sunday and Friday, and

(b) use these equations to calculate the cost of a chicken and the cost of a duck.

(11 marks)

8. The heights of 13 men in centimetres are given below:
163, 162, 160, 165, 160, 167, 170, 167, 174, 176, 178, 179, 178.

(a) Draw a frequency polygon to represent the data.

(b) Determine:

(i) the mean of these heights
(ii) the interquartile range of these heights
(iii) the median height.

(c) State the modal heights.

(d) Calculate the probability that if a man from this sample is chosen at random, his height is at least 174 cm. (16 marks)

9. (a) Construct a triangle *ABC* such that $AB = 7$ cm horizontally, $AC = 5.5$ cm and angle $BAC = 60°$

(b) The triangle *ABC* is reflected in a vertical mirror line *l*, which is 1 cm horizontally away from the point *A*. Construct the image triangle $A'B'C'$ formed by reflecting triangle *ABC* in the line *l*.

(c) Measure and state the length of CC'.

(d) If CC' intersects *l* at *N*, state a relation about *NC*.

(e) State the name of the figure $BCC'B'$.

(10 marks)

10. In an electrical experiment on a mental filament lamp, it is known that the voltage, *V* volts, and resistance, *R* ohms, agree with the formula $R = mV + c$, where *m* and *c* are constants. The following table was obtained in the experiment.

V (volts)	60	70	80	90	100	110	120
R (ohms)	110	125	140	155	170	185	200

(a) Using the data in the table above and a scale of 1 cm to represent 10 units on both the *R*-axis and the *V*-axis, draw a graph to represent the formula.

(b) From your graph determine the values of *m* and *c*.

(c) Hence write down the particular formula for this experiment.

(10 marks)

C.X.C. MODEL EXAMINATION 4
MATHEMATICS
Paper 2 – Basic Proficiency

2 hours 40 minutes
Answer ALL the questions

1. You must not use slide rules, tables or calculators to work your answers to this question. All steps and calculations must be clearly shown to earn credit for your solutions.

Calculate the exact value of

(a) 2.39×6.5 (3 marks)

(b) $\frac{6.78}{1.13}$ (2 marks)

(c) $7\frac{3}{4} - 4\frac{1}{8}$ (3 marks)

2. (a) Given that $p = 5$ and $q = -1$, calculate the value of p^2q^3. (2 marks)

(b) Expand and simplify.

$$(x + y)(x - 3y)$$ (3 marks)

(c) Solve. $7x - 2(3 + x) = 19$ (3 marks)

3. (a) A Tristar left airport *A* at 08:30 h and arrived at airport *B* at 13:15 h the same day. The distance from airport *A* to airport *B* is 3 562.5 km. Calculate the average speed at which the Tristar travelled during the journey. (3 marks)

(b) In July 1987, a Canadian tourist changed CAN $2 500 of her Canadian Travellers' cheques for Trinidad and Tobago currency. She was given the following information:

CAN $1.00 = TT $2.73
12 cents on the dollar is charged for tax on all foreign transactions

Calculate, in Trinidad and Tobago currrency,

(i) the value of CAN $2 500

(ii) the amount of money the Canadian tourist received from the bank after paying the tax.

Note: CAN means Canada.
TT means Trinidad and Tobago. (5 marks)

4. At a market, 8 mangoes and 6 pears cost $25.00; and 4 mangoes and 8 pears cost $30.00.

(a) Let $*m* represent the cost of one mango and $*p* represent the cost of one pear, hence write down a pair of simultaneous equations to represent the information given above.

(b) Calculate:

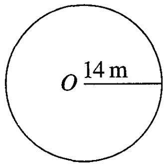
(i) the cost of a mango

(ii) the cost of a pear. (8 marks)

5. In this question use $\pi = \frac{22}{7}$.

The volume of a sphere, $V = \frac{4}{3}\pi r^3$

Area of curved surface of a sphere, $A = 4\pi r^2$



Sphere

(a) The figure (not drawn to scale) represents a sphere of radius 14 m.
Calculate:

(i) the distance around the sphere

(ii) the volume of the sphere.

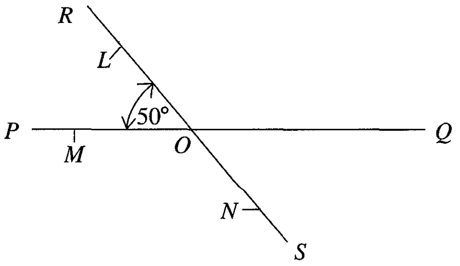
(b) A small model of the sphere is made. The radius of the model is 7 cm.
Calculate:

(i) the scale used to make the model

(ii) the area of the curved surface of the model, given that the area of the curved surface of the sphere is 2 464 m^2.

(10 marks)

6.



The diagram given, (not drawn to scale) shows two line segments PQ and RS intersecting at O. The points L, M and N are equidistant from O. And angle $LOM = 50°$.

(a) L is mapped onto M by means of a reflection in the mirror line l (not shown). Determine the acute angle which l makes with OP. Given a reason for your answer.

(b) M is mapped onto N by means of a reflection in the mirror line k (not shown). Calculate the size of the acute angle which k makes with OP.

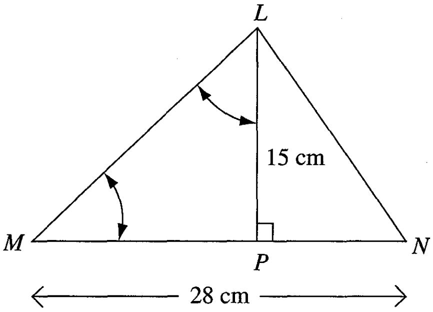
(c) Describe fully a transformation which maps the mirror line k onto the mirror line l.

(6 marks)

7. (a) Draw a triangle PQR in which $PQ = 8$ cm, $PR = 5$ cm and angle $QPR = 65°$. State the length of QR and the magnitude of angle PRQ.

(5 marks)

(b)



In the triangle LMN (not drawn to scale) $MN = 28$ cm. LP is perpendicular to MN. $LP = 15$ cm and angle PLM = angle PML.

(i) State the length of PM. Give a reason for your answer.

(ii) Calculate the length of LN. (4 marks)

8. (a) The customs duty on imported vehicles is 25% of the imported price.

(i) Calculate the customs duty on a car for which the imported price is \$48 600.

(ii) Calculate the imported price of a truck for which the amount paid, inclusive of customs duty is \$106 250. (5 marks)

(b) Charges for electricity in a Caricom country are made up of a fixed fuel charge of 45 cents per unit and an energy charge computed under THREE schemes as follows:

Scheme A. Homes	15 cents per unit
Scheme B. Schools	20 cents per unit
Scheme C. Business Places	30 cents per unit

The meter reading of a certain school reads as follows:

Meter Reading (units)		*Units used*	*Scheme*	*Energy Charge(\$)*	*Fuel Charge(\$)*
Present	*Previous*				
35 826	26 169				

Calculate

(i) the number of units used

(ii) the energy charge

(iii) the fuel charge

(iv) the total amount the school had to pay for the electricity used. (8 marks)

9. Twenty-five students wrote a Mathematics test in which the maximum mark that could be obtained was 10. The mark of each participant is listed below.

8	10	8	9	4
7	4	7	6	8
3	8	2	2	6
0	5	4	5	5
5	1	1	3	7

(a) Calculate the mean mark for the distribution.

(b) Draw the frequency polygon representing the data on graph paper.

(c) Determine the median mark and the semi-interquartile range for the distribution.

(d) Calculate the probability that a competitor chosen at random has a mark greater than 6.

(15 marks)

10. (a) Copy and complete the following table for the function $f(x) = x^2 - 3x - 4$

x	-3	-2	-1	0	1	2	3	4	5	6
$f(x)$	14		0			-6	-4		6	

(b) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of the function $y = f(x)$ for $-3 \leqslant x \leqslant 6$.

(c) On the same diagram, using the same scales as in (b) above, draw the graph of the line $y = 6$.

(d) From your graph determine

(i) the values of x for which $y = 0$,

(ii) the coordinates of the points of intersection of the curve $y = x^2 - 3x - 4$ and the line $y - 6 = 0$. (15 marks)

C.X.C. MODEL EXAMINATION 5
MATHEMATICS
Paper 2 – Basic Proficiency
2 hours 40 minutes
Answer ALL the questions

1. You must not use slide rules, tables or calculators to work your answer to this question. All steps and calculation must be shown to earn credit for your solution.

Calculate the exact value of

$$\frac{24.12 + 36.4}{14.12 - 12.42}$$ (5 marks)

2. Solve the equation

$$\frac{2x + 3}{9} = \frac{3x - 5}{4}$$ (5 marks)

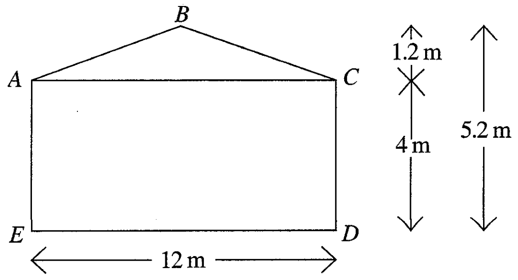
3. A chemical container is $\frac{3}{8}$ full. A chemical is poured in at a rate of 6 litres per minute. After 9 minutes the container is $\frac{3}{5}$ full. Calculate the number of litres of the chemical which the container can hold. (5 marks)

4. A father bought 7 packs of Quik and 5 packs of Orange juice for $27.35. If he had bought 5 packs of Quik and 7 packs of Orange juice at the same grocery then the cost would have been $25.57.

(a) Using q to represent the cost in dollars per pack of Quik and j to represent the cost in dollars per pack of Orange juice, write down a pair of simultaneous equations to represent the information given above.

(b) Calculate:
  (i) the cost per pack of Quik
  (ii) the cost per pack of Orange juice.
(10 marks)

5.



Side of house

The side of a house is shown in the diagram given (not drawn to scale). The side of the house is in the shape of a rectangle $ACDE$ of width 12 m and height 4 m, and an isosceles triangle ABC of altitude 1.2 m.

The length of the house (not shown) is 25 m. Calculate

(a) the total surface area of the side of the house $ABCDE$.

(b) the total surface area of the roof of the house.

(c) the surface area of the front of the house.

(d) the surface area of the floor of the house.

(e) the total surface area of the house.

(f) the volume of the house. (10 marks)

6. A man's annual gross salary is $55 860. He contributes 3% of his salary to a medical scheme and his company contributes 5%. His contribution to the medical scheme is a non-taxable allowance. Other non-taxable allowances and the income tax rates on taxable income are given in the table below.

Non-taxable allowances	*Income tax rates on annual taxable income*
$95 per month for National Insurance	20% on first $4000
$3500 per annum for Personal Allowance	25% on remainder

Calculate for the man:

(a) the monthly amount the company contributes to his medical scheme

(b) the total amount of his annual salary that is not taxed

(c) his annual taxable income

(d) the tax he pays monthly (correct to the nearest cent). (13 marks)

7. The points $A(2, 3)$, $B(4, 5)$ and $C(4, 3)$ are vertices of a triangle ABC. The triangle ABC is reflected in the origin, O to produce the image triangle $A'B'C'$. The original triangle ABC is also reflected in the y-axis to produce the image triangle $A''B''C''$.

(a) Using a scale of 1 cm to represent 1 unit on both axes, draw on graph paper the triangles ABC, $A'B'C'$ and $A''B''C''$.

(b) State the single transformation that will map triangle $A'B'C'$ onto triangle $A''B''C''$.

(c) Determine the length of $C'A''$.

(d) Measure and state the size of angle $A'AA'$.

(10 marks)

8. There are 25 participants in a computer game competition. The score of each participant is listed below.

1	3	5	0	2
2	1	6	5	6
0	3	5	1	1
5	2	1	0	6
1	4	0	3	5

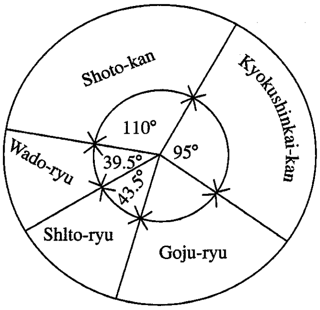
(a) Set up a frequency table for the scores, and hence draw a histogram to illustrate the data.

(b) Calculate the mean score of the distribution.

(c) Determine the modal score and the interquartile range.

(d) Calculate the probability that a competitor chosen at random has a score less than or equal to 5. (15 marks)

9. (a)



Pie-chart

The pie chart given illustrates the different styles of Karate taught to students in a certain country. If 285 000 students were taught Kyokushinkai-kan, calculate the number of students who were taught

(i) Shoto-kan

(ii) Goju-ryu (6 marks)

(b) (i) Using a scale of 1 cm to represent one unit on each axis, draw on graph paper the graph of $y = 3x + 1$.

(ii) On the same graph, draw the line through the points $P(-3, 2)$ and $Q(0, 1)$ and calculate the gradient of PQ.

(iii) State the relationship between the line $y = 3x + 1$ and the line PQ and state the coordinates of the point of intersection of the lines. Hence or otherwise determine the equation of the line PQ.

(9 marks)

10. (a) The rates for posting letters from Trinidad to Guyana are as follows:

Letters not exceeding 10 g	\$0.65
Each additional 10 g or part thereof up to a maximum of 100 g	S0.40
Registration fee for a registered letter	\$2.75

Calculate the cost of posting

(i) an unregistered letter weighing 35 g

(ii) a registered letter weighing 0.1 kg.

(5 marks)

(b) The table below shows the appreciation (i.e. the increase in value) of \$1 from 1 year to 10 years.

Years	12%	13%	14%
1	1.120	1.130	1.140
2	1.254	1.277	1.300
3	1.405	1.443	1.482
4	1.574	1.630	1.689
5	1.762	1.842	1.925
6	1.974	2.082	2.195
7	2.211	2.353	2.502
8	2.476	2.658	2.853
9	2.773	3.004	3.252
10	3.106	3.395	3.707

Compound interest table

Use the table above to determine:

(i) the appreciation of \$1 invested for 5 years at 12% per annum

(ii) the appreciation of \$1 500 invested for 5 years at 12% per annum

(iii) the compound interest earned when \$1 500 is invested for 5 years at 12% per annum

(iv) the compound interest earned when \$1 260 is invested for 8 years at 13% per annum.

(7 marks)

C.X.C. MODEL EXAMINATION 6
MATHEMATICS
Paper 2 – Basic Proficiency
2 hours 40 minutes
Answer ALL the questions

1. You must not use slide rules, tables or calculators to work your answers to this question. All steps and calculation must be shown to earn credit for your solution.

(a) Calculate, correct to two decimal places,
 (i) 0.05×1.2
 (ii) $8(5 - 2.53)$ (3 marks)

(b) Calculate the exact value of

$$\frac{3.60}{0.5 \times 0.12}$$ (3 marks)

2. (a) Express, in the simplest form, 825 g as a fraction of 1 kg. (2 marks)

(b) A chocolate bar of length 15.0 cm is divided in the ratio 3:7. Calculate the length of the shorter piece. (2 marks)

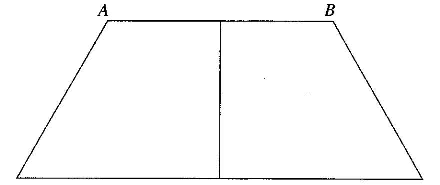
3. (a) Solve the simultaneous equation:

$$3x + y = 9$$
$$-x + y = 1$$ (4 marks)

(b) Simplify:

$$\frac{x+5}{3} - \frac{2x-1}{4}$$ (3 marks)

4.



Car window

The figure *ABCD* is an accurate scale drawing of a car window where *AB* and *DC* represent the top and bottom edges, respectively.

(a) Measure accurately and state, in centimetres,
 (i) the length of *DC*
 (ii) the altitude between *AB* and *DC*.
(2 marks)

(b) Given that $AB = 3.0$ cm, calculate in square centimetres the area of the figure *ABCD*.
(4 marks)

(c) Given that the bottom edge *DC* of the actual car window measures 0.9 m, calculate the scale used in the drawing (2 marks)

(d) Using your answer to part (c), calculate for the actual car window:
 (i) the length of the top edge in metres
 (ii) the actual area of the car window in square metres. (4 marks)

5. In 2004, a teacher's annual income was \$48 360. He paid 3 cents out of every dollar for a medical plan.

(a) Calculate the amount he paid for the medical plan. (2 marks)

Additional Information

Medical plan payments are non-taxable. Other non-taxable allowances and the income tax rates on taxable income are given in the table below.

Non-taxable allowances	*Income tax rates on annual taxable income*
\$37 per month for National Insurance	10% on first \$10 000
\$3 000 per annum for the teacher	25% on next \$10 000
\$1 800 per annum for his wife	40% on remainder
\$1 400 per annum for his children	

(b) Calculate:
 (i) the portion of his income that is not taxed
 (ii) the amount he paid for income tax each month.

(8 marks)

6. A student had \$300. She bought a Physics book, a Mathematics book and a Chemistry book. The Physics book cost twice as much as the Chemistry book. The Mathematics book cost \$9.00 less than the Physics book. She had \$159 left.

(a) Using x to represent the cost (in dollars) of the Chemistry book, state in terms of x, expressions for
  (i) the cost of the Physics book
  (ii) the cost of the Mathematics book
  (iii) the total amount of money spent. (3 marks)

(b) State an equation for the amount of money left. (2 marks)

(c) Hence, determine the cost of a Mathematics book. (3 marks)

7. (a) The regular price of a suit is \$560. During a sale a discount of 30% is given. Calculate the amount a customer pays for the suit by this arrangement. (2 marks)

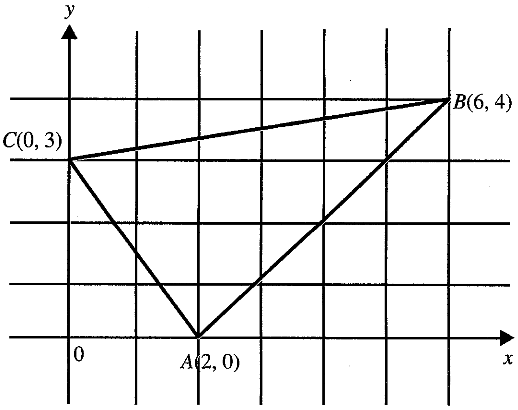
(b) When selling at the discount price, the businessman makes a profit of 75%. Calculate the cost of the suit to the businessman. (3 marks)

(c) Calculate the percentage profit the businessman made when a suit was sold for \$560. (1 mark)

(d) After the sale, the businessman bought 50 suits for \$250 each and sold them to make a profit of 68% on his buying price. Calculate the total profit he made on his set of suits. (4 marks)

8. The coordinates of the vertices of triangle ABC shown are $A(2, 0)$, $B(6, 4)$ and $C(0, 3)$, respectively.

(a) Draw the triangle ABC on graph paper, and using a scale factor of 1.5 and the origin as centre, construct an enlargement $A'B'C'$ of triangle ABC. State the coordinates of the points $A'B'C'$. (3 marks)



(b) After a transformation P, the original triangle ABC is mapped onto triangle $A''B''C''$, where $A''(-2, 0)$, $B''(-6, 4)$ and $C''(0, 3)$ are the vertices of the image. Describe fully the transformation P. (4 marks)

(c) The original triangle ABC is reflected in the x-axis to form $A'''B'''C'''$. State the coordinates of the image $A'''B'''C'''$. (3 marks)

(d) State the relationship between the area of
  (i) triangle ABC and triangle $A'B'C'$
  (ii) triangle ABC and triangle $A''B''C''$
  (iii) triangle ABC and triangle $A'''B'''C'''$ (3 marks)

9. (a)

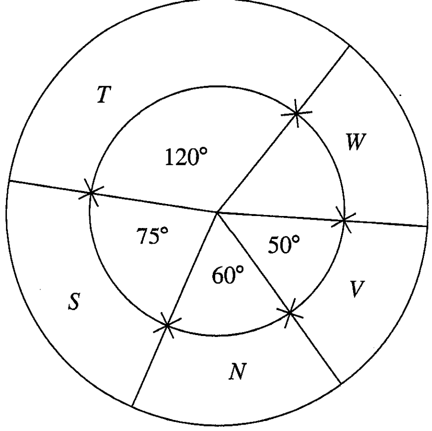
Height (cm)	140	141	142	143	144	145	146
No. of children	2	5	6	8	9	3	1

The table above shows the distribution of the heights of 34 children in a school debating team.

(i) Calculate the mean height and the median height of this distribution.

(ii) Calculate the probability that a child chosen at random is
- less than 143 cm in height
- at least 143 cm in height. (8 marks)

(b)



Pie-chart

The pie-chart above, which is not drawn to scale, represents the amount of money spent by a university on various sports as indicated below:

T: Track
S: Swimming
N: Netball
V: Volleyball
W:Wrestling

The total budget was \$135 000.

(i) Calculate the amount spent on T and W.

(ii) Using a scale of 2 cm to represent \$5 000, draw a bar chart to illustrate the information given in the pie-chart above.

(7 marks)

10. (a) (i) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, plot on graph paper the points $P(4, 6)$ and $Q(2, 2)$.

(ii) Join PQ and calculate the gradient of PQ.

(iii) Produce PQ to cut the y-axis at R. State the coordinates of R. Hence, write down the equation of PQ.

(7 marks)

(b) (i) Given that $y = x^2 - 3$, copy and complete the table below for the range $-3 \leqslant x \leqslant 3$.

x	-3	-2	-1	0	1	2	3
y	6		-2			1	

(ii) Using the same diagram and the same scale as in Part (a) (i), draw the graph of $y = x^2 - 3$ in the given range.

(iii) From your graphs determine the solutions of the equation $x^2 - 3 = 6$

(8 marks)

Logarithms of Numbers

	0	1	2	3	4	5	6	7	8	9
10	.000	004	009	013	017	021	025	029	033	037
11	.041	045	049	053	057	061	064	068	072	076
12	.079	083	086	090	093	097	100	104	107	111
13	.114	117	121	124	127	130	134	137	140	143
14	.146	149	152	155	158	161	164	167	170	173
15	.176	179	182	185	188	190	193	196	199	201
16	.204	207	210	212	215	217	220	223	225	228
17	.230	233	236	238	241	243	246	248	250	253
18	.255	258	260	262	265	267	270	272	274	276
19	.279	281	283	286	288	290	292	294	297	299
20	.301	303	305	307	310	312	314	316	318	320
21	.322	324	326	328	330	332	334	336	338	340
22	.342	344	346	348	350	352	354	356	358	360
23	.362	364	365	367	369	371	373	375	377	378
24	.380	382	384	386	387	389	391	393	394	396
25	.398	400	401	403	405	407	408	410	412	413
26	.415	417	418	420	422	423	425	427	428	430
27	.431	433	435	436	438	439	441	442	444	446
28	.447	449	450	452	453	455	456	458	459	461
29	.462	464	465	467	468	470	471	473	474	476
30	.477	479	480	481	483	484	486	487	489	490
31	.491	493	494	496	497	498	500	501	502	504
32	.505	507	508	509	511	512	513	515	516	517
33	.519	520	521	522	524	525	526	528	529	530
34	.531	533	534	535	537	538	539	540	542	543
35	.544	545	547	548	549	550	551	553	554	555
36	.556	558	559	560	561	562	563	565	566	567
37	.568	569	571	572	573	574	575	576	577	579
38	.580	581	582	583	584	585	587	588	589	590
39	.591	592	593	594	595	597	598	599	600	601
40	.602	603	604	605	606	607	609	610	611	612
41	.613	614	615	616	617	618	619	620	621	622
42	.623	624	625	626	627	628	629	630	631	632
43	.633	634	635	636	637	638	639	640	641	642
44	.643	644	645	646	647	648	649	650	651	652
45	.653	654	655	656	657	658	659	660	661	662
46	.663	664	665	666	667	667	668	669	670	671
47	.672	673	674	675	676	677	678	679	679	680
48	.681	682	683	684	685	686	687	688	688	689
49	.690	691	692	693	694	695	695	696	697	698
50	.699	700	701	702	702	703	704	705	706	707
51	.708	708	709	710	711	712	713	713	714	715
52	.716	717	718	719	719	720	721	722	723	723
53	.724	725	726	727	728	728	729	730	731	732
54	.732	733	734	735	736	736	737	738	739	740

Logarithms of Numbers

	0	1	2	3	4	5	6	7	8	9
55	.740	741	742	743	744	744	745	746	747	747
56	.748	749	750	751	751	752	753	754	754	755
57	.756	757	757	758	759	760	760	761	762	763
58	.763	764	765	766	766	767	768	769	769	770
59	.771	772	772	773	774	775	775	776	777	777
60	.778	779	780	780	781	782	782	783	784	785
61	.785	786	787	787	788	789	790	790	791	792
62	.792	793	794	794	795	796	797	797	798	799
63	.799	800	801	801	802	803	803	804	805	806
64	.806	807	808	808	809	810	810	811	812	812
65	.813	814	814	815	816	816	817	818	818	819
66	.820	820	821	822	822	823	823	824	825	825
67	.826	827	827	828	829	829	830	831	831	832
68	.833	833	834	834	835	836	836	837	838	838
69	.839	839	840	841	841	842	843	843	844	844
70	.845	846	846	847	848	848	849	849	850	851
71	.851	852	852	853	854	854	855	856	856	857
72	.857	858	859	859	860	860	861	862	862	863
73	.863	864	865	865	866	866	867	867	868	869
74	.869	870	870	871	872	872	873	873	874	874
75	.875	876	876	877	877	878	879	879	880	880
76	.881	881	882	883	883	884	884	885	885	886
77	.886	887	888	888	889	889	890	890	891	892
78	.892	893	893	894	894	895	895	896	897	897
79	.898	898	899	899	900	900	901	901	902	903
80	.903	904	904	905	905	906	906	907	907	908
81	.908	909	910	910	911	911	912	912	913	913
82	.914	914	915	915	916	916	917	918	918	919
83	919	920	920	921	921	922	922	923	923	924
84	.924	925	925	926	926	927	927	928	928	929
85	.929	930	930	931	931	932	932	933	933	934
86	.934	935	936	936	937	937	938	938	939	939
87	.940	940	941	941	942	942	943	943	943	944
88	.944	945	945	946	946	947	947	948	948	949
89	.949	950	950	951	951	952	952	953	953	954
90	.954	955	955	956	956	957	957	958	958	959
91	.959	960	960	960	961	961	962	962	963	963
92	.964	964	965	965	966	966	967	967	968	968
93	.968	969	969	970	970	971	971	972	972	973
94	.973	974	974	975	975	975	976	976	977	977
95	.978	978	979	979	980	980	980	981	981	982
96	.982	983	983	984	984	985	985	985	986	986
97	.987	987	988	988	989	989	989	990	990	991
98	.991	992	992	993	993	993	994	994	995	995
99	.996	996	997	997	997	998	998	999	999	1.000

Antilogarithms

	0	1	2	3	4	5	6	7	8	9
.00	100	100	100	101	101	101	101	102	102	102
.01	102	103	103	103	103	104	104	104	104	104
.02	105	105	105	105	106	106	106	106	107	107
.03	107	107	108	108	108	108	109	109	109	109
.04	110	110	110	110	111	111	111	111	112	112
.05	112	112	113	113	113	114	114	114	114	115
.06	115	115	115	116	116	116	116	117	117	117
.07	117	118	118	118	119	119	119	119	120	120
.08	120	121	121	121	121	122	122	122	122	123
.09	123	123	124	124	124	124	125	125	125	126
.10	126	126	126	127	127	127	128	128	128	129
.11	129	129	129	130	130	130	131	131	131	132
.12	132	132	132	133	133	133	134	134	134	135
.13	135	135	136	136	136	136	137	137	137	138
.14	138	138	139	139	139	140	140	140	141	141
.15	141	142	142	142	143	143	143	144	144	144
.16	145	145	145	146	146	146	147	147	147	148
.17	148	148	149	149	149	150	150	150	151	151
.18	151	152	152	152	153	153	153	154	154	155
.19	155	155	156	156	156	157	157	157	158	158
.20	158	159	159	160	160	160	161	161	161	162
.21	162	163	163	163	164	164	164	165	165	166
.22	166	166	167	167	167	168	168	169	169	169
.23	170	170	171	171	171	172	172	173	173	173
.24	174	174	175	175	175	176	176	177	111	177
.25	178	178	179	179	179	180	180	181	181	182
.26	182	182	183	183	184	184	185	185	185	186
.27	186	187	187	188	188	188	189	189	190	190
.28	191	191	191	192	192	193	193	194	194	195
.29	195	195	196	196	197	197	198	198	199	199
.30	200	200	200	201	201	202	202	203	203	204
.31	204	205	205	206	206	207	207	207	208	208
.32	209	209	210	210	211	211	212	212	213	213
.33	214	214	215	215	216	216	217	217	218	218
.34	219	219	220	220	221	221	222	222	223	223
.35	224	224	225	225	226	226	227	228	228	229
.36	229	230	230	231	231	232	232	233	233	234
.37	234	235	236	236	237	237	238	238	239	239
.38	240	240	241	242	242	243	243	244	244	245
.39	245	246	247	247	248	248	249	249	250	251
.40	251	252	252	253	254	254	255	255	256	256
.41	257	258	258	259	259	260	261	261	262	262
.42	263	264	264	265	265	266	267	267	268	269
.43	269	270	270	271	272	272	273	274	274	275
.44	275	276	277	277	278	279	279	280	281	281
.45	282	282	283	284	284	285	286	286	287	288
.46	288	289	290	290	291	292	292	293	294	294
.47	295	296	296	297	298	299	299	300	301	301
.48	302	303	303	304	305	305	306	307	308	308
.49	309	310	310	311	312	313	313	314	315	316

Antilogarithms

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
.50	316	317	318	318	319	320	321	321	322	323
.51	324	324	325	326	327	327	328	329	330	330
.52	331	332	333	333	334	335	336	337	337	338
.53	339	340	340	341	342	343	344	344	345	346
54	347	348	348	349	350	351	352	352	353	354
.55	355	356	356	357	358	359	360	361	361	362
.56	363	364	365	366	366	367	368	369	370	371
.57	372	372	373	374	375	376	377	378	378	379
.58	380	381	382	383	384	385	385	386	387	388
.59	389	390	391	392	393	394	394	395	396	397
.60	398	399	400	401	402	403	404	405	406	406
.61	407	408	409	410	411	412	413	414	415	416
.62	417	418	419	420	421	422	423	424	425	426
.63	427	428	429	430	431	432	433	434	435	436
.64	437	438	439	440	441	442	443	444	445	446
.65	447	448	449	450	451	452	453	454	455	456
.66	457	458	459	460	461	462	463	465	466	467
.67	468	469	470	471	472	473	474	475	476	478
.68	479	480	481	482	483	484	485	486	488	489
.69	490	491	492	493	494	495	497	498	499	500
.70	501	502	504	505	506	507	508	509	511	512
.71	513	514	515	516	518	519	520	521	522	524
.72	525	526	527	528	530	531	532	533	535	536
.73	537	538	540	541	542	543	545	546	547	548
.74	550	551	552	553	555	556	557	558	560	561
.75	562	564	565	566	568	569	570	571	573	574
.76	575	577	578	579	581	582	583	585	586	587
.77	589	590	592	593	594	596	597	598	600	601
.78	603	604	605	607	608	610	611	612	614	615
.79	617	618	619	621	622	624	625	627	628	630
.80	631	632	634	635	637	638	640	641	643	644
.81	646	647	649	650	652	653	655	656	658	659
.82	661	662	664	665	667	668	670	671	673	675
.83	676	678	679	681	682	684	685	687	689	690
.84	692	693	695	697	698	700	701	703	705	706
.85	708	710	711	713	714	716	718	719	721	723
.86	724	726	728	729	731	733	735	736	738	740
.87	741	743	745	746	748	750	752	753	755	757
.88	759	760	762	764	766	767	769	771	773	774
.89	776	778	780	782	783	785	787	789	791	793
.90	794	796	798	800	802	804	805	807	809	811
.91	813	815	817	818	820	822	824	826	828	830
.92	832	834	836	838	839	841	843	845	847	849
.93	851	853	855	857	859	861	863	865	867	869
.94	871	873	875	877	879	881	883	885	887	889
.95	891	893	895	897	899	902	904	906	908	910
.96	912	914	916	918	920	923	925	927	929	931
.97	933	935	938	940	942	944	946	948	951	953
.98	955	957	959	962	964	966	968	971	973	975
.99	977	979	982	984	986	989	991	993	995	998

Squares from 1 to 10

	0	1	2	3	4	5	6	7	8	9
1.0	1.00	1.02	1.04	1.06	1.08	1.10	1.12	1.14	1.17	1.19
1.1	1.21	1.23	1.25	1.28	1.30	1.32	1.35	1.37	1.39	1.42
1.2	1.44	1.46	1.49	1.51	1.54	1.56	1.59	1.61	1.64	1.66
1.3	1.69	1.72	1.74	1.77	1.80	1.82	1.85	1.88	1.90	1.93
1.4	1.96	1.99	2.02	2.04	2.07	2.10	2.13	2.16	2.19	2.22
1.5	2.25	2.28	2.31	2.34	2.37	2.40	2.43	2.46	2.50	2.53
1.6	2.56	2.59	2.62	2.66	2.69	2.72	2.76	2.79	2.82	2.86
1.7	2.89	2.92	2.96	2.99	3.03	3.06	3.10	3.13	3.17	3.20
1.8	3.24	3.28	3.31	3.35	3.39	3.42	3.46	3.50	3.53	3.57
1.9	3.61	3.65	3.69	3.72	3.76	3.80	3.84	3.88	3.92	3.96
2.0	4.00	4.04	4.08	4.12	4.16	4.20	4.24	4.28	4.33	4.37
2.1	4.41	4.45	4.49	4.54	4.58	4.62	4.67	4.71	4.75	4.80
2.2	4.84	4.88	4.93	4.97	5.02	5.06	5.11	5.15	5.20	5.24
2.3	5.29	5.34	5.38	5.43	5.48	5.52	5.57	5.62	5.66	5.71
2.4	5.76	5.81	5.86	5.90	5.95	6.00	6.05	6.10	6.15	6.20
2.5	6.25	6.30	6.35	6.40	6.45	6.50	6.55	6.60	6.66	6.71
2.6	6.76	6.81	6.86	6.92	6.97	7.02	7.08	7.13	7.18	7.24
2.7	7.29	7.34	7.40	7.45	7.51	7.56	7.62	7.67	7.73	7.78
2.8	7.84	7.90	7.95	8.01	8.07	8.12	8.18	8.24	8.29	8.35
2.9	8.41	8.47	8.53	8.58	8.64	8.70	8.76	8.82	8.88	8.94
3.0	9.00	9.06	9.12	9.18	9.24	9.30	9.36	9.42	9.49	9.55
3.1	9.61	9.67	9.73	9.80	9.86	9.92	9.99	10.05	10.11	10.18
3.2	10.24	10.30	10.37	10.43	10.50	10.56	10.63	10.69	10.76	10.82
3.3	10.89	10.96	11.02	11.09	11.16	11.22	11.29	11.36	11.42	11.49
3.4	11.56	11.63	11.70	11.76	11.83	11.90	11.97	12.04	12.11	12.18
3.5	12.25	12.32	12.39	12.46	12.53	12.60	12.67	12.74	12.82	12.89
3.6	12.96	13.03	13.10	13.18	13.25	13.32	13.40	13.47	13.54	13.62
3.7	13.69	13.76	13.84	13.91	13.99	14.06	14.14	14.21	14.29	14.36
3.8	14.44	14.52	14.59	14.67	14.75	14.82	14.90	15.98	15.05	15.13
3.9	15.21	15.29	15.37	15.44	15.52	15.60	15.68	15.76	15.84	15.92
4.0	16.00	16.08	16.16	16.24	16.32	16.40	16.48	16.56	16.65	16.73
4.1	16.81	16.89	16.97	17.06	17.14	17.22	17.31	17.39	17.47	17.56
4.2	17.64	17.72	17.81	17.89	17.98	18.06	18.15	18.23	18.32	18.40
4.3	18.49	18.58	18.66	18.75	18.84	18.92	19.01	19.10	19.18	19.27
4.4	19.36	19.45	19.54	19.62	19.71	19.80	19.89	20.98	20.07	20.16
4.5	20.25	20.34	20.43	20.52	20.61	20.70	20.79	20.88	21.98	21.07
4.6	21.16	21.25	21.34	21.44	21.53	21.62	21.72	21.81	21.90	22.00
4.7	22.09	22.18	22.28	22.37	22.47	22.56	22.66	22.75	22.85	22.94
4.8	23.04	23.14	23.23	23.33	23.43	23.52	23.62	23.72	23.81	23.91
4.9	24.01	24.11	24.21	24.30	24.40	24.50	24.60	24.70	24.80	24.90
5.0	25.00	25.10	25.20	25.30	25.40	25.50	25.60	25.70	25.81	25.91
5.1	26.01	26.11	26.21	26.32	26.42	26.52	26.63	26.73	26.83	26.94
5.2	27.04	27.14	27.25	27.35	27.46	27.56	27.67	27.77	27.88	27.98
5.3	28.09	28.20	28.30	28.41	28.52	28.62	28.73	28.84	28.94	29.05
5.4	29.16	29.27	29.38	29.48	29.59	29.70	29.81	29.92	30.03	30.14

Squares from 1 to 10

	0	1	2	3	4	5	6	7	8	9
5.5	30.25	30.36	30.47	30.58	30.69	30.80	30.91	31.02	31.14	31.25
5.6	31.36	31.47	31.58	31.70	31.81	31.92	32.04	32.15	32.26	32.38
5.7	32.49	32.60	32.72	32.83	32.95	33.06	33.18	33.29	33.41	33.52
5.8	33.64	33.76	33.87	34.99	34.11	34.22	34.34	34.46	34.57	34.69
5.9	34.81	34.93	35.05	35.16	35.28	35.40	35.52	35.64	35.76	35.88
6.0	36.00	36.12	36.24	36.36	36.48	36.60	36.72	36.84	36.97	37.09
6.1	37.21	37.33	37.45	37.58	37.70	37.82	37.95	38.07	38.19	38.32
6.2	38.44	38.56	38.69	38.81	38.94	39.06	39.19	39.31	39.44	39.56
6.3	39.69	39.82	39.94	40.07	40.20	40.32	40.45	40.58	40.70	40.83
6.4	40.96	41.09	41.22	41.34	41.47	41.60	41.73	41.86	42.99	42.12
6.5	42.25	42.38	42.51	42.64	42.77	42.90	43.03	43.16	43.30	43.43
6.6	43.56	43.69	43.82	43.96	44.09	44.22	44.36	44.49	44.62	44.76
6.7	44.89	45.02	45.16	45.29	45.43	45.56	45.70	45.83	46.97	46.10
6.8	46.24	46.38	46.51	46.65	46.79	46.92	47.06	47.20	47.33	47.47
6.9	47.61	47.75	47.89	48.02	48.16	48.30	48.44	48.58	48.72	48.86
7.0	49.00	49.14	49.28	49.42	49.56	49.70	49.84	49.98	50.13	50.27
7.1	50.41	50.55	50.69	50.84	50.98	51.12	51.27	51.41	51.55	51.70
7.2	51.84	51.98	52.13	52.27	52.42	52.56	52.71	52.85	53.00	53.14
7.3	53.29	53.44	53.58	53.73	53.88	54.02	54.17	54.32	54.46	54.61
7.4	54.76	54.91	55.06	55.20	55.35	55.50	55.65	55.80	56.95	56.10
7.5	56.25	56.40	56.55	56.70	56.85	57.00	57.15	57.30	57.46	57.61
7.6	57.76	57.91	58.06	58.22	58.37	58.52	58.68	58.83	59.98	59.14
7.7	59.29	59.44	59.60	59.75	59.91	60.06	60.22	60.37	60.53	60.68
7.8	60.84	61.00	61.15	61.31	61.47	61.62	61.78	61.94	62.09	62.25
7.9	62.41	62.57	62.73	62.88	63.04	63.20	63.36	63.52	63.68	63.84
8.0	64.00	64.16	64.32	64.48	64.64	64.80	64.96	65.12	65.29	65.45
8.1	65.61	65.77	65.93	66.10	66.26	66.42	66.59	66.75	66.91	67.08
8.2	67.24	67.40	67.57	67.73	67.90	68.06	68.23	68.39	68.56	68.72
8.3	68.89	69.06	69.22	69.39	69.56	69.72	69.89	70.06	70.22	70.39
8.4	70.56	70.73	70.90	71.06	71.23	71.40	71.57	71.74	71.91	72.08
8.5	72.25	72.42	72.59	72.76	72.93	73.10	73.27	73.44	73.62	73.79
8.6	73.96	74.13	74.30	74.48	74.65	74.82	75.00	75.17	75.34	75.52
8.7	75.69	75.86	76.04	76.21	76.39	76.56	76.74	76.91	77.09	77.26
8.8	77.44	77.62	77.79	77.97	78.15	78.32	78.50	78.68	78.85	79.03
8.9	79.21	79.39	79.57	79.74	79.92	80.10	80.28	80.46	80.64	80.82
9.0	81.00	81.18	81.36	81.54	81.72	81.90	82.08	82.26	82.45	82.63
9.1	82.81	82.99	83.17	83.36	83.54	83.72	83.91	84.09	84.27	84.46
9.2	84.64	84.82	85.01	85.19	85.38	85.56	85.75	85.93	86.12	86.30
9.3	86.49	86.68	86.86	87.05	87.24	87.42	87.61	87.80	87.98	88.17
9.4	88.36	88.55	88.74	88.92	89.11	89.30	89.49	89.68	89.87	90.06
9.5	90.25	90.44	90.63	90.82	91.01	91.20	91.39	91.58	91.78	91.97
9.6	92.16	92.35	92.54	92.74	92.93	93.12	93.32	93.51	93.70	93.90
9.7	94.09	94.28	94.48	94.67	94.87	95.06	95.26	95.45	95.65	95.84
9.8	96.04	96.24	96.43	96.63	96.83	97.02	97.22	97.42	97.61	97.81
9.9	98.01	98.21	98.41	98.60	98.80	99.00	99.20	99.40	99.60	99.80

Square Roots from 1 to 10

	0	1	2	3	4	5	6	7	8	9
1.0	1.00	1.00	1.01	1.01	1.02	1.02	1.03	1.03	1.04	1.04
1.1	1.05	1.05	1.06	1.06	1.07	1.07	1.08	1.08	1.09	1.09
1.2	1.10	1.10	1.10	1.11	1.11	1.12	1.12	1.13	1.13	1.14
1.3	1.14	1.14	1.15	1.15	1.16	1.16	1.17	1.17	1.17	1.18
1.4	1.18	1.19	1.19	1.20	1.20	1.20	1.21	1.21	1.22	1.22
1.5	1.22	1.23	1.23	1.24	1.24	1.24	1.25	1.25	1.26	1.26
1.6	1.26	1.27	1.27	1.28	1.28	1.28	1.29	1.29	1.30	1.30
1.7	1.30	1.31	1.31	1.32	1.32	1.32	1.33	1.33	1.33	1.34
1.8	1.34	1.35	1.35	1.35	1.36	1.36	1.36	1.37	1.37	1.37
1.9	1.38	1.38	1.39	1.39	1.39	1.40	1.40	1.40	1.41	1.41
2.0	1.41	1.42	1.42	1.42	1.43	1.43	1.44	1.44	1.44	1.45
2.1	1.45	1.45	1.46	1.46	1.46	1.47	1.47	1.47	1.48	1.48
2.2	1.48	1.49	1.49	1.49	1.50	1.50	1.50	1.51	1.51	1.51
2.3	1.52	1.52	1.52	1.53	1.53	1.53	1.54	1.54	1.54	1.55
2.4	1.55	1.55	1.56	1.56	1.56	1.57	1.57	1.57	1.57	1.58
2.5	1.58	1.58	1.59	1.59	1.59	1.60	1.60	1.60	1.61	1.61
2.6	1.61	1.62	1.62	1.62	1.62	1.63	1.63	1.63	1.64	1.64
2.7	1.64	1.65	1.65	1.65	1.66	1.66	1.66	1.66	1.67	1.67
2.8	1.67	1.68	1.68	1.68	1.69	1.69	1.69	1.69	1.70	1.70
2.9	1.70	1.71	1.71	1.71	1.71	1.72	1.72	1.72	1.73	1.73
3.0	1.73	1.73	1.74	1.74	1.74	1.75	1.75	1.75	1.75	1.76
3.1	1.76	1.76	1.77	1.77	1.77	1.77	1.78	1.78	1.78	1.79
3.2	1.79	1.79	1.79	1.80	1.80	1.80	1.81	1.81	1.81	1.81
3.3	1.82	1.82	1.82	1.82	1.83	1.83	1.83	1.84	1.84	1.84
3.4	1.84	1.85	1.85	1.85	1.85	1.86	1.86	1.86	1.87	1.87
3.5	1.87	1.87	1.88	1.88	1.88	1.88	1.89	1.89	1.89	1.89
3.6	1.90	1.90	1.90	1.91	1.91	1.91	1.91	1.92	1.92	1.92
3.7	1.92	1.93	1.93	1.93	1.93	1.94	1.94	1.94	1.94	1.95
3.8	1.95	1.95	1.95	1.96	1.96	1.96	1.96	1.97	1.97	1.97
3.9	1.97	1.98	1.98	1.98	1.98	1.99	1.99	1.99	1.99	2.00
4.0	2.00	2.00	2.00	2.01	2.01	2.01	2.01	2.02	2.02	2.02
4.1	2.02	2.03	2.03	2.03	2.03	2.04	2.04	2.04	2.04	2.05
4.2	2.05	2.05	2.05	2.06	2.06	2.06	2.06	2.07	2.07	2.07
4.3	2.07	2.08	2.08	2.08	2.08	2.09	2.09	2.09	2.09	2.10
4.4	2.10	2.10	2.10	2.10	2.11	2.11	2.11	2.11	2.12	2.12
4.5	2.12	2.12	2.13	2.13	2.13	2.13	2.14	2.14	2.14	2.14
4.6	2.14	2.15	2.15	2.15	2.15	2.16	2.16	2.16	2.16	2.17
4.7	2.17	2.17	2.17	2.17	2.18	2.18	2.18	2.18	2.19	2.19
4.8	2.19	2.19	2.20	2.20	2.20	2.20	2.20	2.21	2.21	2.21
4.9	2.21	2.22	2.22	2.22	2.22	2.22	2.23	2.23	2.23	2.23
5.0	2.24	2.24	2.24	2.24	2.24	2.25	2.25	2.25	2.25	2.26
5.1	2.26	2.26	2.26	2.26	2.27	2.27	2.27	2.27	2.28	2.28
5.2	2.28	2.28	2.28	2.29	2.29	2.29	2.29	2.30	2.30	2.30
5.3	2.30	2.30	2.31	2.31	2.31	2.31	2.32	2.32	2.32	2.32
5.4	2.32	2.33	2.33	2.33	2.33	2.33	2.34	2.34	2.34	2.34

Square Roots from 1 to 10

	0	1	2	3	4	5	6	7	8	9
5.5	2.35	2.35	2.35	2.35	2.35	2.36	2.36	2.36	2.36	2.36
5.6	2.37	2.37	2.37	2.37	2.37	2.38	2.38	2.38	2.38	2.39
5.7	2.39	2.39	2.39	2.39	2.40	2.40	2.40	2.40	2.40	2.41
5.8	2.41	2.41	2.41	2.41	2.42	2.42	2.42	2.42	2.42	2.43
5.9	2.43	2.43	2.43	2.44	2.44	2.44	2.44	2.44	2.45	2.45
6.0	2.45	2.45	2.45	2.46	2.46	2.46	2.46	2.46	2.47	2.47
6.1	2.47	2.47	2.47	2.48	2.48	2.48	2.48	2.48	2.49	2.49
6.2	2.49	2.49	2.49	2.50	2.50	2.50	2.50	2.50	2.51	2.51
6.3	2.51	2.51	2.51	2.52	2.52	2.52	2.52	2.52	2.53	2.53
6.4	2.53	2.53	2.53	2.54	2.54	2.54	2.54	2.54	2.55	2.55
6.5	2.55	2.55	2.55	2.56	2.56	2.56	2.56	2.56	2.57	2.57
6.6	2.57	2.57	2.57	2.57	2.58	2.58	2.58	2.58	2.58	2.59
6.7	2.59	2.59	2.59	2.59	2.60	2.60	2.60	2.60	2.60	2.61
6.8	2.61	2.61	2.61	2.61	2.62	2.62	2.62	2.62	2.62	2.62
6.9	2.63	2.63	2.63	2.63	2.63	2.64	2.64	2.64	2.64	2.64
7.0	2.65	2.65	2.65	2.65	2.65	2.66	2.66	2.66	2.66	2.66
7.1	2.66	2.67	2.67	2.67	2.67	2.67	2.68	2.68	2.68	2.68
7.2	2.68	2.69	2.69	2.69	2.69	2.69	2.69	2.70	2.70	2.70
7.3	2.70	2.70	2.71	2.71	2.71	2.71	2.71	2.71	2.72	2.72
7.4	2.72	2.72	2.72	2.73	2.73	2.73	2.73	2.73	2.73	2.74
7.5	2.74	2.74	2.74	2.74	2.75	2.75	2.75	2.75	2.75	2.75
7.6	2.76	2.76	2.76	2.76	2.76	2.77	2.77	2.77	2.77	2.77
7.7	2.77	2.78	2.78	2.78	2.78	2.78	2.79	2.79	2.79	2.79
7.8	2.79	2.79	2.80	2.80	2.80	2.80	2.80	2.81	2.81	2.81
7.9	2.81	2.81	2.81	2.82	2.82	2.82	2.82	2.82	2.82	2.83
8.0	2.83	2.83	2.83	2.83	2.84	2.84	2.84	2.84	2.84	2.84
8.1	2.85	2.85	2.85	2.85	2.85	2.85	2.86	2.86	2.86	2.86
8.2	2.86	2.87	2.87	2.87	2.87	2.87	2.87	2.88	2.88	2.88
8.3	2.88	2.88	2.88	2.89	2.89	2.89	2.89	2.89	2.89	2.90
8.4	2.90	2.90	2.90	2.90	2.91	2.91	2.91	2.91	2.91	2.91
8.5	2.92	2.92	2.92	2.92	2.92	2.92	2.93	2.93	2.93	2.93
8.6	2.93	2.93	2.94	2.94	2.94	2.94	2.94	2.94	2.95	2.95
8.7	2.95	2.95	2.95	2.95	2.96	2.96	2.96	2.96	2.96	2.96
8.8	2.97	2.97	2.97	2.97	2.97	2.97	2.98	2.98	2.98	2.98
8.9	2.98	2.98	2.99	2.99	2.99	2.99	2.99	2.99	3.00	3.00
9.0	3.00	3.00	3.00	3.00	3.01	3.01	3.01	3.01	3.01	3.01
9.1	3.02	3.02	3.02	3.02	3.02	3.02	3.03	3.03	3.03	3.03
9.2	3.03	3.03	3.04	3.04	3.04	3.04	3.04	3.04	3.05	3.05
9.3	3.05	3.05	3.05	3.05	3.06	3.06	3.06	3.06	3.06	3.06
9.4	3.07	3.07	3.07	3.07	3.07	3.07	3.08	3.08	3.08	3.08
9.5	3.08	3.08	3.09	3.09	3.09	3.09	3.09	3.09	3.10	3.10
9.6	3.10	3.10	3.10	3.10	3.10	3.11	3.11	3.11	3.11	3.11
9.7	3.11	3.12	3.12	3.12	3.12	3.12	3.12	3.13	3.13	3.13
9.8	3.13	3.13	3.13	3.14	3.14	3.14	3.14	3.14	3.14	3.14
9.9	3.15	3.15	3.15	3.15	3.15	3.15	3.16	3.16	3.16	3.16
10.0	3.16									

Square Roots from 10 to 100

	0	1	2	3	4	5	6	7	8	9
10	3.16	3.18	3.19	3.21	3.22	3.24	3.26	3.27	3.29	3.30
11	3.32	3.33	3.35	3.36	3.38	3.39	3.41	3.42	3.44	3.45
12	3.46	3.48	3.49	3.51	3.52	3.54	3.55	3.56	3.58	3.59
13	3.61	3.62	3.63	3.65	3.66	3.67	3.69	3.70	3.71	3.73
14	3.74	3.75	3.77	3.78	3.79	3.81	3.82	3.83	3.85	3.86
15	3.87	3.89	3.90	3.91	3.92	3.94	3.95	3.96	3.97	3.99
16	4.00	4.01	4.02	4.04	4.05	4.06	4.07	4.09	4.10	4.11
17	4.12	4.14	4.15	4.16	4.17	4.18	4.20	4.21	4.22	4.23
18	4.24	4.25	4.27	4.28	4.29	4.30	4.31	4.32	4.34	4.35
19	4.36	4.37	4.38	4.39	4.40	4.42	4.43	4.44	4.45	4.46
20	4.47	4.48	4.49	4.51	4.52	4.53	4.54	4.55	4.56	4.57
21	4.58	4.59	4.60	4.62	4.63	4.64	4.65	4.66	4.67	4.68
22	4.69	4.70	4.71	4.72	4.73	4.74	4.75	4.76	4.77	4.79
23	4.80	4.81	4.82	4.83	4.84	4.85	4.86	4.87	4.88	4.89
24	4.90	4.91	4.92	4.93	4.94	4.95	4.96	4.97	4.98	4.99
25	5.00	5.01	5.02	5.03	5.04	5.05	5.06	5.07	5.08	5.09
26	5.10	5.11	5.12	5.13	5.14	5.15	5.16	5.17	5.18	5.19
27	5.20	5.21	5.22	5.22	5.23	5.24	5.25	5.26	5.27	5.28
28	5.29	5.30	5.31	5.32	5.33	5.34	5.35	5.36	5.37	5.38
29	5.39	5.39	5.40	5.41	5.42	5.43	5.44	5.45	5.46	5.47
30	5.48	5.49	5.50	5.50	5.51	5.52	5.53	5.54	5.55	5.56
31	5.57	5.58	5.59	5.59	5.60	5.61	5.62	5.63	5.64	5.65
32	5.66	5.67	5.67	5.68	5.69	5.70	5.71	5.72	5.73	5.74
33	5.74	5.75	5.76	5.77	5.78	5.79	5.80	5.81	5.81	5.82
34	5.83	5.84	5.85	5.86	5.87	5.87	5.88	5.89	5.90	5.91
35	5.92	5.92	5.93	5.94	5.95	5.96	5.97	5.97	5.98	5.99
36	6.00	6.01	6.02	6.02	6.03	6.04	6.05	6.06	6.07	6.07
37	6.08	6.09	6.10	6.11	6.12	6.12	6.13	6.14	6.15	6.16
38	6.16	6.17	6.18	6.19	6.20	6.20	6.21	6.22	6.23	6.24
39	6.24	6.25	6.26	6.27	6.28	6.28	6.29	6.30	6.31	6.32
40	6.32	6.33	6.34	6.35	6.36	6.36	6.37	6.38	6.39	6.40
41	6.40	6.41	6.42	6.43	6.43	6.44	6.45	6.46	6.47	6.47
42	6.48	6.49	6.50	6.50	6.51	6.52	6.53	6.53	6.54	6.55
43	6.56	6.57	6.57	6.58	6.59	6.60	6.60	6.61	6.62	6.63
44	6.63	6.64	6.65	6.66	6.66	6.67	6.68	6.69	6.69	6.70
45	6.71	6.72	6.72	6.73	6.74	6.75	6.75	6.76	6.77	6.77
46	6.78	6.79	6.80	6.80	6.81	6.82	6.83	6.83	6.84	6.85
47	6.86	6.86	6.87	6.88	6.88	6.89	6.90	6.91	6.91	6.92
48	6.93	6.94	6.94	6.95	6.96	6.96	6.97	6.98	6.99	6.99
49	7.00	7.01	7.01	7.02	7.03	7.04	7.04	7.05	7.06	7.06
50	7.07	7.08	7.09	7.09	7.10	7.11	7.11	7.12	7.13	7.13
51	7.14	7.15	7.16	7.16	7.17	7.18	7.18	7.19	7.20	7.20
52	7.21	7.22	7.22	7.23	7.24	7.25	7.25	7.26	7.27	7.27
53	7.28	7.29	7.29	7.30	7.31	7.31	7.32	7.33	7.33	7.34
54	7.35	7.36	7.36	7.37	7.38	7.38	7.39	7.40	7.40	7.41

Square Roots from 10 to 100

	0	1	2	3	4	5	6	7	8	9
55	7.42	7.42	7.42	7.44	7.44	7.45	7.46	7.46	7.47	7.48
56	7.48	7.49	7.50	7.50	7.51	7.52	7.52	7.53	7.54	7.54
57	7.55	7.56	7.56	7.57	7.58	7.58	7.59	7.60	7.60	7.61
58	7.62	7.62	7.63	7.64	7.64	7.65	7.66	7.66	7.67	7.67
59	7.68	7.69	7.69	7.70	7.71	7.71	7.72	7.73	7.73	7.74
60	7.75	7.75	7.76	7.77	7.77	7.78	7.78	7.79	7.80	7.80
61	7.81	7.82	7.82	7.83	7.84	7.84	7.85	7.85	7.86	7.87
62	7.87	7.88	7.89	7.89	7.90	7.91	7.91	7.92	7.92	7.93
63	7.94	7.94	7.95	7.96	7.96	7.97	7.97	7.98	7.99	7.99
64	8.00	8.01	8.01	8.02	8.02	8.03	8.04	8.04	8.05	8.06
65	8.06	8.07	8.07	8.08	8.09	8.09	8.10	8.11	8.11	8.12
66	8.12	8.13	8.14	8.14	8.15	8.15	8.16	8.17	8.17	8.18
67	8.19	8.19	8.20	8.20	8.21	8.22	8.22	8.23	8.23	8.24
68	8.25	8.25	8.26	8.26	8.27	8.28	8.28	8.29	8.29	8.30
69	8.31	8.31	8.32	8.32	8.33	8.34	8.34	8.35	8.35	8.36
70	8.37	8.37	8.38	8.38	8.39	8.40	8.40	8.41	8.41	8.42
71	8.43	8.43	8.44	8.44	8.45	8.46	8.46	8.47	8.47	8.48
72	8.49	8.49	8.50	8.50	8.51	8.51	8.52	8.53	8.53	8.54
73	8.54	8.55	8.56	8.56	8.57	8.57	8.58	8.58	8.59	8.60
74	8.60	8.61	8.61	8.62	8.63	8.63	8.64	8.64	8.65	8.65
75	8.66	8.67	8.67	8.68	8.68	8.69	8.69	8.70	8.71	8.71
76	8.72	8.72	8.73	8.73	8.74	8.75	8.75	8.76	8.76	8.77
77	8.77	8.78	8.79	8.79	8.80	8.80	8.81	8.81	8.82	8.83
78	8.83	8.84	8.84	8.85	8.85	8.86	8.87	8.87	8.88	8.88
79	8.89	8.89	8.90	8.91	8.91	8.92	8.92	8.93	8.93	8.94
80	8.94	8.95	8.96	8.96	8.97	8.97	8.98	8.98	8.99	8.99
81	9.00	9.01	9.01	9.02	9.02	9.03	9.03	9.04	9.04	9.05
82	9.06	9.06	9.07	9.07	9.08	9.08	9.09	9.09	9.10	9.10
83	9.11	9.12	9.12	9.13	9.13	9.14	9.14	9.15	9.15	9.16
84	9.17	9.17	9.18	9.18	9.19	9.19	9.20	9.20	9.21	9.21
85	9.22	9.22	9.23	9.24	9.24	9.25	9.25	9.26	9.26	9.27
86	9.27	9.28	9.28	9.29	9.30	9.30	9.31	9.31	9.32	9.32
87	9.33	9.33	9.34	9.34	9.35	9.35	9.36	9.36	9.37	9.38
88	9.38	9.39	9.39	9.40	9.40	9.41	9.41	9.42	9.42	9.43
89	9.43	9.44	9.44	9.45	9.46	9.46	9.47	9.47	9.48	9.48
90	9.49	9.49	9.50	9.50	9.51	9.51	9.52	9.52	9.53	9.53
91	9.54	9.54	9.55	9.56	9.56	9.57	9.57	9.58	9.58	9.59
92	9.59	9.60	9.60	9.61	9.61	9.62	9.62	9.63	9.63	9.64
93	9.64	9.65	9.65	9.66	9.66	9.67	9.67	9.68	9.69	9.69
94	9.70	9.70	9.71	9.71	9.72	9.72	9.73	9.73	9.74	9.74
95	9.75	9.75	9.76	9.76	9.77	9.77	9.78	9.78	9.79	9.79
96	9.80	9.80	9.81	9.81	9.82	9.82	9.83	9.83	9.84	9.84
97	9.85	9.85	9.86	9.86	9.87	9.87	9.88	9.88	9.89	9.89
98	9.90	9.90	9.91	9.91	9.92	9.92	9.93	9.93	9.94	9.94
99	9.95	9.95	9.96	9.96	9.97	9.97	9.98	9.98	9.99	9.99

Natural Sines

Degrees	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	0.000	002	003	005	007	009	010	012	014	016
1	.017	019	021	023	024	026	028	030	031	033
2	.035	037	038	040	042	044	045	047	049	051
3	.052	054	056	058	059	061	063	065	066	068
4	.070	071	073	075	077	078	080	082	084	085
5	0.087	089	091	092	094	096	098	099	101	103
6	.105	106	108	110	111	113	115	117	118	120
7	.122	124	125	127	129	131	132	134	136	137
8	.139	141	143	144	146	148	150	151	153	155
9	.156	158	160	162	163	165	167	168	170	172
10	0.174	175	111	179	181	182	184	186	187	189
11	.191	193	194	196	198	199	201	203	204	206
12	.208	210	211	213	215	216	218	220	222	223
13	.225	227	228	230	232	233	235	237	239	240
14	.242	244	245	247	249	250	252	254	255	257
15	0.259	261	262	264	266	267	269	271	272	274
16	.276	277	279	281	282	284	286	287	289	291
17	.292	294	296	297	299	301	302	304	306	307
18	.309	311	312	314	316	317	319	321	322	324
19	.326	327	329	331	332	334	335	337	339	340
20	0.342	344	345	347	349	350	352	353	355	357
21	.358	360	362	363	365	367	368	370	371	373
22	.375	376	378	379	381	383	384	386	388	389
23	.391	392	394	396	397	399	400	402	404	405
24	.407	408	410	412	413	415	416	418	419	421
25	0.423	424	426	427	429	431	432	434	435	437
26	.438	440	442	443	445	446	448	449	451	452
27	.454	456	457	459	460	462	463	465	466	468
28	.469	471	473	474	476	477	479	480	482	483
29	.485	486	488	489	491	492	494	495	497	498
30	0.500	502	503	505	506	508	509	511	512	514
31	.515	517	518	520	521	522	524	525	527	528
32	.530	531	533	534	536	537	539	540	542	543
33	.545	546	548	549	550	552	553	555	556	558
34	.559	561	562	564	565	566	568	569	571	572
35	0.574	575	576	578	579	581	582	584	585	586
36	.588	589	591	592	593	595	596	598	599	600
37	.602	603	605	606	607	609	610	612	613	614
38	.616	617	618	620	621	623	624	625	627	628
39	.629	631	632	633	635	636	637	639	640	641
40	0.643	644	645	647	648	649	651	652	653	655
41	.656	657	659	660	661	663	664	665	667	668
42	.669	670	672	673	674	676	677	678	679	681
43	.682	683	685	686	687	688	690	691	692	693
44	.695	696	697	698	700	701	702	703	705	706

Natural Sines

Degrees	**.0**	**.1**	**.2**	**.3**	**.4**	**.5**	**.6**	**.7**	**.8**	**.9**
45	0.707	708	710	711	712	713	714	716	717	718
46	.719	721	722	723	724	725	727	728	729	730
47	.731	733	734	735	736	737	738	740	741	742
48	.743	744	745	747	748	749	750	751	752	754
49	.755	756	757	758	759	760	762	763	764	765
50	0.766	767	768	769	771	772	773	774	775	776
51	.777	778	779	780	782	783	784	785	786	787
52	.788	789	790	791	792	793	794	795	797	798
53	.799	800	801	802	803	804	805	806	807	808
54	.809	810	811	812	813	814	815	816	817	818
55	0.819	820	821	822	823	824	825	826	827	828
56	.829	830	831	832	833	834	835	836	837	838
57	.839	840	841	842	842	843	844	845	846	847
58	.848	849	850	851	852	853	854	854	855	856
59	.857	858	859	860	861	862	863	863	864	865
60	0.866	867	868	869	869	870	871	872	873	874
61	.875	875	876	877	878	879	880	880	881	882
62	.883	884	885	885	886	887	888	889	889	890
63	.891	892	893	893	894	895	896	896	897	898
64	.899	900	900	901	902	903	903	904	905	906
65	0.906	907	908	909	909	910	911	911	912	913
66	.914	914	915	916	916	917	918	918	919	920
67	.921	921	922	923	923	924	925	925	926	927
68	.927	928	928	929	930	930	931	932	932	933
69	.934	934	935	935	936	937	937	938	938	939
70	0.940	940	941	941	942	943	943	944	944	945
71	.946	946	947	947	948	948	949	949	950	951
72	.951	952	952	953	953	954	954	955	955	956
73	.956	957	957	958	958	959	959	960	960	961
74	.961	962	962	963	963	964	964	965	965	965
75	0.966	966	967	967	968	968	969	969	969	970
76	.970	971	971	972	972	972	973	973	974	974
77	.974	975	975	976	976	976	977	977	977	978
78	.978	979	979	979	980	980	980	981	981	981
79	.982	982	982	983	983	983	984	984	984	985
80	0.985	985	985	986	986	986	987	987	987	987
81	.988	988	988	988	989	989	989	990	990	990
82	.990	991	991	991	991	991	992	992	992	992
83	.993	993	993	993	993	994	994	994	994	994
84	.995	995	995	995	995	995	996	996	996	996
85	0.996	996	996	997	997	997	997	997	997	997
86	.998	998	998	998	998	998	998	998	998	999
87	.999	999	999	999	999	999	999	999	999	999
88	0.999	999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
89	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
90	1.000									

Natural Cosines

Degrees	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	1.000	000	000	000	000	000	000	000	000	000
1	1.000	000	000	000	000	000	000	000	000	0.999
2	0.999	999	999	999	999	999	999	999	999	999
3	.999	999	998	998	998	998	998	998	998	998
4	.998	997	997	997	997	997	997	997	996	996
5	0.996	996	996	996	996	995	995	995	995	995
6	.995	994	994	994	994	994	993	993	993	993
7	.993	992	992	992	992	991	991	991	991	991
8	.990	990	990	990	989	989	989	988	988	988
9	.988	987	987	987	987	986	986	986	985	985
10	0.985	985	984	984	984	983	983	983	982	982
11	.982	981	981	981	980	980	980	979	979	979
12	.978	978	977	977	977	976	976	976	975	975
13	.974	974	974	973	973	972	972	972	971	971
14	.970	970	969	969	969	968	968	967	967	966
15	0.966	965	965	965	964	964	963	963	962	962
16	.961	961	960	960	959	959	958	958	957	957
17	.956	956	955	955	954	954	953	953	952	952
18	.951	951	950	949	949	948	948	947	947	946
19	.946	945	944	944	943	943	942	941	941	940
20	0.940	939	938	938	937	937	936	935	935	934
21	.934	933	932	932	931	930	930	929	928	928
22	.927	927	926	925	925	924	923	923	922	921
23	.921	920	919	918	918	917	916	916	915	914
24	.914	913	912	911	911	910	909	909	908	907
25	0.906	906	905	904	903	903	902	901	900	900
26	.899	898	897	896	896	895	894	893	893	892
27	.891	890	889	889	888	887	886	885	885	884
28	.883	882	881	880	880	879	878	877	876	875
29	.875	874	873	872	871	870	869	869	868	867
30	0.866	865	864	863	863	862	861	860	859	858
31	.857	856	855	854	854	853	852	851	850	849
32	.848	847	846	845	844	843	842	842	841	840
33	.839	838	837	836	835	834	833	832	831	830
34	.829	828	827	826	825	824	823	822	821	820
35	0.819	818	817	816	815	814	813	812	811	810
36	.809	808	807	806	805	804	803	802	801	800
37	.799	798	797	795	794	793	792	791	790	789
38	.788	787	786	785	784	783	782	780	779	778
39	.777	776	775	774	773	772	771	769	768	767
40	0.766	765	764	763	762	760	759	758	757	756
41	.755	754	752	751	750	749	748	747	745	744
42	.743	742	741	740	738	737	736	735	734	733
43	.731	730	729	728	727	725	724	723	722	721
44	.719	718	717	716	714	713	712	711	710	708

Natural Cosines

Degrees	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
45	0.707	706	705	703	702	701	700	698	697	696
46	.695	693	692	691	690	688	687	686	685	683
47	.682	681	679	678	677	676	674	673	672	670
48	.669	668	669	665	664	663	661	660	659	667
49	.656	655	653	652	651	649	648	647	645	644
50	0.643	641	640	639	637	636	635	633	632	631
51	.629	628	627	625	624	623	621	620	618	617
52	.616	614	613	612	610	609	607	606	605	603
53	.602	600	599	598	596	595	593	592	591	589
54	.588	586	585	584	582	581	579	578	576	575
55	0.574	572	571	569	568	566	565	564	562	561
56	.559	558	556	555	553	552	550	549	548	546
57	.545	543	542	540	539	537	536	534	533	531
58	.530	528	527	525	524	522	521	520	518	517
59	.515	514	512	511	509	508	506	505	503	502
60	0.500	498	497	495	494	492	491	489	488	486
61	.485	483	482	480	479	477	476	474	473	471
62	.469	468	466	465	463	462	460	459	457	456
63	.454	452	451	449	448	446	445	443	442	440
64	.438	437	435	434	432	431	429	427	426	424
65	0.423	421	419	418	416	415	413	412	410	408
66	.407	405	404	402	400	399	397	396	394	392
67	.391	389	388	386	384	383	381	379	378	376
68	.375	373	371	370	368	367	365	363	362	360
69	.358	357	355	353	352	350	349	347	345	344
70	0.342	340	339	337	335	334	332	331	329	327
71	.326	324	322	321	319	317	316	314	312	311
72	.309	307	306	304	302	301	299	297	296	294
73	.292	291	289	287	286	284	282	281	279	287
74	.276	274	272	271	269	267	266	264	262	261
75	0.259	257	255	254	252	250	249	247	245	244
76	.242	240	239	237	235	233	232	230	228	227
77	.225	223	222	220	218	216	215	213	211	210
78	.208	206	204	203	201	199	198	196	194	193
79	.191	189	187	186	184	182	181	179	177	175
80	0.174	172	170	168	167	165	163	162	160	158
81	.156	155	153	151	150	148	146	144	143	141
82	.139	137	136	134	132	131	129	127	125	124
83	.122	120	118	117	115	113	111	110	108	106
84	.105	103	101	099	098	096	094	092	091	089
85	0.087	085	084	082	080	078	077	075	073	071
86	.070	068	066	065	063	061	059	058	056	054
87	.052	051	049	047	045	044	042	040	038	037
88	.035	033	031	030	028	026	024	023	021	019
89	.017	016	014	012	010	009	007	005	003	002
90	0.000									

Natural Tangents

Degrees	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	0.000	002	003	005	007	009	010	012	014	016
1	.017	019	021	023	024	026	028	030	031	033
2	.035	037	038	040	042	044	045	047	049	051
3	.052	054	056	058	059	061	063	065	066	068
4	.070	072	073	075	077	079	080	082	084	086
5	0.087	089	091	093	095	096	098	100	102	103
6	.105	107	109	110	112	114	116	117	119	121
7	.123	125	126	128	130	132	133	135	137	139
8	.141	142	144	146	148	149	151	153	155	157
9	.158	160	162	164	166	167	169	171	173	175
10	0.176	178	180	182	184	185	187	189	191	193
11	.194	196	198	200	202	203	205	207	209	211
12	.213	214	216	218	220	222	224	225	227	229
13	.231	233	235	236	238	240	242	244	246	247
14	.249	251	253	255	257	259	260	262	264	266
15	0.268	270	272	274	275	277	279	281	283	285
16	.287	289	291	292	294	296	298	300	302	304
17	.306	308	310	311	313	315	317	319	321	323
18	.325	327	329	331	333	335	337	338	340	342
19	.344	346	348	350	352	354	356	358	360	362
20	0.364	366	368	370	372	374	376	378	380	382
21	.384	386	388	390	392	394	396	398	400	402
22	.404	406	408	410	412	414	416	418	420	422
23	.424	427	429	431	433	435	437	439	441	443
24	.445	447	449	452	454	456	458	460	462	464
25	0.466	468	471	473	475	477	479	481	483	486
26	.488	490	492	494	496	499	501	503	505	507
27	.510	512	514	516	518	521	523	525	527	529
28	.532	534	536	538	541	543	545	547	550	552
29	.554	557	559	561	563	566	568	570	573	575
30	0.577	580	582	584	587	589	591	594	596	598
31	.601	603	606	608	610	613	615	618	620	622
32	.625	627	630	632	635	637	640	642	644	647
33	.649	652	654	657	659	662	664	667	669	672
34	.675	677	680	682	685	687	690	692	695	698
35	0.700	703	705	708	711	713	716	719	721	724
36	.727	729	732	735	737	740	743	745	748	751
37	.754	756	759	762	765	767	770	773	776	778
38	.781	784	787	790	793	795	798	801	804	807
39	.810	813	816	818	821	824	827	830	833	836
40	0.839	842	845	848	851	854	857	860	863	866
41	.869	872	875	879	882	885	888	891	894	897
42	.900	904	907	910	913	916	920	923	926	929
43	.933	936	939	942	946	949	952	956	959	962
44	.966	969	972	976	979	983	986	990	993	997

Natural Tangents

Degrees	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
45	1.000	003	007	011	014	018	021	025	028	032
46	.036	039	043	046	050	054	057	061	065	069
47	.072	076	080	084	087	091	095	099	103	107
48	.111	115	118	122	126	130	134	138	142	146
49	.150	154	159	163	167	171	175	179	183	188
50	1.192	196	200	205	209	213	217	222	226	230
51	.235	239	244	248	253	257	262	266	271	275
52	.280	285	289	294	299	303	308	313	317	322
53	.327	332	337	342	347	351	356	361	366	371
54	.376	381	387	392	397	402	407	412	418	423
55	1.428	433	439	444	450	455	460	466	471	477
56	.483	488	494	499	505	511	517	522	528	534
57	.540	546	552	558	564	570	576	582	588	594
58	.600	607	613	619	625	632	638	645	651	658
59	.664	671	678	684	691	698	704	711	718	725
60	1.732	739	746	753	760	767	775	782	789	797
61	.804	811	819	827	834	842	849	857	865	873
62	.881	889	897	905	913	921	929	937	946	954
63	1.963	971	980	988	997	2.006	2.014	2.023	2.032	2.041
64	2.050	059	069	078	087	097	106	116	125	135
65	2.145	154	164	174	184	194	204	215	225	236
66	.246	257	267	278	289	300	311	322	333	344
67	.356	367	379	391	402	414	426	438	450	463
68	.475	488	500	513	526	539	552	565	578	592
69	.605	619	633	646	660	675	689	703	718	733
70	2.747	762	778	793	808	824	840	856	872	888
71	2.904	921	937	954	971	989	3.006	3.024	3.042	3.060
72	3.078	096	115	133	152	172	191	211	230	251
73	.271	291	312	333	354	376	398	420	442	465
74	.487	511	534	558	582	606	630	655	681	706
75	3.732	758	785	812	839	867	895	923	952	981
76	4.011	041	071	102	134	165	198	230	264	297
77	4.331	366	402	437	474	511	548	586	625	665
78	4.705	745	787	829	872	915	959	5.005	5.050	5.097
79	5.145	193	242	292	343	396	449	503	558	614
80	5.671	730	789	850	912	976	6.041	6.107	6.174	6.243
81	6.314	386	460	535	612	691	772	855	940	7.026
82	7.115	207	300	396	495	596	700	806	916	8.028
83	8.144	264	386	513	643	777	915	9.058	9.205	9.357
84	9.514	9.677	9.845	10.02	10.20	10.39	10.58	10.78	10.99	11.20
85	11.43	11.66	11.91	12.16	12.43	12.71	13.00	13.30	13.62	13.95
86	14.30	14.67	15.06	15.46	15.89	16.35	16.83	17.34	17.89	18.46
87	19.08	19.74	20.45	21.20	22.02	22.90	23.86	24.90	26.03	27.27
88	28.64	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08
89	57.29	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0

Answers

Exercise 1a

1. $A = \{2, 4, 6, 8, 10, 12\}$
2. $B = \{17, 19, 23, 29\}$
3. $C = \{15, 20, 25, 30, 35, 40, 45\}$
4. $X = \{11, 12, 13, 14, 15, 16, 17, 18, 19\}$
5. $Y = \{a, c, e, h, i, m, s, t\}$
6. $Z = \{2, 3, 5, 7, 11, 13, 17, 19\}$
7. $D = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$
8. $E = \{4, 6, 8, 10, 12, 14, 16\}$
9. $F = \{5, 7, 9, 11, 13\}$
10. $H = \{a, b, c, d, e\}$
11. $P =$ the set of prime numbers less than 18
12. $Q =$ the set of multiples of 5 between 25 and 45 inclusive
13. $R =$ the set of the last four letters in the English alphabet
14. $S =$ the set of squares of whole numbers from 1 to 7 inclusive
15. $T =$ the set of odd numbers from 15 to 25 inclusive
    $=$ the set of odd numbers from 13 to 27 exclusive

Exercise 1b

1. Turtle $\in$ {living things}
2. Brazil $\notin$ {Asian countries}
3. Orange $\in$ {fruits}
4. Electricity $\notin$ {living things}
5. Mathematics $\notin$ {school subjects}
6. Curry $\notin$ {cars}
7. Carite $\in$ {fish}
8. Sparrow, Chalkdust, Denise Plummer $\in$ {calypso singers} (for example)
9. Grape $\notin$ {animals}
10. Zero $\notin$ {natural numbers}
11. Physics is a science subject.
12. French is not a science subject.
13. Cricket belongs to the set of team games.
14. Albert does not belong to the set of girls' names.
15. 1, 3 and 5 belong to the set of odd numbers.
16. 2, 4 and 6 do not belong to the set of odd numbers.

17. $n(A) = 7$ **18.** $n(B) = 8$
19. $n(C) = 6$ **20.** $n(P) = 7$
21. $n(Q) = 8$ **22.** $n(R) = 6$
23. $n(X) = 5$ **24.** $n(Y) = 6$
25. $n(Z) = 6$

Exercise 1c

1. Finite **2.** Infinite
3. Null **4.** Infinite
5. Infinite **6.** Finite
7. Null **8.** Null
9. Infinite **10.** Finite

Exercise 1d

1. $U =$ {whole numbers} or $U =$ {natural numbers}
2. $U =$ {geometrical instruments}
3. $U =$ {integers}
4. $U =$ {prime numbers}
5. $U =$ {whole numbers}

Exercise 1e

1. (a) $\{3, 6, 9\} \subset \{3, 6, 9, 12, 15, 18, 21\}$.
   (b) $\{9, 12\} \subset \{3, 6, 9, 12, 15, 18, 21\}$.
   (c) $\{3, 12, 21\} \subset \{3, 6, 9, 12, 15, 18, 21\}$.
   (d) $\{9, 12\} \not\subset \{3, 6, 9\}$.
   (e) $\{9, 12\} \not\subset \{3, 12, 21\}$.
   (f) $\{3, 6, 9\} \not\subset \{3, 12, 21\}$.
2. $\{2, 4\} \subset \{2, 4, 6, 8\} \subset \{2, 4, 6, 8, 10, 12, 14\}$. True.
3. $\{3, 5\} \not\subset \{3, 7, 9\} \not\subset \{3, 7, 11, 13, 15\}$. False.
4. $\{p\}, \{q\}, \{r\}, \{s\}, \{p, q\}, \{p, r\}, \{p, s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, r, s\}, \{p, q, s\}, \{q, r, s\}$ = Proper subsets of A.
   $\{p, q, r, s\}, \{\ \}$
5. (a) $\{2, 5, 7, 11, 19, 41\}$
   (b) $\{1, 5, 7, 11, 19, 35, 39, 41\}$
   (c) $\{2, 4, 8, 10, 22, 54\}$
   (d) $\{2, 4, 8, 10, 22, 54\}$
   (e) $\{39, 54\}$
   (f) $\{1, 39\}$
6. (a) $\{4, 16, 25\}$
   (b) $\{8, 27, 64\}$
7. $S = 16$ subsets.
8. $S = 32$ subsets.
9. $S = 512$ subsets.

10. T **11.** T **12.** T **13.** T
14. F **15.** F **16.** F **17.** F
18. F **19.** F **20.** F

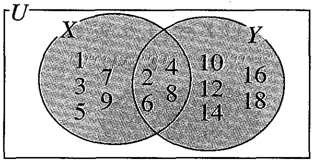
Exercise 1f

1. $X = Y$ **2.** $A \neq B$
3. $E = F$ **4.** $C \neq D$
5. $P \neq Q$

6. $\{2, 4, 6, 8\} = \{6, 8, 2, 4\}$ and $\{2, 4, 6, 8\} \rightleftharpoons \{6, 8, 2, 4\}$
7. $\{2, 4, 6, 8\} \rightleftharpoons \{p, q, r, s\}$
8. $\{2, 4, 6, 8\} \neq \{2, 4\}$
9. $\{2, 4, 6, 8\} \neq \{6, 2, 4\}$
10. $\{a, e, i, o, u\} = \{e, i, o, u, a\}$ and $\{a, e, i, o, u\} \rightleftharpoons \{e, i, o, u, a\}$
11. $\{a, e, i, o, u\} \rightleftharpoons \{2, 3, 5, 7, 11\}$
12. $\{a, e, i\} \neq \{2, 3, 5, 7, 11\}$
13. $\{a, e, i, o, u\} \neq \{2, 3, 5, 7\}$

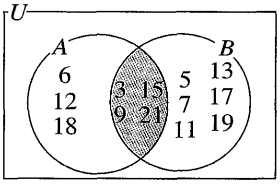
Exercise 1g

1.



(a) $X \cup Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 16, 18\}$
(b) $X \cap Y = \{2, 4, 6, 8\}$

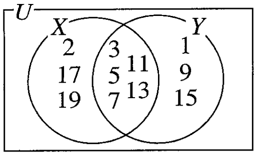
2.



$A \cap B = \{3, 9, 15, 21\}$

3. (a) $A \cup B$ (b) $(A \cup B)'$
(c) $A \cap B$ (d) $(A \cap B)'$
(e) A (f) A'
(g) $A \subset B = A$
(h) $A \cup B$ or $A \cap B = \{\}$

4. (a)



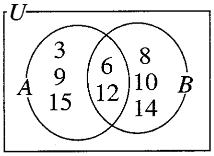
(b) $X \cap Y = \{3, 5, 7, 11, 13\}$

(c) $X \cup Y = \{1, 2, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$

(d) $X \cap Y' = \{2, 17, 19\}$

(e) $Y \cap X' = \{1, 9, 15\}$

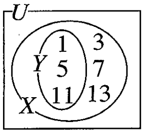
5. (a)



$A \cup B = \{3, 6, 8, 9, 10, 12, 14, 15\}$

$A \cap B = \{6, 12\}$

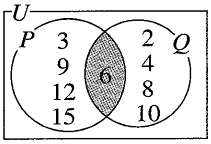
(b)



$X \cup Y = \{1, 3, 5, 7, 11, 13\}$

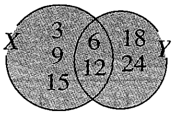
$X \cap Y = \{1, 5, 11\}$

6.

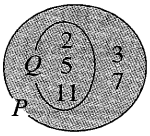


$P \cap Q = \{6\}$

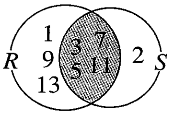
7. (a) $X \cup Y = \{3, 6, 9, 12, 15, 18, 24\}$



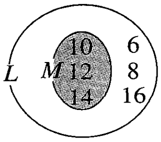
(b) $P \cup Q = \{2, 3, 5, 7, 11\}$



8. (a) $R \cap S = \{3, 5, 7, 11\}$



(b) $L \cap M = \{10, 12, 14\}$



9. $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 15, 16\}$

10. $A = \{2, 3, 5, 8\}$

11. $B = \{4, 7, 9\}$

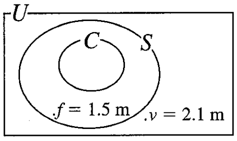
12. $A \cap B = \{\ \}$

13. $A \cup B = \{2, 3, 4, 5, 7, 8, 9\}$

14. $(A \cup B)' = \{1, 6, 11, 13, 15, 16\}$

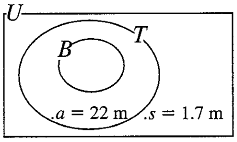
15. $A \cap B' = \{2, 3, 5, 8\}$

16.



$U = \{\text{students}\}$

17.



$U = \{\text{students}\}$

18. {rhombuses} $\cap$ {rectangles} $\neq \{\ \}$

19. {parallelograms} $\cap$ {squares} $\neq \{\ \}$

20. {squares} $\cap$ {rectangles} $\neq \{\ \}$

21. {rhombuses} $\cap$ {parallelograms} $\neq \{\ \}$

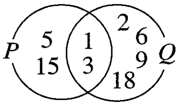
22. {kites} $\cap$ {trapeziums} $= \{\ \}$

23. {trapeziums} $\cap$ {parallelograms} $= \{\ \}$

24. $P = \{1, 3, 5, 15\}$

$Q = \{1, 2, 3, 6, 9, 18\}$

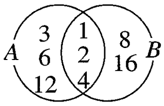
$P \cap Q = \{1, 3\}$



25. $A = \{1, 2, 3, 4, 6, 12\}$

$B = \{1, 2, 4, 8, 16\}$

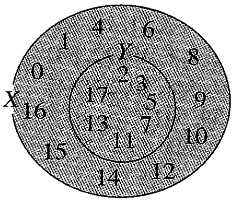
$A \cup B = \{1, 2, 3, 4, 6, 8, 12, 16\}$



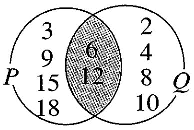
26. $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17\}$

$Y = \{2, 3, 5, 7, 11, 13, 17\}$

$X \cup Y = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17\}$



27.



$P = \{3, 6, 9, 12, 15, 18\}$

$Q = \{2, 4, 6, 8, 10, 12\}$

$P \cap Q = \{6, 12\}$

Exercise 1h

1. (a) $n(A \cap B') = 12$
(b) $n(B \cap A') = 13$

2. (a) $n(P \cap Q') = 21$
(b) $n(Q \cap P') = 15$

3. (a) $n(D \cap C') = 19$ students
(b) $n(C \cap D') = 6$ students

4. (a) $n(V \cap G') = 5$ students
(b) $n(G \cap V') = 13$ students

5. (a) $n(B \cap T') = 19$ students
(b) $n(T \cap B') = 15$ students

6. (a) $x = 8$
(b) $n(A \cap B') = 12$
(c) $n(B \cap A') = 10$

7. (a) $x = 12$
(b) $n(P \cap Q') = 15$
(c) $n(Q \cap P') = 19$

8. (a) $n(F \cap S) = 8$ students
(b) $n(F \cap S') = 23$ students
(c) $n(S \cap F') = 15$ students

9. (a) $n(K \cap J) = 5$ experts
(b) $n(K \cap J') = 21$ experts
(c) $n(J \cap K') = 18$ experts

10. (a) $n(R \cap W) = 18$ athletes
(b) $n(R \cap W') = 13$ athletes
(c) $n(W \cap R') = 47$ athletes

Exercise 2a

1. (a) 32 (b) 512 (c) 1 024

2. (a) 81 (b) 10
(c) 100 000

3. (a) 216 (b) 343 (c) 6 561

4. (a) 72 (b) 707

5. (a) 625 (b) 1 152
(c) 469 (d) 504

6. (a) 200 (b) 882
(c) 496 (d) 98

7. 288

8. (a) 144
(b) 3 108

9. 1 764 **10.** 648

11. (a) 3 888 (b) 90
(c) 4 096 (d) 1

12. (a) 8^{10} (b) 1 (c) $\frac{1}{9}$ (d) 5^9
13. $2^3 \times 3^2 \times 5^3$ **14.** $2^3 \times 3^2 \times 5^2$
15. (a) 18^4 (b) $2^3 \times 3^3 \times 5^4$
16. (a) $2^3 \times 3^2 \times 5^2$
(b) $3 \times 6^3 \times 7^2 \times 9^4$
17. (a) 3^3 (b) $2^6 = 4^3 = 8^2$
18. (a) 2^5 (b) $2^4 \times 3^3 \times 5^3$
19. (a) $2^4 = 4^2$ (b) $3^4 = 9^2$
20. (a) 3^{11} (b) 10^9
21. (a) 10^4 (b) $9^1 = 9$
22. (a) 18^5 (b) $2^2 \times 3^3 \times 5^4$
23. (a) 4^2 (b) 7^4
24. 4^5
25. (a) 1 (b) 6
(c) 1 (d) $\frac{1}{8}$
26. (a) $2^6 = 64$ (b) $3^6 = 729$
27. (a) $5^8 = 390625$
(b) $8^6 = 262144$
28. (a) $10^{15} = 1000000000000000$
(b) $7^6 = 117649$
29. 1 **30.** $5 \times 3^2 \times 2 = 90$

Exercise 2b

1. (a) 9 (b) 25 (c) 231
2. (a) 1 (b) 12 (c) 11
3. (a) 8 (b) 24 (c) 170
4. (a) 23 (b) 37 (c) 37
5. (a) 82 (b) 94 (c) 146
6. (a) 11 (b) 54 (c) 61
7. (a) 15 (b) 12 (c) 15
8. (a) 7 (b) 22 (c) 32
9. (a) 18 (b) 21 (c) 33
10. (a) −3 (b) 10 (c) 30
11. (a) 5 (b) 14 (c) 35
12. (a) 9 (b) 17 (c) 22

Exercise 2c

1. $88 = 1 \times 88 = 2 \times 44 = 4 \times 22 = 8 \times 11$
2. $18 = 1 \times 18 = 2 \times 9 = 3 \times 6$
3. (i) $36 = 1 \times 36 = 2 \times 18 = 3 \times 12 = 4 \times 9 = 6 \times 6$
(ii) $100 = 1 \times 100 = 2 \times 50 = 4 \times 25 = 5 \times 20 = 10 \times 10$
4. {1, 2, 4, 8, 11, 22, 44, 88}
5. {1, 3, 5, 15}
6. $28 = 1 \times 28 = 2 \times 14 = 4 \times 7$
7. {1, 2, 3, 6, 7, 14, 21, 42}
8. {1, 5, 11, 55}
9. $16 = 4 \times 4$ **10.** $49 = 7 \times 7$
11. $81 = 9 \times 9$ **12.** $144 = 12 \times 12$
13. $15 = 3 \times 5$ **14.** $28 = 2 \times 14$
15. $34 = 2 \times 17$
16. $40 = 2 \times 20 = 4 \times 10 = 5 \times 8$
17. {4, 16, 64} **18.** {8, 64}

Exercise 2d

1. {2, 3, 5, 7, 11}
2. {2, 3, 5, 7, 11, 13, 17}
3. {2, 3, 5, 7, 11, 13, 17, 19, 23}
4. {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31}
5. {31, 37, 41, 43, 47, 53, 59}
6. {43, 47, 53, 59, 61, 67}
7. {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97}
8. $2^3 \times 5 \times 19$ **9.** $2^4 \times 3^2 \times 5$
10. $2 \times 3^2 \times 19$ **11.** $2 \times 3 \times 5^3$
12. $2^3 \times 3^2 \times 5$ **13.** $2^2 \times 3^3 \times 5$
14. $2^3 \times 3^2 \times 7$ **15.** $2^8 \times 3$
16. $3^2 \times 5 \times 7$ **17.** $3^2 \times 5^2 \times 7$
18. $3^3 \times 5^2 \times 7$
19. {15, 20, 25, 30, 35, 40, 45}
20. {13, 26, 39, 52, 65, 78, 91}
21. {35, 42, 49, 56, 63}
22. {8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88}
23. {8, 12, 16, 20, 24, 28, 32, 36}
24. {42, 48, 54, 60, 66}
25. {9, 18, 27, 36, 45, 54}
26. {10, 20, 30, 40, 50, 60, 70}
27. {2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24}
28. {6, 9, 12, 15, 18, 21, 24}

Exercise 2e

1. {2, 4, 6, 8, 10, 12, 14, 16}
2. {12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34}
3. {30, 32, 34, 36, 38, 40, 42, 44}
4. {2, 4, 6, 8, 10, 12, 14, 16, 18, 20}
5. {1, 3, 5, 7, 9, 11, 13, 15, 17}
6. {15, 17, 19, 21, 23, 25, 27, 29, 31, 33}
7. {23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43}
8. {1, 3, 5, 7, 9, 11, 13, 15, 17}
9. {20, 22, 24, 26, 28, 30, 32, 34}
10. {22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44}
11. {33, 35, 37, 39, 41, 43, 45, 47}
12. {53, 55, 57, 59, 61, 63, 65, 67, 69, 71, 73}

Exercise 2f

1. H.C.F. = 12 **2.** H.C.F. = 6
3. H.C.F. = 5 **4.** H.C.F. = 12
5. H.C.F. = 5
6. H.C.F. = 50 $l = 50$ cm
7. H.C.F. = 25 $l = 25$ cm
8. H.C.F. = 30 $l = 30$ cm
9. H.C.F. = 13
10. L.C.M. $= 2^5 \times 3 \times 5 = 480$
11. L.C.M. $= 2 \times 3^2 = 18$
12. L.C.M. $= 2^2 \times 3^2 \times 5^2 \times 7$
$= 6300$
13. L.C.M. $= 2^4 \times 3 \times 5 = 240$
14. L.C.M. = 20
15. L.C.M. = 25. 25 ¢
16. L.C.M. = 50. 50 ¢
17. L.C.M. $= 2^3 \times 3^2 \times 5 = 360$.
360 pupils. 12 classes
18. L.C.M. = 30. 30 sweets

Exercise 2g

1. 375, 1875 **2.** 5, 4
3. 7, 9 **4.** 10, 12
5. 3, 1 **6.** 9, 11
7. 192, 768 **8.** 6, 2
9. 7, 9 **10.** 64, 125
11. 49, 64 **12.** 81, 121
13. 100, 144 **14.** 15, 14
15. 6, 9 **16.** $-\frac{1}{4}, -\frac{1}{8}$

Exercise 2h

1. (a) $1 \times 2^2 + 1 \times 2^0$
(b) $1 \times 2^3 + 1 \times 2^1 + 1 \times 2^0$
(c) $1 \times 2^4 + 1 \times 2^3 + 1 \times 2^1 + 1 \times 2^0$
(d) $1 \times 2^5 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^0$
2. (a) $1 \times 2^{-1} + 1 \times 2^{-2}$
(b) $1 \times 2^{-1} + 1 \times 2^{-3}$
(c) $1 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3}$
(d) $1 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-4}$
3. (a) $1 \times 2^1 + 1 \times 2^0 + 1 \times 2^{-1}$
(b) $1 \times 2^2 + 1 \times 2^0 + 1 \times 2^{-2}$
(c) $1 \times 2^3 + 1 \times 2^2 + 1 \times 2^0 + 1 \times 2^{-1} + 1 \times 2^{-2}$
(d) $1 \times 2^4 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 + 1 \times 2^{-1} + 1 \times 2^{-3}$

- (b) $1 \times 3^2 + 2 \times 3^1 + 1 \times 3^0$
- (c) $2 \times 3^3 + 1 \times 3^1 + 2 \times 3^0$
- (d) $2 \times 3^4 + 1 \times 3^3 + 1 \times 3^1 + 2 \times 3^0$
- **5.** (a) $2 \times 3^{-1} + 1 \times 3^{-2}$
  - (b) $2 \times 3^{-1} + 1 \times 3^{-2} + 2 \times 3^{-3}$
  - (c) $1 \times 3^{-1} + 2 \times 3^{-2} + 1 \times 3^{-4}$
  - (d) $1 \times 3^{-1} + 2 \times 3^{-2} + 1 \times 3^{-3} + 2 \times 3^{-5}$
- **6.** (a) $2 \times 3^1 + 1 \times 3^0 + 1 \times 3^{-2}$
  - (b) $1 \times 3^2 + 2 \times 3^1 + 1 \times 3^0 + 1 \times 3^{-1} + 1 \times 3^{-2}$
  - (c) $2 \times 3^3 + 1 \times 3^2 + 2 \times 3^1 + 1 \times 3^0 + 1 \times 3^{-2}$
  - (d) $2 \times 3^4 + 1 \times 3^3 + 2 \times 3^2 + 1 \times 3^0 + 1 \times 3^{-1} + 2 \times 3^{-3}$
- **7.** (a) $3 \times 4^1 + 1 \times 4^0$
  - (b) $2 \times 4^2 + 1 \times 4^1 + 3 \times 4^0$
  - (c) $1 \times 4^3 \times 3 \times 4^1 + 2 \times 4^0$
  - (d) $3 \times 4^4 + 1 \times 4^3 + 2 \times 4^2 + 3 \times 4^0$
- **8.** (a) $3 \times 4^{-1} + 1 \times 4^{-2}$
  - (b) $1 \times 4^{-1} + 3 \times 4^{-2} + 2 \times 4^{-3}$
  - (c) $3 \times 4^{-1} + 1 \times 4^{-2} + 2 \times 4^{-3} + 3 \times 4^{-4}$
  - (d) $2 \times 4^{-2} + 1 \times 4^{-3} + 3 \times 4^{-4} + 1 \times 4^{-5}$
- **9.** (a) $2 \times 4^1 + 1 \times 4^0 + 3 \times 4^{-1}$
  - (b) $1 \times 4^2 + 3 \times 4^1 + 2 \times 4^0 + 1 \times 4^{-1} + 2 \times 4^{-2}$
  - (c) $2 \times 4^3 + 3 \times 4^1 + 1 \times 4^0 + 3 \times 4^{-1} + 1 \times 4^{-2} + 2 \times 4^{-3}$
  - (d) $3 \times 4^4 + 1 \times 4^3 + 2 \times 4^1 + 1 \times 4^0 + 2 \times 4^{-1} + 1 \times 4^{-2} + 3 \times 4^{-3}$
- **10.** (a) $4 \times 5^1 + 1 \times 5^0$
  - (b) $3 \times 5^2 + 1 \times 5^1 + 4 \times 5^0$
  - (c) $2 \times 5^3 + 3 \times 5^1 + 4 \times 5^0$
  - (d) $1 \times 5^4 + 3 \times 5^3 + 4 \times 5^2 + 2 \times 5^1 + 1 \times 5^0$
- **11.** (a) $4 \times 5^{-1} + 3 \times 5^{-2}$
  - (b) $4 \times 5^{-1} + 1 \times 5^{-2} + 2 \times 5^{-3}$
  - (c) $3 \times 5^{-1} + 4 \times 5^{-3} + 1 \times 5^{-4}$
  - (d) $4 \times 5^{-1} + 1 \times 5^{-2} + 3 \times 5^{-3} + 2 \times 5^{-5}$

$\times 5^{-2}$

- (b) $1 \times 5^2 + 4 \times 5^0 + 3 \times 5^{-1} + 2 \times 5^{-2}$
- (c) $2 \times 5^3 + 4 \times 5^2 + 1 \times 5^1 + 3 \times 5^0 + 3 \times 5^{-2}$
- (d) $1 \times 5^4 + 3 \times 5^3 + 4 \times 5^2 + 2 \times 5^0 + 1 \times 5^{-1} + 4 \times 5^{-3}$
- **13.** (a) $5 \times 6^1 + 4 \times 6^0$
  - (b) $4 \times 6^2 + 5 \times 6^1 + 1 \times 6^0$
  - (c) $3 \times 6^3 + 5 \times 6^2 + 4 \times 6^0$
  - (d) $2 \times 6^4 + 5 \times 6^2 + 1 \times 6^1 + 3 \times 6^0$
- **14.** (a) $5 \times 6^{-1} + 1 \times 6^{-2}$
  - (b) $4 \times 6^{-1} + 1 \times 6^{-2} + 5 \times 6^{-3}$
  - (c) $1 \times 6^{-2} + 4 \times 6^{-3} + 3 \times 6^{-4}$
  - (d) $3 \times 6^{-1} + 4 \times 6^{-2} + 1 \times 6^{-3} + 5 \times 6^{-5}$
- **15.** (a) $5 \times 6^1 + 3 \times 6^0 + 2 \times 6^{-1}$
  - (b) $4 \times 6^2 + 5 \times 6^1 + 1 \times 6^0 + 3 \times 6^{-1} + 2 \times 6^{-2}$
  - (c) $3 \times 6^3 + 4 \times 6^2 + 5 \times 6^1 + 1 \times 6^{-2} + 4 \times 6^{-3}$
  - (d) $4 \times 6^4 + 5 \times 6^2 + 1 \times 6^1 + 3 \times 6^0 + 2 \times 6^{-1} + 5 \times 6^{-3}$
- **16.** (a) $6 \times 7^1 + 5 \times 7^0$
  - (b) $5 \times 7^2 + 6 \times 7^0$
  - (c) $4 \times 7^3 + 6 \times 7^2 + 1 \times 7^1 + 3 \times 7^0$
  - (d) $6 \times 7^4 + 3 \times 7^3 + 4 \times 7^1 + 5 \times 7^0$
- **17.** (a) $6 \times 7^{-1} + 5 \times 7^{-2}$
  - (b) $1 \times 7^{-1} + 4 \times 7^{-2} + 5 \times 7^{-3}$
  - (c) $4 \times 7^{-1} + 6 \times 7^{-2} + 5 \times 7^{-4}$
  - (d) $5 \times 7^{-1} + 1 \times 7^{-2} + 6 \times 7^{-3} + 4 \times 7^{-5}$
- **18.** (a) $6 \times 7^0 + 1 \times 7^{-1} + 4 \times 7^{-2}$
  - (b) $5 \times 7^1 + 6 \times 7^{-1} + 3 \times 7^{-3}$
  - (c) $4 \times 7^2 + 6 \times 7^1 + 2 \times 7^0 + 3 \times 7^{-1} + 1 \times 7^{-3} + 5 \times 7^{-4}$
  - (d) $6 \times 7^3 + 5 \times 7^2 + 4 \times 7^0 + 1 \times 7^{-2} + 3 \times 7^{-3}$
- **19.** (a) $7 \times 8^1 + 4 \times 8^0$
  - (b) $6 \times 8^2 + 7 \times 8^0$

- (d) $5 \times 8^4 + 7 \times 8^3 + 6 \times 8^2 + 3 \times 8^1 + 2 \times 8^0$
- **20.** (a) $7 \times 8^{-1} + 1 \times 8^{-2}$
  - (b) $6 \times 8^{-1} + 7 \times 8^{-2} + 3 \times 8^{-3}$
  - (c) $5 \times 8^{-1} + 7 \times 8^{-3} + 2 \times 8^{-4}$
  - (d) $7 \times 8^{-2} + 1 \times 8^{-3} + 5 \times 8^{-4} + 4 \times 8^{-5}$
- **21.** (a) $7 \times 8^0 + 6 \times 8^{-1} + 1 \times 8^{-2}$
  - (b) $6 \times 8^1 + 5 \times 8^0 + 7 \times 8^{-1} + 1 \times 8^{-3}$
  - (c) $5 \times 8^2 + 7 \times 8^1 + 6 \times 8^{-1} + 2 \times 8^{-2}$
  - (d) $4 \times 8^3 + 6 \times 8^2 + 7 \times 8^1 + 3 \times 8^0 + 7 \times 8^{-1} + 1 \times 8^{-2} + 4 \times 8^{-4}$
- **22.** (a) $8 \times 9^1 + 4 \times 9^0$
  - (b) $7 \times 9^2 + 6 \times 9^1 + 8 \times 9^0$
  - (c) $5 \times 9^3 + 4 \times 9^2 + 3 \times 9^1 + 8 \times 9^0$
  - (d) $7 \times 9^3 + 6 \times 9^2 + 8 \times 9^1$
- **23.** (a) $4 \times 9^{-1} + 8 \times 9^{-2}$
  - (b) $7 \times 9^{-1} + 6 \times 9^{-2} + 3 \times 9^{-3}$
  - (c) $8 \times 9^{-1} + 4 \times 9^{-2} + 1 \times 9^{-4}$
  - (d) $7 \times 9^{-1} + 4 \times 9^{-3} + 5 \times 9^{-4} + 8 \times 9^{-5}$
- **24.** (a) $7 \times 9^1 + 6 \times 9^0 + 8 \times 9^{-1}$
  - (b) $5 \times 9^3 + 4 \times 9^1 + 8 \times 9^0 + 5 \times 9^{-2}$
  - (c) $8 \times 9^3 + 1 \times 9^2 + 6 \times 9^1 + 1 \times 9^{-1} + 3 \times 9^{-2} + 4 \times 9^{-3}$
  - (d) $7 \times 9^4 + 6 \times 9^3 + 8 \times 9^2 + 4 \times 9^{-1} + 5 \times 9^{-2} + 1 \times 9^{-3} + 3 \times 9^{-4}$
- **25.** (a) $9 \times 10^1 + 8 \times 10^0$
  - (b) $9 \times 10^2 + 8 \times 10^1 + 7 \times 10^0$
  - (c) $8 \times 10^3 + 6 \times 10^2 + 9 \times 10^1 + 5 \times 10^0$
  - (d) $7 \times 10^4 + 8 \times 10^3 + 9 \times 10^2 + 3 \times 10^0$
- **26.** (a) $9 \times 10^{-1} + 6 \times 10^{-2}$
  - (b) $8 \times 10^{-1} + 9 \times 10^{-2} + 5 \times 10^{-3}$
  - (c) $7 \times 10^{-1} + 6 \times 10^{-2} + 8 \times 10^{-3}$
  - (d) $9 \times 10^{-1} + 7 \times 10^{-3} + 6 \times 10^{-4} + 3 \times 10^{-5}$

27. (a) $9 \times 10^0 + 5 \times 10^{-2}$
(b) $9 \times 10^1 + 5 \times 10^0 + 1 \times 10^{-1} + 3 \times 10^{-2}$
(c) $8 \times 10^2 + 9 \times 10^1 + 4 \times 10^0 + 4 \times 10^{-2} + 3 \times 10^{-3}$
(d) $7 \times 10^3 + 6 \times 10^2 + 4 \times 10^1 + 9 \times 10^{-1} + 8 \times 10^{-2} + 1 \times 10^{-3} + 3 \times 10^{-4}$

Exercise 2i

1. (a) 101_2 (b) $1\,000_2$ (c) $1\,010_2$ (d) $10\,011_2$
2. (a) $1\,000\,011_2$ (b) $1\,001\,110_2$ (c) $10\,111\,001_2$ (d) $101\,010\,101_2$
3. (a) $110\,110\,011_2$ (b) $111\,100\,111_2$ (c) $111\,111\,011_2$ (d) $111\,111\,110_2$
4. (a) 5_{10} (b) 14_{10} (c) 23_{10} (d) 59_{10}
5. (a) 0.875_{10} (b) 0.875_{10} (c) $0.906\,25_{10}$ (d) $0.968\,75_{10}$
6. (a) 3.25_{10} (b) 5.75_{10} (c) 15.25_{10} (d) 7.75_{10}
7. (a) $10\,000_2$ (b) $10\,101_2$ (c) $10\,000_2$ (d) $11\,001_2$
8. (a) $100\,100_2$ (b) $100\,100_2$ (c) $1\,001\,000_2$ (d) $101\,110_2$
9. (a) $101\,010_2$ (b) $101\,010_2$ (c) $101\,010_2$ (d) $101\,000_2$
10. (a) $1\,101\,100_2$ (b) $1\,101\,100_2$ (c) $10\,001\,010_2$ (d) $10\,110\,010_2$
11. (a) 10_2 (b) 10_2 (c) 11_2 (d) 11_2
12. (a) 10_2 (b) 10_2 (c) $1\,100_2$ (d) $1\,010_2$
13. (a) 10_2 (b) $100\,000_2$ (c) $1\,010_2$ (d) $100\,100_2$
14. (a) $1\,110_2$ (b) $110\,111_2$ (c) $101\,101_2$ (d) $1\,101\,001_2$
15. (a) $1\,111_2$ (b) $100\,011_2$ (c) $1\,001\,101_2$ (d) $1\,011\,011_2$
16. (a) $11\,011\,001_2$ (b) $1\,011\,111_2$ (c) $1\,010\,111_2$ (d) $1\,110\,011_2$

Exercise 2j

1. (a) 140_5 (b) 232_5 (c) 324_5 (d) 403_5
2. (a) $1\,442_5$ (b) $2\,033_5$ (c) $2\,344_5$ (d) $11\,342_5$
3. (a) 19_{10} (b) 21_{10} (c) 44_{10} (d) 116_{10}
4. (a) 0.384_{10} (b) 0.776_{10} (c) 0.856_{10} (d) $0.476\,8_{10}$
5. (a) 23.44_{10} (b) 19.28_{10} (c) 39.216_{10} (d) 89.568_{10}
6. (a) 132_5 (b) $1\,030_5$ (c) $1\,140_5$ (d) 432_5
7. (a) $3\,420_5$ (b) $3\,221_5$ (c) $13\,233_5$ (d) $13\,030_5$
8. (a) 224_5 (b) 134_5 (c) 22_5 (d) 20_5
9. (a) 102_5 (b) 444_5 (c) 113_5 (d) 200_5
10. (a) $1\,410_5$ (b) $4\,444_5$ (c) $4\,441_5$ (d) $10\,111_5$
11. (a) $3\,133_5$ (b) $22\,411_5$ (c) $100\,414_5$ (d) $43\,441_5$

Exercise 2k

1. (a) 124_8 (b) 135_8 (c) 150_8 (d) 211_8
2. (a) 367_8 (b) 600_8 (c) $1\,511_8$ (d) $1\,710_8$
3. (a) 39_{10} (b) 93_{10} (c) 286_{10} (d) 423_{10}
4. (a) $0.937\,5_{10}$ (b) $0.406\,25_{10}$ (c) $0.687\,5_{10}$ (d) $0.968\,75_{10}$
5. (a) $28.390\,625_{10}$ (b) $39.781\,25_{10}$ (c) $69.531\,25_{10}$ (d) $159.968\,75_{10}$
6. (a) 105_8 (b) 116_8 (c) 425_8 (d) 555_8
7. (a) $1\,553_8$ (b) $4\,161_8$ (c) $4\,127_8$ (d) $12\,432_8$
8. (a) 12_8 (b) 22_8 (c) 6_8 (d) 135_8
9. (a) 576_8 (b) $3\,374_8$ (c) 520_8 (d) $1\,037_8$
10. (a) $1\,640_8$ (b) $2\,624_8$ (c) $2\,614_8$ (d) $20\,337_8$
11. (a) $703\,675_8$ (b) $1\,266\,422_8$ (c) $2\,436\,060_8$ (d) $2\,627\,056_8$

Exercise 2l

1. (a) 101_3 (b) $1\,021_3$ (c) $11\,022_3$
2. (a) 10_3 (b) 100_3 (c) 2_3
3. (a) $1\,120_3$ (b) $12\,222_3$ (c) $1\,222\,111_3$
4. (a) 103_4 (b) $1\,020_4$ (c) $10\,111_4$
5. (a) 10_4 (b) 33_4 (c) $1\,021_4$
6. (a) $1\,300_4$ (b) $21\,102_4$ (c) $1\,111\,110_4$
7. (a) 40_5 (b) 300_5 (c) $4\,433_5$
8. (a) 10_5 (b) 31_5 (c) 41_5
9. (a) $2\,010_5$ (b) $21\,322_5$ (c) $242\,242_5$
10. (a) 105_6 (b) 513_6 (c) $11\,330_6$
11. (a) 32_6 (b) 233_6 (c) $1\,205_6$
12. (a) $2\,430_6$ (b) $22\,010_6$ (c) $444\,212_6$
13. (a) 125_7 (b) 623_7 (c) $4\,621_7$
14. (a) 32_7 (b) 115_7 (c) 251_7
15. (a) $3\,230_7$ (b) $40\,201_7$ (c) $110\,311_7$
16. (a) 140_8 (b) $1\,225_8$ (c) $6\,020_8$
17. (a) 32_8 (b) 141_8 (c) $1\,312_8$
18. (a) $4\,470_8$ (b) $20\,316_8$ (c) $1\,266\,530_8$
19. (a) 167_9 (b) $1\,241_9$ (c) $12\,760_9$
20. (a) 11_9 (b) 253_9 (c) $3\,382_9$
21. (a) $2\,730_9$ (b) $25\,148_9$ (c) $2\,631\,370_9$

Exercise 3a

1. 56 551 **2.** 25
3. 7 **4.** 12
5. 33 810 **6.** 71
7. 17 **8.** 24
9. (a) 20 (b) 0
10. (a) 13 (b) ∞. Meaningless operation
11. (a) 60 (b) 11
12. (a) 26 (b) 6
13. (a) 5 (b) 10
14. $250
15. 4

Exercise 3b

1. 430 ¢ **2.** 38 cm
3. 569 patties **4.** 696 h
5. $24 **6.** 200 km
7. 108 steps **8.** 80 boxes
9. $8 **10.** 800 ¢
11. 3 985 ¢ **12.** 1 368 cars
13. 480 min **14.** 17 cents
15. $175 **16.** 97 marks
17. 30 marbles **18.** 131 123
19. 186 students **20.** 1 080 h
21. 16 mangoes **22.** 16 maxi taxis
23. 415 ¢ **24.** 240 steps
25. 2 000 ¢ **26.** 683 ¢
27. $7 958 **28.** 276 girls
29. 2 014 m **30.** 75 kg
31. 20 ¢ **32.** 46 computers
33. 49 min **34.** 280 ¢
35. 106 cm **36.** 63 min
37. 245 ¢ **38.** 19 video games
39. 120 ¢ **40.** 2 926
41. 138 cm **42.** 120 km
43. 100 steps **44.** $105
45. 507 ¢ **46.** 20 packets
47. 850 ¢ **48.** 67 hamburgers
49. 31 marbles **50.** 178 ¢
51. 2 415 ¢ **52.** 420 km
53. 92 steps **54.** 120 boxes
55. $104 **56.** 910 ¢
57. 310 ¢

Exercise 3c

1. (a) $\frac{11}{20}$ (b) $\frac{3}{5}$
2. (a) $\frac{15}{22}$ (b) $\frac{6}{17}$
3. (a) $\frac{7}{15}$ (b) $\frac{41}{35}$
4. (a) $\frac{1}{12}$ (b) $\frac{5}{33}$
5. (a) 60 m (b) 25 m
6. (a) 219 days (b) 10 h
7. (a) $\frac{6}{5}$ (b) 15 days
8. (a) $\frac{3}{4}$ (b) $\frac{1}{4}$
9. (a) $\frac{5}{8}$ (b) $\frac{1}{8}$
10. (a) $\frac{1}{3}$ (b) $\frac{5}{12}$
11. (a) 60 *l* (b) 9 h
12. (a) 55 (b) $\frac{3}{2}$
13. (a) $\frac{5}{4}$ (b) $\frac{23}{18}$
14. (a) $\frac{11}{24}$ (b) $\frac{1}{10}$
15. (a) $\frac{3}{2}$ (b) $\frac{13}{30}$
16. (a) $\frac{1}{11}$ (b) $\frac{2}{9}$
17. (a) $\frac{7}{12}$ (b) $\frac{3}{20}$
18. (a) $\frac{17}{12}$ (b) $\frac{24}{25}$
19. (a) $\frac{17}{30}$ (b) $\frac{25}{18}$
20. (a) $\frac{13}{12}$ (b) $\frac{1}{15}$
21. $\frac{10}{3}$

Exercise 3d

1. 14 **2.** 2
3. 18 times **4.** 36
5. $\frac{17}{30}$. $\frac{13}{30}$
6. (a) $\frac{1}{73}$ (b) $\frac{7}{20}$
7. (a) $\frac{2}{7}$ (b) $\frac{9}{35}$
8. $\frac{5}{12}$ **9.** $\frac{3}{7}$
10. (a) $\frac{11}{15}$ (b) $\frac{4}{15}$
11. (a) $\frac{7}{8}$ (b) $\frac{1}{8}$
(c) $\frac{5}{8}$ (d) $\frac{3}{8}$
12. $\frac{17}{5}$ kg. $\frac{10}{17}$
13. 16 **14.** 9
15. 70 ¢ **16.** 45 **17.** 15
18. $\frac{7}{12}$ **19.** $\frac{53}{365}$
20. (a) $\frac{5}{21}$ (b) $\frac{4}{21}$
(c) $\frac{11}{21}$
21. (a) $\frac{5}{16}$ (b) $\frac{3}{16}$ (c) $\frac{15}{16}$
22. 360 min = 6 h **23.** $\frac{7}{45}$
24. 20 **25.** $\frac{27}{45}$ **26.** 24 lengths
27. 15 thirds **28.** $\frac{18}{5}$
29. (a) $\frac{7}{24}$ (b) $\frac{3}{4}$

Exercise 3e

1. (a) $3\frac{5}{12}$ (b) $1\frac{1}{6}$
2. (a) $4\frac{44}{45}$ (b) $5\frac{3}{28}$
3. (a) $3\frac{3}{16}$ (b) $5\frac{1}{4}$
4. (a) $3\frac{5}{9}$ (b) $2\frac{5}{8}$
5. (a) $18\frac{13}{18}$ (b) 7
6. (a) $8\frac{5}{8}$ (b) $9\frac{17}{36}$
7. (a) $\frac{1}{3}$ (b) $\frac{2}{11}$
8. (a) $\frac{91}{136}$ (b) $1\frac{1}{3}$
9. (a) $14\frac{11}{15}$ (b) 9
10. (a) $18\frac{71}{72}$ (b) 6
11. (a) $26\frac{5}{6}$ (b) $2\frac{1}{4}$
12. (a) $1\frac{9}{17}$ (b) $\frac{80}{213}$
13. (a) $3\frac{13}{30}$ (b) $7\frac{1}{2}$
14. (a) $7\frac{49}{90}$ (b) $2\frac{22}{39}$
15. (a) $5\frac{11}{21}$ (b) $-\frac{7}{58}$
16. (a) $2\frac{5}{8}$ (b) $3\frac{5}{12}$
17. (a) 7 (b) $\frac{29}{31}$
18. (a) $1\frac{1}{2}$ (b) $1\frac{16}{27}$
19. (a) $\frac{2}{11}$ (b) $1\frac{1}{6}$
20. (a) $14\frac{2}{27}$ (b) $2\frac{5}{8}$
21. (a) $2\frac{1}{2}$ (b) $\frac{338}{405}$
22. (a) $2\frac{4}{25}$ (b) $1\frac{1}{6}$
23. (a) $2\frac{3}{8}$ (b) $2\frac{6}{7}$
24. (a) $1\frac{3}{8}$ (b) $-4\frac{11}{25}$
25. (a) $6\frac{2}{3}$ (b) $3\frac{5}{6}$
26. (a) $1\frac{2}{5}$ (b) $\frac{3}{25}$

27. (a) $\frac{7}{30}, \frac{1}{2}, \frac{2}{3}, \frac{4}{5}$
(b) $\frac{4}{5}, \frac{2}{3}, \frac{1}{2}, \frac{7}{30}$
28. $\frac{1}{2}, \frac{13}{22}, \frac{27}{44}, \frac{7}{11}$
29. $\frac{1}{2}, \frac{7}{12}, \frac{3}{4}, \frac{5}{6}$
30. (a) $\frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{7}{10}$
(b) $\frac{7}{10}, \frac{2}{3}, \frac{3}{5}, \frac{1}{2}$
31. $\frac{3}{7}, \frac{5}{8}, \frac{3}{4}, \frac{7}{9}$ **32.** $14\frac{2}{3}$
33. $2\frac{8}{9}$ **34.** 12
35. $1\frac{1}{3}$ min **36.** $1\frac{1}{2}$
37. 135 kg **38.** $1\frac{1}{2}$ min
39. $\frac{1}{3}$ **40.** $\frac{5}{6}$
41. $22\frac{1}{2}$ min **42.** 70 sheets
43. $140\frac{1}{2}$ g **44.** $187\frac{1}{2}$ g
45. (a) $\frac{3}{10} > \frac{1}{4}$ (b) $\frac{8}{11} < \frac{9}{10}$
46. $\frac{4}{15} > \frac{1}{5}$ **47.** 140 g
48. $\frac{13}{15}. \frac{2}{15}$
49. (a) $\frac{7}{9} > \frac{5}{8}$ (b) $\frac{2}{3} > \frac{5}{6}$

Exercise 3f

1. (a) 5 639.6 (b) 0.000 061 345
2. (a) 37 580 (b) 0.542
3. 0.008 24
4. (a) 0.34 (b) 3.4
(c) 34
5. (a) 1.531 (b) 0.1531
(c) 0.015 31
6. (a) 7 500 (b) 507.1
(c) 47.3 (d) 6 870
7. (a) 39 710 (b) 0.004 36
8. (a) 497.1 (b) 0.010 32
9. (a) 0.062 742 8 (b) 0.000 094 3
10. (a) 8 470
(b) 0.000 004 531
11. 0.12 **12.** $\frac{17}{2\,000}$
13. 0.375 **14.** $\frac{7}{100}$
15. 290 **16.** 1.05
17. $\frac{2}{25}$ **18.** 0.262 87
19. 0.004 5 **20.** 0.072
21. 0.16 **22.** 0.008 2
23. 0.875
24. 0.09 **25.** $\frac{4}{5}$
26. $\frac{19}{20}$ **27.** 0.625
28. $\frac{17}{20}$ **29.** $\frac{7}{8}, \frac{9}{10}$, 0.95
30. 0.6, $\frac{5}{8}, \frac{2}{3}$
31. 5 430 000 000
32. 0.000 000 017 54
33. (a) $\frac{807}{10\,000}$ (b) $9\frac{7}{100}$
(c) $17\frac{3}{4}$ (d) $15\frac{1}{4}$
34. (a) $\frac{3}{8}$ (b) $\frac{18}{25}$
(c) $\frac{63}{2\,000}$ (d) $\frac{1}{6\,250}$

Exercise 3g

1. (a) 39.091 (b) 4.302
2. 0.29 **3.** 7.38
4. 4.8 **5.** 16.1
6. 4.08 **7.** 14.06
8. 57.3 **9.** 8.11 **10.** 8.5
11. (a) 142 (b) 0.017 04
12. 7.65 kg **13.** 12.225
14. (a) 54.852 (b) 5.410 9
15. (a) 14.835 (b) 5
16. (a) 15.745 (b) 6
17. (a) 6.9 (b) 0.410 5
18. 20.536 **19.** 20.74
20. 25.03 **21.** 51.37
22. 32.52 **23.** 64.8
24. 7.1 **25.** 0.154
26. 4.71 **27.** 9.1
28. 12.65 kg **29.** 1.1
30. 5.7 **31.** 0.001 19
32. 0.000 350 5 **33.** $\frac{69}{200}$
34. 0.875 **35.** 22.25
36. 32.45 **37.** 0.14
38. (a) 1 399.2 (b) 38.5
(c) 0.032 376 (d) 236.538
39. $364.50 **40.** 13.2 kg

Exercise 3h

1. $7.03 **2.** $22.33
3. $3.20 **4.** $5.45
5. $1.30 **6.** $P = 47.2$ cm
7. $36.37 **8.** 6.5 cm
9. $3.11 **10.** $P = 43.1$ cm
11. 5.8 cm **12.** $58.75
13. 3.47 cm **14.** $5.16
15. 2 g **16.** $4.37
17. $10.67
18. (a) $5.73 (b) $1.78
19. $P = 34.2$ cm **20.** $P = 24.8$ cm
21. $P = 62.4$ cm **22.** $P = 30.8$ cm
23. $P = 61.7$ cm **24.** $l = 4.1$ cm
25. $l = 9.3$ cm **26.** $l = 17.3$ cm
27. $l = 70.74$ cm

Exercise 3i

1. 45 **2.** 0.056
3. 4.5 **4.** 0.26
5. 1.2 **6.** 1.01
7. 2 **8.** 0.81
9. 30 **10.** 9
11. (a) 15.448 (b) 15.4
12. 0.09 **13.** 15
14. 5.81 **15.** 3
16. 25 **17.** 2.85
18. 10 **19.** 0.2
20. (a) 9.835 (b) 9.8
21. 4 **22.** 2.2
23. 30 **24.** 4.07
25. 14.23 **26.** $2\frac{2}{3}$
27. 14 **28.** 35
29. 24.102 **30.** 1.6
31. 119
32. (a) 0.16 (b) 6
(c) 6.2 (d) $28\frac{1}{3}$

Exercise 3j

1. (a) 10 (b) 2
(c) 6 (d) 6
2. (a) 16 (b) 24
(c) 40 (d) 75
3. (a) 346 (b) 472
(c) 213 (c) 839
4. (a) 175 (b) 54
(c) 1 896 (d) 347
5. (a) 800 (b) 39
(c) 350 (d) 47
6. (a) 60 (b) 50
(c) 60 (d) 50
7. (a) 13 tens (b) 12 tens
(c) 13 tens (d) 12 tens

8. (a) 3540 (b) 4780
(c) 9440 (d) 2900
9. (a) 4500 (b) 8000
(c) 3800 (d) 8400
10. (a) 31 hundreds
(b) 32 hundreds
(c) 31 hundreds
(d) 32 hundreds
11. (a) 71400 (b) 85700
(c) 97500 (d) 21800
12. 157500
13. (a) 260 (b) 380
(c) 420 (d) 450
14. (a) 500 (b) 300
(c) 600 (d) 200
15. (a) 14800 (b) 600
(c) 5200 (d) 3000

Exercise 3k

1. 25.035 **2.** 15.40
3. (a) 5.13 (b) 0.09
(c) 4.00
4. (a) 5.1 (b) 286.60
(c) 0.0039 (d) 0.009
5. (a) 5.1 (b) 289.60
(c) 0.0039 (d) 0.009
6. (a) 10 (b) 9.8
(c) 9.781
7. 34.85 **8.** $0.\dot{7}1428\dot{5}$
9. 2.733 **10.** $0.68\dot{6}$
11. $3.\dot{3}\dot{6}$ **12.** 12.071
13. (a) 743.03 (b) 0.001
14. (a) 1.33 (b) 43.2
15. $0.\dot{6}$
16. (a) $0.\dot{4}\dot{5}$ (b) $0.\dot{9}\dot{0}$
17. (a) 0.45 (b) 0.91
18. (a) 0.5333 (b) 0.8667
19. (a) $0.5\dot{3}$ (b) $0.8\dot{6}$
20. (a) 0.77778 (b) 0.88889
21. (a) 1.86 (b) 1.14
22. (a) 22.388 (b) 0.084
23. (a) $3.\dot{5}$ (b) $4.\dot{6}\dot{3}$
(c) $2.\dot{8}5714\dot{2}$
24. (a) $\frac{1}{2} = 0.50$ $\frac{3}{5} = 0.60$
$0.47, \frac{1}{2}, \frac{3}{5}$
(b) $\frac{3}{4} = 0.75$ $\frac{2}{5} = 0.40$
$\frac{2}{5}, \frac{3}{4}, 0.76$
(c) $0.\dot{6} = 0.67$ $\frac{7}{10} = 0.70$
$\frac{4}{5} = 0.80$ $0.\dot{6}, \frac{7}{10}, \frac{4}{5}$
(d) $0.\dot{3} = 0.33$ $\frac{1}{4} = 0.25$
$\frac{2}{5} = 0.40$ $\frac{1}{4}, 0.\dot{3}, \frac{2}{5}$

Exercise 3l

1. 19.4 **2.** 0.0055
3. (a) 475.8 (b) 59.1
(c) 0.000068 (d) 0.05
4. (a) 478.8 (b) 58.1
(c) 0.000068 (d) 0.05
5. 10000 **6.** 0.051
7. 0.295
8. (a) 47
(b) 5000 (c) 0.07 (d) 37.9
9. (a) 816.095 (b) 816.10
(c) 816.1 (d) 816
(e) 820 (f) 800
10. (a) 0.00783615 (b) 0.0078362
(c) 0.007836 (d) 0.00784
(e) 0.0078 (f) 0.008
11. (a) $2\frac{4}{25}$ (b) 2.16
12. 405.3 **13.** 743.03
14. 0.001376 **15.** 1350
16. 0.9083

Exercise 3m

1. (a) 7.438×10^3
(b) 1.2149×10^4
2. (a) 4.79×10^{-3}
(b) 9.431×10^{-2}
3. (a) 1.578×10^1
(b) 2.2409×10^2
4. (a) 8.4708×10^2
(b) 1.24363×10^4
5. (a) 4.798×10^{-2}
(b) 3.45×10^{-5}
6. (a) 9.352×10^{-3}
(b) (i) 9.35×10^{-3}
(ii) 9.4×10^{-3}
7. (a) 3.85×10^9
(b) 7.308×10^{-9}
8. (a) 4.4×10^{-5}
(b) 9.48×10^{-8}
(c) 4.94×10^6
(d) 9.8×10^3
9. (a) 3.9×10^{-3}
(b) 8.8×10^{-3}
10. (a) 4.3×10^{-5}
(b) 9.53×10^{-8}
(c) 4.93×10
(d) 9.8×10^3
11. 7.805×10^{-3}
12. (a) 7.36×10^5
(b) 4.35×10^1
(c) 4.37×10^{-3}
13. (a) 6.3021×10^2
(b) 6.279×10^1
(c) 8.05×10^{-2}
14. 3.7×10^4 **15.** 5×10^{-3}

Exercise 3n

1. (a) ±0.05 m (b) 15.65 m
(c) 15.55 m
(d) (15.6 ± 0.05) m
2. (a) ±0.005 cm
(b) 128.925 cm
(c) 128.915 cm
(d) (128.92 ± 0.005) cm
3. (a) (125.6 ± 0.05) cm
(b) (18.53 ± 0.005) cm
(c) (9.237 ± 0.0005) cm
4. (a) 2347.8 ± 0.05
(b) 145.94 ± 0.005
(c) 768.149 ± 0.0005
5. (a) ±0.05 cm (b) 60.3 cm
(c) 60.4 cm (d) 60.2 cm
(e) (60.3 ± 0.1) cm
6. (a) ±0.005 cm
(b) 213.55 cm (c) 213.56 cm
(d) 213.54 cm
(e) (213.55 ± 0.01) cm
7. (a) (17.2 ± 0.1) cm
(b) (20.86 ± 0.01) cm
(c) (161.902 ± 0.001) cm
8. (a) 20.6 ± 0.1
(b) 178.64 ± 0.01
(c) 221.981 ± 0.001
9. (a) 0.05 cm (b) 353.06 cm^2
(c) 355.09 cm^2 (d) 351.04 cm^2
(e) (353.06 ± 2.03) cm^2
10. (a) ±0.005 cm
(b) 152.204 cm^2
(c) 152.292 cm^2
(d) 152.116 cm^2
(e) 152.204 ± 0.088 cm^2
11. (a) (66.75 ± 0.82) cm^2
(b) (199.616 ± 0.143) cm^2
(c) (3726.9203 ± 0.0774) cm^2
12. (a) (120.28 ± 1.11) g
(b) (2433.209 ± 0.613) g
(c) (6355.708 ± 0.888) g

13. (5.7 ± 0.1) kg
14. (8.7 ± 0.1) kg
15. (a) (9 ± 1) kg
(b) (5.1 ± 0.1) cm
(c) (16.8 ± 0.1) kg
(d) (12.09 ± 0.01) mm
16. (a) (4.9 ± 0.1) mg
(b) (0.78 ± 0.01) kg
(c) (7.9 ± 0.1) cm
(d) (6.12 ± 0.01) km
17. 2.70 ± 0.06
18. 2.50 ± 0.01
19. (a) 0.30 ± 0.01
(b) 0.50 ± 0.01
(c) 5.25 ± 0.20
(d) 6.13 ± 0.23
20. (a) 0.88 ± 0.01
(b) 3.20 ± 0.01
(c) 5.80 ± 0.23
(d) 6.00 ± 0.03
21. (a) 9.25 cm, 10.55 cm and 11.35 cm
9.15 cm, 10.45 cm and 11.25 cm
(b) 31.15 cm. 30.85 cm
22. (a) 5.5 cm 4.5 cm
(b) (i) 166.375 cm^3
(ii) 91.125 cm^3
(c) (i) 41.375 cm^3
(ii) 33.875 cm^3
23. (a) (i) $C = 23.55 \times 10^{-6}$ m
(ii) $C_{\min} = 23.39 \times 10^{-6}$ m
(b) $C_{\max} = 23.\ 71 \times 10^{-6}$.
$b = \pm 0.16 \times 10^{-6}$ m
$(C \pm b) \times 10^{-6}$ m $=$
$(23.55 \pm 0.16) \times 10^{-6}$ m

Exercise 3o

1. (a) 21 175 (b) 211.75
(c) 2117.5 (d) 211 750
2. (a) 9600 (b) 96 000
(c) 960 000 (d) 9 600 000
3. (a) 296 375 (b) 296.375
(c) 2 963.75 (d) 29 637.5
4. (a) 123 000 (b) 1 230 000
(c) 12 300 000 (d) 123 000 000
5. (a) 216 875 (b) 216.875
(c) 2 168.75 (d) 21 687.5
6. (a) 30 625 (b) 306 250
(c) 3 062 500 (d) 30 625 000
7. (a) 145 827 (b) 1 471 527
(c) 208 065 (d) 2 138 565
8. (a) 84 739 (b) 839 839
(c) 58 446 (d) 574 146
9. (a) 23.36 (b) 2 336
(c) 233.6 (d) 2.336
10. (a) 138.88 (b) 13.888
(c) 1.388 8 (d) 0.138 88
11. (a) 67.768 (b) 67 768
(c) 6 776.8 (d) 677.68
12. (a) 6 718.776 (b) 671.877 6
(c) 67.187 76 (d) 6.718 776
13. (a) 5.136 (b) 5 136
(c) 513.6 (d) 51.36
14. (a) 1 678.12 (b) 167.812
(c) 16.7812 (d) 1.678 12

Exercise 3p

1. 20:1 **2.** 7:45
3. (a) 4:1 (b) 1:4
(c) 1:5 (d) 4:5
4. 33:40
5. $n{:}1 = \frac{4}{7}{:}1$ **6.** 14:5
7. 197:100
8. 43:25 **9.** 3:1
10. 2:5 **11.** 15:2
12. 7:20 **13.** 1:4
14. 1:9 **15.** 3:7

Exercise 3q

1. 312 m **2.** \$10.80
3. \$338 **4.** \$1 260
5. \$490 **6.** \$154
7. (a) 7.5 *l* (b) 3.75 *l*
8. 160 g **9.** $13\frac{1}{3}$ cm
10. (a) $1\frac{1}{3}$ km (b) $\frac{1}{45}$ km

Exercise 3r

1. \$25 000. \$40 000. \$10 000.
2. \$1 372.
3. \$10 500. \$15 000. \$19 500.
4. (a) 17 cm (b) 42.5 cm
5. 50 g. 30 g. 70 g.
6. \$1 125
7. \$10 000. \$20 000. \$30 000.
8. \$1 000. \$600. \$400.
9. 15 chocolates. 20 chocolates. 25 chocolates.
10. (a) \$3 000 (b) \$1 500
(c) 30%
11. 49 cm
12. (a) \$2 250 (b) \$1 200
(c) $A{:}J{:}S = 7{:}8{:}15$
(d) $23\frac{1}{3}\%$
13. (a) \$3 000 (b) \$900
(c) 20%
14. (a) \$1 750 (b) \$900
(c) $A{:}J{:}S = 17{:}18{:}35$
(d) $24\frac{2}{7}\%$
15. (a) \$7 500 (b) \$2 500
(c) 20%
16. \$1 400 **17.** 288 g

Exercise 3s

1. \$8.25 \$123.75
2. \$11.25 **3.** $58\frac{1}{3}$ h
4. \$29.25 **5.** \$1.75
6. 50 *l* **7.** 11 h **8.** \$32.24
9. \$198 **10.** (a) 7.5 *l* (b) 3.75 *l*
11. 12 *l* **12.** \$816
13. 1.125 kg raisins 1.125 kg prunes
1.125 kg currants 250 g mixed peel
$3\frac{3}{4}$ tablespoons ground cinnamon
562.5 g glace cherries
5 cups non-alcoholic red wine
5 cups cherry brandy
1.125 kg granulated sugar
1.125 kg butter
5 tablespoons baking powder
25 large eggs
562.5 g brown sugar
$1\frac{1}{4}$ cups of boiling water
14. \$137.80
15. (a) 3 bottles orange juice
1 bottle tangerine juice
1 kg sugar
3 bottles grapefruit juice
(b) $\frac{3}{4}$ bottle orange juice
$\frac{1}{4}$ bottle tangerine juice
$\frac{1}{4}$ kg sugar
$\frac{3}{4}$ bottle of grapefruit juice

16. $73.23

17. (a) $2.85 (b) $7.95
(c) $8.28 (d) 2050 g

18. (a) $2.70 (b) $8.46
(c) $13.50

19. (a) $6.96 (b) $169.36

20. (a) $4.93 (b) $50.75
(c) $65.54 (d) 60 marbles

Exercise 3t

1. 6 days **2.** 20 days
3. 90 rev./min **4.** 18 sweets
5. 6 hours **6.** 8 days
7. 9 hours **8.** $7\frac{1}{5}$ days
9. 5 weeks **10.** 9 weeks
11. 96 rev./min

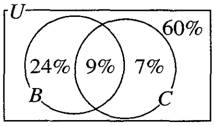
Exercise 3u

1. 0.165 **2.** $113\frac{1}{3}$%
3. 84.5%

4.

	Common fraction	Percentage	Decimal fraction
(a)	$\frac{3}{5}$	60%	0.6
(b)	$\frac{9}{20}$	45%	0.45
(c)	$\frac{7}{20}$	35%	0.35

5. 9% **6.** 60%



7. 98.5% **8.** 108 pages
9. 371 students **10.** $30 375
11. 27 teachers **12.** 9 m
13. $13\frac{1}{3}$% **14.** 78 shops
15. $52.50

16.

	Common fraction	Percentage	Decimal fraction
(a)	$\frac{1}{8}$	12.5%	0.125
(b)	$\frac{13}{20}$	65%	0.65
(c)	$\frac{47}{50}$	94%	0.94

17. 12%

18.

	Common Fraction	Percentage	Decimal fraction
(a)	$\frac{5}{8}$	62.5%	0.625
(b)	$\frac{7}{10}$	70%	0.7
(c)	$\frac{17}{20}$	85%	0.85

19. 85% **20.** 525 children
21. $27\frac{7}{9}$% **22.** 16 170 females
23. 90 pages **24.** 45%
25. 107 pages **26.** 66 shops
27. 20.25 m **28.** $4\frac{4}{9}$%
29. 42 shops **30.** $81\frac{2}{3}$%

Exercise 3v

1. (a) 567 marks (b) $70\frac{7}{8}$ marks
2. $73\frac{1}{8}$ marks **3.** 156 cm
4. (a) 383 marks (b) 76.6 marks
5. 169.2 cm
6. 155 km/day, $12.40 per day,
$0.08 per km
7. 596 marks **8.** $7.92
9. 2 runs **10.** 1 291.5 km
11. 5 marks **12.** $155\frac{1}{9}$ cm
13. $116\frac{2}{3}$ cl
14. (a) $1 377.50 (b) $1 765.50
(c) $388
15. 71 runs
16. 15 years 2 months
17. 46.5 marks

Exercise 3w

1. (a) 1 (b) 4 (c) 9
(d) 64 (e) 81
2. (a) 144 (b) 169 (c) 225
(d) 324 (e) 361
3. (a) 0.01 (b) 0.04 (c) 0.16
(d) 0.49 (e) 0.64
4. (a) 5.29 (b) 6.76 (c) 7.29
(d) 7.84 (e) 8.41
5. (a) 0.000 9 (b) 0.001 6
(c) 0.002 5 (d) 0.004 9
(e) 0.008 1
6. (a) 0.000 001 (b) 0.000 025
(c) 0.000 036 (d) 0.000 064
(e) 0.000 081
7. (a) 237.16 (b) 372.49
(c) 462.25 (d) 778.41
(e) 846.81
8. (a) 14 745.245 (b) 15 823.124
(c) 18 409.062 (d) 21 103.373
(e) 22 305.423
9. (a) 0.022 5 (b) 0.084 1
(c) 0.115 6 (d) 0.504 1
(e) 0.902 5
10. (a) 233 264 530
(b) 367 795 680
(c) 591 413 760
(d) 1 268 713 200
(e) 1 558 275 600
11. (a) 0.000 225 (b) 0.000 529
(c) 0.002 209 (d) 0.007 921
(e) 0.009 025
12. (a) 0.000 001 96
(b) 0.000 003 61
(c) 0.000 014 44
(d) 0.000 033 64
(e) 0.000 088 36
13. (a) 1.21 (b) 1.44
(c) 2.25 (d) 2.89
(e) 3.61
14. (a) 8.58 (b) 14.75
(c) 33.18 (d) 70.73
(e) 95.06
15. (a) 216 (b) 640
(c) 1 163 (d) 2 107
(e) 8 724
16. (a) 20 400 (b) 61 000
(c) 282 000 (d) 568 500
(e) 900 600
17. (a) 1 990 000 (b) 10 050 000
(c) 31 580 000 (d) 66 750 000
(e) 87 240 000
18. (a) 0.021 (b) 0.047 1
(c) 0.101 8 (d) 0.377
(e) 0.508 4
19. (a) 0.000 19 (b) 0.000 61
(c) 0.001 756 (d) 0.004 02
(e) 0.005 112
20. (a) 9 cm^2 (b) 64 cm^2
(c) 100 cm^2
21. (a) 28.09 cm^2 (b) 40.96 cm^2
(c) 92.16 cm^2
22. (a) 237.16 cm^2
(b) 561.69 cm^2
(c) 1 288.81 cm^2
23. (a) 11 088.09 cm^2
(b) 20 678.44 cm^2
(c) 30 905.64 cm^2

24. (a) 5 476 mm^2 (b) 6 889 mm^2 (c) 8 464 mm^2
25. (a) 6.25 m^2 (b) 12.96 m^2 (c) 33.64 m^2
26. (a) 49 (b) 16
27. (a) 9 (b) 9
28. (a) 0.81 (b) 1.44
29. (a) 1.96 (b) 0.108 9
30. (a) 25 (b) 36
31. (a) $y = 0.25$ (b) $y = 2.25$ (c) $y = 30.25$ (d) $y = 42.25$
32. (a) $y = 0.25$ (b) $y = 2.25$ (c) $y = 30.25$ (d) $y = 42.25$
33. (a) $y = 2.25$ (b) $y = 6.25$ (c) $y = 12.25$ (d) $y = 20.25$
34. (a) $y = 6.25$ (b) $y = 12.25$ (c) $y = 20.25$ (d) $y = 2.25$
35. (a) $y = 6.25$ (b) $y = 12.25$ (c) $y = 6.25$ (d) $y = 12.25$

Exercise 3x

1. (a) 1 (b) 10 (c) 100 (d) 0.1 (e) 0.01
2. (a) 4 (b) 40 (c) 400 (d) 0.4 (e) 0.04
3. (a) 6 (b) 60 (c) 0.6 (d) 0.06 (e) 0.006
4. (a) 7 (b) 70 (c) 0.7 (d) 0.07 (e) 0.007
5. (a) 12 (b) 120 (c) 1 200 (d) 0.12 (e) 0.012
6. (a) 15 (b) 150 (c) 0.15 (d) 0.015 (e) 0.001 5
7. (a) 1.2 (b) 1.3 (c) 1.4 (d) 1.8 (e) 1.9
8. (a) 8.6 (b) 8.7 (c) 8.9 (d) 9.5 (e) 9.8
9. (a) 275 (b) 289 (c) 290 (d) 294 (e) 297
10. (a) 143.5 (b) 152.3 (c) 161.4 (d) 175.6 (e) 199.7
11. (a) 0.51 (b) 0.63 (c) 0.74 (d) 0.86 (e) 0.97
12. (a) 0.045 (b) 0.047 (c) 0.069 (d) 0.078 (e) 0.093
13. (a) 1.67 (b) 2.39 (c) 2.51 (d) 2.80 (e) 3.06
14. (a) 4.29 (b) 5.46 (c) 6.97 (d) 8.89 (e) 9.26
15. (a) 11.4 (b) 18.6 (c) 20.6 (d) 25.5 (e) 26.9
16. (a) 38.3 (b) 49.8 (c) 79.8 (d) 86.9 (e) 91.3
17. (a) 136 (b) 160 (c) 194 (d) 214 (e) 291
18. (a) 0.497 (b) 0.617 (c) 0.826 (d) 0.861 (e) 0.876
19. (a) 0.116 (b) 0.157 (c) 0.195 (d) 0.220 (e) 0.269
20. (a) 0.057 3 (b) 0.067 9 (c) 0.076 4 (d) 0.083 2 (e) 0.088 5
21. (a) 18 (b) 24 (c) 40
22. (a) 63 (b) 48 (c) 56
23. (a) 112 (b) 90 (c) 288
24. (a) 220 (b) 84 (c) 360
25. (a) 480 (b) 390 (c) 588
26. (a) 5 cm (b) 6 cm
27. (a) 7 cm (b) 8 cm
28. (a) 11 cm (b) 12 cm
29. (a) 17 mm (b) 19 mm
30. (a) 25.6 mm (b) 31.5 mm
31. (a) ±2 (b) ±5
32. (a) ±6 (b) ±7
33. (a) ±2.5 (b) ±3.7
34. (a) ±4.5 (b) ±5.9
35. (a) ±6.3 (b) ±9.5
36. (a) $x = \pm 1.6$ (b) $x = \pm 2.2$ (c) $x = \pm 2.7$
37. (a) $x = \pm 3.2$ (b) $x = \pm 3.5$ (c) $x = \pm 3.9$
38. (a) $x = \pm 4.2$ (b) $x = \pm 4.5$ (c) $x = \pm 4.7$
39. (a) $x = \pm 5.0$ (b) $x = \pm 5.2$ (c) $x = \pm 5.5$
40. (a) $x = \pm 5.7$ (b) $x = \pm 5.9$ (c) $x = \pm 6.1$
41. (a) $x = \pm 6.3$ (b) $x = \pm 6.5$ (c) $x = \pm 6.7$
42. (a) $x = \pm 6.9$ (b) $x = \pm 7.1$ (c) $x = \pm 7.2$
43. (a) $x = \pm 7.4$ (b) $x = \pm 7.6$ (c) $x = \pm 7.7$
44. (a) $x = \pm 7.9$ (b) $x = \pm 8.1$ (c) $x = \pm 8.2$
45. (a) $x = \pm 8.4$ (b) $x = \pm 8.5$ (c) $x = \pm 8.7$

Exercise 3y

1. (a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$
2. (a) $2\frac{1}{2}$ (b) $1\frac{1}{3}$ (c) $1\frac{3}{5}$
3. (a) 12 (b) 38 (c) 57
4. (a) $1\frac{2}{3}$ (b) $1\frac{2}{7}$ (c) $1\frac{4}{9}$
5. (a) 0.015 4 (b) 0.011 2 (c) 0.010 5
6. (a) 7.30 (b) 6.33 (c) 5.10
7. (a) 0.001 47 (b) 0.001 30 (c) 0.001 12
8. (a) 0.001 54 (b) 0.001 28 (c) 0.001 02
9. (a) 28.8 (b) 13.0 (c) 10.2
10. (a) 719 (b) 206 (c) 114
11. (a) 0.667 (b) 0.149 (c) 0.101
12. (a) 0.178 (b) 0.128 (c) 0.105
13. (a) 0.002 88 (b) 0.001 53 (c) 0.001 02
14. (a) 6.90 (b) 2.60 (c) 1.30
15. (a) 68.0 (b) 20.9 (c) 15.7

Exercise 3z

1. (a) 0.076 (b) $2\frac{6}{7}$
2. (a) $3\frac{1}{3}$
(b) (i) 0.030 15 (ii) 0.030 2 (iii) 3.015×10^{-2}
(c) (i) 350 boys (ii) 7:12
3. (a) (i) (9 ± 0.5) m (ii) 46.75 m^2 (iii) 32 m

Exercise 4a

1. (a) 4 510 mm (b) 37 500 mm
2. (a) 78.5 mm (b) 249 mm
3. (a) 1 820 000 km (b) 9 153 000 km
4. (a) 810 mm (b) 9.4 mm
5. (a) 630 000 km (b) 750 mm
6. (a) 85 cm (b) 9.45 cm
7. (a) 450 cm (b) 93 cm
8. (a) 810 000 cm (b) 8 000 cm
9. (a) 7.43 cm (b) 92.1 cm
10. (a) 41 000 cm (b) 580 cm
11. (a) 4.875 cm (b) 9.421 cm
12. (a) 3.94 m (b) 48.69 m
13. (a) 4 510 m (b) 24 300 m
14. (a) 39.423 m (b) 0.493 m
15. (a) 8 450 m (b) 0.047 9 m
16. (a) 0.512 475 km (b) 0.769 861 km
17. (a) 0.283 72 km (b) 0.117 86 km
18. (a) 68.475 km (b) 3.147 km

19. (a) 2.479 km (b) 0.496 km
20. (a) 894 km
(b) 0.000 045 3 km

Exercise 4b

1. (a) 7 000 000 mm^2
(b) 9 000 000 mm^2
2. (a) 4 500 000 mm^2
(b) 8 300 000 mm^2
3. (a) 210 000 mm^2
(b) 650 000 mm^2
4. (a) 8 412 000 mm^2
(b) 3 173 000 mm^2
5. (a) 20 000 cm^2 (b) 90 000 cm^2
(c) 50 000 cm^2
6. (a) 36 000 cm^2 (b) 81 000 cm^2
7. (a) 5 100 cm^2 (b) 8 300 cm^2
8. (a) 93 140 cm^2 (b) 47 130 cm^2
9. (a) 8.475 m^2 (b) 12.341 m^2
10. (a) 0.009 345 m^2
(b) 0.017 834 m^2
11. (a) 48.3 m^2 (b) 34.17 m^2
12. (a) 0.847 5 m^2 (b) 0.314 9 m^2
13. (a) 45.735 km^2 (b) 37.412 km^2
14. (a) 1.425 km^2 (b) 8.375 km^2
(c) 8.5 km^2
15. (a) 0.647 km^2 (b) 0.312 km^2
16. (a) 0.000 345 km^2
(b) 0.000 849 km^2
17. (a) 34.7 ha (b) 83.9 ha
18. (a) 8.5 ha (b) 3.9 ha
19. (a) 0.947 5 ha (b) 0.713 5 ha
20. (a) 0.076 8 ha (b) 0.084 7 ha

Exercise 4c

1. (a) 7 000 000 000 mm^3
(b) 9 000 000 000 mm^3
2. (a) 400 000 000 mm^3
(b) 500 000 000 mm^3
3. (a) 8 400 000 000 mm^3
(b) 9 700 000 000 mm^3
4. (a) 75 310 000 000 mm^3
(b) 84 190 000 000 mm^3
5. (a) 5 000 000 cm^3
(b) 8 000 000 cm^3
6. (a) 600 000 cm^3
(b) 900 000 cm^3
7. (a) 7 100 000 cm^3
(b) 4 900 000 cm^3
8. (a) 27 150 000 cm^3
(b) 34 970 000 cm^3
9. (a) 9.495 m^3 (b) 84.763 m^3
10. (a) 0.645 m^3 (b) 0.321 m^3
11. (a) 9.847 m^3 (b) 4.134 m^3
12. (a) 0.935 m^3 (b) 0.347 m^3

Exercise 4d

1. (a) 5 *l* (b) 9 *l*
2. (a) 3 *l* (b) 8 *l*
(c) 6 *l* (d) 3.5 *l*
3. (a) 0.480 95 *l* (b) 0.793 84 *l*
4. (a) 4.765 8 *l* (b) 2.175 3 *l*
5. (a) 39.47 *l* (b) 45.763 *l*
6. (a) 4 000 cm^3 (b) 8 000 cm^3
7. (a) 5 610 cm^3 (b) 45 300 cm^3
8. (a) 8 750 ml (b) 24 900 ml
9. (a) 15 810 ml (b) 19 480 ml
10. (a) 35 800 ml (b) 105 740 ml
11. (a) 12 m^3 (b) 16 m^3
12. (a) 0.685 7 m^3 (b) 0.947 8 m^3
(c) 7 m^3
13. (a) 9.475 84 m^3
(b) 4.349 15 m^3
14. (a) 15.378 m^3 (b) 12.147 m^3
15. (a) 6.470 4 m^3 (b) 1.384 5 m^3
16. (a) 42 500 cl (b) 94 300 cl
17. (a) 4 870 cl (b) 8 320 cl
18. (a) 43.8 cl (b) 57.4 cl
19. (a) 5.13 cl (b) 3.415 cl
20. (a) 4.5 cl (b) 17.84 cl

Exercise 4e

1. (a) 5 000 g (b) 9 000 g
2. (a) 470 g (b) 891 g
3. (a) 4 390 g (b) 7 149 g
4. (a) 8 479.5 g (b) 3 147.6 g
5. (a) 4 000 mg (b) 7 000 mg
6. (a) 490 mg (b) 950 mg
7. (a) 1 500 mg (b) 8 900 mg
8. (a) 34 780 mg (b) 49 170 mg
9. (a) 5 000 kg (b) 7 000 mg
10. (a) 800 kg (b) 900 kg
11. (a) 4 310 kg (b) 7 640 kg
12. (a) 43 290 kg (b) 84 170 kg
13. (a) 147.435 t (b) 849.135 t
14. (a) 15.768 t (b) 24.140 t
15. (a) 8.471 t (b) 3.178 t
16. (a) 0.947 t (b) 0.835 t

Exercise 4f

1. 1.609 km
2. 1.13 pg
3. 31.56 Ms
4. 45.9 mA
5. 5.4 kK **6.** 4.848 μrad
7. 3.084 Mm **8.** 4.65 ag
9. 3.6 Gs **10.** 4.8 nK
11. 9.461×10^{15} m
12. 5.08×10^{-6} g
13. 3.5×10^{7} s
14. 5.93×10^{9} K
15. 2.909×10^{-4} rad
16. 1.055×10^{6} J
17. 7.457×10^{6} W
18. 3.048×10^{6} g
19. 2.54×10^{-6} m
20. 1.475×10^{-9} s

Exercise 4g

1. $A = 25.2$ cm^2
2. $A = 187.5$ mm^2
3. $A = 71.8$ cm^2 **4.** $A = 31.1$ m^2
5. $A = 100.9$ mm^2
6. $A = 161.3$ m^2
7. $A = 14.7$ cm^2 **8.** $h = 3.5$ m
9. $b = 10$ cm **10.** $h = 7.2$ mm
11. $b = 5$ cm **12.** $b = 6\frac{2}{3}$ cm
13. $h = 6$ mm
14. $A = 194.9$ cm^2
15. $A = 67.5$ mm^2
16. $A = 4.8$ m^2
17. $A = 110.9$ cm^2
18. $A = 1\,120$ mm^2
19. $A = 114.4$ cm^2
20. $A = 631$ mm^2
21. $A = 936.4$ cm^2
22. $A = 2\,540$ mm^2
23. $A = 240$ cm^2

Exercise 4h

1. (a) $A = 36$ cm^2, $P = 24$ cm
(b) $A = 60$ cm^2, $P = 34$ cm
2. 16 squares **3.** 28 squares
4. 2 400 tiles **5.** 11 250 tiles
6. $A = 15$ m^2, \$12.00
7. $A = 13\,500$ m^2, $P = 480$ m
8. $A = 135$ m^2, \$81.00
9. (a) $A = 100\,000$ cm^2
(b) $A = 2\,250$ m^2
10. $b = 7\frac{1}{6}$ cm
11. $l = 9$ cm
12. (a) $A = 30\,000$ cm^2
(b) $A = 10$ m^2
(c) $A = 100\,000$ m^2
(d) $A = 216$ mm^2

13. (a) $b = 3$ cm, $A = 12$ cm^2
(b) $l = 5$ cm, $P = 16$ cm
14. (a) $b = 4$ cm, $A = 28$ cm^2
(b) $l = 9$ cm, $A = 27$ cm^2
(c) $b = 8$ cm, $P = 40$ cm
(d) $l = 11$ cm, $P = 36$ cm
15. $b = 4$ cm **16.** $l = 12$ cm
17. $b = 16$ cm **18.** $P = 30$ cm, $A = 30$ cm^2
19. $A = 97$ m^2
20. (a) $A = 128$ m^2, $P = 52$ m
(b) $A = 46$ cm^2, $P = 42$ cm
21. $A = 34$ cm^2
22. (a) $A = 56$ cm^2
(b) $A = 91$ cm^2
23. $A = 126$ cm^2
24. $A = 296$ m^2, 1 184 tiles
25. $A = 8.71$ m^2
26. $A = 112$ mm^2
27. (a) 176 cm^2
(b) $A = 171$ cm^2
(c) $A = 54$ m^2

Exercise 4i

1. $A = 54.4$ cm^2, $P = 34$ cm
2. $A = 70.5$ cm^2, $P = 37.6$ cm
3. $A = 124.5$ mm^2, $P = 50.8$ cm
4. $A = 473.9$ mm^2, $P = 97.2$ cm
5. $A = 105.8$ cm^2
6. $A = 132.5$ cm^2
7. $A = 65$ cm^2
8. $A = 53.8$ cm^2
9. $A = 180$ cm^2
10. $A = 72$ cm^2
11. $A = 28.2$ cm^2
12. $A = 838.1$ mm^2
13. (a) $h = 6.5$ cm
(b) $b = 5$ cm
14. (a) $b = 9.5$ mm
(b) $h = 5.2$ mm
15. $h = 6$ cm
16. $b = 9.5$ mm
17. $A = 56$ cm^2
18. $A = 93.6$ cm^2
19. $A = 386.3$ mm^2
20. $A = 538.6$ cm^2

Exercise 4j

1. (a) $r = 4$ cm
(b) $C = 25.1$ cm
(c) $A = 50.2$ cm^2
2. $r = 10$ cm
3. $C = 59.7$ cm, $A = 283.6$ cm^2
4. $C = 12.6$ cm, $A = 12.6$ cm^2
5. $r = 3.5$ cm
6. (a) $l = 11$ cm
(b) $A = 77$ cm^2
7. (a) $P = 24.6$ cm
(b) $A = 37.7$ cm^2
8. $C = 176$ cm
9. $A = 188$ cm^2
10. $P = 42\frac{2}{3}$ cm
11. $C = 132$ cm, 4 times
12. $r = 16.7$ cm
13. $A = 3\,773$ mm^2
14. (a) $P = 28.7$ cm
(b) $A = 51.3$ cm^2
15. $A = 256.6$ cm^2
16. (a) $A = 38.5$ cm^2
(b) $A = 12.8$ cm^2
(c) $A = 5.3$ cm^2
(d) $A = 7.5$ cm^2
17. (a) $A = 25.7$ cm^2
(b) $l = 7.3$ cm
18. (a) $A = 102\frac{2}{3}$ cm^2
(b) $l = 29\frac{1}{3}$ cm
19. (a) (i) $A = 75.5$ cm^2
(ii) $A = 48.0$ cm^2
(iii) $A = 27.5$ cm^2
(b) (i) $C = 61.6$ cm
(ii) $l = 46.2$ cm
20. (a) (i) $A = 193.6$ mm^2
(ii) $A = 123.2$ mm^2
(iii) $A = 70.4$ mm^2
(b) (i) $C = 98.7$ mm
(ii) $l = 74.0$ mm
21. (a) $A = 687$ mm^2
(b) $l = 73.4$ mm
22. (a) $l = 19.9$ m (b) $A = 37.8$ m^2

Exercise 4k

1. $A = 1\,940$ m^2
2. $A = 1\,790$ cm^2
3. $A = 42$ cm^2
4. $l = d = 17.8$ m
5. $r = 63.6$ cm
6. $A = 392.9$ cm^2
7. $A = 314.3$ cm^2 **8.** $P = 46.9$ cm
9. $A = 84$ cm^2 **10.** $A = 24$ m^2
11. $A = 168$ cm^2 **12.** $A = 38.5$ m^2
13. $A = 3\,013.5$ m^2 **14.** $A = 45$ cm^2
15. $A = 38.5$ cm^2 **16.** $A = 5.4$ cm^2
17. $A = 814.2$ cm^2
18. $A = 546.3$ cm^2
19. $A = 52$ cm^2

Exercise 4l

1. (a) T.S.A. = 249 cm^2
$V = 150$ cm^3
(b) m = 1 069.5 g
2. (a) $V = 90$ cm^3 (b) m = 801 g
3. (a) T.S.A. = 156 cm^2
(b) $V = 72$ cm^3
4. (a) $BD = 10$ cm
(b) T.S.A. = 336 cm^2
(c) $V = 288$ cm^2
5. (a) $BD = 5$ cm
(b) T.S.A. = 120 cm^2
(c) $V = 54$ cm^3
(d) $DBC = 36.9°$
6. (a) $BD = 26$ cm
(b) T.S.A. = 1 800 cm^2
(c) $V = 3\,120$ cm^3
(d) $BDC = 22.6°$
7. (a) $BD = 10$ cm
(b) T.S.A. = 408 cm^2
(c) $V = 360$ cm^3
(d) $DBC = 53.1°$

Exercise 4m

1. (a) 1 600 cubes (b) m = 2 270 g
2. (a) $V = 9\,576$ cm^3
(b) $V = 10\,500$ m^3
(c) $V = 160$ m^3
3. $V = 24$ m^3 = 24 000 l
4. (a) $V = 36$ cm^3 (b) $V = 2$ m^3
(c) $V = 2\,500\,000\,000$ mm^3
5. $V = 105$ m^3
6. $V = 512$ colognes
7. 90 people
8. 48 pupils
9. $V = 432$ m^3 = 432 000 l
10. 729 cubes **11.** 12 800 cubes
12. $V = 252$ cm^3, $A = 282$ cm^2
13. $V = 60$ m^3 **14.** $V = 729$ cm^3
15. 216 boxes **16.** 24 people
17. (a) $V = 30\,000\,000$ cm^3
(b) $V = 30$ m^3 (c) $V = 30\,000$ l
18. $V = 360$ m^3 **19.** 945 cm^3
20. $A = 1\,350$ cm^3, $V = 121\,500$ cm^3, $V = 364\,500$ cm^3.

Exercise 4n

1. $V = 1.4$ cm^3
2. $V = 282.8$ cm^3

3. $V = 2\,771.2 \text{ cm}^3$
C.S.A. $= 791.8 \text{ cm}^2$
4. $V = 36\,949.9 \text{ cm}^3$
5. (a) $V = 24\,640 \text{ cm}^3$
(b) $h = 30 \text{ cm}$
6. (a) C.S.A. $= 660 \text{ m}^2$
(b) $V = 2\,310 \text{ m}^3$
7. (a) $V = 1\,256.8 \text{ cm}^3$
(b) $m = 11\,235.8 \text{ g} = 11.2 \text{ kg}$
(c) C.S.A. $= 754.1 \text{ cm}^2$
8. $V = 18\,317.9 \text{ cm}^3$
9. $V = 7\,540.8 \text{ mm}^3$
10. $A = 50.1 \text{ cm}^2$
11. $V = 268.8 \text{ cm}^3$
$V = 1\,803.6 \text{ cm}^3$
(b) $V = 45 \text{ m}^3$
12. (a) $V = 3\,380 \text{ cm}^3$
13. $V = 2\,611.8 \text{ cm}^3$
14. $A = 22.3 \text{ cm}^2$ $V = 446 \text{ cm}^3$
15. $V = 216 \text{ cm}^3$
16. $V = 180 \text{ cm}^3$
17. (a) $V = 510 \text{ cm}^3$
(b) $V = 76.8 \text{ m}^3$
(c) $V = 800 \text{ cm}^3$
18. $A = 814.2 \text{ cm}^2$ $V = 20\,355 \text{ cm}^3$
19. $A = 546.3 \text{ cm}^2$ $V = 16\,389 \text{ cm}^3$
20. $A = 1\,214.2 \text{ cm}$ $V = 30\,355 \text{ cm}^3$
21. (a) T.S.A. $= 55.2 \text{ m}^2$
(b) $V = 1\,380 \text{ m}^3$
22. (a) $P = 124.3 \text{ m}$
(b) $A = 911.5 \text{ m}^2$
(c) $V = 252 \text{ m}^3$
23. (a) $P = 101.4 \text{ m}$
(b) $A = 603.1 \text{ m}^2$
(c) $V = 108 \text{ m}^3$
24. (a) (i) $V = 972 \text{ cm}^3$
(ii) $V = 672 \text{ cm}^3$
(iii) $V = 880 \text{ cm}^3$
(b) (i) m $= 2\,624.4$ g
(ii) m $= 1\,814.4$ g
(iii) m $= 2\,376$ g

Exercise 4o

1. $V = 30 \text{ cm}^3$
2. $V = 1\,000.4 \text{ mm}^3$
3. (a) $V = 24.5 \text{ cm}^3$
(b) m $= 86.5$ g
4. (a) $V = 361.7 \text{ cm}^3$
(b) m $= 3.80$ kg
5. $V = 42 \text{ cm}^3$
6. $V = 384 \text{ cm}^3$
7. (a) $V = 36.9 \text{ cm}^3$
(b) m $= 212.2$ g
8. (a) $V = 777.9 \text{ cm}^3$
(b) m $= 5.7$ kg
9. $V = 57.0 \text{ cm}^3$
10. $V = 300 \text{ cm}^3$, $r = 3.7 \text{ cm}$
11. (a) $V = 119.1 \text{ cm}^3$
(b) m $= 2.6$ kg
12. (a) $V = 774.7 \text{ cm}^3$
(b) m $= 8.8$ kg
13. (a) $V = 301.6 \text{ cm}^3$
(b) C.S.A. $= 188.5 \text{ cm}^2$
14. (a) $V = 2\,375.4 \text{ cm}^3$
(b) C.S.A. $= 1\,091.5 \text{ cm}^2$
15. $V = 1\,055.7 \text{ cm}^3$
16. (a) C.S.A. $= 77 \text{ cm}^3$
(b) $V = 78.3 \text{ cm}^3$
17. $V = 23\,760 \text{ cm}^3$
18. $V = 7\,392 \text{ cm}^3$

Exercise 4p

1. (a) $V = 2\,572.8 \text{ cm}^3$
(b) C.S.A. $= 908 \text{ cm}^2$
(c) $C = 40\,029.1 \text{ km}$
2. $V = 1.08 \times 10^{12} \text{ km}^3$,
C.S.A. $= 5.10 \times 10^8 \text{ km}^2$
3. $V = 3\,080 \text{ cm}^3$
4. (a) $V = 462 \text{ cm}^3$
(b) 2 ice-cubes
5. (a) (i) $V = 18\,500 \text{ cm}^3$
(ii) 18.5 l (iii) $V = 493 \text{ cm}^3$
(b) 24 globes
6. m $= 35$ kg
7. (a) T.S.A. $= 414.7 \text{ cm}^2$
(b) $V = 754 \text{ cm}^3$
8. T.S.A. $= 154 \text{ cm}^2$
9. (a) $C = 44 \text{ m}$
(b) $V = 1\,437.3 \text{ m}^3$
10. (a) $A = 933.2 \text{ cm}^2$
(b) $V = 2\,545 \text{ cm}^3$
11. (a) $V = 9.05 \times 10^{20} \text{ m}^3$
(b) $S = 6.63 \times 10^3 \text{ kg/m}^3$

Exercise 4q

1. $t = 1$ h 18 min
2. $t = 4$ h 7 min
3. $t = 2$ h 20 min
4. $t = 1$ h 40 min
5. $t = 1$ h 45 min
6. $t = 53$ min
7. $t = 3$ h 15 min
8. $t = 6$ h 6 min
9. $t = 3$ h 18 min
10. $t = 3$ h 50 min
11. $t = 3$ h 42 min
12. $t = 2$ h 54 min
13. $t = 6$ h 50 min
14. $t = 2$ h 35 min
15. $t = 1$ h 40 min
16. $t = 8$ h 59 min
17. $t = 4$ h 50 min
18. $t = 1$ h 50 min
19. $t = 16$ h 3 min
20. $t = 15$ h 40 min
21. $t = 14$ h 25 min
22. $t = 3$ h 23 min
23. $t = 1$ h 24 min
24. $t = 1$ h 14 min
25. $t = 4$ h 47 min
26. $t = 1$ h 39 min
27. $t = 1$ h 45 min
28. $t = 3$ h 15 min
29. $t = 5$ h 53 min
30. $t = 1$ h 26 min
31. $t = 3$ h 50 min
32. $t = 3$ h 42 min
33. $t = 2$ h 3 min
34. $t = 7$ h 55 min
35. $t = 3$ h 35 min
36. $t = 5$ h 30 min
37. $t = 2$ h 14 min
38. $t = 47$ min
39. $t = 1$ h 12 min
40. $t = 2$ h 53 min

Exercise 4r

1. $s = 60$ km/h
2. $d = 390$ km **3.** $t = 4$ h
4. $d = 21\frac{3}{7}$ km/h
5. (a) $t = 50$ min
(b) $t = 1$ h 30 min
6. (a) $d = 20$ km
(b) $s = 60$ km/h
7. (a) $d = 26\frac{1}{4}$ km/h
(b) $s = 37\frac{1}{2}$ km/h
8. $s = 12$ km/h
9. $s = 45$ km/h
10. $d = 180$ km
11. $t = 6$ h
12. $d = 252$ km $s = 84$ km/h
13. $t = 35$ min $s = 84$ km/h
14. (a) $t = 2$ h
(b) $s = 68\frac{2}{11}$ km/h
(c) 10:50 h

15. (a) $a = 5$ h 10 min
$b = 2635$ km
$c = 50$ min
$d = 672$ km/h
(b) $s = 523.2$ km/h
16. (a) $d = 12.75$ km
(b) 12.4 km/h
17. $s = 750$ km/h
18. (a) $t = 5.0 \times 10^2$ s
(b) $d = 4 \times 10^8$ m
19. $t = 60$ $s = 1$ min, 21:01 h
20. $s = 32$ km/h
21. $d = 39$ km
22. $s = 32$ km/h
23. $s = 48$ km/h

Exercise 4s

1. $s = 10.5$ m.p.h.
2. $s = 29$ m.p.h.
3. $s = 37.5$ m.p.h.
4. $s = 78.5$ m.p.h.
5. $s = 92.5$ m.p.h.
6. $s = 39.2$ km/h
7. $s = 63.2$ km/h
8. $s = 128$ km/h
9. $s = 200$ km/h
10. $s = 256$ km/h
11. $s = 5.25$ m/s
12. $s = 20$ m/s
13. $s = 25.5$ m/s
14. $s = 33.5$ m/s
15. $s = 45.5$ m/s
16. $s = 18$ km/h
17. $s = 55.8$ km/h
18. $s = 88.2$ km/h
19. $s = 117$ km/h
20. $s = 144$ km/h

Exercise 4t

1. $l = 16$ m
2. $l = 59$ mm
3. $l = 353$ cm
4. $l = 17$ m
5. $l = 336$ mm
6. $l = 471$ cm
7. $l = 10.7$ m
8. $l = 61.3$ mm
9. $l = 213$ cm
10. $l = 18.5$ m
11. $A = 24$ m^2
12. $A = 2200$ cm^2
13. $A = 7800$ mm^2
14. $A = 20$ m^2
15. $A = 851$ cm^2
16. $A = 380$ mm^2
17. $A = 29$ m^2
18. $A = 6100$ cm^2
19. $A = 1400$ mm^2
20. $A = 64$ m^2

Exercise 4u

1. (a) 255 km (b) 260 km
(c) 90 km
2. (a) 80 km (b) 100 km
(c) 245 km
3. (a) 220 km (b) 200 km
(c) 375 km
4. (a) 95 km (b) 55 km
(c) 490 km
5. (a) 70 km (b) 155 km
(c) 40 km
6. (a) 450 km (b) 460 km
(c) 330 km
7. 25.5 km **8.** 14 km
9. 67 km **10.** 41.5 km
11. 76.5 km **12.** 53 km
13. 33 km **14.** 36 km
15. 27 km **16.** 51.5 km
17. 54 km **18.** 42.5 km
19. 12.5 cm **20.** 7.1 cm
21. 1.1 cm **22.** 13.9 cm
23. 5.9 cm **24.** 0.9 cm
25. 7.2 cm **26.** 6.6 cm
27. 10.5 cm **28.** 7.3 cm
29. (a) 1275 m (b) 1215000 m^2
(c) 121.5 ha
30. (a) 900000 cm^2
(b) 1:1000000
(c) 1:1000
31. (a) 86400 cm^2 (b) 1:1600
32. 1.35 m^2
33. (a) 50 m (b) 2500 m^2
(c) 125000 m^3
34. (a) 30 m (b) 900 m^2
(c) 27000 m^3
35. (a) 1:50 (b) 2464 cm^2
(c) $11498\frac{2}{3}$ cm^3

Exercise 4v

1. (i) 36000 m $= 3.6 \times 10^4$ m
2. 687500 cm^3, 111 cones,
3739 cm^3
3. (i) 22.56 m^2 (ii) 564 tiles
4. $t = 0.0025$ cm
(i) $A = 1000$ cm^2
(ii) $t \propto \frac{1}{A}$ (iii) $t = \frac{0.5}{A}$
5. (i) 7.65 mm (ii) 18165 m^2
6. (i) $d = 70$ m
(ii) (a) $YZ = 77$ m
(b) $XY = 22$ m
7. (i) 3 h 10 min
(ii) (a) 36000 cm^3
(b) 108000 cm^3
8. (i) 2.6 m^2
(ii) (a) 0.15 m^3
(b) 1.5×10^{-1} m^3
(c) 90 kg
9. (ii) (a) 3 windows
(b) 1250 m^2
10. (a) (i) 12000 cm^3
(ii) 12 *l*
(iii) 310 cm^3
(b) 11 balls
11. 400 *l*
12. (a) 60 cm^2 (b) 39 cm^2
(c) 63 cm^2 (d) 547 cm^2
13. (a) (i) $AB = 5$ cm
(ii) 4.3 cm
(b) $A = 28.8$ cm^2
(c) 1:50
(d) (i) $ED = 3.5$ m
(ii) 7.2 m^2
14. (a) (i) 60000 cm^3
(ii) 20 cm
15. (a) (i) $BC = 12.56$ m
(ii) 62.8 m^2
(b) 8.14 rolls = 9 rolls
(c) 9420000 cm^3
16. (a) 62.5 km
(b) (i) 151200 cm^3
(ii) 151.2 *l*
(iii) 20 cm
17. (i) 136 cm^2 (ii) 10 cm
18. (i) 30.8 cm
(ii) 75.5 cm^2
(iii) 9.4 cm^2
19. $m = 21$ kg
20. $V = 720$ cm^3

Exercise 5a

1. \$2680 **2.** \$1860
3. \$4853 **4.** \$6521
5. \$12560 **6.** \$78516
7. \$61788 **8.** \$58092
9. \$109764 **10.** \$50148

11. $58 788 **12.** $61 296
13. $2 928 **14.** $3 045
15. 3 879

Exercise 5b

1. $205.20
2. $41\frac{1}{4}$ h. $299.06
3. $250
4. 75 h. $656.25 **5.** $748
6. 38 h **7.** 35 h
8. 80 h **9.** 76 h
10. 75 h **11.** $6.35
12. $8.75 **13.** $9.85
14. $8.45 **15.** $5.39
16. $7.54

Exercise 5c

1. $390.00 **2.** $1 006.40
3. (a) $224.00 (b) $63.00
(c) $67.20 (d) $75.60
(e) $429.80
4. $649.60 **5.** (a) $584
(b) $551.15 (c) $1 135.15
6. (a) $9.25 (b) 8 h
7. 10 h **8.** $8.90
9. 44 h
10. (a) $8.36
(b) 15 h

Exercise 5d

1. $1 750 $5 250
2. $36.90 $161.90
3. $76.25 $216.25
4. $210 $435
5. (a) $3 450 (b) $2 500
6. (a) $28.50 (b) $178.50
7. (a) $43.24 (b) $218.24
8. (a) $493.75 (b) $1 073.75
9. (a) $171 (b) $6 840
10. (a) $207.55 (b) $5 930

Exercise 5e

1. (a) $4 075 (b) $21 525
(c) $8 610 (d) $717.50
2. $43 070. $10 767.50
3. $32 370. $6 474
4. (a) $102.50 (b) $5 838
(c) $18 762 (d) $349.21
5. (a) $7 233.20 (b) $22 226.80
(c) $5 356.70
6. (a) $105 (b) $5 256
(c) $19 944 (d) $398.83
7. $29 340
8. (a) $2 400 (b) $14 600
(c) $17 000 (d) $11 000
9. (a) $160 (b) $5 792
(c) $32 608 (d) $662.67
10. (a) (i) $6 300
(ii) $39 300
(iii) $12 055
(iv) $33 545
(b) (i) $3 300
(ii) $42 300
(iii) $13 105
(iv) $32 495

Exercise 5f

1. 25%
2. (a) $75 (b) 27.3%
3. 50%
4. Profit = 91.3%
5. (a) $141.75 (b) 15%
6. (a) $989 (b) $731
7. (a) $12 060.16 (b) $10 552.64
8. (a) $1 218.75 (b) $1 096.88
9. (a) $1 618.50 (b) $1 431.75
10. (a) $1 619.15 (b) $1 329.77

Exercise 5g

1. $68.57 **2.** $1 460
3. $2 250 **4.** $6:00 $7.50
5. $12 480 **6.** $1 080
7. $150 **8.** 66 shops
9. 50.4 kg **10.** $2 450
11. (a) $6.00
(b) 900 *l*
(c) $2 520 more
12. 58.24 kg **13.** 33 teachers
14. $316.80 **15.** 506 students
16. 100 kg
17. (a) $1.63
(b) 810 *l*
(c) $11.70 less
18. (a) 2% (b) 17.6%
19. $245 **20.** 44.4%
21. $204.75 **22.** $1 560
23. $12 600 **24.** $582.40
25. $315 **26.** $2 509.09
27. $1 380 **28.** $598
29. $7 700 **30.** $7 500
31. (a) $6 135.86
(b) $5 957.14

Exercise 5h

1. $106.25 **2.** $492.80
3. (a) $3 072 (b) $2 764.80
4. $931 **5.** $204.75
6. $345
7. (a) $2 083.33 (b) $2 250
(c) $1 875
8. $106.25 **9.** $139.40
10. (a) $2 250 (b) $2 430
11. (a) $1 460 (b) $1 733.75

Exercise 5i

1. $3 220 **2.** $2 378.88
3. $1 110 **4.** $4 370
5. $6 670
6. (a) $2 550 (b) $12 000
7. $865 **8.** $1 300
9. $1 190.25 **10.** $41 328
11. (a) $4 200 (b) $55 800
12. (a) $4 375 (b) $45 600

Exercise 5j

1. (a) $5 775. $4 050. $1 725.
(b) 38.3%
2. (a) $19 040 (b) 13 866.67
3. (a) $1 517 (b) $1 050
4. $2 242 $1 652.17
5. (a) $2 550 (b) $3 466
(c) 30.5%
6. (a) $1 501.50 (b) $59 895
7. (a) $3 200. $2 250. $950.
(b) 38%
8. (a) $2 990 (b) 2 133.33
(c) 25.2%
9. (a) $446.25 (b) $3 480.75
(c) $505.75
(d) % difference = 17%
10. (a) (i) $446.25
(ii) $252.88 (iii) $3 480.75
(b) (i) $2 618 (ii) $862.75
(iii) 29%
11. (a) $3 088.80 (b) $212.40
(c) $388.80 (d) 14.4%
12. (a) $3 271.75 (b) $426.75
(c) 20%
13. (a) $3 252 (b) $2 117.39

Exercise 5k

1. $13 500 **2.** $22 500
3. (a) $157 500 (b) $148 750
4. (a) $131 250 (b) $123 750

5. (a) $27 500 (b) $247 500
(c) $1 020 900
6. (a) $7 600 (b) $87 400
(c) $251 100
7. (a) $18 500 (b) $166 500
(c) $491 040 (d) $324 540
8. (a) $18 750 (b) $106 250
(c) $255 000 (d) $148 750
9. (a) $35 250 (b) $199 750
(c) $678 960 (d) $714 210
10. (a) $9 500 (b) $85 500
(c) $218 880 (d) $228 380

Exercise 5l

1. $945 **2.** $1 125
3. $805 **4.** $2 164.80
5. $1 734.75
6. $0.33 in the $1 = 33%
7. $0.43 in the $1 = 43%
8. (a) $4 000 000
(b) $0.20 in the $1 = 20%
9. (a) $91 695 000
(b) $0.20 in the $1 = 20%
10. $0.25 in the $1 = 25%
11. $500 **12.** $365
13. $60 579 310 **14.** $5 500
15. $5 750

Exercise 5m

1. $83.43 **2.** $154.38
3. $4 714.94 **4.** $2 127.50
5. $6 441.60

Exercise 5n

1. $60.55 **2.** $82.80
3. $3 945.92 **4.** $104.75
5. $37 808.55

Exercise 5o

1. (a) $0.90 (b) $3.35
2. (a) $2.80 (b) $4.95
3. (a) $0.80 (b) $4.05
4. (a) $0.80 (b) $5.55
5. (a) $1.75 (b) $9.10

Exercise 5p

1. 36 units **2.** 7.5 units
3. 3.6 units **4.** 60 units
5. 0.3125 units **6.** 105 kWh
7. 7.5 kWh **8.** 8.4 kWh
9. 1.14 kWh **10.** 37.8 kWh
11. $185.82 **12.** $157.64
13. $232.67 **14.** $296.70
15. $358.95 **16.** $52.24
17. $139.27 **18.** $371.49
19. (a) 193 units (b) $28.95
(c) $70.40 (d) $124.35
20. (a) 9 210 kWh (b) $3 223.50
(c) $1 657.80 (d) $4 881.30
21. (a) 13 564 units (b) $2 712.80
(c) $6 103.80 (d) $8 816.60
22. (a) 125 kWh (b) 22.50
(c) $56.25 (d) $78.75
(e) $70.88
23. $275.43
24. (a) 19 000 kWh (b) $1 520
(c) $4 750 (d) $6 270
25. $279.10 **26.** $536.13
27. (a) 24 948 kWh
(b) $4 989.60 (c) $8 731.80
(d) $13 721.40 (e) $12 349.26

Exercise 5q

1. $296.75 **2.** $224.99
3. $41.20 **4.** $39.80
5. $148.93 **6.** $239.05
7. $333.60 **8.** $248.80
9. (a) $594.30 (b) $97.74
(c) $109.21 (d) $837.25
10. $370.00 **11.** $481.45
12. $252.31
13. (a) (i) 114 calls (ii) $17.80
(iii) $118.50 (iv) $37.75
(b) 103 calls

Exercise 5r

1. (a) TT $5 827.50
(b) US $425
2. (a) TT $4 410
(b) BDS $582.5
3. (a) J $12 200 (b) US $50
(c) EC $135
4. (a) TT $5 670
(b) CAN $756.80
5. (a) EC $53.75 (b) US $314
6. (a) BDS $1 350 (b) US $300
7. (a) TT $768.75 (b) EC $2 000
8. (a) TT $916.50
(b) TT $6.00 (c) $6 127.50
9. EC $3 267
10. (a) TT $7 560 (b) TT $6 912
11. J $5 875
12. (a) TT $7 380 (b) CAN $125
13. BEL $69.50
14. (a) BDS $1 500
(b) US $500
15. (a) TT $6 110 (b) US $350
16. (a) £350 (b) TT $1 121.10
17. (a) BDS $2 250
(b) US $505.05
18. TT $7 320
19. (a) TT $31.50 (b) 5 500 yen
20. (a) TT $73.44
(b) TT $43.20. TT $7 227.36
21. US $150

Exercise 5s

1. (a) $I = \$3\,840$
(b) $A = \$13\,440$
2. (a) $I = \$240$
(b) $A = \$840$
3. $A = \$558$
4. (a) $I = \$360$
(b) $A = \$1\,860$
(c) $51.67
5. (a) $I = \$4\,042.50$
(b) $A = \$14\,542.50$
6. (a) $I = \$1\,989$
(b) $A = \$9\,639$
(c) $200.81
7. (a) $I = \$5\,415.75$
(b) $A = \$17\,865.75$
8. (a) $I = \$2\,536.50$
(b) $A\ \$7\,876.50$
(c) $131.28
9. (a) $I = \$161$
(b) $A = \$3\,841$
(c) $640.17
10. (a) $I = \$356.40$
(b) $A = \$6\,116.40$
(c) $679.60
11. $P = \$980$ **12.** $P = \$500$
13. $P = \$900$ $A = \$942$
14. $P = \$1\,500$ **15.** $P = \$9\,760$
16. $P = \$12\,500$ **17.** $P = \$8\,500$
18. $P = \$5\,340$ **19.** $P = \$8\,560$
20. $P = \$3\,200$
21. $R = 6.25\%$ p.a.
22. $R = 5\%$ p.a.
23. $R = 12.5\%$ p.a.
24. $R = 8.8\%$ p.a.
25. $R = 7.5\%$ p.a.
26. $R = 10.8\%$ p.a.
27. $R = 9.5\%$ p.a.
28. $R = 8.5\%$ p.a.
29. $R = 12.5\%$ p.a.

30. $R = 12.25\%$ p.a.
31. $T = 5$ years
32. $T = 10$ years
33. $T = 7$ years
34. $T = 4.5$ years
35. $T = 5$ years
36. $T = 4$ years
37. $T = 5$ years
38. (a) (i) $1 501.50
(ii) $59 895
(b) (i) $65 835 (ii) $1 828.75
39. (a) $3 272 (b) $A = \$3414$
(c) H.P. better. $142
40. $225 **41.** $A = \$6442.88$
42. (a) $A = \$6450$
(b) $A = \$6442.88$
(c) (a) was the better investment $7.12
43. (a) $I = \$1050$
(b) $P = \$10000$
44. $A = \$9757.80$
45. $4 360 $362.97
46. $420
47. $A = \$500000$
48. $I = \$150150$
49. $A = \$530100$
50. $I = \$129600$
51. $A = \$1340625$
52. $I = \$708750$
53. $A = \$542500$
54. $A = \$681615$
55. $I = \$507000$

Exercise 5t

1. C.I. = $86.94
2. C.I. = $135.03
3. C.I. = $2 207.55
4. C.I. = $5 631.94
5. C.I. = $5 635.72
6. (a) S.I. = $216
(b) C.I. = $199.68 $16.32. Simple interest better.
7. $A = \$8590.50$
8. S.I. = $1 260 C.I. = $1 417.25 Compound interest better.
9. (a) $A = \$16897.27$
(b) $A = \$16100$
(c) $797.27. Compound interest better.
10. S.I. = $1 050 C.I. = $1 251.83 $201.83. Compound interest better.
11. S.I. = $2 137.50
C.I. = $1 341.08 $796.42. Simple interest better.
12. S.I. = $525 C.I. = $562.61 $37.61. Compound interest better.
13. S.I. = $2 520 C.I. = $3 408.38 $888.38. Compound interest better.
14. S.I. = $2 422.50
C.I. = $2 870.84 $448.34. Compound interest better.
15. $A = \$282960$
16. C.I. = $101 844
17. $A = \$334225$
18. C.I. = $95 742
19. $A = \$110520$
20. C.I. = $89 625
21. $A = \$107175$
22. C.I. = $337 575
23. $A = \$150937.50$
24. $A = \$55939.95$
25. C.I. = $208 187.50

Exercise 5u

1. $A = \$21930.52$
2. $A = \$22692.47$
3. $A = \$19798.29$
4. $A = \$5566.01$
5. $A = \$25509.17$
6. $A = \$25793.25$
7. $A = \$48600$
8. (a) $A = \$66.55$
(b) $A = \$36.45$
9. $A = \$42930.92$
10. $A = \$9654.87$
11. (a) $A = \$43581.20$
(b) 72.25% (c) 27.75%
12. (a) $A = \$53461.48$
(b) 68.1% (c) 31.9%
13. $A = \$12649.32$ $2 110.68
14. $A = \$6591.12$ $2 878.88
15. $A = \$27481.50$ $71 018.50
16. $A = \$3870$ $4 730
17. $A = \$67654.50$ $91 345.50
18. $A =$ M $1.47 M $1.03

Exercise 5v

1. (ii) $7.50
2. (i) (a) J $79.20 (b) G $74.63
(ii) (a) $125.00 (b) $135.00
3. (i) $125 60 (ii) $3 624
4. (i) $29.37
(ii) (a) $5.59 (b) $31.39
(c) $160.39 (d) $2 224.39
5. (i) (a) $120.00 (b) $3 280.00
(ii) $3 360.00
(iii) H.P. is less costly. $80.00
6. (ii) $1 950.00
7. (i) J $450.00
(ii) US $150.00
(iii) EC $45.00
8. (i) $1 080 (ii) $2 550
(iii) $3 630 (iv) $3 748
9. (a) (i) $1 680 (ii) $1 250
(b) (i) BDS $1 200
(ii) US $500
10. (a) (i) $2 550 (ii) $12 000
(b) (i) 21 053 units
(ii) $6 315.90
(iii) $9 473.85
(iv) $15 789.75
11. (a) $32 (b) $20
(c) 100% (d) $1 400
12. (b) (i) 3 h
(ii) $41\frac{1}{4}$ km/h
(iii) 13:35 h
13. (a) BEL $281.60
(b) (i) $2 000 (ii) $9 000
14. (a) $1 344.40 (b) $30.00
(c) (i) $3 673.60 (ii) $5 320
15. (a) (i) 77 calls (ii) $9.40
(iii) $60.00 (iv) $32.50
(b) 95 calls
16. (a) (i) 3 h 20 min
(ii) $473\frac{1}{3}$ m.p.h.
(b) 18:20 h
17. (a) (i) $2 400 (ii) $120
(iii) $720 (iv) 4
(b) 15%
18. (i) (a) $125 (b) $135

Exercise 6a

1. $9x$ **2.** $12x + y$
3. $11x - y$
4. $\frac{3}{4}xy = \frac{3xy}{4}$
5. $\frac{5xy}{z}$ **6.** $\frac{7xy - 5}{3z}$
7. $\frac{1}{2}xy - 5z = \frac{xy}{2} - 5z$
8. $\frac{3x - 4y}{7z}$
9. $(x + y)^2$ **10.** $(x + y)^3$

11. $\frac{3a + 4b}{2c}$
12. $\frac{1}{2}ab + 3c = \frac{ab}{2} + 3c$
13. $(3a - 2b)^2$
14. $(2a - 3b)^3$
15. $\frac{9ab - 5c}{d}$
16. Seven times a number x.
17. Nine times a number x plus a second number y.
18. Five times a number x minus a second number y.
19. Half the product of two numbers x and y.
20. The square of the difference of two numbers x and y.
21. The cube of the difference of two numbers x and y.
22. Double a number a added to thrice a second number b divided by four times a third number c.
23. Half the product of two numbers a and b minus thrice a third number c.
24. The square of four times a number a take away thrice a second number b.
25. The cube of thrice a number a take away four times a second number b.
26. $\frac{5x - 4y}{2z}$
27. $\frac{x + y}{4z} = \frac{1}{4}\frac{x + y}{z}$

Exercise 6b

1. 7
2. 6
3. 6
4. 3
5. 7
6. 11
7. 1
8. 1
9. 2
10. 7
11. 5
12. 3
13. 18
14. 32
15. 24
16. -12
17. -24
18. -60
19. 1
20. 1
21. 2
22. 8
23. 5
24. 6
25. 3
26. 2
27. $1\frac{1}{2}$
28. 8
29. 1
30. 12
31. 16
32. 24
33. 4
34. 11
35. 6
36. 17
37. 16
38. 25
39. 9
40. 10
41. 4
42. 7
43. 4
44. 1
45. 7
46. 20
47. 25
48. $1\frac{7}{9}$
49. $2\frac{1}{3}$
50. $2\frac{3}{4}$
51. 1
52. 31
53. $\frac{1}{4}$
54. 30
55. 9
56. $12\frac{1}{2}$
57. $1\frac{9}{14}$
58. $3\frac{1}{3}$
59. -43
60. -53
61. 4
62. 9
63. 16
64. 8
65. -27
66. 64
67. 20
68. 27
69. 32
70. 32
71. -135
72. 128
73. 18
74. -54
75. -12
76. 16
77. 32
78. -48
79. 576
80. 36
81. -216
82. 64
83. -36
84. 54
85. 128
86. 1280
87. -168
88. 0
89. 28
90. 176
91. -76
92. 68
93. -72
94. -3
95. $4\frac{1}{2}$
96. $-10\frac{2}{3}$
97. $-5\frac{1}{3}$
98. $6\frac{3}{4}$
99. $-7\frac{1}{9}$
100. $-3\frac{3}{4}$
101. $-10\frac{2}{25}$
102. 18
103. $5\frac{5}{9}$
104. -25
105. -4
106. $5\frac{25}{27}$
107. -128
108. -200

Exercise 6c

1. $8x$
2. $11x$
3. $12y$
4. $15y$
5. $11a$
6. $16p$
7. $6x$
8. $3x$
9. $5x$
10. $4y$
11. $7r$
12. $7p$
13. $-4x$
14. $-4x$
15. $-3y$
16. $-7p$
17. $-r$
18. $-s$
19. $15x$
20. $14x$
21. $18x$
22. $12x$
23. $14x$
24. $14p$
25. $-15x$
26. $-20x$
27. $-14x$
28. $-10y$
29. $-15y$
30. $-8q$
31. $16x$
32. $8x$
33. $9x$
34. $2x$
35. $7y$
36. $8p$
37. $10x$
38. $12x$
39. $12x$
40. $10x$
41. $5x$
42. $13x$
43. $-2x$
44. $4y$
45. $10x$
46. $p + r$
47. $a + 2b + 3c$
48. $x + 2y$
49. $p + 3d - 2c$
50. $a - 3b + 4c$
51. $5x + 14y + 9z$
52. $8x + 15y + 6z$
53. $7x + 13y + z$
54. $10x + 15y - z$
55. $5x + 19y - 6z$
56. $4x + 13y - z$
57. $12x^2 + 4y^2$
58. $6x^2 - y^2$
59. $7x^2 + 4y^2$
60. $7x^2 + 3y^2$
61. $4x^2 - 4y^2$
62. $6x^2 - 3y^2$
63. $4a^2b - ab^2 + 3a^2b^2$
64. $3a^2b + 2ab^2 + 2a^2b^2$
65. $3x^2 + 4y^2 + 5x^3$
66. $2x^3 + 2y^3 + 3x^4$
67. $4p^2 + 14q^2 - 7pq^2$
68. $3r^2 + 2s^2 + 5r^2s$
69. $4.8p^2q + 5.6pq^3$
70. $4.9r^2s - 6.3rs^2$
71. $1.6x^2 + 2.1y^2$
72. $3.4p^2 + 2.4q^2$
73. $5.2x + 0.3y$
74. $2.3x + 1.8y$
75. $2x - 12xy + 8$
76. $5x^2 + 5xy + 8y - 5$
77. $7xy + 11x^2y^2$
78. $2.3x^3 + 2.2x^2 + 1.1x + 12.4$
79. $3x^2 - 2xy + 7y - 4$
80. $9x - 13y - 1$

81. $(a + b - c)x$
82. $x + 5y + 5$
83. $3x^2 + 7x + 3y - 7$
84. $(a - c)x + (d - b)x^2 + ey$

Exercise 6d

1. $6x$
2. $15y$
3. $28p$
4. $12x$
5. $28y$
6. $25q$
7. $6xy$
8. $12xy$
9. $28pq$
10. $20x^2$
11. $21y^2$
12. $36p^2$
13. $-8x$
14. $-15y$
15. $-30p$
16. $-12x$
17. $-35y$
18. $-27p$
19. $6x$
20. $35x$
21. $56x$
22. $-12xy$
23. $-20xy$
24. $-40pq$
25. $-6xy$
26. $-28xy$
27. $-30xy$
28. $12xy$
29. $40xy$
30. $54pq$
31. $3y$
32. $3x$
33. $3p$
34. $-5x$
35. $-4x$
36. $-8q$
37. $6xy$
38. $6pq$
39. $4qr$
40. $-4xy$
41. $-4xy$
42. $-9pq$
43. $8xy$
44. $14xy$
45. $14xy$
46. $9x^2$
47. $12x^2$
48. $9p^2$
49. $6x^2$
50. $10x^2$
51. $7x^2$
52. $-12x^2$
53. $-8x^2$
54. $-3x^2$
55. $-22x^2$
56. $-16x^2$
57. $-12p^2$
58. $3pqr$
59. $30pqr$
60. $30xyz$
61. $-40xyz$
62. $-30pqr$
63. $-84rst$
64. $30xyz$
65. $84pqr$
66. $48xyz$
67. $60xyz$
68. $84xyz$
69. $60pqr$
70. $42xyz$
71. $108pqr$
72. $105lmn$
73. $24x^2y$
74. $30xy^2$
75. $30p^2r$
76. $60xy^2$
77. $60pq^2$
78. $30rs^2$
79. $24x^2y$
80. $60y^2z$
81. $42p^2q$
82. $24x^2y$
83. $24x^2y$
84. $60y^2z$
85. $24x^2y$
86. $24p^2q$
87. $24r^2s$
88. $60xy^2$
89. $105pq^2$
90. $48p^2q$
91. $-60x^2y$
92. $-84pq^2$
93. $-60x^2y$
94. x^3
95. $24x^3$
96. $24p^3$
97. $-60x^3$
98. $-36y^3$
99. $-72p^3$
100. $140x^3$
101. $90y^3$
102. $96p^3$
103. $140x^3$
104. $180y^3$
105. $360p^3$
106. $30x^3$
107. $108y^3$
108. $224p^3$
109. $20p^3q^4$
110. $36x^5y^3$
111. $21r^3s^5$
112. $24x^5y^3$
113. $35x^3y^4$
114. $20x^4y^3$
115. $-35x^4y^2$
116. $-72x^3y^3$
117. $-36x^3y^4$
118. $-36x^3y^3$
119. $-27x^3y^3$
120. $-72p^3q^4$
121. $30p^4q^5$
122. $60x^4y^5$
123. $140x^6y^5$
124. $-60x^4y^5$
125. $-108x^4y^4$
126. $-60p^4q^5$
127. $60x^4y^4$
128. $140x^4y^5$
129. $160p^5q^6$
130. $60x^6y^4$
131. $180x^6y^5$
132. $216p^5q^6$
133. $-15q^4r^3$
134. $-30m^5n^4$
135. $-30m^4n^4$
136. $4a$
137. $13x$
138. $11y$
139. $-4x$
140. $-3y$
141. $-3p$
142. $7\frac{a}{b}$
143. $8\frac{x}{y}$
144. $12\frac{p}{q}$
145. $3\frac{x}{y}$
146. $27\frac{y}{a}$
147. $6\frac{p}{q}$
148. $-7\frac{x}{y}$
149. $-11\frac{p}{q}$
150. $-19\frac{x}{p}$
151. $-4\frac{x}{y}$
152. $-3\frac{p}{q}$
153. $-3\frac{p}{q}$
154. 1
155. 1
156. -1
157. -1
158. $3\frac{x^2z^2}{y}$
159. $7\frac{y^2z^2}{x}$
160. $9p^2qr^2$
161. $11\frac{y^2}{x^2}$
162. $12p^2q^2$
163. $13r^3s^3$
164. $-7x^2yz^2$
165. $-10p^2q^2r^2$
166. $-15x^2yz^2$
167. $-8x^2yz$
168. $-12\frac{p^2}{q^3}$
169. $-13p^3q^2$
170. $\frac{7}{3}x^2y^2z^2$
171. $-3\frac{x^2}{y^2}$
172. $\frac{3}{2}\frac{y^2z^3}{x}$
173. $\frac{1}{2}ab^2$
174. $\frac{1}{4}\frac{s}{t^2}$

Exercise 6e

1. $2x + 6$
2. $3x + 12$
3. $5x + 35$
4. $6y + 18$
5. $9p + 45$
6. $2x - 6$
7. $3x - 15$
8. $5x - 35$
9. $6y - 24$
10. $9p - 27$
11. $-3x - 12$
12. $-4x - 20$
13. $-8x - 24$
14. $-9y - 45$
15. $-10p - 30$
16. $-4x + 12$
17. $-5x + 35$
18. $-6x + 54$
19. $-8y + 32$
20. $-9p + 27$
21. $3x + 3y$
22. $8x + 8y$
23. $9x + 9y$
24. $7p + 7q$
25. $10r + 10s$
26. $4x - 4y$
27. $5x - 5y$
28. $8x - 8y$
29. $7p - 7q$
30. $12r - 12s$
31. $-5x - 5y$
32. $-7x - 7y$
33. $-8x - 8y$
34. $-9p - 9q$
35. $-15r - 15s$
36. $-3x + 3y$
37. $-42 + 4y$
38. $-5x + 5y$
39. $-6p + 6q$
40. $-13r + 13s$
41. $3x + \frac{3}{2}$
42. $3x + 2$
43. $5x + 1$
44. $6x + 7y$
45. $7x + y$
46. $x - 2$
47. $x - 2$
48. $9x - 1$
49. $10x - y$
50. $12x - y$
51. $-2x + 3$
52. $-4x + 1$
53. $-5x + 1$
54. $-7x + y$
55. $-8x + y$
56. $-9x - 1$
57. $-10x - 1$
58. $-12x - 1$
59. $-7x - 1$
60. $-6x - 2$
61. $10x + 15$
62. $21x + 28$
63. $40x + 24$
64. $63y + 36$
65. $36p + 24$
66. $6x - 9$
67. $28x - 20$
68. $40x - 35$
69. $30y - 18$
70. $63p - 35$
71. $-15x - 10$
72. $-30x - 18$
73. $-56x - 40$
74. $-27y - 45$
75. $-50p - 70$
76. $-42x + 18$
77. $-21y + 28$
78. $-32y + 40$
79. $-45x + 63$
80. $-30p + 40$
81. $12x + 9y$
82. $15x + 10y$
83. $30x + 42y$
84. $24x + 32y$
85. $63x + 45y$
86. $-15x - 6y$
87. $-24x - 18y$
88. $-21x - 35y$
89. $-32x - 24y$
90. $-45x - 72y$
91. $-20x + 12y$
92. $-15x + 25y$
93. $-56x + 21y$
94. $-56x + 16y$
95. $-72p + 27p$
96. $15x - 6y$
97. $56x - 24y$
98. $72x - 45y$
99. $36p - 24q$

100. $65r - 52s$ **101.** $8x + 8y$
102. $7x + 7y$ **103.** $12x + 12y$
104. $15x + 15y$ **105.** $12p + 12q$
106. $8x - 8y$ **107.** $7x - 7y$
108. $9x - 9y$ **109.** $11p - 11q$
110. $11p - 11q$ **111.** $4x + 12y$
112. $6x + 12y$ **113.** $3x + 11y$
114. $3x + 13y$ **115.** $2p + 10q$
116. $x - y$ **117.** $x - y$
118. $3x - 3y$ **119.** $5p - 5q$
120. $5r - 5s$ **121.** $24x + 17y$
122. $21x + 25y$ **123.** $33x + 43y$
124. $42x + 21y$ **125.** $76x + 32y$
126. $9x + 6y$ **127.** $24x + 3y$
128. $23x + 13y$ **129.** $17x - y$
130. $25x - 8y$ **131.** $-14x - 5y$
132. $-3x - 12y$ **133.** $10x - 9y$
134. $5x - 5y$ **135.** $39x - 20y$
136. $-6x - 17y$ **137.** $-12x + 9y$
138. $3y$ **139.** $22x - 9y$
140. $27x - 6y$
141. $4x + 1\frac{1}{2}$ **142.** $5x + \frac{2}{3}$
143. $2x + \frac{3}{5}$ **144.** $15x + 2$
145. $16x + 2$ **146.** $-2x - 1\frac{1}{2}$
147. 0 **148.** $-2x - \frac{1}{4}$
149. $4x$ **150.** $-3x + \frac{2}{3}$
151. $\frac{3}{4}$ **152.** $-7x + 1\frac{1}{5}$
153. $\frac{3}{7}$ **154.** $x + \frac{7}{12}$
155. $-2x + \frac{4}{9}$ **156.** 0
157. $\frac{1}{6}$ **158.** $2x + \frac{3}{10}$
159. $-\frac{1}{4}$ **160.** $6\frac{1}{2}x - \frac{4}{7}$
161. $5x - 2y$ **162.** $9n - m$
163. $3y - 2x$ **164.** $2p - 6r$
165. $12x + 15$ **166.** $15x - 20$
167. $6x + 7$
168. $2x + \frac{1}{2}y - 1\frac{5}{6}$
169. $4a + 14$ **170.** $m + 5n$
171. $6x - 15$
172. $\frac{1}{3}x + \frac{2}{5}y - 2\frac{1}{3}$
173. $4x - 27$
174. $a^2 - 3ab - 2b^2$
175. $-10x + 8x^2 + 6x^3$
176. $8x - y + 3$
177. $8x + 5y - 7$ **178.** $3x + 3$
179. $4x - 4$ **180.** $24rt - 16rs$
181. $12xy + 3xz$ **182.** $2x - 15y$
183. $7x - y$ **184.** $30x - 16$
185. $13x + 7y$ **186.** $31x - 11$
187. $7x - y$

Exercise 6f

1. 3 **2.** 1 **3.** 159
4. (a) (i) 11 (ii) 15 (b) $x = 100$
5. (a) 14 (b) 30
6. $\sqrt{14}$ **7.** 69 **8.** 5
9. 19 **10.** -37 **11.** 58
12. 42 **13.** -39
14. (a) 11 (b) 123 **15.** 75

Exercise 6g

1. $6(x + y)$ **2.** $9(x + y)$
3. $m(x + y)$ **4.** $q(x + y)$
5. $z(x + y)$ **6.** $5(x - y)$
7. $7(x - y)$ **8.** $8(x - y)$
9. $m(x - y)$ **10.** $r(x - y)$
11. $-3(x - y)$ **12.** $-7(x - y)$
13. $-8(x - y)$ **14.** $-p(x - y)$
15. $-q(x - y)$ **16.** $-5(x + y)$
17. $-6(x + y)$ **18.** $-7(x + y)$
19. $-p(x + y)$ **20.** $-r(x + y)$
21. $5a(5a + 1)$ **22.** $6p(6\mathrm{p} + 1)$
23. $7r(7r + 1)$ **24.** $8x(8x + 1)$
25. $12a(12a + 1)$ **26.** $3p(3p - 1)$
27. $4r(4r - 1)$ **28.** $9x(9x - 1)$
29. $12y(12y - 1)$ **30.** $13x(13x - 1)$
31. $-5r(5r + 1)$ **32.** $-7y(7y + 1)$
33. $-8s(8s + 1)$ **34.** $-9x(9x + 1)$
35. $-11p(11p + 1)$
36. $-2x(2x - 1)$ **37.** $-3y(3y - 1)$
38. $-4r(4r - 1)$ **39.** $-5p(5p - 1)$
40. $-8s(8\mathrm{s} - 1)$ **41.** $5(5x + 2)$
42. $12(3x + 1)$ **43.** $7(7x + 2)$
44. $16(4x + 1)$ **45.** $9(9y + 2)$
46. $2(2x - 1)$ **47.** $3(3x - 2)$
48. $8(2x - 1)$ **49.** $5(5y - 2)$
50. $18(2p - 1)$ **51.** $-7(7p + 3)$
52. $-27(3x + 1)$
53. $-10(10x + 3)$
54. $-11(11x + 5)$
55. $-12(12x + 3)$ **56.** $-5(5 - 2)$
57. $-18(2x - 1)$ **58.** $-7(7x - 3)$
59. $-8(8y - 3)$ **60.** $-9(9y - 4)$
61. $4(4x + 1)$ **62.** $5(5y + 1)$
63. $6(6p + 1)$ **64.** $8(8q + 1)$
65. $10(10r + 1)$ **66.** $5(5x - 1)$
67. $6(6y - 1)$
68. $7(7p - 1)$ **69.** $9(9y - 1)$
70. $11(11r - 1)$ **71.** $-5(5x + 1)$
72. $-7(7y + 1)$ **73.** $-8(8p + 1)$
74. $-10(10r + 1)$
75. $-12(12s + 1)$
76. $-2(2x + 1)$ **77.** $-3(3y + 1)$
78. $-5(5p + 1)$ **79.** $-6(6r + 1)$
80. $-8(8s + 1)$
81. $5w(p + 3q + 4r)$
82. $3r(s + 3t + 6u)$
83. $8p(q + 2r + 3s)$
84. $9a(9b + 3c + 4d)$
85. $36p(a + 2b + 4c)$
86. $-5l(x - 3y - 5z)$
87. $-8p(q - 3r - 4s)$
88. $-9a(b - 4c - 5d)$
89. $-5r(2a - 5b - 7c)$
90. $-12r(t - 2u - 3v)$
91. $-5r(a + 2b - 3c)$
92. $-6r(x + 6y - 3z)$
93. $-7p(x + 3y - 4z)$
94. $-8l(x + 3y - 4z)$
95. $-9g(k + 3l - 2m)$
96. $-3p(a + 3b + 6c)$
97. $-5g(l + 5m + 2n)$
98. $-7a(r + 3s + 2t)$
99. $-8t(x + 2y + 3z)$
100. $-9l(9a + 4b + 5c)$
101. $9(x - 3y^3)$ **102.** $5(x + 1)$
103. $6(3x - 1)$ **104.** $3x(x - 9)$
105. $4x^2(x^2 + 4)$
106. $4bc(5a - 2d)$ **107.** $4g(h_1 - h_2)$
108. $\frac{1}{3}\pi r^2(4r - h)$
109. $5\pi(R^2 + 2r^2)$ **110.** $6(x - 3y^2)$
111. $7a(c - 2d)$
112. $3(3y^2 - 2y + 1)$
113. $\frac{1}{3}\pi r^2(h - 4r)$

Exercise 6h

1. $a^2b^3c^4$ **2.** p^3q^3
3. x^4y^3 **4.** $l^4m^2n^3$
5. x^4y^3 **6.** $a^3b^3c^2$
7. $p^3q^3r^4$ **8.** $x^2y^3z^3$
9. $l^3m^2n^2$ **10.** $x^4y^3z^3$
11. $3abc$ **12.** $5p^2qr^3$
13. $4x^2y^2z$ **14.** $8l^2m^2n^3$
15. $9x^2y^2z^3$
16. $\frac{a^2b}{5c^2} = \frac{1}{5}\frac{a^2b}{c^2}$
17. $\frac{3pq^2}{r^2} = 3\frac{pq^2}{r^2}$

18. $\dfrac{2x^2y^2}{7z^2} = \dfrac{2}{7}\dfrac{x^2y^2}{z^2}$

19. $\dfrac{5l^2m^2}{7n^2} = \dfrac{5}{7}\dfrac{l^2m^2}{n^2}$

20. $\dfrac{7x^3y^2}{z^3} = \dfrac{7x^3y^2}{z^3}$

Exercise 6i

1. $5xy(3x - 2y^2)$
2. $6r^2s^3(3 + 2rs)$
3. $7pr^2(p^2 - 2r)$
4. $8lm^2(lm + 2)$
5. $27x^3y^2(3xy - 1)$
6. $2ab(2a + b - 4)$
7. $2x^2(1 - 5y + 4y^2)$
8. $5x^2y^2(2x - y + 3x^2y^3)$
9. $9p^2q^2(2p + 3q - 4p^2q)$
10. $7x^3y^2(3xy^3 - 1 + 2y^2)$
11. $\dfrac{p^2m}{7}\left(pm + \dfrac{m^2}{3} - \dfrac{p^2}{2}\right)$
12. $\dfrac{l^2m}{5}\left(\dfrac{m}{3} + \dfrac{1}{2} - \dfrac{lm}{4}\right)$
13. $\dfrac{3lm}{n}\left(3l + \dfrac{m}{n} + \dfrac{2lm}{n^2}\right)$
14. $\dfrac{l^2m^2}{5pn^2}\left(\dfrac{1}{5} - \dfrac{2l}{pn} + \dfrac{3m}{2p^2}\right)$
15. $\dfrac{3m^2}{5p^2n^2}\left(\dfrac{m^2}{n} + \dfrac{3m}{2p} - \dfrac{2}{5p^2n^3}\right)$
16. $7x(7x - 1)$
17. $3xy^2(6x^2 - 2x + y^3)$

Exercise 6j

1. $(a + b)(x + y)$
2. $(m + n)(x + y)$
3. $(x + y)(p + q)$
4. $(a + 3b)(x + y)$
5. $(r + s)(a + b)$
6. $(x - 3y)(2a + b)$
7. $(2x - y)(a + b)$
8. $(m - 4n)(p + q)$
9. $(m - 2n)(x + y)$
10. $(3x - y)(r + s)$
11. $(5a - 3b)(x - y)$
12. $(3p - 7q)(x - y)$
13. $(4r - 3s)(x - y)$
14. $(4a - 5b)(r - s)$
15. $(7a - 3b)(l - m)$
16. $(a + b)(x - y)$
17. $(3a + 2b)(x - y)$
18. $(5a + 3b)(x - y)$
19. $(4p + 3q)(r - s)$
20. $(7p + 3q)(l - m)$
21. $(3x + 1)(2ab + 3pq)$
22. $(4x + 3)(3lm - 2ab)$
23. $(5x - 3)(4pq - 3ab)$
24. $(7x - 5)(5lm - 3pq)$
25. $(4x - 7)(3ab - 2pq)$
26. $(4q - 3r)(px - 1)$
    $= (1 - px)(3r - 4q)$
27. $(2x + 1)(5x - y - 1)$
28. $(2b + 3c)(x + y)$
29. $(52 - y)(3x - 1)$
30. $2(3x + y)(x - 1)$
31. $(a + 2b)(2c - a)$
32. $(a - 2c)(2a + b)$
33. $(p - q)(l + m)$
34. $(a + 5b)(p - q)$
35. $(r - s)(a - b)$
36. $(y + z)(p + y)$
37. $2(x - 2)(5 - y)$
38. $(2r - s)(p - q)$
39. $(4p + q)(r - 2s)$
40. $(mn - pq)(5x + 1)$
41. $(a - 2b)(x + y)$
42. $(x + y)(p - q)$
43. $(5x - 1)(mn - pq)$
44. $(b + 2c)(a - d)$
45. $(2x - 1)(3x + y - 1)$
46. $(a - 2p)(3 + t)$.

Exercise 6k

1. x
2. $\dfrac{3}{5}x$
3. $\dfrac{7}{20}x$
4. $-\dfrac{1}{18}x$
5. $-\dfrac{5}{12}x$
6. $\dfrac{53}{10x}$
7. $\dfrac{27}{35x}$
8. $\dfrac{8}{9x}$
9. $\dfrac{7}{16x}$
10. $\dfrac{37}{30x}$
11. $\dfrac{14 + 5q}{16pq}$
12. $\dfrac{6 + 7y}{10xy}$
13. $\dfrac{20 - 9q}{15pq}$
14. $\dfrac{16 - 21y}{56xy}$
15. $\dfrac{18 - 8y}{34xy}$
16. $\dfrac{21x^2 + 20y^2}{35xy}$
17. $\dfrac{8x^2 + 6y^2}{14xy}$
18. $\dfrac{35x^2 - 24y^2}{56xy}$
19. $\dfrac{10x^2 - 21y^2}{18xy}$
20. $\dfrac{8x^2 + 6y^2}{9xy}$
21. $\dfrac{35xz - 3y}{5z}$
22. $\dfrac{56xz + 5y}{7z}$
23. $\dfrac{3x + 28yz}{7y}$
24. $\dfrac{8x - 63yz}{9y}$
25. $\dfrac{32xz - 3y}{8z}$
26. $\dfrac{7m - 26n}{21}$
27. $\dfrac{19m - 8n}{10}$
28. $\dfrac{9x - 4y}{24}$
29. $\dfrac{2x - 3y}{4}$
30. $\dfrac{9x + 5y}{8}$
31. $\dfrac{41x + 62}{35}$
32. $\dfrac{-7x + 114}{45}$
33. $\dfrac{-38x - 22}{21}$
34. $\dfrac{2x + 49}{40}$
35. $\dfrac{4 - 11x}{24}$
36. $\dfrac{5x + 32}{12}$
37. $\dfrac{-8x - 13}{15}$
38. $\dfrac{x - 7}{20}$
39. $\dfrac{4x + 11}{6}$
40. $\dfrac{19x - 12}{40}$
41. $\dfrac{11}{35x}$
42. $\dfrac{27 + 8q}{15pq}$
43. $\dfrac{25x + 27}{24}$
44. $\dfrac{-19x - 17}{21}$
45. $\dfrac{19}{15pq}$
46. $\dfrac{9x - 3}{20}$
47. $\dfrac{-17x - 20}{24}$
48. $\dfrac{58x + 1}{10}$
49. $\dfrac{8x^2 - 5y^2}{6xy}$
50. $\dfrac{43m + n}{21}$
51. $\dfrac{11m + 39n}{15}$
52. $\dfrac{2m + 16n}{15}$
53. $\dfrac{23x + 29}{36}$
54. $\dfrac{11x + 1}{15}$

Exercise 6l

1. $\dfrac{b^2x^2}{ay}$
2. $\dfrac{c}{ab}$
3. $\dfrac{p^3r}{q^2}$
4. $\dfrac{ab^2y}{x}$
5. pqr^2
6. $\dfrac{6a^2cm}{b^3}$
7. $\dfrac{10s^2}{r}$
8. $\dfrac{3a^2c^2z}{y^4}$
9. $\dfrac{a^2c^2}{b}$
10. $\dfrac{x^2y^2}{a^2bc}$
11. $\dfrac{p^2qx}{y^2}$
12. bmn
13. $\dfrac{a^2x^2y}{bc}$
14. $\dfrac{pr}{q^2}$
15. $\dfrac{lpx}{my}$
16. $2bmn^2$
17. $\dfrac{2ar}{b^2s}$
18. $\dfrac{5q^2r}{2ps}$

19. $\frac{6pqx}{ry}$ **20.** $\frac{9lmx}{2pqy}$
21. $\frac{15pqr}{s}$ **22.** $\frac{3ac^3d}{2b^2}$
23. $6x^3yz$ **24.** $\frac{az^2}{4cy^2}$
25. $\frac{8axy^2}{9cz^2}$ **26.** $\frac{1}{4}ay$

Exercise 6m

1. $x = -7$ **2.** $x = 4$
3. $x = 9$ **4.** $x = 5$
5. $x = 7$ **6.** $x = 5$
7. $x = 5$ **8.** $y = 5$
9. $x = 7$ **10.** $c = 11$
11. $y = 7$ **12.** $b = 4$
13. $x = 15$ **14.** $x = 7$
15. $x = 14$ **16.** $x = 4$
17. $x = 8$ **18.** $x = 4$
19. $x = 9$ **20.** $y = 9$
21. $a = 6$ **22.** $p = 8$
23. $x = 4$ **24.** $y = 4.5$
25. $x = 10$ **26.** $x = 1\frac{1}{6}$
27. $x = \frac{14}{45}$ **28.** $x = 2\frac{1}{5}$
29. $x = 3$ **30.** $x = 3$
31. $y = 1$ **32.** $x = 2$
33. $x = \frac{1}{2}$ **34.** $x = 2$
35. $x = 3$ **36.** $x = 1$
37. $x = 1$ **38.** $x = -25$
39. $x = 4$ **40.** $x = 4$
41. $x = 6$ **42.** $x = 1$
43. $x = 2$ **44.** $x = 2$
45. $x = 1\frac{1}{2}$ **46.** $x = 2$
47. $x = 2$ **48.** $x = 4$
49. $y = 7$ **50.** $y = 3\frac{2}{3}$
51. $x = 1$ **52.** $x = 3$
53. $x = -2$ **54.** $y = 1$
55. $x = 2$ **56.** $x = 2$
57. $x = 1$ **58.** $x = 2$
59. $x = 2$ **60.** $x = 2\frac{1}{2}$
61. $x = 4$ **62.** $x = 3$
63. $x = 1\frac{3}{4}$ **64.** $x = \frac{4}{5}$
65. $x = 3$ **66.** $x = 5$
67. $x = 4$ **68.** $x = 5$
69. $x = 5$ **70.** $x = 1$
71. $x = 5$ **72.** $x = 2$
73. $x = -\frac{1}{2}$ **74.** $x = -\frac{1}{7}$
75. $x = 5$ **76.** $x = \frac{3}{14}$
77. $x = 2$ **78.** $x = -\frac{1}{6}$
79. $x = 3$ **80.** $x = 5$
81. $x = 6$ **82.** $x = 2$
83. $x = 3$ **84.** $x = 7$
85. $x = 6$ **86.** $x = 15$
87. $x = \frac{1}{6}$ **88.** $x = 1$
89. $x = 2$ **90.** $x = 3$
91. $x = 3$ **92.** $x = 10$
93. $x = 40$ **94.** $x = 5$
95. $x = 4$ **96.** $x = \frac{1}{2}$
97. $x = 5$ **98.** $x = 3$
99. $x = 148$ **100.** $x = 4$
101. $m = 2$ **102.** $x =6$
103. $x = 6$ **104.** $x = 3$
105. $x = 3$ **106.** $x = 6$
107. $x = 15$ **108.** $x = -1$
109. $x = \frac{3}{14}$ **110.** $x = 5$
111. $p = 6$

Exercise 6n

1. $\{x: x \geqslant 3\}$ **2.** $\{x: x < 3\}$
3. $\{x: x > 7\}$ **4.** $\{y: y \leqslant 9\}$
5. $\{y: y \geqslant 7\}$ **6.** $\{x: x \leqslant 13\}$
7. $\{x: x \geqslant 12\}$ **8.** $\{y: y < 7\}$
9. $\{y: y < 13\}$ **10.** $\{y: y > 17\}$
11. $\{x: x > 4\}$ **12.** $\{x: x \leqslant 4\}$
13. $\{x: x \geqslant 9\}$ **14.** $\{x: x < 3\}$
15. $\{y: y > 2\}$ **16.** $\{y: y \geqslant -5\}$
17. $\{x: x > 20\}$
18. $\{x: x \geqslant 13.5\}$
19. $\{x: x < -6.5\}$
20. $\{y: y \leqslant -4.8\}$ **21.** $\{y: y < 3.5\}$
22. $(x: x \geqslant -5\}$ **23.** $\{x: x \geqslant 3\}$
24. $(x: x \geqslant 16\}$ **25.** $\{x: x \leqslant 7\}$
26. $\{x: x \leqslant -3\}$ **27.** $\{x: x < -7\}$
28. $\{x: x > -2\}$ **29.** $\{x: x \geqslant 2\}$
30. $\{x: x \leqslant -7\}$ **31.** $\left\{x: x > 2\frac{1}{3}\right\}$
32. $\{x: x < 8\}$
33. $\{x: x > 17.5\}$
34. $\{x: x \geqslant 2\}$ **35.** $\left\{y: y \geqslant \frac{4}{7}\right\}$
36. $\{x: x < 27\}$ **37.** $\left\{x: x \leqslant 2\frac{1}{9}\right\}$
38. $\left\{x: x > \frac{9}{13}\right\}$
39. $\{x: x < 10.5\}$
40. $\{x: x > -25\}$

Exercise 6o

1. $x = 3, y = 7$
2. $x = 3, y = 7$
3. $x = 2, y = -1$
4. $x = -\frac{1}{4}, y = -1$
5. $x = 3, y = 1$
6. $x = 3, y = 4$
7. $x = 4, y = -3$
8. $x = -\frac{3}{10}, y = 4\frac{9}{10}$
9. $x = 11, y = 39$
10. $x = 2, y = 3$
11. $x = 6, y = 7$
12. $x = 99, y = 45$
13. $x = -\frac{1}{4}, y = -1$
14. $x = \frac{1}{2}, y = 1$
15. $x = 3, y = \frac{1}{2}$
16. $x = 3, y = 5$
17. $x = 1, y = 2$
18. $x = 2, y = 2\frac{2}{3}$
19. $x = 3, y = 5$
20. $x = -2, y = 7$
21. $x = 4, y = 3$
22. $x = 2, y = 3$
23. $x = 12, y = 4$
24. $x = 3, y = -2$
25. $x = 2, y = 1\frac{1}{2}$
26. $x = 3, y = 1$
27. $x = 5, y = 3$
28. $x = 3, y = 7$
29. $x = 2, y = 3$
30. $x = 4, y = 1$
31. $x = 2, y = 3$
32. $x = 0, y = 3$
33. $x = 1, y = -2$
34. $x = -1, y = 2$
35. $x = 12, y = 2$
36. $x = 2, y = 5$
37. $x = 3, y = 4$
38. $x = -2, y = 5$
39. $x = 13, y = 36$
40. $x = 2, y = -4$
41. $x = 8.25, y = 1.75$
42. $x = 1.95, y = 1.30$
43. $x = 2.25, y = 1.80$
44. $x = 1.8, y = 2.5$
45. $x = 0.50, y = 0.75$
46. $x = 2.5, y = 1.5$

47. $x = 2.75, y = 1.50$

48. $x = 3.5, y = 1.0$

49. $x = 2.25, y = 1.50$

50. $x = 2.80, y = 1.25$

51. $x = 125, y = 105$

52. $x = 1.75, y = 5.50$

53. $x = 0.5, y = 1.5$

54. $x = 2.25, y = 1.75$

55. $x = 6.5, y = 1.5$

56. $x = 1, y = 3$

57. $x = 3, y = 5$

58. $x = 3, y = \frac{1}{9}$

59. $x = \frac{1}{2}, y = \frac{3}{4}$

60. $x = 3, y = -1$

61. $x = 2, y = 1$

62. $(x, y) = (3, 4)$

63. $(x, y) = (2, -1)$

64. $(x, y) = (-2, 3)$

65. $(x, y) = (2, 5)$

Exercise 6p

1. $2n + 7 = 25$ Number = 9

2. $\frac{1}{2}n = 9$ Number = 18

3. $\frac{1}{3}p = 10$ Perimeter = 30 cm

4. $2n = 9$ Number = 4.5

5. $2n + 7 = 23$ Number = 8

6. $2n + 5 = 13$ Number = 4

7. $2n - 3 = 12$ Number = 7.5

8. $3n + 3 = 33$ Number = 10

9. $4(n - 6) = 36$ Number = 15

10. $n + 5 = 25$ Number = 20

11. $\frac{n}{2} = 7$ Number = 14

12. $n + 7 = 15$ Number = 8

13. $n - 5 = 9$ Number = 14

14. $n - 6 = 4$ Number = 10

15. $2n = 15$ Number = 7.5

16. $8n = 32$ Number = 4

17. $\frac{1}{5}n + \frac{1}{2}n = 14$ Number = 20

18. $\frac{1}{4}n + \frac{3}{5}n = 34$ Number = 40

19. $6x = 30$ $x = 5$ cm

20. $2x + 10 = 29$ $x = 9\frac{1}{2}$ cm

21. $2x + 8 = 46$ $x = 19$ cm

22. $4x + 10 = 58$ Length = 17cm
Width = 12 cm

23. $4x - 14 = 50$ Length = 16 cm
Width = 9 cm

24. $x = 45°, (x - 25)° = 20°,$
$(2x + 40)° = 130°, 30°.$

25. $x = 40°, (x - 5)° = 35°,$
$(x + 15)° = 55°,$
$(2x + 10)° = 90°.$

26. $x = 40°, (2x + 5)° = 85°,$
$(x - 10)° = 30°, 65°.$

27. $x = 30°, (2x + 20)° = 80°,$
$(x + 25)° = 55°,$
$(2x - 15)° = 45°.$

28. 18, 20 and 22

29. 17, 19 and 21

30. Number = 4

31. 32, 34 and 36

32. 41, 43 and 45

33. Number = 2

34. 44, 46 and 48

35. 39, 41 and 43

36. Number = 5

37. Number = 1

38. Number = 2

39. $12 - x = 6 + x$ $x = \$3.00$

40. (a) $\$(3x + 11) = \$26$
(b) $x = \$5$

41. 4 books at \$6.50 each
5 books at \$6.00 each

42. 8 articles at \$2.25 each
6 articles at \$2.00 each

43. 7 fruits at \$1.50 each
11 fruits at \$2.00 each

44. 3 eggnogs and 2 peanut punches

45. $x = 8$ cassettes

46. (a) Total mass = $(9 + 1.2x)$ kg
(b) $x = 10$ articles

47. Cost per shirt = \$55

48. 6 ~ \$20 notes

49. Length = 8 m Breadth = 5 m

50. (a) (i) $\$4x$ (ii) $\$(x - 15)$
(b) $\$(6x - 15) = \$195, x = \$35$

Exercise 6q

1. $b_{\max} = 11$ cm

2. $l_{\max} = 19$ cm

3. $b_{\max} = 12.5$ m

4. $l_{\max} = 17.5$ m

5. $h_{\max} = 17$ cm

6. $b_{\max} = 24$ cm

7. $h_{\max} = 18$ m

8. $b_{\max} = 24$ cm

9. $l_{\max} = 13$ m

10. $b_{\max} = 17$ m

11. $h_{\max} = 25$ cm

12. $l_{\max} = 12$ m

13. 7 books **14.** 4 chocolates

15. 12 chocolates **16.** 7 books

Exercise 6r

1. Price per hot dog = \$1.75
Price per hamburger = \$5.50

2. $10c + 4d = 274$ -----①
$4c + 3d = 160$ -----②
Cost per chicken = \$13
Cost per duck = \$36

3. No. of \$6 books bought = 5
No. of \$6.50 books bought = 4

4. No. of \$5 notes used = 7
No. of \$20 notes used = 6

5. No. of \$5.50 books bought = 5
No. of \$8.50 books bought = 3

6. No. of eggnogs bought = 3
No. of peanut punches bought = 2

7. No. of oranges bought = 5
No. of apples bought = 3

8. No. of Snack A bought = 2
No. of Snack B bought = 4

9. (a) Cost per pack of grapefruit juice = \$2.75
(b) Cost per pack of orange juice = \$1.50

10. (a) $2r + 3p = 10.00$ -----①
$5r + 2p = 19.50$ -----②
(b) Cost per roti = \$3.50
(c) Cost per pie = \$1.00

11. $7m + 6p = 12.50$ -----①
$5m + 8p = 14.50$ -----②
(a) Cost of a mango = \$0.50
(b) Cost of a pear = \$1.50

12. (a) $5c + 3s = 16.65$ -----①
$3c + 7s = 19.35$ -----②
(b) (i) Cost per pack of chocolate milk = \$2.25
(ii) Cost per pack of strawberry milk = \$1.80

13. (a) $7m + 5j = 20.15$ -----①
$5m + 7j = 18.85$ -----②
(b) Cost of a pack of milk = \$1.95
(c) Cost per pack of orange juice = \$1.30

14. (a) $3c + 4jw = 13.40$ -----①
$4c + 3jw = 14.95$ -----②
(b) Cost per pack of coconut water = \$2.80
(c) Cost per pack of watermelon juice = \$1.25

15. (a) Price of a hot dog = \$2.25
(b) Price of a juice = \$1.50

16. $2r + 3p = 17.50$ -----①
$4r + 3p = 30.50$ -----②
(a) Cost of a roti = \$6.50
(b) Cost of a pattie = \$1.50

17. (a) $7m + 9w = 31.50$ -----①
$13m + 6w = 39.75$ -----②
(b) (i) Cost of a pack of milk = \$2.25
(ii) Cost of a bottle of water = \$1.75

18. (a) $5x + 7y = 26.50$ -----①
$9x + 6y = 31.20$ -----②
(b) (i) Cost of a pack of X = \$1.80
(ii) Cost of a pack of Y = \$2.50

19. (a) Cost of a roti = \$8.25
(b) Cost of a bottled drink = \$1.75

20. $3d + 4s = 795$ -----①
$d - s = 20$ -----②
Cost for a dress = \$125
Cost for a shirt = \$105

21. Price of a Mathematics book = \$27.00
Price of an English book = \$18.00

22. $x = 99$ and $y = 45$

23. $x = 4$ and $y = 6$

24. The original fraction = $\frac{5}{7}$

25. $s = 8$ and $j = 3$

26. $s = 4$ and $d = 2$

27. $x = 12$ and $y = 4$

Exercise 6s

1. 2^7 **2.** 3^8 **3.** 5^6
4. 9^7 **5.** 7^9 **6.** x^5
7. y^7 **8.** a^P **9.** m^q
10. z^n **11.** 2^8 **12.** 3^9
13. 5^9 **14.** 7^7 **15.** 8^9
16. x^6 **17.** y^8 **18.** z^9
19. p^{10} **20.** q^{13} **21.** x^7y^5
22. $x^{11}y^5$ **23.** $p^{11}q^7$ **24.** p^7q^9
25. a^7b^5 **26.** $p^5q^3r^4$ **27.** $P^3q^5r^5$
28. $x^7y^5z^3$ **29.** $x^9y^5z^7$ **30.** $a^6b^5c^5$
31. 6^3 **32.** 7^3 **33.** 8^2
34. 9^5 **35.** 10^5 **36.** a^4
37. x^4 **38.** m^3 **39.** p^3
40. q^2 **41.** x^2y **42.** x^3y^2
43. p^2q **44.** r^3s **45.** m^3n^3
46. $p^3q^3r^2$ **47.** p^3q^5r **48.** $x^2y^2z^2$
49. l^2m^2n **50.** a^3bc^2 **51.** 5^{12}
52. 6^{10} **53.** 8^{20} **54.** 7^{15}
55. 9^{16} **56.** x^6 **57.** p^{20}
58. m^{25} **59.** n^{14} **60.** r^{24}
61. x^6y^9 **62.** $x^{15}y^{10}$ **63.** $p^{16}q^{12}$
64. $p^{30}q^{12}$ **65.** $a^{21}b^{35}$ **66.** $\frac{8m^9}{27n^6}$
67. $\frac{9x^{10}}{25y^6}$ **68.** $\frac{64p^{21}}{729q^{15}}$
69. $\frac{16r^{20}}{625s^{12}}$
70. $\frac{25l^8}{49m^6}$ **71.** 1 **72.** 1
73. 1 **74.** 1 **75.** 1
76. $\frac{1}{5^4}$ **77.** $\frac{1}{7^5}$ **78.** $\frac{1}{x^3}$
79. $\frac{1}{y^7}$ **80.** $\frac{1}{a^9}$ **81.** 8^{-3}
82. 10^{-5} **83.** x^{-4} **84.** a^{-7}
85. m^{-9} **86.** 1 **87.** 4
88. 7 **89.** 9 **90.** 12
91. 1 **92.** 3 **93.** 4
94. 6 **95.** 7 **96.** 16
97. 729 **98.** 25 **99.** 16
100. 1000 **101.** 32 **102.** 8
103. 243 **104.** 8 **105.** 64

Exercise 6t

1. $6x^5$ **2.** $12x^8$ **3.** $12x^7$
4. $10x^7$ **5.** $15x^9$ **6.** $6x^4y^2$
7. $12x^4y$ **8.** $15x^5y^3$ **9.** $12x^5y^4$
10. $28x^5y^3$ **11.** $24r^{10}$ **12.** $15a^8$
13. $24x^4y^5$ **14.** $8x^6y^4$ **15.** $60x^5y^3$
16. $\frac{5}{2}x^4$ **17.** $6a^4$ **18.** $10a^2$
19. $\frac{5}{3}a$ **20.** $\frac{1}{a^2}$ **21.** $125x^6y^3$
22. $81x^{12}y^8$ **23.** $64x^6y^9$
24. $\frac{16a^4b^6}{9c^2}$
25. $\frac{27x^9y^6}{125z^3}$ **26.** $3x^2$
27. $4y^3$ **28.** $5xy^2$ **29.** $3x^2y$
30. $2xy^3$
31. x^8 **32.** x^3 **33.** x^4
34. x^3 **35.** x^3 **36.** $15xy$
37. $15x^2y$ **38.** $12xy$ **39.** $8\frac{y^2}{x}$
40. $\frac{21}{xy^2}$ **41.** $\frac{1}{27p^{12}}$ **42.** $\frac{1}{16x^{10}}$
43. $\frac{1}{5x^2}$ **44.** $\frac{1}{4x^2}$ **45.** $\frac{1}{3x^4}$
46. $5 - 5a^2$ **47.** $7 + \frac{7}{2a}$
48. $5a^4 - \frac{5}{a^3}$
49. $8a^4 - 8a^2$
50. $7a - \frac{7}{a^2}$ **51.** 4
52. $\frac{1}{3}$ **53.** 4
54. $\frac{1}{25}$ **55.** 64 **56.** 32
57. 5 **58.** $\frac{1}{6}$ **59.** $\frac{1}{8}$
60. 81 **61.** 27 **62.** 21
63. 30 **64.** 60 **65.** 64
66. $2\frac{2}{3}$ **67.** $1\frac{1}{4}$ **68.** $\frac{5}{9}$
69. $1\frac{1}{2}$ **70.** $\frac{7}{25}$ **71.** 10
72. 35 **73.** 6 **74.** 12
75. 35 **76.** $\frac{1}{49}$ **77.** $\frac{1}{32}$
78. $\frac{1}{81}$ **79.** $\frac{1}{125}$ **80.** $\frac{1}{512}$
81. $\frac{1}{64}$ **82.** $\frac{1}{4096}$ **83.** $\frac{1}{25}$
84. $\frac{1}{216}$ **85.** $\frac{1}{81}$ **86.** $\frac{1}{2}$
87. $\frac{1}{4}$ **88.** $\frac{1}{3}$ **89.** $\frac{1}{3}$
90. $\frac{1}{2}$ **91.** $\frac{1}{9}$ **92.** $\frac{1}{125}$
93. $\frac{1}{32}$ **94.** $\frac{1}{27}$ **95.** $\frac{1}{49}$
96. $\frac{5}{7}$ **97.** $\frac{4}{9}$ **98.** $\frac{16}{25}$
99. $3\frac{3}{8}$ **100.** $1\frac{61}{64}$
101. $x^6 = \frac{1}{25}$ **102.** $x^{10} = \frac{1}{64}$
103. $y^{12} = \frac{1}{100}$ **104.** $y^{14} = \frac{1}{169}$
105. $y^4 = \frac{1}{1728}$

Exercise 6u

1. $x = 2\frac{1}{3}$ **2.** $x = 2\frac{1}{2}$
3. $x = 1$ **4.** $x = 1\frac{1}{3}$
5. $x = 2$ **6.** $p = 1\frac{4}{7}$
7. $q = -\frac{1}{4}$ **8.** $p = 3$
9. $r = 14$ **10.** $s = 4$
11. $x = -1\frac{1}{2}$ **12.** $x = 1\frac{1}{4}$
13. $x = \frac{3}{8}$ **14.** $x = 1$

15. $p = -3\frac{1}{2}$
16. $x = 6$
17. $x = -1$
18. $x = -1\frac{1}{3}$
19. $x = -\frac{5}{6}$
20. $x = -\frac{3}{4}$
21. $x = -\frac{1}{2}$

Exercise 6v

1. 1.47×10^2
2. 2.53×10^2
3. 7.68×10^2
4. 8.25×10^3
5. 9.49×10^3
6. 7.54×10^4
7. 1.24×10^5
8. 8.47×10^5
9. 9.46×10^6
10. 7.68×10^7
11. 4.31×10^{-2}
12. 7.83×10^{-3}
13. 4.85×10^{-3}
14. 7.61×10^{-4}
15. 4.87×10^{-4}
16. 3.25×10^{-5}
17. 1.87×10^{-5}
18. 3.12×10^{-6}
19. 4.90×10^{-6}
20. 1.85×10^{-7}
21. 1.1×10^2
22. 1.2×10^2
23. 1.5×10^2
24. 1.7×10^2
25. 1.9×10^2
26. 1.8×10^2
27. 2.3×10^3
28. 2.5×10^4
29. 1.7×10^3
30. 2.4×10^4
31. 2.0×10^{-5}
32. 5.0×10^{-6}
33. 1.2×10^{-4}
34. 1.5×10^{-5}
35. 2.1×10^{-7}
36. 2.5×10^{-2}
37. 2.5×10^{-2}
38. 6.0×10^{-2}
39. 5.0×10^{-3}
40. 8.75×10^{-3}
41. 4.0×10^1
42. 5.0×10^1
43. 6.0×10^1
44. 2.0×10^2
45. 3.0×10^2
46. 4.0×10^2
47. 5.0×10^2
48. 7.0×10^2
49. 8.0×10^2
50. 9.0×10^2
51. 3.0×10^{-3}
52. 5.0×10^{-3}
53. 6.0×10^{-3}
54. 2.0×10^{-4}
55. 4.0×10^{-4}
56. 7.5×10^{-2}
57. 8.0×10^{-2}
58. 6.25×10^{-2}
59. 1.2×10^2
60. 2.25×10^2
61. 4.82×10^4
62. 6.57×10^5
63. 6.12×10^6
64. 8.39×10^7
65. 9.04×10^8
66. 3.87×10^3
67. 4.85×10^4
68. 5.97×10^5
69. 7.15×10^6
70. 8.59×10^7
71. 1.44×10^6
72. 2.25×10^8
73. 2.89×10^{10}
74. 3.61×10^{12}
75. 4.0×10^{14}
76. 3.6×10^7
77. 2.5×10^9
78. 2.25×10^{12}
79. 4.9×10^{15}
80. 9.0×10^{16}

Exercise 6w

1. 0.090
2. 0.673
3. 1.538
4. 1.762
5. 2.680
6. 2.729
7. 3.836
8. 3.979
9. 4.097
10. 4.926
11. 5.540
12. 5.864
13. $\bar{1}.540$
14. $\bar{1}.920$
15. $\bar{2}.855$
16. $\bar{2}.970$
17. $\bar{3}.870$
18. $\bar{3}.920$
19. $\bar{4}.588$
20. $\bar{4}.911$
21. 2.21
22. 9.57
23. 22.2
24. 72.9
25. 518
26. 822
27. 2960
28. 7280
29. 43 600
30. 86 500
31. 679 000
32. 444 000
33. 0.239
34. 0.304
35. 0.0171
36. 0.0262
37. 0.004 72
38. 0.008 63
39. 0.000 207
40. 0.000 412

Exercise 6x

1. 213
2. 442
3. 17.2
4. 0.309
5. 283
6. 6.30
7. 28.8
8. 1.64
9. 724
10. 12 800
11. 2630
12. 200 000
13. 27.9
14. 15.2
15. 76.3
16. 23.8
17. 0.252
18. 0.0492
19. 0.426
20. 0.002 65
21. 33.1
22. 501
23. 4.32
24. 423
25. 0.858
26. 0.750
27. 12.1
28. 0.111
29. 0.168
30. 2.59
31. 0.204
32. 2.95
33. 3.49
34. 4.80
35. 4660
36. 72 500

Exercise 6y

1. $x = 2.68$
2. $x = 0.797$
3. $x = 0.571$
4. $x = 0.158$
5. $x = 0.154$
6. $x = 3.02$
7. $x = 1.19$
8. $x = 1.04$
9. $x = 1.08$
10. $x = 0.717$
11. $y = 0.778$
12. $y = 0.301$
13. $y = 0.100$
14. $y = -0.176$
15. $y = -0.151$
16. $x = 100$
17. $x = 316$
18. $x = 10$
19. $x = 31.6$
20. $x = 10000$

Exercise 6z

1. (i) $\$(4x - 3)$ (ii) $x < 9.25$
(iii) \$9.00 and \$15.00
2. (i) $y \geqslant x$ and $x > -3$
(ii) e.g. $A(1, 3)$, $B(0, 2)$ and $C(-3, 4)$
3. (i) $\{m: m \geqslant 2\}$
4. (a) $3c + 2b = 299$
$4c + b = 272$
(b) $c = 49$ ¢ and $b = 76$ ¢
5. (a) (i) $\$3x$
(ii) $\$(x - 12)$
(b) $5x - 12 = 63, x = \$15$
6. $x = -3\frac{1}{3}$
7. (a) 42
(b) $5x + 5y - 5$
(c) $\frac{2x - 21}{36}$
8. (a) 648 persons
(b) \$626.64
(c) \$4595.36
9. (b) (i) $x + 4y = 292$
$2x + 5y = 482$
(ii) $x = \$156$ and $y = \$34$
10. (a) $12c^5$ (b) 1
(c) $3x(2 + 3x)$
(d) (i) $2x + 8$
(ii) $\frac{5a - 2}{6}$
11. (a) (i) $\$(x + 12)$
(ii) $5x + 24 = 619$
(iii) $r = \$131$

Exercise 7a

1. (a) $A = 18$ units^2
(b) $A = 4$ units^2
2. ABC is a scalene triangle
3. $ABCD$ is a parallelogram
4. (b) Parallelogram
(c) $PQ = SR$ and $PS = QR$
(d) $PQ // SR$ and $PS // QR$
(e) $\hat{P} = \hat{R} = 106°$and $Q = S = 74°$
5. (b) Rectangle
(c) 10 cm
(d) 73° and 107°
(e) $X(5, 9)$
(f) $Y(8, 5)$
6. (b) $H(-5, -11)$, $I(-3, -7)$ $J(1, 1)$, $K(3, 5)$, $L(5, 9)$, $M(7, 13)$ and $N(a, 2a - 1)$
7. Rectangle

8. Scalene triangle

9. (a) Trapezium (c) No
(d) No (e) No

10. (b) Right-angled isosceles triangle
(c) $AB = CB$
(d) $\hat{A} = \hat{C} = 45°$ and $\hat{B} = 90°$
(e) $O(0, 0)$

11. $A = 14$ units^2

12. $A = 12.5$ units^2

13. $A = 4$ units^2

14. (b) Trapezium
(c) $A = 14$ units^2

15. (a) Square
(b) $AC = BD = 8$ cm

16. (a) Parallelogram
(b) $PR = 7.3$ cm, $QS = 10.8$ cm

17. (a) Trapezium
(b) $JK = 6.3$ cm, $LM = 8.5$ cm

Exercise 7c

1. $x < 21$
2. $y > 57$
3. $p \leq q$
4. $r \geq s$
5. $a < b$
6. $b \leq 103$
7. $r > j$
8. $N \geq M$
9. $Z > 0$
10. $T \geq 33°C$
11. $x < y$ cm
12. $t < -7°C$
13. $w \leq 1.2m$
14. $x \geq 1m$
15. $p < 2q$
16. $r > 3s$
17. $x \geq 2y$
18. $l \leq 3m$
19. $2a > b$
20. $2p \geq 3q$

Exercise 7d

1. $t > 3$
2. $s > 150$
3. $a < 12000$
4. $s \leq 25$
5. $k \leq 13$
6. $p \geq 9$
7. $c \geq 35$
8. $I \leq 1.2$
9. $u \geq 5.76\,t$
10. $s < 15$
11. $T \leq 38$
12. $m < 64$
13. $c \geq 6p$
14. $W \geq 2R$
15. $i \leq 25$
16. $F < \frac{1}{2}f$
17. $t > \frac{3}{2}l$
18. $M < \frac{1}{4}m$
19. $a > 225000$
20. $t \geq \frac{3}{2}l$

Exercise 7e

1. $30 \leq T \leq 34$
2. $6.50 \leq p \leq 8.50$
3. $3.4 \leq w \leq 4.7$
4. $99 < T < 102$
5. $15.7 < l < 18.5$
6. $125 < m < 137$
7. $25 < s < 40$
8. $175 < d \leq 250$
9. $1.5 < i \leq 4$
10. $2.25 \leq c < 3.25$
11. $75.00 \leq b < 124.00$
12. $350 \leq l < 975$
13. $49.5 \leq w \leq 54.3$
14. $40 \leq s \leq 160$
15. $40 \leq w \leq 75$

Exercise 7f

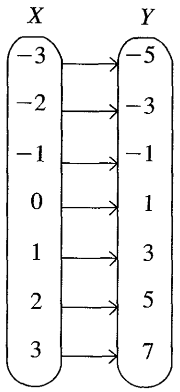
1. $\{x: x \geq 3\}$
2. $\{x: x \geq 2.5\}$
3. $\{x: x \geq 0\}$
4. $\{x: x \geq -2\}$
5. $\{x: x \geq -3.5\}$
6. $\{x: x < 4\}$
7. $\{x: x < 3.5\}$
8. $\{x: x < 0\}$
9. $\{x: x < -1.5\}$
10. $\{x: x < -4.5\}$
11. $\{x: -1 \leq x \leq 2\}$
12. $\{x: 1 \leq x \leq 4\}$
13. $\{x: 0 \leq x \leq 3.5\}$
14. $\{x: -5 \leq x \leq -1\}$
15. $\{x: -5.5 \leq x \leq -0.5\}$
16. $\{x: 1 < x < 5\}$
17. $\{x: 0 < x < 4.5)$
18. $\{x: -1 < x < 3\}$
19. $\{x: -2.5 < x < 2\}$
20. $\{x: -3.5 < x < -0.5\}$
21. $\{x: 1 < x < 4\}$
22. $\{x: 0 < x < 5\}$
23. $\{x: -1.5 < x < 3\}$
24. $\{x: -2.5 < x < 1\}$
25. $\{x: -4.5 < x < -1\}$
26. $\{x: 1 \leq x < 5\}$
27. $\{x: 0 \leq x < 4\}$
28. $\{x: -0.5 \leq x < 3\}$
29. $\{x: -2.5 \leq x < 2.5\}$
30. $\{x: -4.5 \leq x < 0.5\}$

Exercise 7g

1. $\{x: x \geq 0\}$
2. $\{x: x \geq 4\}$
3. $\{x: x \geq -2.5\}$
4. $\{y: y \leq 0\}$
5. $\{y: y \leq 3\}$
6. $\{y: y \leq -4.5\}$
7. $\{x: x < 2\}$
8. $\{x: x < 3.5\}$
9. $\{x: x < -1\}$
10. $\{y: y < 1\}$
11. $\{y: y < 2.5\}$
12. $\{y: y < -2.5\}$
13. $\{x: 1 \leq x \leq 3\}$
14. $\{x: 0 \leq x \leq 2.5\}$
15. $\{x: -3 \leq x \leq 1.5\}$
16. $\{y: 1 \leq y \leq 3\}$
17. $\{y: 0 \leq y \leq 2.5\}$
18. $\{y: -3 \leq y \leq 1.5\}$
19. $\{y: 1 < y < 5\}$
20. $\{y: 0.5 < y < 3.5\}$
21. $\{y: -3 < y < -0.5\}$
22. $\{x: 1 < x < 3\}$
23. $\{x: -0.5 < x < 2.5\}$
24. $\{x: -3 < x < 1.5\}$
25. $\{x: 0 < x: \leq 3\}$
26. $\{x: -1 < x \leq 2.5\}$
27. $\{x: -2.5 < x \leq 3.5\}$
28. $\{y: 0 < y \leq 3\}$
29. $\{y: -2 < y \leq 3\}$
30. $\{y: -3.5 < y \leq 1)$
31. $\{x: 0 \leq x < 4\}$
32. $\{x: -0.5 \leq x < 3\}$
33. $\{x: -3.5 \leq x < 2\}$
34. $\{y: 0 \leq y < 3\}$
35. $\{y: -1.5 \leq y < 2\}$
36. $\{y: -2.5 \leq y < 3\}$

Exercise 7h

1. (a)

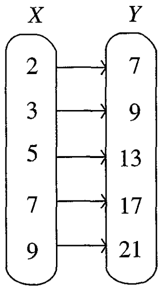


(b) 1 – 1 relation

2. (a) $Y = \{-9, -6, -3, 0, 3, 6, 9\}$
(b) $Y = \{9, 4, 1, 0)$
(c) $Y = \{12, 6, 2, 0\}$

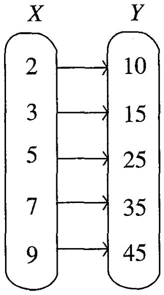
3. (a) $Y = \{3, 4, 5, 6\}$
(b) $Y = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 1\frac{1}{2}\right\}$
(c) $Y = \left\{\frac{1}{4}, \frac{1}{2}, 1, 2, 4\right\}$

4. (a)



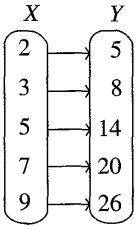
1 − 1 relation

(b)



1 – 1 relation

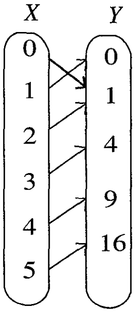
(c)



1 − 1 relation

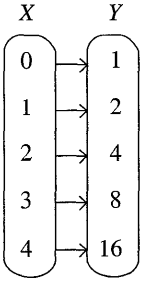
5. $x \longrightarrow 3x$

6.



m − 1 relation

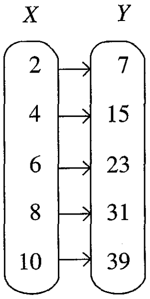
7.



1 − 1 relation

8. (a) $x \longrightarrow 4x - 1$

(b)



1 − 1 relation

9. (a) $2 \longrightarrow 8$

(b) $3 \longrightarrow 27$

10. (a) $7 \longrightarrow -5$

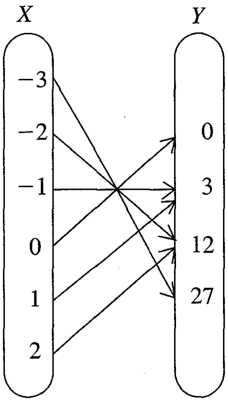
(b) $9 \longrightarrow -7$

11. {(1, 3.5), (2, 4), (3, 4.5), (4, 5), (5, 5.5), (6, 6), (7, 6.5), (8, 7), (9, 7.5), (10, 8)}

12. {(0, 0.5), (1, 3.5), (2, 6.5), (3, 9.5), (4, 12.5), (5, 15.5), (6, 18.5)}

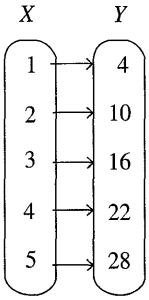
Exercise 7i

1. (a)



(b) m – 1 relation

2. (a)



(b) 1 − 1 relation

3. (a) (i) $g(-3) = 10$

(ii) $g(-1) = 2$

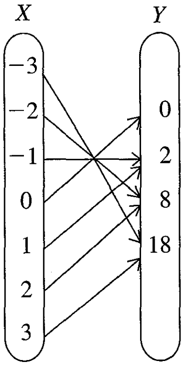
(iii) $g(0) = 1$

(iv) $g(2) = 5$

(b) $x = \pm 2$

(c) $x = 4$

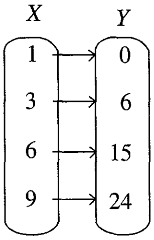
4. (a)



(b) m − 1 relation

(c) Yes

5. (a)



(b) 1 − 1 relation

6. (a) 2 → 12

(b) 4 → 22

(c) $x = 3$

(d) $x = 7$

7. Yes

(a) $f(1) = 6$

(b) $f(2) = 9$

(c) $f(4) = 21$

$x = 4$

8. $x = 9$

9. (a) $K\left(\frac{1}{2}\right) = -1$

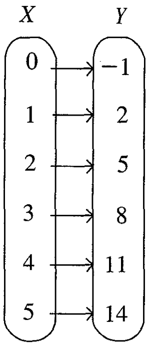
(b) $K(-1) = -4$

(c) $K\left(\frac{1}{3}\right) = 0$

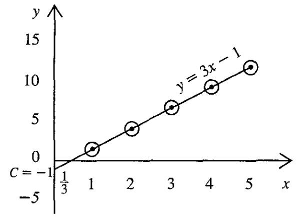
(d) $K(-2) = -3\frac{1}{2}$

10. {(1, 2), (2, 2.5), (3, 3), (4, 3.5), (5, 4), (6, 4.5), (7, 5), (8, 5.5), (9, 6)}

11. (a)



(b)



(c) The relation is a function 1 − 1 relation or vertical line test.

12. (a) $f(3) = 5\frac{1}{2}$

(b) $f(-2) = -4\frac{1}{2}$

(c) $f(0) = -\frac{1}{2}$

13. (a) Yes (b) No

(c) Yes (d) No

14. (a) (i) $f(0) = -1$

(ii) $f(2) = 15$

(iii) $f(-1) = 0$

(b) A smooth curve called a parabola

(c) Minimum value. Because the coefficient of x^2 is positive.

15. (i) $g(3) = 1\frac{3}{5}$

(ii) $h(2) = 1\frac{7}{8}$

16. (a) No (b) No

(c) No (d) Yes

17. (a) $Y = x.\, 3^x$

(b) $Y = 324$

18. (a) No

(b) Yes

(c) No

Exercise 7k

1. (a) $y = ax^2$ (b) $y = ax^3$

(c) $y = ax^{\frac{1}{2}} = a\sqrt{x}$

(d) $y = ax^{\frac{1}{3}} = a\sqrt[3]{x}$

2. (a) $y = \frac{a}{x^2}$ (b) $y = \frac{a}{x^3}$

(c) $y = \frac{a}{\sqrt{x}} = \frac{a}{x^{\frac{1}{2}}} = ax^{-\frac{1}{2}}$

(d) $y = \frac{a}{\sqrt[3]{x}} = \frac{a}{x^{\frac{1}{3}}} = ax^{-\frac{1}{3}}$

3. (a) $y = ax^2$, $a = \frac{1}{8}$ and $y = 4\frac{1}{2}$

(b) $y = \frac{a}{\sqrt{x}}$, $a = 4$ and $y = 1.63$

4. (a) $y = a\sqrt{x}$, $a = 8.93$ and $y = 26.8$

(b) Plot y against $\sqrt{x}$.

5. (a) $y = \frac{a}{x^3}$, $a = 216$ and $y = 1.$

(b) Plot y against $\frac{1}{x^3}$.

6. $y = 14.$

7. $x = 2.$

8. $y = 72.$

9.

y	7.5	3.3	1.9	1.2	0.8	0.6	0.5	0.4
$\frac{1}{x^2}$	0.250	0.111	0.063	0.040	0.028	0.020	0.016	0.012

Plot y against $\frac{1}{x^2}$.

Exercise 7m

1. (a) $m = 2$

(b) $m = 2$

2. (a) (i) $AB = 4.5$ units

(ii) $BC = 4.5$ units

(b) (i) $X(2, 4)$

(ii) $Y(4, 8)$

3. (a) $PQ = 8.9$ units

(b) $PR = 7.6$ units

4. (a) (i) $X(1, 4)$

(ii) $Y = \left(\frac{1}{2}, \frac{1}{2}\right)$

(b) (i) $m = \frac{1}{2}$

(ii) $m = -\frac{3}{7}$

5. (a) $AB = 7.6$ units

(b) $m = -\frac{3}{7}$

(c) $X\left(1\frac{1}{2}, -5\frac{1}{2}\right)$

6. $m = -\frac{3}{4}$

7. (a) $PQ = 12.5$ units

(b) $X = \left(0, 1\frac{1}{2}\right)$

(c) $m = 1\frac{5}{6}$

8. (a) $AB = 12.4$ units

(b) $X\left(2, 3\frac{1}{2}\right)$

(c) $m = \frac{1}{4}$

9. (a) $LM = 13.6$ units

(b) $X\left(1, -\frac{1}{2}\right)$

(c) $m = -1\frac{3}{8}$

10. (a) Parallelogram

(b) $AB = 8.6$ units

(c) $X\left(-1\frac{1}{2}, -1\frac{1}{2}\right)$

(d) $m = \frac{5}{7}$

Exercise 7n

1. $m = 2$, $c = 1$ and $y = 2x + 1$

2. $m = 2$, $c = 4$ and $y = 2x + 4$

3. $m = \frac{2}{5}$, $c = 4\frac{3}{5}$ and $y = \frac{2}{5}x + 4\frac{3}{5}$

4. (b) $m = -1$

(c) $c = 2$, Point = $(0, 2)$

(d) $y = -x + 2$

5. (a) (i) $AB = 4.47$ units

(ii) $m = -2$

(iii) $X(5, 5)$

(iv) $c = 15$

(b) $y = -2x + 15$

6. (a) (i) $PQ = 14.1$ units

(ii) $m = 1$

(iii) $X(1, -2)$

(b) (i) $c = -3$

(ii) $y = x - 3$

7. (a) (i) $m = \frac{1}{3}$ (ii) $c = 5$

(b) $y = \frac{1}{3}x + 5$

8. $m = \frac{4}{5}$ and $c = -2\frac{4}{5}$.

$y = \frac{4}{5}x - 2\frac{4}{5}$

9. $m = \frac{2}{3}$ and $c = 4$. $y = \frac{2}{3}x + 4$.

10. $m = -\frac{2}{5}$ and $c = 0$. $y = -\frac{2}{5}x$

11. (b) $m = \frac{3}{2}$

(c) $N(0, 4)$. $y = \frac{3}{2}x + 4$

(d) $X\left(3, 8\frac{1}{2}\right)$

Exercise 7o

1. (a) $m = -4$ and $c = 17$
(b) $y = -4x + 17$

2. (a) $m = 4$ and $c = 17$
(b) $y = 4x + 17$

3. (a) (i) $PQ = 8.25$ units
(ii) $M(1, 8)$ (iii) $m = \frac{1}{4}$
(b) $y = \frac{1}{4}x + 7\frac{3}{4}$

4. (a) (i) $PQ = 4.12$ units
(ii) $M\left(-3, 1\frac{1}{2}\right)$
(iii) $m = -\frac{1}{4}$
(b) $y = -\frac{1}{4}x + \frac{3}{4}$

5. (a) (i) $PQ = 10$ units
(ii) $m = 1\frac{1}{3}$ (iii) $X(1, 1)$
(iv) $c = -\frac{1}{3}$
(b) $y = 1\frac{1}{3}x - \frac{1}{3}$

6. (a) (i) $AB = 4.47$ units
(ii) $m = -\frac{1}{2}$ (iii) $X(4, 4)$
(b) $y = 2x - 4$

7. (b) $m = -\frac{2}{3}$ (c) $c = 0$.
Point $= O(0, 0)$
(d) $y = -\frac{2}{3}x$
(e) (i) $AB = 7.21$ units
(ii) $X(0, 0)$.

8. (a) $P(0, 3)$
(b) $m = -1\frac{1}{2}$
(c) $Q(2, 0)$.

9. $m = 1$ and $c = 6$, $y = x + 6$

10. $m = -\frac{3}{5}$ and $c = \frac{2}{5}$,
$y = -\frac{3}{5}x + \frac{2}{5}$

11. (a) (i) $m = 2$ (ii) $c = -1$
(iii) $y = 2x - 1$
(b) (i) $y = -4$ (ii) $y = -1$
(iii) $y = 3$

12. (a) (i) $x = -1$ (ii) $x = 1$
(iii) $x = 2$
(b) (i) $m = -3, c = 2$

13. $y = 3x + 4$

14. $y = -2.5x + 8.25$

Exercise 7p

1. (a) $y = 7x - 2$ and $y = 7x + 5$
$m_1 = m_2 = 7$. Lines are parallel.
(b) $y = -2x + \frac{5}{2}$ and
$y = \frac{1}{2}x + 4$
$m_1 = -2$ and $m_2 = \frac{1}{2}$, Lines are perpendicular

2. $y = 3x - \frac{5}{2}$ and
$y = -\frac{1}{3}x + \frac{7}{3}$
$m_1 = 3$ and $m_2 = -\frac{1}{3}$. Lines are perpendicular

3. $m_1 = m_3 = -3$. Lines are parallel.

4. $y = \frac{3}{2}x - 4$ ----- ①
$y = -\frac{3}{2}x + 2$ ----- ②
$y = \frac{3}{2}x - 2$ ----- ③
(a) $m_1 = m_3 = \frac{3}{2}$. Lines ① and ③ are parallel.
(b) $m_1 = \frac{3}{2}$ and $m_2 = -\frac{2}{3}$.
Lines ① and ② are perpendicular.
$m_2 = -\frac{2}{3}$ and $m_2 = \frac{3}{2}$.
Lines ② and ③ are perpendicular.

5. (a) $m_1 = \frac{5}{3}$ and $m_2 = -\frac{3}{5}$.
Lines are perpendicular.
(b) $m_1 = m_2 = \frac{3}{2}$.
Lines are parallel.
(c) $m_1 = \frac{6}{5}$ and $m_2 = \frac{4}{5}$.
Lines are neither parallel nor perpendicular.

6. (a) $AB = 9.06$ units
(b) $p = 7$ and $q = -1$,
$C(7, -1)$.
(c) $X\left(5, 1\frac{1}{2}\right)$

7. (a) $AC = 8.94$ units
(b) $m_1 = -\frac{1}{2}$ and $m_2 = 2$.
$m_1 \times m_2 = -1$.
(c) Gradient of BC, $m_1 = \frac{1}{3}$
Gradient of AD, $m_2 = \frac{1}{3}$
$m_1 = m_2 = \frac{1}{3}$

8. (a) (i) $AB = 4.47$ units
(ii) $m_1 = -\frac{1}{2}$
(iii) $X(4, 4)$
(b) $m_2 = 2$

9. (a) $AB = 7.28$ units
(b) $a = 10$ and $b = -1$.
$C(10, -1)$.

10. $y = \frac{2}{5}x + \frac{1}{3}$ and $y = \frac{2}{5}x - 1$
$m_1 = m_2 = \frac{2}{5}$. Lines are parallel.

11. (a) (i) 7.8 units
(ii) $X\left(2, -\frac{1}{2}\right)$
(iii) $m = \frac{5}{6}$
(iv) $c = -2\frac{1}{6}$
(v) $x = 2\frac{3}{5}$
(vi) $y = \frac{5}{6}x - 2\frac{1}{6}$
(b) $y = -\frac{6}{5}x + \frac{19}{10}$
Point $= \frac{3}{4}$

12. $y = \frac{5}{2}x - 2$

13. $m_1 = 2, m_2 = 2, m_3 = -2$ and
$m_4 = \frac{1}{2}$.
$y = 2x + 5$ and
$2y = 4x - 9$ are parallel.

Exercise 7q

1. (a) (i) $y = 4$ (ii) $y = -2$
(b) $m = -3$ (c) $c = 1$

2. (a) (i) $x = 0$ (ii) $x = 3$
(b) $m = \frac{1}{2}$ (c) $x = 3$

3. (b) $m = 2$ (c) $c = -1$
(d) $x = 2$

4. (b) $m = 4$ (c) $c = -3$
(d) $x = \frac{7}{8}$

5. (a) $x = 3$ (b) $x = 1$
(c) $x = 2$ (d) $x = 4$

6. (a) $x = 3$ (b) $x = 1$
(c) $x = 7\frac{1}{2}$ (d) $x = -2$

7. (a) $x = 4$ (b) $x = 2$
(c) $x = 4$ (d) $x = 4$

8. (a) $x = 1\frac{1}{6}$ (b) $x = 5$
(c) $x = \frac{1}{6}$ (d) $x = 0$

9. (a) $x = 5\frac{1}{2}$ (b) $x = 0$
(c) $x = -2$ (d) $x = \frac{3}{4}$

10. (a) $x = -\frac{1}{3}$
(b) $x = -1\frac{1}{2}, y = -3\frac{1}{3}$
(c) $x = 4, y = 4\frac{3}{4}$
(d) $x = -1\frac{1}{2}, y = 1\frac{1}{6}$

Exercise 7r

1. (a) $x = 3, y = 4$
(b) $x = 4, y = 3$
(c) $x = 3, y = 5$
(d) $x = -25, y = -97$

2. (a) $x = 2, y = 3$
(b) $x = -2, y = 7$
(c) $x = 2.25, y = 1.8$
(d) $x = 3.5, y = 1.0$

3. (a) $x = 3, y = 5$
(b) $x = 6, y = 2$
(c) $x = 0.5, y = 1$
(d) $x = 2, y = 3$

4. (a) $x = 1, y = 7$
(b) $x = 5, y = 2$
(c) $x = 1, y = -4$
(d) $x = 3, y = 4$

5. (a) $x = 1$ (b) $x = 5$
(c) $x = 1$ (d) $x = -\frac{1}{7}$
(e) $x = 5$

6. (a) $x = 2, y = 3$
(b) $x = 3, y = 4$
(c) $x = 1, y = 2$
(d) $x = 2.25, y = 1.5$

7. (a) $x = 2, y = 2\frac{2}{3} = 2.67$
(b) $x = 3, y = 1$
(c) $x = 1, y = -2$
(d) $x = -1, y = 2$

8. (a) $x = 15$ (b) $x = 5$
(c) $x = 3, y = 5$ (d) $x = 6$

9. (a) $x = 40$
(b) $x = 3, y = 1$
(c) $x = 3, y = 5$
(d) $x = 2, y = -1$

10. (a) $x = 4, y = 1$
(b) $x = 2, y = -1$
(c) $x = 2, y = 3$
(d) $x = 2, y = 3$

11. (a) $x = -1, y = 2$
(b) $x = 1, y = -2$
(c) $x = 6, y = 7$
(d) $x = 99, y = 45$

12. (a) $x = 13, y = 36$
(b) $x = 5, f(x) = y = 1$
(c) $x = 4, f(x) = y = -3$
(d) $x = -2, f(x) = y = 7$

13. (a) $x = 4, f(x) = y = 1$
(b) $x = -0.25,$
$f(x) = y = -0.25$
(c) $x = -0.25, f(x) = y = -1$
(d) $x = 3, f(x) = y = 7$

14. (a) $x = 148, f(x) = y = 89$
(b) $x = 0, f(x) = y = 3$
(c) $x = 2, f(x) = y = 2\frac{2}{3}$
(d) $x = 0.5, f(x) = y = 1$

15. (a) $x = 3, y = 7$
(b) $x = 5, y = 3$
(c) $x = 3, y = 1$
(d) $x = 0.5, y = 1.5$

16. (a) $x = -0.25, y = -1$
(b) $x = 0.5, y = 1$
(c) $x = 3, y = 0.5$
(d) $x = -0.25, y = -1$

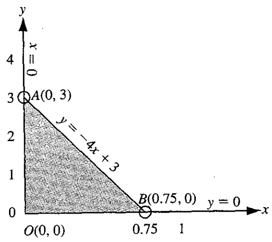
17. $x = 3, y = 7$
18. $x = 2, y = 5$
19. $x = 1, y = 1.5$
20. $x = 1.5, y = 4.5$
21. $x = 2.5, y = 1.5$
22. $x = 0.5, y = 0.75$
23. $(-0.3, 4.9)$
24. $x = 2$
25. $x = 3$

Exercise 7s

1. $\{(x, y): y \geqslant 2x + 3\}$
2. $\{(x, y): y \geqslant 3x + 1\}$
3. $\{(x, y): y \geqslant 5x + 3\}$
4. $\{(x, y): y \geqslant -4x + 1\}$
5. $\{(x, y): y \geqslant -2x + 3\}$
6. $\{(x, y): y \geqslant -5x + 4\}$
7. $\{(x, y): y \geqslant 2x - 5\}$
8. $\{(x, y): y \geqslant 3x - 2\}$
9. $\{(x, y): y \geqslant 4x - 1\}$
10. $\{(x, y): y \geqslant -3x - 2)$
11. $\{(x, y): y \geqslant -4x - 1\}$
12. $\{(x, y):y \geqslant -5x - 3\}$
13. $\{(x, y): y > x + 3\}$
14. $\{(x, y): y > 2x + 1\}$
15. $\{(x, y): y > 3x + 2\}$
16. $\{(x, y): y > -x + 2\}$
17. $\{(x, y): y > -2x + 3\}$
18. $\{(x, y): y > -3x + 1\}$
19. $\{(x, y): y > 4x - 3\}$
20. $\{(x, y): y > 3x - 2\}$
21. $\{(x, y): y > 5x - 4\}$
22. $\{(x, y): y > -x - 1\}$
23. $\{(x, y): y > -2x - 3\}$
24. $\{(x, y): y > -3x - 4\}$
25. $\{(x, y): y \leqslant 4x + 3\}$
26. $\{(x, y): y \leqslant 5x + 4\}$
27. $\{(x, y): y \leqslant 6x + 1\}$
28. $\{(x, y): y \leqslant 7x - 1\}$
29. $\{(x, y): y \leqslant 8x - 3\}$
30. $\{(x, y): y \leqslant 6x - 5\}$
31. $\{(x, y): y \leqslant -5x + 4\}$
32. $\{(x, y): y \leqslant -7x + 3\}$
33. $\{(x, y): y \leqslant -8x + 5\}$
34. $\{(x, y): y \leqslant -9x - 5\}$
35. $\{(x, y): y \leqslant -8x - 7\}$
36. $\{(x, y): y \leqslant -7x - 6\}$
37. $\{(x, y): y < 4x + 3\}$
38. $\{(x, y): y < 5x + 7\}$
39. $\{(x, y): y < 8x + 5\}$
40. $\{(x, y): y < 7x - 3\}$
41. $\{(x, y): y < 8x - 7\}$
42. $\{(x, y): y < 9x - 8\}$
43. $\{(x, y): y < -8x + 5\}$
44. $\{(x, y): y < -9x + 7\}$
45. $\{(x, y): y < -10x + 9\}$
46. $\{(x, y): y < -11x - 7\}$
47. $\{(x, y): y < -12x - 1\}$
48. $\{(x, y): y < -13x - 9\}$

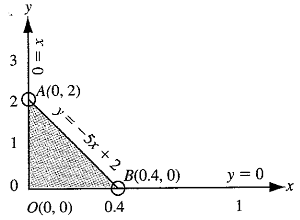
Exercise 7t

1.



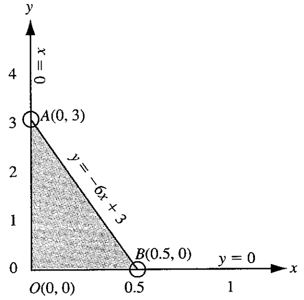
$A(0, 3), B(0.75, 0)$ and $O(0, 0)$.

2.



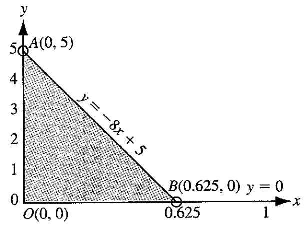
$A(0, 2)$, $B(0.4, 0)$ and $O(0, 0)$.

3.



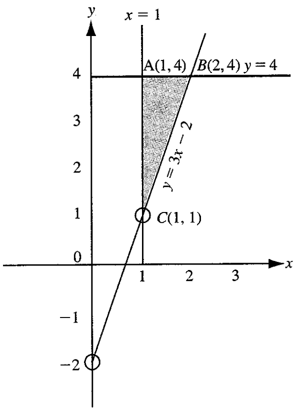
$A(0, 3)$, $B(0.5, 0)$ and $C(0, 0)$.

4.



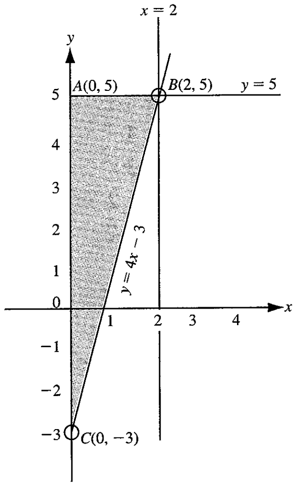
$A(0, 5)$, $B(0.625, 0)$ and $O(0, 0)$.

5.



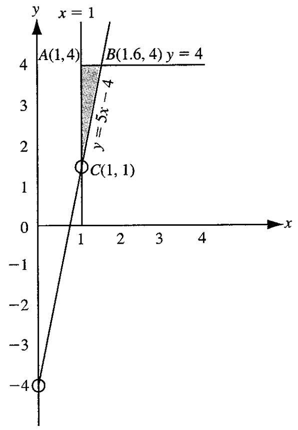
$A(1, 4)$, $B(2, 4)$ and $C(1, 1)$

6.



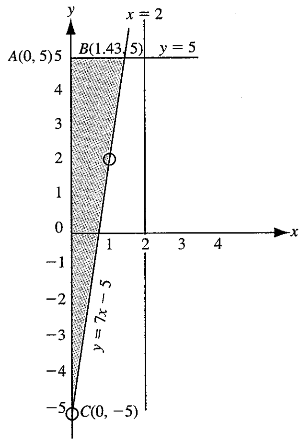
$A(0, 5)$, $B(2, 5)$ and $C(0, -3)$.

7.



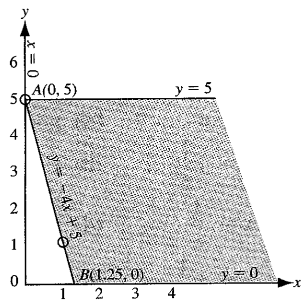
$A(1, 4)$, $B(1.6, 4)$ and $C(1, 1)$.

8.



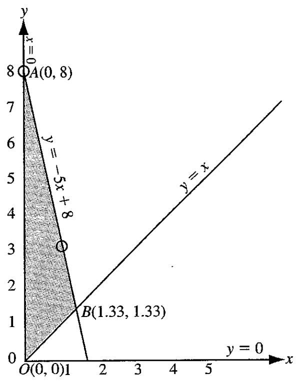
$A(0, 5)$, $B(1.43, 5)$ and $C(0, -5)$.

9.



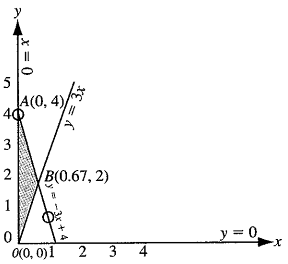
$A(0, 5)$ and $B(1.25, 0)$.

10.



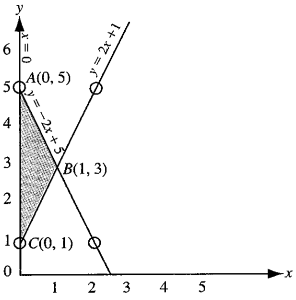
$A(0, 8)$, $B(1.33, 1.33)$ and $O(0, 0)$.

11.



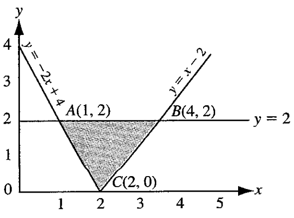
$A(0, 4)$, $B(0.67, 2)$ and $O(0, 0)$.

12.



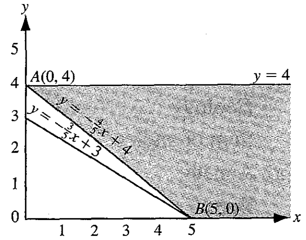
$A(0, 5)$, $B(1, 3)$ and $C(0, 1)$.

13.



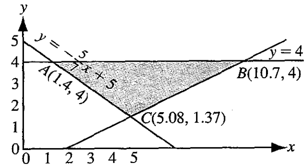
$A(1, 2)$, $B(4, 2)$ and $C(2, 0)$.

14.



$A(0, 4)$ and $B(5, 0)$.

15.



$A(1.4, 4)$, $B(10.7, 4)$ and $C(5.08, 1.37)$.

Exercise 7u

1. (b) $\{x: -4 < x < 2\}$.

2. (b) $\{x: -4 < x < 2\}$.

3. (b) $\{x: x < -1 \text{ and } x > 3\} = \{x: -1 \leqslant x \leqslant 3\}'$.

4. (b) $\{x: x < -1 \text{ and } x > 3\} = \{x: -1 \leqslant x \leqslant 3\}'$.

5. (b) (i) $\{x: x < -3 \text{ and } x > -\frac{1}{2}\} = = \{x: -3 \leqslant x \leqslant -\frac{1}{2}\}'$
(ii) $\{x: -3 < x < -\frac{1}{2}\}$.

6. (b) (i) $\{x: -3 < x < \frac{1}{2}\}$
(ii) $\{x: x < -3 \text{ and } x > \frac{1}{2}\} = \{x: -3 \leqslant x \leqslant \frac{1}{2}\}'$.

7. (b) (i) $\{x: -4 < x < 6\}$
(ii) $\{x: x < -4 \text{ and } x > 6\} = \{x: -4 \leqslant x \leqslant 6\}'$.

8. (b) (i) $\{x: < -4 \text{ and } x > 3\} = \{x - 4 \leqslant x \leqslant 3\}'$.
(ii) $\{x: -4 < x < 3\}$.

Exercise 7v

1. $x = -\frac{1}{3}$ and $x = 5$

2. $x = 0.18$ and $x = 2.8$

3. $x = 1$ and $x = 1.5$

4. $x = -0.5$ and $x = 5$

5. $x = -\frac{1}{3}$ and $x = 5$

6. $x = -0.64$ and $x = 1.24$

7. $x = -2.18$ and $x = 0.18$

8. $x = 0.75$ and $x = 1$

9. $x = -0.43$ and $x = 1.18$

10. $x = -2$ and $x = 0.2$

11. $x = -0.2$ and $x = 4$

12. $x = -0.25$ and $x = 5$

13. $x = -0.85$ and $x = 1.65$

14. $x = 1.21$ and $y = -3.71$

15. $x = 0.29$ and $x = 0.6$

16. $r = -0.95$ (neglect). $r = 16.95$

17. $l = -3.5$ (neglect). $l = 10$

18. $n = -11$ (neglect). $n = 2$

19. $x = -10$ (neglect). $x = 2$

20. $x = -1.5$ and $x = 0.25$

21. $x = -0.5$ and $x = 1.6$

22. $x = -82$ (neglect). $x = 2$

23. $x = -7$ (neglect). $x = 2$

24. (b) (i) $x = -2.54$ and $x = 3.54$
(ii) $x = -3$ and $x = 4$

25. (a) $x = -1.5$ and $x = 4$
(b) $x = -1.11$ and $x = 3.61$
(c) $x = 0.22$ and $x = 2.28$
(d) $x = -1.57$ and $x = 4.07$

26. (a) $x = -\frac{1}{3}$ and $x = 1$
(b) $x = -1\frac{1}{3}$ and $x = 2$

27. (a) $x = -\frac{1}{3}$ and $x = 1$
(b) $x = -1\frac{1}{3}$ and $x = 2$

28. (a) $x = -2.62$ and $x = -0.38$
(b) $x = 0.5$ and $x = 1.5$
(c) $x = -1$ and $x = 0.6$

29. (a) $x = -3$ and $x = 0$
(b) $x = 7$ and $x = 9$
(c) $x = -0.75$ and $x = -0.4$
(d) $x = -1.75$ and $x = 0.6$

30. (a) $x = -4$ and $x = 3$
(b) $x = -5$ and $x = -2$
(c) $x = \pm 1.6$
(d) $x = 0$ and $x = 0.375$

31. (a) $x = -1.67$ and $x = -0.4$
(b) $x = 0.6$ and $x = 1.25$
(c) $x = -5$ and $x = -2$

32. (a) $x = -2.5$ and $x = 8$
(b) $x = -1\frac{1}{3}$ and $x = 1$
(c) $x = -3.5$ and $x = 3$
(d) $x = -3.5$ and $x = -0.2$

33. (a)

x	−3	−1	0	2
y	2	−2	−1	7

(c) $(-3, 2)$ and $(1, 2)$
(d) $x = -3$ and $x = 1$

34. (a)

x	−3	−1	0	2
y	2	0	−1	7

(c) $(-4, 7)$ and $(2, 7)$
(d) $x = -4$ and $x = 2$

35. (a)

x	−3	−1	0	2
y	−2	−4	−2	8

(c) $(-3, -2)$ and $(0, -2)$
(d) $x = -3$ and $x = 0$

36. (a)

x	−2	−1	0	1
y	−1	−3	−1	5

(c) $(-3, 5)$ and $(1, 5)$
(d) $x = 1$ and $x = -3$

37. (a)

x	−2	−1	0	3
y	13	4	1	28

(c) $x = 0$
(d) $x = \pm 1.41$

38. (a) $x = -0.68$ and $x = 3.68$
(b) $y_{\text{min}} = -9.5$

39. (a) $x = -3$ and $x = 0.5$
(b) $y_{\text{min}} = -6.125$

40. (a) $y = 10$
(b) y incorrect $= 4$
(c) y incorrect $= 8$
(d) $B = 3$ and $C = -2$.
$y = 10 + 3x - 2x^2$

Exercise 7w

1. (b) $m = 2$
(c) $c = 15$
(d) $s = 2t + 15$

2. (b) (i) $m = 1.27 \times 10^{-3}$
(ii) $c = -9.5 - 10^{-5}$
(c) $Q = 1.27 \times 10^{-3}$.
$h = -9.5 \times 10^{-5}$

3. (b) Yes. $f \alpha \frac{1}{l}$.

4. (b) Yes. $\mu \alpha$ concentration

5. (b) (i) $C = 0.45\ uF$, $C = 0.65\ uF$ and $C = 0.95\ uF$
(c) (i) $T = 0.6$ second, $T = 1.4$ second and $T = 1.6$ second.

6. (b) Slope $= 1.14$ and $\lambda = 5.7 \times 10^{-7}$ m

7. (b) $m = 2$ and $c = 12$.
$I = 2V + 12$

8. (b) Yes. $\log I_A \alpha \log V_A$.

9. Cooling curve.

10. Characteristic curve.

11. Characteristic curve.

12. $d - T$ curve.

13. (b) A quadratic curve or parabola.
(c) When $d = 0$, then $t = -3$ seconds and $t = 1$ second.
(d) $(t + 3)(t - 1) = 0$.

14. (b) A quadratic curve or parabola.
(c) When $d = 0$, then $t = 1$ second and $t = 2$ seconds.
(d) $(t - 1)(t - 2) = 0$.

15. (b) A quadratic curve or parabola.
(c) When $d = 0$, then $t = -3$ seconds and $t = 5$ seconds.
(d) $(t + 3)(t - 5) = 0$.

Exercise 7x

1.

x	-1	0	2
y	9	10	0

$x = -1$ when $y = 9$
$x = 1.5$ when $y = 4$

2.

x	-1	1	2
y	$\frac{1}{2}$	2	4

$x = -1.7$ and $x = 2$

3. (iii) (a) 5.8 min
(b) $2\frac{5}{12}$ km

4. (b) $b = \$8.80$

5. (i)

x	-2	0	1	2
y	-2	-2	1	6

(iii) $A(-3.45, 3)$ and $B(1.45, 3)$
(iv) $x = -3.45$ and $x = 1.45$

6. (b) $m = -1.5$ (c) $C(0, 2)$
(d) $y = -1.5x + 2$
(e) $x = 1.2$ when $y = 0.2$

7. (ii) $m = -\frac{1}{2}$
Line PQ and line $y = 2x + 1$ are perpendicular to each other.
(iii) $(0, 1)$. $y = -\frac{1}{2}x + 1$

8. (a)

x	-2	0	1	2	3
$f(x)$	0	-10	-9	-4	5

(d) (i) $x = -2$ and $x = 2.5$
(ii) $A(-1.5, -4)$ and $B(2, -4)$

9. (a) (ii) $m = 2$
(iii) $R(0, 4)$. $y = 2x + 4$
(b) (i)

x	-2	-1	0	3
y	7	4	3	12

(iii) $x = -2$ and $x = 2$

10. (a) (i) when $x = 0$, then $y = 2$
when $x = 3$, then $y = 4$
(ii) $m = \frac{2}{3}$
(iii) $x = -3$
(b) (i)

x	-1	2	4
y	$3\frac{1}{2}$	-1	-4

(c) $x = 0, y = 2$

11. (a) (i) $y = 2x^3$
(ii) $y = 128$

12. (b) (i) $x \to 3x + 1$
(ii) $x = 13$. $y = 7$

Exercise 8a

1. (a) Total amount = \$96 million
(b) Wages and salaries = 3.7 cm
Health = 2.3 cm
Education = 1.4 cm
Agriculture = 1.9 cm
Communication = 0.3 cm
Total height = 9.6 cm

2. (a) Total number of teachers = 140 teachers
(b) Mathematics = 1.6 cm
English = 3.5 cm
History = 4.1 cm
Science = 3.7 cm
Modern languages = 1.1 cm
Total length = 14.0 cm

3. (a) \$85 at \$1.00 = 85 notes
\$125 at \$5.00 = 25 notes
\$150 at \$10.00 = 15 notes
\$420 at \$20.00 = 21 notes
\$500 at \$100.00 = 5 notes
Total number of notes = 151 notes

4. 182 times

Exercise 8d

1. (a) 60 students
(b) Physics = 54°
Chemistry = 90°
Biology = 114°
Mathematics = 72°
Geology = 30°
Sum of angles = 360°

2. (a) Annette = \$10000
Betty = \$20000
Carol = \$30000
Total sum = \$60000
(b) Annette = 60°
Betty = 120°
Carol = 180°
Sum of angles = 360°

3. (b) English = 60°
Mathematics = 96°
History = 72°
Geography = 48°
French = 48°
Spanish = 36°
Sum of angles = 360°

4. (c) Labourers = 162°
Operators = 87°
Supervisors = 54°
Transportation = 57°
Sum of angles = 360°

5. (a) \$468 million
(b) \$195 million
(c) 25%
(d) 41.7%

6. (a) \$470880 (b) \$117720
(c) 53.6%

7. (a) $B = \$61160$.
$W = \$125100$

8. (a) \$39.9 million
(b) \$38 million

Exercise 8e

1. (b) (i) (9–11) years
(ii) (11–13) years
(iii) (15–17) years and (17–19) years
(iv) (19–21) years

2. (b) (i) 1985 to 1986 and 1987 to 1988
(ii) Largest increase = \$58M
(iii) 1989 to 1990 and 1992 to 1993
(iv) Largest decrease = \$52M

3. (b) So that the infant mortality can be compared from year to year.
  (c) (i) 1939–1944
    (ii) 1944–1949
    (iii) Largest decrease = 23 infants per 1000 births

4. (b) (i) 1972–1973. Increase = 330 employees
    (ii) 1971–1972. Decrease = 280 employees
    (iii) 1974–1975, 1975–1976 and 1981–1982. Increase or decrease = ±10 employees

5. (b) So that the faults reported can be compared from month to month
  (c) (i) May–June. Increase = 2 faults per hundred stations reported.
    (ii) April–May and August–September. Increase = 0.1 fault per hundred stations reported.
    (iii) March–April. Decrease = 2.1 faults per hundred stations reported.

Exercise 8f

1. Quantitative - continuous
2. Quantitative - continuous
3. Quantitative - discrete
4. Qualitative
5. Qualitative
6. Quantitative - discrete
7. Quantitative - continuous
8. Quantitative - discrete
9. Quantitative - discrete
10. Quantitative - continuous
11. Quantitative - continuous
12. Quantitative - discrete
13. Quantitative - continuous
14. Quantitative - discrete
15. Quantitative - continuous
16. Quantitative - continuous
17. Quantitative - continuous
18. Qualitative
19. Qualitative
20. Qualitative

Exercise 8g

1. (a)

Shoe size	4	5	6	7	8	9
Frequency	4	9	3	6	5	3

  (b) Relative frequency = 0.1
  (c) P(pupil wears a size 7 shoe) = 0.2

2. (a)

Number of sessions absent	0	1	2	3	4	5	6	7	8	9	10
Frequency	15	3	1	4	7	2	0	2	0	1	1

  (b) (i) P(student absent for 6 sessions) = 0
    (ii) P(student absent < 2 sessions) = $\frac{1}{2}$
    (iii) P(student absent > 7 sessions) = $\frac{1}{18}$

3. (a)

Score	0	1	2	3	4	5	6
Frequency	4	6	3	3	1	5	3

  (b) (i) P(participant's score < 2) = 0.4
    (ii) P(participant's score = 5) = 0.2
    (iii) P(participant's score ⩾ 4) = 0.36

4. (a)

Height (cm)	*Frequency*
150	1
151	5
152	10
153	16
154	10
155	6
156	2
Total frequency = 50	

  (b) P(student's height ⩽ 153 cm) = 0.64
  (c) P(student's height > 153 cm) = 0.36

5. (a)

Number of tickets bought per person for a calypso show	*Frequency*
1	12
2	35
3	44
4	18
5	8
6	3
Total frequency = 120	

  (b) Relative frequency = 24.2%
  (c) P(person bought < 4 tickets) = 0.758

6. (a)

Stem length (cm)	*Frequency*
25	2
26	9
27	10
28	12
29	20
30	19
31	13
32	15
Total frequency = 100	

  (b) Relative frequency = 0.47
  (c) P(stem length ⩽ 29 cm) = 0.53

Exercise 8h

1. (b) (i) P(number of children < 3) = 0.2
    (ii) P(number of children > 3) = 0.5
    (iii) P(number of children = 3) = 0.3

2. (b) (i) P(pupil's shoe size < 5) = $\frac{2}{15}$
    (iii) P(pupil's shoe size > 8) = $\frac{1}{10}$

3. (b) P(pupil absent ⩾ 6 sessions) = $\frac{1}{9}$

4. (b) (i) P(student < 152 cm) = 0.12
    (ii) P(student > 154 cm) = 0.16

5. (b) P(stem length < 28 cm) = 0.21

6. (b) *P*(person purchased 4 tickets) = 0.15

Exercise 8i

1. (b) Area = 30 pupils
(c) *P*(pupil wears a size 5 or 6) = 0.4

2. (b) Area = 30 pupils
(c) *P*(number of children is 4 or 5) = 0.4

3. (b) Area = 50 participants
(c) *P*(participant scored 7 or 8 points) = 0.16

4. (b) Area = 50 students
(c) *P*(student is 153 cm or 154 cm tall) = 0.52

5. (b) Area = 120 persons
(c) *P*(person bought 2 or 3 tickets) = $\frac{79}{120}$

6. (a)

Mark	1	2	3	4	5	6	7	8	9	10
Frequency	3	2	2	5	5	6	4	2	1	0

(c) (i) *P*(student received < 5 marks) = 0.4
(ii) *P*(student received ⩾ 5 marks) = 0.6

7. (a)

Score	0	1	2	3	4	5	6
Frequency	5	7	4	4	1	6	3

(c) *P*(competitor score < 4) = $\frac{2}{3}$

8. (a)

Class mark	0	1	2	3	4	5	6	7	8	9	10
Frequency	1	1	2	3	3	4	5	6	5	3	2

(b) *P*(child watched T.V. ⩾ 7 hours) = $\frac{16}{35}$

Exercise 8j

1. (a)

Income per hour in dollars	*Frequency*
0– 4	10
5– 9	3
10–14	2
15–19	20
20–24	16
25–29	9
Total frequency = 60	

(b) (i) *P*(family earned < 15 dollars per hour) = 0.25
(ii) *P*(family earned ⩾ 15 dollars per hour) = 0.75

2. (a)

Height (cm)	*Frequency*
130–134	2
135–139	6
140–144	19
145–149	14
150–154	4
155–159	3
160–164	2
Total frequency = 50	

(b) *P*(child height is between (145–149 cm)) = 0.28

3. (a)

Mass (kg)	*Frequency*
50– 59	5
60– 69	9
70– 79	28
80– 89	33
90– 99	17
100–109	8
Total frequency = 100	

(b) *P*(person has a mass between 80 kg and 99 kg inclusive) = 0.5

4. (a)

Marks	*Frequency*
1– 5	7
6–10	10
11–15	13
16–20	14
21–25	22
26–30	18
31–35	15
36–40	9
41–45	7
46–50	5
Total frequency = 120	

(b) *P*(candidate mark > 35) = $\frac{7}{40}$

5.

Class interval ($)	*Theoretical class intervals ($)*
0– 4	$0 \leqslant x < 4.5$
5– 9	$4.5 \leqslant x < 9.5$
10–14	$9.5 \leqslant x < 14.5$
15–19	$14.5 \leqslant x < 19.5$
20–24	$19.5 \leqslant x < 24.5$
25–29	$24.5 \leqslant x < 29.5$

6.

Class intervals (cm)	*Theoretical class intervals (cm)*
130–134	$129.5 \leqslant x < 134.5$
135–139	$134.5 \leqslant x < 139.5$
140–144	$139.5 \leqslant x < 144.5$
145–149	$144.5 \leqslant x < 149.5$
150–154	$149.5 \leqslant x < 154.5$
155–159	$154.5 \leqslant x < 159.5$
160–164	$159.5 \leqslant x < 164.5$

7.

Class intervals (kg)	*Theoretical class intervals (kg)*
50– 59	$49.5 \leqslant x < 59.5$
60– 69	$59.5 \leqslant x < 69.5$
70– 79	$69.5 \leqslant x < 79.5$
80– 89	$79.5 \leqslant x < 89.5$
90– 99	$89.5 \leqslant x < 99.5$
100–109	$99.5 \leqslant x < 109.5$

8.

Class intervals (marks)	*Theoretical class intervals (marks)*
1– 5	$0.5 \leqslant x < 5.5$
6–10	$5.5 \leqslant x < 10.5$
11–15	$10.5 \leqslant x < 15.5$
16–20	$15.5 \leqslant x < 20.5$
21–25	$20.5 \leqslant x < 25.5$
26–30	$25.5 \leqslant x < 30.5$
31–35	$30.5 \leqslant x < 35.5$
36–40	$35.5 \leqslant x < 40.5$
41–45	$40.5 \leqslant x < 45.5$
46–50	$45.5 \leqslant x < 50.5$

9.

Class intervals ($)	*Class mid-points ($)*
0– 4	2
5– 9	7
10–14	12
15–19	17
20–24	22
25–29	27

10.

Class intervals (cm)	*Class mid-points (cm)*
130–134	132
135–139	137
140–144	142
145–149	147
150–154	152
155–159	157
160–164	162

11.

Class intervals (kg)	*Class mid-marks (kg)*
50– 59	54.5
60– 69	64.5
70– 79	74.5
80– 89	84.5
90– 99	94.5
100–109	104.5

12.

Class intervals (marks)	*Class values (marks)*
1–5	3
6–10	8
11–15	13
16–20	18
21–25	23
26–30	28
31–35	33
36–40	38
41–45	43
46–50	48

13. The width of the class intervals = \$5

14. The unit size of the class intervals = 5 cm

15. The class size of the class intervals = 10 kg

16. The width of the class intervals = 5 marks

Exercise 8k

1. (b) Relative frequency = 0.33

2. (b) *P*(candidate's mark < 25.5) = 0.56

3. (b) 24%

4. (b) 82%

5. (b) *P*(rod > 124.5 mm) = 0.35

Exercise 8l

1. (b) 100 candidates

2. (b) 100 persons

3. (b) 100 candidates

4. (b) 50 pupils

5. (b) 120 people

Exercise 8m

1. $\bar{x} = 3.6$

2. $\bar{x} = 5$

3. (a) 567 marks

(b) $\bar{x} = 70\frac{7}{8}$ marks

4. $\bar{x} = 169.15$ cm

5. (a) $\bar{x} =$ TT \$47.66 M

(b) $\bar{x} =$ TT \$21.84 M

6. $\bar{x} = 153.6$ cm

7. $\bar{x} = 156$ cm

8. $\bar{x} = 3.9$ children ≈ 4 children

9. $\bar{x} = 2.9$ tickets ≈ 3 tickets

10. $\bar{x} = 29.23$ cm

11. $\bar{x} = 4.45$ marks

12. $\bar{x} = 5$

13. (a) $\bar{x} = 164$ cm

(b) $\bar{x} = 164$ cm

14. (a) $\bar{x} = 52.5$ kg

(b) $\bar{x} = 54$ kg

15. (a) $\bar{x} = 72.5$ marks

(b) $\bar{x} = 75$ marks

Exercise 8n

1. $Q_2 = 4$

2. $Q_2 = 5$

3. $Q_2 = 154$ cm

4. $Q_2 = 156$ cm

5. $Q_2 = 68$ kg

6. (a) $Q_2 = 4$ marks

(b) *P*(pupil's mark ⩽ 5) = 0.7

7. (a) $Q_2 = 5$

(b) *P*(participant score < 6) = 0.66

8. (a) $Q_2 = 4$ children

(b) *P*(family has > 5 children) $= \frac{1}{6}$

9. (a) $Q_2 = 29$ cm

10. (a)

Shoe size	4	5	6	7	8	9
Frequency	4	9	3	6	5	3

(b) $Q_2 =$ size 6

Exercise 8o

1. Mode = 4

2. Mode = 5

3. Modal height = 154 cm

4. Modal height = 156 cm

5. Mode = 3 children

6. Modal mark = 4 marks

7. Modal shoe size = size 5

8. (b) (i) Mode = 3 children

(ii) $Q_2 = 3.5$ children

(iii) $\bar{x} = 3.7$ children

9. (b) (i) $\bar{x} = 2\frac{4}{9}$ sessions

(ii) $Q_2 = 1.5$ sessions

(iii) Mode = 0

10. (a) $Q_2 = 45$ marks

(b) $\bar{x} = 46$ marks

(c) Mode = 45 marks

(d) (i) *P*(candidate's mark ≤ 45) $= \frac{72}{125}$

(ii) *P*(candidate's mark ≥ 75) $= \frac{21}{125}$

11. (a) $\bar{x} = 153.1$ cm

(b) $Q_2 = 153$ cm

(c) Modal height = 153 cm

12. (a)

Class mark	*Frequency*
1	10
2	13
3	29
4	22
5	11
6	5
$n = \Sigma f = 90$	

(b) $\bar{x} = 3.3$ hours

(c) *P*(person watched T.V. ⩾ 4 hours) $= \frac{19}{45}$

13. (a) (i) Amount invested in 1985 = \$3 300

Amount invested in 1988 = \$4 800

(ii) $\bar{x} = \$4\,300$

(iii) Amount invested in 1990 = \$5 800

(iv) $\Theta = 80.4°$

(b) *P*(second ball is also green) $= \frac{1}{3}$

14. (b) *P*(teacher is an English teacher) = 0.26

(c) $\Theta = 91.2°$

(d) (i) 96 teachers

(ii) $\bar{x} = 28.5$ teachers

15. (a)

Types of notes	*Number of notes*
$1.00	79
$5.00	16
$10.00	35
$20.00	20
$50.00	10
$100.00	7
$n = \Sigma f = 167$	

(c) $Q_2 = \$5.00$
(d) (i) P(note is a five - dollar) $= \frac{16}{167}$
(ii) P(note is NOT a hundred - dollar) $= \frac{160}{167}$

16. (a) Mode = 5 (b) $Q_2 = 4$
(d) 406 minutes
(c) $\bar{x}$ = 4 minutes
(f) (i) P(student late by exactly 6 minutes) = 0.16
(ii) P(student late $\geqslant$ 6 minutes) = 0.26
(iii) P(student late $<$ 6 minutes) = 0.74

17. (a) $\bar{x}$ = 31.5 minutes
(b) Q_2 = 35 minutes
(c) Mode = 40 minutes. Mode - most popular
(d) P(child waits $\geqslant$ half an hour) = 0.55

18. (a) Increase = $1.5 M
(b) (i) $\bar{x}$ = $3.2 M
(ii) Q_2 =S3.5 M
(c) P(year's expenditure $>$ $3.5 M) = 0.4
(d) Θ 1980 = 45°
Θ 1981 = 78.75°
Θ 1982 = 33.75°
Θ 1983 = 112.5°
Θ 1984 = 90°
Total = 360°

19. (a) $\bar{x}$ = 14.5 years
Q_2 = 14 years
(b) (i) P(child $<$ 15 years of age) = 0.55
(ii) P(child $\geqslant$ 15 years of age) = 0.45

Exercise 8p

1. (a) Range = 5 (b) *I.Q.R.* = 3
(c) *S.I.Q.R.* = 1.5

2. (a) Range = 6 (b) *I.Q.R.* = 3
(c) *S.I.Q.R.* = 1.5

3. (a) Range = 19 cm
(b) *I.Q.R.* = 14.5 cm
(c) *S.I.Q.R.* = 7.25 cm

4. (a) Range = 8 kg
(b) *I.Q.R.* = 4 kg
(c) *S.I.Q.R.* = 2 kg

5. (a) Range = 10 marks
(b) Q_2 = 5 marks.
S.I.Q.R. = 2.5 marks
(c) $\bar{x}$ = 5.08 marks
(d) P(student's mark $>$ 7) = 0.28

6. (a) Range = 5 marks
(b) Q_2 = 2 marks.
S.I.Q.R. = 1 mark
(c) $\bar{x}$ = 2.48 marks

7. (a) Range = 8
(b) *S.I.Q.R.* = 1

8. (a) (i) Q_1 = 53.5 kg
(ii) Q_3 = 57 kg
(b) (i) *I.Q.R.* = 3.5 kg
(ii) *S.I.Q.R.* = 1.75 kg
(c) Range = 10 kg

9. (a) Range = 10 cm
(b) (i) *I.Q.R.* = 4 cm
(ii) *S.I.Q.R.* = 2 cm

10. (a) range = 10 mm
(b) (i) *I.Q.R.* = 4 mm
(ii) *S.I.Q.R.* = 2 mm

Exercise = 8q

1. (a) $P(T) = \frac{1}{2}$
(b) $P(H) = \frac{1}{2}$

2. (a) $P(6) = \frac{1}{6}$
(b) $P(\text{number} > 4) = \frac{1}{3}$

3. (a) P(multiple of 2) $= \frac{1}{2}$
(b) P(odd number) $= \frac{1}{2}$

4. (a) P(prime number) $= \frac{1}{2}$
(b) $P(\text{number} < 5) = \frac{2}{3}$

5. (a) P(blue marble) $= \frac{4}{5}$
(b) P(yellow marble) $= \frac{1}{9}$

6. (a) P(green marble) $= \frac{1}{3}$
(b) P(blue marble) $= \frac{1}{2}$

7. (a) P(Guyanese coin) $= \frac{3}{5}$
(b) P(Jamaican coin) $= \frac{2}{5}$
(c) P(Guyanese coin or Jamaican coin) = 1

8. (a) P(king) $= \frac{1}{13}$
(b) P(black card) $= \frac{1}{2}$

9. (a) P(Ace of Hearts) $= \frac{1}{52}$
(b) P(Joker) = 0

10. (a) P(red Jack) $= \frac{1}{26}$
(b) P(King or Queen) $= \frac{2}{13}$
(c) P(King, Queen or Jack) $= \frac{3}{13}$

11. (a) P(vowel) $= \frac{7}{19}$
(b) $P(M) = \frac{3}{19}$
(c) $P(O) = \frac{1}{19}$

12. (a) P(multiple of 5) $= \frac{1}{5}$
(b) P(prime number) $= \frac{9}{25}$

13. (a) P(boy perfect) $= \frac{3}{5}$
(b) P(girl perfect) $= \frac{2}{5}$

14. (a) P($1 note) $= \frac{3}{5}$
(b) P($5 note) $= \frac{3}{10}$
(c) P($10 note) $= \frac{1}{10}$

15. (a) P($100 note) $= \frac{2}{15}$
(b) P($20 note or $100 note) $= \frac{5}{18}$
(c) P($10 note or $20 note or $100 note) $= \frac{4}{9}$

16. P(pencil is unsharpened) $= \frac{5}{8}$

17. (a) P(2 800 cc car) $= \frac{1}{4}$
(b) P(1 500 cc car or 1 300 cc car) $= \frac{3}{4}$

18. (a) P(square) $= \frac{1}{10}$
(b) P(cube) $= \frac{1}{25}$
(c) P(divisible by 5) $= \frac{1}{5}$

19. (a) $P(\text{blue or green}) = \frac{3}{10}$

(b) $P(\text{red or black}) = \frac{7}{10}$

20. (a) $P(\text{Western or Romance})$ $= \frac{9}{16}$

(b) $P(\text{Mystery or Comedy})$ $= \frac{7}{16}$

Exercise 8r

1. (i)

x	f
1	17
2	22
3	29
4	18
5	9
6	5
$n = \Sigma f = 100$	

(ii) $\bar{x} = 2.95$ h

(iii) P(person watched T.V. $\geqslant 5$ hours $= \frac{7}{50} = 0.14$)

2. (i)

x	f
0	4
1	6
2	3
3	3
4	1
5	5
6	3
$n = \Sigma f = 25$	

(iii) $Q_2 = 2$. $I.Q.R. = 4$

(iv) $P(\text{competitor's score} > 4)$ $= \frac{8}{25}$

3. (i)

x	f
1	2
2	3
3	5
4	8
5	10
6	12
7	15
8	18
9	15
10	12
$n = \Sigma f = 100$	

(ii) Mode = 8 $\quad Q_2 = 7$
$\bar{x} = 6.8$

(iii) Mode size.
Most popular size

(iv) P(pair of shoes is a size 6) $= \frac{3}{25} = 0.12$

4. (a) $I.Q.R. = 13$ cm

(b) (ii) $Q_2 = 47$ kg

(iii) $P(\text{boy weighs} \leqslant 41 \text{ kg})$ $= \frac{17}{60}$

5. (a) (i) \$3 200 and \$4 400

(ii) $\bar{x} = \$4000$

(iii) \$5 200

(iv) 79.2°

(b) P(second ball is yellow) $= \frac{1}{3}$

6. (a) (i) \$30 million

(ii) \$17.5 million

7. (a)

Value	*Type of note*	*Number of notes*
\$38	\$1.00	38
\$20	\$2.00	10
\$90	\$5.00	18
\$250	\$10.00	25
\$300	\$20.00	15
\$400	\$100.00	4

(c) $Q_2 = \$5.00$

(d) (i) P(note is a ten-dollar note) $= \frac{5}{22}$

(ii) P(note is not a hundred dollar note) $= \frac{53}{55}$

8. (a) (i) $\bar{x} = 13.6$ years.
$Q_2 = 13.5$ years

(ii) $P(\text{child} < 15 \text{ years old})$ $= \frac{7}{10} = 0.7$
$P(\text{child} \geqslant 15 \text{ years})$ $= \frac{3}{10} = 0.3$

(b) (i) $S = \$9000$ and $B = \$16000$

(ii) $W = \$27000$, $E = \$10000$ and $M = \$10000$

9. (b) P(teacher is an English teacher) $= \frac{1}{5}$

(c) 108°

(d) (i) 93 teachers

(ii) $\bar{x} = 27$ teachers

10. (a) (i) $a = \frac{1}{4}$, $b = \$1800$, $c = \$675$ and $d = \frac{1}{8}$

(ii) 120° and 75°

(b) (ii) $P(\text{student's mark} < 4)$ $= \frac{7}{30}$

11. (b) (i) \$367 200

(ii) \$60 180 (iii) $\frac{11}{18}$

12. (a) Mode = 5 min

(b) $Q_2 = 4.5$ min

(d) 372 min

(e) $\bar{x} = 4$ min

(f) (i) P(student late 5 min) $= \frac{2}{9}$

(ii) $P(\text{student late} \geq 5 \text{ min})$ $= \frac{1}{2}$

Exercise 9a

1. (a) $\frac{2}{3}$ rev.
(b) $\frac{3}{4}$ rev. (c) $\frac{7}{12}$ rev.
2. (a) $\frac{3}{4}$ rev. (b) $\frac{5}{6}$ rev.
3. (a) $\frac{2}{3}$ rev. (b) $\frac{1}{2}$ rev.
4. (a) $\frac{5}{12}$ rev. (b) $\frac{1}{4}$ rev.
5. (a) 4 (b) 6
6. (a) 3 (b) 8
7. (a) 9 (b) 1
8. (a) 12 (b) 12
9. (a) 3 rt. ∠s (b) 3 rt. ∠s
(c) 4 rt. ∠s
10. (a) 3 rt. ∠s (b) 2 rt. ∠s
(c) 3 rt. ∠s
11. (a) 2 rt. ∠s (b) 1 rt. ∠
12. (a) 1 rt. ∠ (b) 2 rt. ∠s
13. (a) N (b) E
14. (a) W (b) N
15. (a) N (b) S
16. (a) E (b) N
17. (a) 3 rt. ∠s (b) 2 rt. ∠s
18. (a) 3 rt. ∠s (b) 3 rt. ∠s
19. (a) 1 rt. ∠ (b) 2 rt. ∠s
20. (a) 3 rt. ∠s (b) 1 rt. ∠
21. (a) 270° (b) 180°
(c) 180°
22. (a) 200° (b) 234°
(c) 270°
23. (a) 210° (b) 306° (c) 450°

24. (a) 320° (b) 342°
(c) 315°
25. (a) 210° (b) 150°
(c) 45° (d) 165°

Exercise 9b

1. (a) Reflex angle
(b) Acute angle
(c) Obtuse angle
(d) Right angle
2. (a) Reflex angle
(b) Acute angle
(c) Obtuse angle
3. (a) Obtuse angle
(b) Reflex angle
(c) Acute angle
(d) Right angle
4. (a) Obtuse angle
(b) Acute angle
(c) Right angle
(d) Reflex angle
5. (a) Obtuse angle
(b) Reflex angle
(c) Right angle
(d) Acute angle
6. (a) $\hat{x} = 145^\circ$
(b) $\hat{a} = 75^\circ$, $\hat{b} = 105^\circ$ and $\hat{c} = 75^\circ$
(c) $\hat{y} = 115^\circ$
(d) $\hat{z} = 105^\circ$
7. (a) $\hat{t} = 130^\circ$
(b) $\hat{x} = 35^\circ$
8. (a) $\hat{d} = 130^\circ$ (b) $\hat{y} = 40^\circ$
9. (a) $\hat{d} = 140^\circ$, $\hat{e} = 40^\circ$ and $\hat{f} = 140^\circ$
(b) $\hat{g} = 155^\circ$ and $\hat{h} = 180^\circ$
10. (a) $\hat{s} = 100^\circ$
(b) $\hat{e} = 35^\circ$
11. (a) $\hat{x} = 155^\circ$
(b) $\hat{a} = 95^\circ$, $\hat{b} = 85^\circ$ and $\hat{c} = 95^\circ$
(c) $\hat{y} = 145^\circ$
(d) $\hat{z} = 115^\circ$
12. $\hat{b} = \hat{d} = \hat{f} = \hat{h} = 55$ and $\hat{a} = \hat{c} = \hat{e} = \hat{g} = 125^\circ$
13. (a) $\hat{d} = 115^\circ$
(b) $\hat{r} = 90^\circ$
(c) $\hat{a} = 105^\circ$, $\hat{b} = 75^\circ$ and $\hat{c} = 105^\circ$
(d) $\hat{p} = 125^\circ$, $\hat{q} = 55^\circ$ $\hat{r} = 125^\circ$ and $\hat{s} = 55^\circ$
14. (a) $\hat{p} = 116^\circ$
(b) $\hat{a} = 115^\circ$, $\hat{b} = 65^\circ$ and $\hat{c} = 115^\circ$
(c) $\hat{e} = 136^\circ$, $\hat{f} = 44^\circ$, $\hat{g} = 136^\circ$ and $\hat{h} = 44^\circ$
15. (a) $\hat{d} = 50^\circ$
(b) $\hat{e} = 35^\circ$
(c) $\hat{f} = 65^\circ$
16. (a) $\hat{a} = 45^\circ$ and $\hat{b} = 75^\circ$
(b) $\hat{r} = 30^\circ$, $\hat{t} = 60^\circ$ and $\hat{s} = 270^\circ$
(c) $\hat{p} = 40^\circ$
17. (a) $\hat{p} = 120^\circ$, $\hat{q} = 60^\circ$ and $\hat{r} = 120^\circ$
(b) $\hat{a} = 50^\circ$, $\hat{b} = 110^\circ$ and $\hat{c} = 60^\circ$
(c) $\hat{l} = 110^\circ$
18. (a) $\hat{p} = 328^\circ$
(b) $\hat{q} = 27^\circ$
(c) $\hat{r} = 145^\circ$, $\hat{s} = 35^\circ$ and $\hat{t} = 145^\circ$
(d) $\hat{l} = 80^\circ$
19. (a) $\hat{x} =120^\circ$
(b) $\hat{y} = 55^\circ$
(c) $\hat{z} = 70^\circ$
(d) $\hat{a} = 35^\circ$
20. (a) $\hat{l} = 98^\circ$, $\hat{m} = 82^\circ$ and $\hat{n} = 98^\circ$
(b) $\hat{p} = 65^\circ$
(c) $\hat{q} =145^\circ$
(d) $\hat{r} = 80^\circ$
21. (a) $\hat{r} = 45^\circ$ and $\hat{s} = 50^\circ$
(b) $\hat{x} = 42^\circ$
22. (a) $\hat{x} = 95^\circ$
(b) $\hat{p} = 145^\circ$ and $\hat{q} = 35^\circ$
(c) $\hat{x} = 53^\circ$ and $\hat{y} = 127^\circ$
(d) $\hat{p} = 55^\circ$, $\hat{q} = 125^\circ$ and $\hat{r} = 55^\circ$
23. (a) $\hat{a} = 85^\circ$
(b) $\hat{b} = 165^\circ$
(c) $\hat{c} = 90^\circ$
(d) $\hat{d} = 50^\circ$
24. (a) $\hat{p} = 335^\circ$
(b) $\hat{q} = 248^\circ$
25. (a) $\hat{r} = 55^\circ$, $\hat{s} = 70^\circ$ and $\hat{t} = 35^\circ$
(b) $\hat{p} = 130^\circ$, $\hat{q} = 50^\circ$, $\hat{r} = 120^\circ$ and $\hat{s} = 60^\circ$
26. (a) $\hat{x} = 120^\circ$ and $\hat{y} = 60^\circ$
(b) $\hat{p} = 110^\circ$
27. $\hat{x} = 22.5^\circ$ **28.** $\hat{x} = 45^\circ$
29. $\hat{y} = 60^\circ$ **30.** $\hat{x} = 30^\circ$

Exercise 9c

1. (a) $\hat{x} = 34^\circ$
(b) $\hat{y} = 140^\circ$
(c) $\hat{z} = 336^\circ$
2. (a) $\hat{p} = 37^\circ$
(b) $\hat{q} = 124^\circ$
(c) $\hat{r} = 297^\circ$
(d) $\hat{s} = 30^\circ$
3. (a) $\hat{x} = 330^\circ$
(b) $\hat{y} = 240^\circ$
(c) $\hat{z} = 315^\circ$
4. (a) $\hat{p} = 30^\circ$
(b) $\hat{q} = 150^\circ$
(c) $\hat{r} = 323^\circ$
(d) $\hat{s} = 50^\circ$
5. (a) $\hat{x} = 338^\circ$
(b) $\hat{y} = 241^\circ$
(c) $\hat{z} = 140^\circ$
6. (a) $\hat{p} = 335^\circ$
(b) $\hat{q} = 240^\circ$
(c) $\hat{r} = 220^\circ$
35. 4 cm each
36. 5 cm each
37. 6 cm each
38. 4.3 cm each
39. 4.7 cm each
40. 5.1 cm each
41. 3.25 cm each
42. 4.75 cm each
43. 6.25 cm each
44. 18° each
45. 24° each
46. 31° each
47. 26.6° each
48. 37.8° each
49. 43.9° each
50. 14.75° each
51. 23.75° each
52. 39.25° each

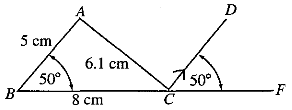
Exercise 9e

1. $\hat{x} = 65^\circ$ **2.** $\hat{y} = 27^\circ$
3. $\hat{z} = 61.7^\circ$ **4.** $\hat{x} = 44.2^\circ$
5. $\hat{y} = 54.3^\circ$
6. $\hat{x} = 60^\circ$ and $\hat{y} = 30^\circ$
7. $\hat{p} = 90^\circ$ and $\hat{q} = 45^\circ$
8. $\hat{x} = 60^\circ$
9. $B\hat{A}C = 110^\circ$ **10.** $L\hat{N}M = 25^\circ$
11. $\hat{x} = 136.3^\circ$ **12.** $\hat{y} = 45.2^\circ$
13. $\hat{d} = 25^\circ$ and $\hat{e} = 130^\circ$
14. $\hat{d} = 35^\circ$ and $\hat{e} = 145^\circ$
15. $\hat{r} = 42^\circ$ and $\hat{s} = 68^\circ$

16. $\hat{x} = 50°$ and $\hat{y} = 70°$
17. $\hat{x} = 50°$
18. $\hat{x} = 45°$ and $\hat{y} = 75°$
19. $A\hat{C}E = 70°$
20. $\hat{x} = 144°$ and $\hat{y} = 123°$
21. $C\hat{D}B = 113°$
22. $\hat{p} = 135°$, $\hat{q} = 45°$, $\hat{r} = 130°$, $\hat{s} = 50°$ and $\hat{t} = 85°$
23. $\hat{x} = 45°$ and $\hat{y} = 135°$
24. $\hat{t} = 60°$

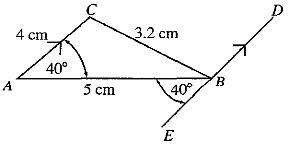
Exercise 9f

1. (b) $A\hat{C}B = 90°$
2. (b) $K\hat{M}L = 115°$
3. (b) $P\hat{Q}R = 30.8°$
4. (b) $T\hat{U}V = 53.1°$
5. (b) $Q\hat{P}R = 41.2°$
6. (b) $AC = 6.3$ cm
7. (b) $PQ = 13.7$ cm
8. (b) $LN = 6.4$ cm
9. (b) $KL = 6.5$ cm
10. (b) $q = 9$ cm
11. (b) $\hat{C} = 22.3°$
12. (b) $AC = 6$ cm
13. (b) $AB = 14.5$ cm
14. (b) $\hat{C} = 90°$
15. (b) $\hat{R} = 30°$
16. (b) $\hat{C} = 100°$
17. (b) $\hat{R} = 30°$
18. (b) $M\hat{N}L = 105°$
19. (b) $\hat{R} = 30°$
20. (b) $\hat{M} = 75°$
21. (b) $\hat{C} = 95°$
22. (b) $\hat{N} = 65°$
23. (b) $AB = 8$ cm
24. (b) $P\hat{R}Q = 36.9°$
25. (b) $B\hat{A}C = 60°$
26. (b) $KL = 4.5$ cm
27. (b) $RQ = 7.2$ cm
28. (b) $XY = 6.3$ cm
29.



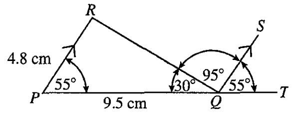
$AC = 6.1$ cm
$D\hat{C}F = 50°$

30.



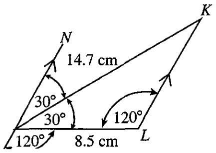
$BC = 3.2$ cm
$A\hat{B}E = 40°$

31.



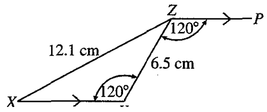
$PR = 4.8$ cm
$R\hat{Q}S = 95°$
$S\hat{Q}T = 55°$

32.



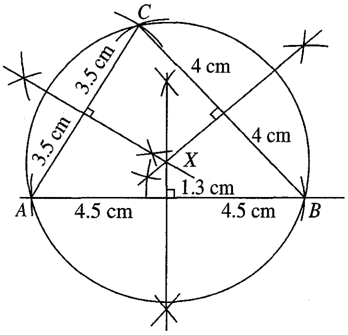
$MK = 14.7$ cm
$K\hat{M}N = 30°$
$L\hat{M}X = 120°$

33.



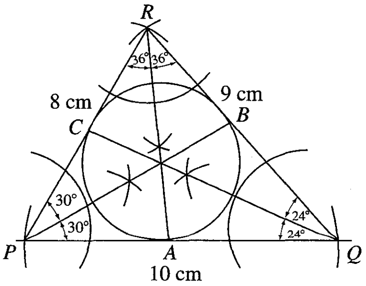
$XZ = 12.1$ cm
$P\hat{Z}Y = 120°$

34. $PS = 4.5$ cm
35. (c) $XC = 2$ cm and $XD = 5$ cm
36. (a)



(b) $XD = 1.3$ cm
(c) We have constructed the circumscribed circle of $\triangle ABC$ or the circumcircle of $\triangle ABC$

37. (a)



(c) $XA = XB = XC = 2.5$ cm
(d) We have constructed the inscribed circle of $\triangle PQR$.

Exercise 9g

1. Yes. S.S.S.
2. Yes. A.A.S.
3. Yes. S.A.S.
4. Yes. A.A.S.
5. Yes. S.A.S.
6. No
7. Yes. R.H.S.
8. No.
9. Yes. R.H.S.
10. Yes. S.S.S.
11. Yes. A.A.S.
12. No
13. $\triangle AXY \equiv \triangle AZY$ (R.H.S.). So $AX = AZ$
14. $\triangle ABD \equiv \triangle ACD$ (S.A.S.). So $AB = AC$
15. $\triangle ABD \equiv \triangle CBD$ (S.S.S.).
16. $\triangle PQS \equiv \triangle RQS$ (S.S.S.). So $P\hat{Q}S = R\hat{Q}S$ and $P\hat{S}Q = R\hat{S}Q$
17. $\triangle AOC \equiv \triangle BOD$ (S.A.S.). So $AC = BD$
18. $\triangle AOB \equiv \triangle DOC$ (A.A.S.). So $AO = DO$ and $BO = CO$
19. $\triangle NCB \equiv \triangle MBC$ (S.A.S.). So $NC = MB$
20. $\triangle OAX \equiv \triangle OBY$ (R.H.S.). So $AX = BY$
21. (a) $\triangle PAS \equiv \triangle RCQ$ (S.A.S.)
(b) $\triangle QBP \equiv \triangle SDR$ (S.A.S.)
(c) From (a) $PS = RQ$
(d) From (b) $PQ = RS$
22. (a) $\triangle PQS \equiv \triangle RSQ$ (S.S.S.)
(b) $\triangle PQR \equiv \triangle RSP$ (S.S.S.)
23. (a) (i) $\triangle ROS \equiv \triangle POQ$ (A.A.S.)
(ii) $RO = PO$
(iii) $SO = QO$
(b) (i) $\triangle POS \equiv \triangle ROQ$ (A.A.S.)
(ii) $PO = RO$
(iii) $QO = SO$
24. (a) $\triangle DOC \equiv \triangle BOA$ (A.A.S.). So $D\hat{O}C = B\hat{O}A$
(b) $\triangle AOD \equiv \triangle COB$ (A.A.S.). So $A\hat{O}D = C\hat{O}B$

25. (a) $\triangle ABD \equiv \triangle CBD$ (S.S.S.).
So $A\hat{B}D = C\hat{B}D$ and
$A\hat{D}B = C\hat{D}B \Rightarrow \hat{B}$ and $\hat{D}$
are bisected by BD
(b) $\triangle ABC \equiv \triangle ADC$ (S.S.S.).
So $B\hat{A}C = D\hat{A}C$ and
$B\hat{C}A = D\hat{C}A \Rightarrow \hat{A}$ and $\hat{C}$
are bisected by AC

Exercise 9h

1. $\hat{P} = \hat{L} = 120°$, $\hat{Q} = \hat{M} = 35°$ and $\hat{R} = \hat{N} = 25°$.
$\frac{\triangle LMN}{\triangle PQR}$. (A.A.A.)

2. $\hat{A} = \hat{P} = 40°$, $\hat{B} = \hat{Q} = 120°$ and $\hat{C} = \hat{R} = 20°$
$\frac{\triangle PQR}{\triangle ABC}$. (A.A.A.)

3. $\frac{\triangle COD}{\triangle BOA}$. (A.A.A.)

4. $\frac{\triangle POQ}{\triangle SOR}$. (A.A.A.)

5. $\frac{\triangle ROS}{\triangle POQ}$. (A.A.A.)

6. (a) $\frac{\triangle PQR}{\triangle ABC}$. (A.A.A.)
(b) $k = 13$
(c) (i) $r = 78$ cm
(ii) $q = 130$ cm
(d) $A_2 = 4056 \text{ cm}^2$

7. (a) $\frac{\triangle XYZ}{\triangle KLM}$. (A.A.A.)
(b) $k = 7$
(c) (i) $x = 28$ cm
(ii) $z = 21$ cm
(d) $A_2 = 294 \text{ cm}^2$

8. (a) $\frac{\triangle POQ}{\triangle MON}$. (A.A.A.)
(b) $k = 2$
(c) (i) $OP = 26$ cm
(ii) $OQ = 24$ cm
(iii) $PQ = 10$ cm
(d) $A_2 = 120 \text{ cm}^2$

9. (a) $\frac{\triangle XOY}{\triangle POQ}$. (A.A.A.)
(b) $x = 10$ cm
(c) $A_1 = 60 \text{ cm}^2$

10. (a) $\frac{\triangle MOL}{\triangle POQ}$. (A.A.A.)
(b) (i) $x = 8$ cm
(ii) $y = 17$ cm
(c) $A_1 = 60 \text{ cm}^2$

Exercise 9i

1. (a) $AC = 10$ cm
(b) $LM = 3$ cm
(c) $PR = 11.5$ cm

2. $AD = h = 23.0$ cm

3. (a) $BD = 8$ cm
(b) $AC = 25.2$ cm

4. (a) $PN = LP = 10$ cm.
$\triangle PLN$ isosceles
(b) (i) $LN = 14.1$ cm
(ii) $LM = 11.2$ cm

5. (a) $PN = LP = 13$ cm.
$\triangle PLN$ isosceles
(b) (i) $LM = 13.9$ cm
(ii) $LN = 18.4$ cm

6. (a) $h = 2.5$ cm
(b) $AC = 3.5$ cm

7. (a) $AB = 10$ cm
(b) $CD = 17.5$ cm

8. $AC = 18.3$ m

9. $PQ = 11.3$ m

10. $YZ = 11.3$ m

11. $h = 4.7$ cm

12. $h = 13.7$ cm

13. (a) $QP = ML = 8$ cm.
$\triangle MLO \equiv \triangle QPO$ (R.H.S.)
(b) $MO = QO = 10$ cm

14. (a) (i) $MR = 21.8$ cm
(ii) $RS = 43.6$ cm
(b) $P = 218$ cm

15. $BF = 2.7$ m

Exercise 9j

1. $\hat{x} = 41°$

2. $\hat{d} = 60°$

3. $\hat{d} = 70°$ and $\hat{e} = 125°$

4. $\hat{d} = 60°$

5. $\hat{a} = 60°$ and $\hat{b} = 115°$

6. $\hat{d} = 40°$

7. $\hat{e} = 75°$ and $\hat{f} = 120°$

8. $\hat{x} = 149°$ and $\hat{y} = 72°$

9. $\hat{x} = 38°$ and $\hat{y} = 115°$

10. $\hat{a} = 90°$, $\hat{b} = 70°$, $\hat{c} = 110°$, $\hat{d} = 30°$, $\hat{e} = 100°$ and $\hat{f} = 70°$

11. $\hat{x} = 36°$

12. $\hat{x} = 42°$

13. $\hat{x} = 120°$

14. $\hat{y} = 127°$

15. $\hat{a} = 70°$, $\hat{b} = 42°$, $\hat{c} = 147°$, $\hat{d} = 101°$ and $\hat{p} = 33°$

16. 105° **17.** 120°

18. 55° **19.** 95°

Exercise 9k

1. (b) $AC = BD = 10.1$ cm

2. (b) $PR = QS = 11.3$ cm

3. (b) $KM = LN = 9$ cm

4. (b) $WY = XZ = 12.7$ cm

5. (b) $JL = KM = 13.9$ cm

6. (b) $AC = BD = 10.6$ cm

7. (b) $PR = QS = 11.9$ cm

8. (b) $KM = LN = 12.9$ cm

9. (b) $JL = KM = 15$ cm

10. (b) $WY = XZ = 16.5$ cm

11. (b) $QS = 10.4$ cm

12. (b) $BD = 8.5$ cm

13. (b) $BD = 9.3$ cm

14. (b) $QS = 6.1$ cm

15. (b) $P\hat{S}Q = 24.9°$

16. (b) $P\hat{S}Q = 23.8°$

17. (b) $P\hat{S}Q = 32.1°$

18. (b) $QS = 5.8$ cm

19. (b) $NL = 11.3$ cm

20. (b) $A\hat{B}D = 67.5°$

21. (b) $W\hat{Z}X = 15°$

22. (b) $A\hat{D}B = 30°$

23. (b) (i) $K\hat{N}L = 31°$
(ii) $K\hat{M}N = 59°$

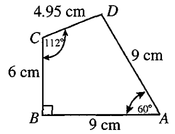
24. (b) (i) $S\hat{P}R = 56.3°$
(ii) $Q\hat{S}R = 33.7°$

25. (b) (i) $AB = 9.2$ cm
(ii) $BC = 9.2$ cm

26. (b) (i) $PQ = 7.2$ cm
(ii) $RQ = 7.2$ cm

27. (b) (i) $L\hat{K}N = 108.7°$
(ii) $L\hat{M}N = 108.7°$

28. (a)

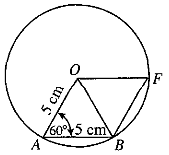


(b) (i) $DC = 4.95$
(ii) $B\hat{C}D = 112°$

29. (a) $BC = 8$ cm
(b) $PR = 12.2$ m

30. (a) $AD = 6$ m
(b) $KM = 9.43$ m

31.



Exercise 9l

1. (a) (i) $S = 32$ rt. ∠s
(ii) $S = 2880°$
(b) (i) Int. $\angle = 160°$
(ii) Ext. $\angle = 20°$

2. (a) (i) $S = 34$ rt. ∠s
(ii) $S = 3060°$
(b) (i) Int. $\angle = 161.1°$
(ii) Ext. $\angle = 18.9°$

3. (a) (i) $S = 38$ rt. ∠s
(ii) $S = 3420°$
(b) (i) Int. $\angle = 162.9°$
(ii) Ext. $\angle = 17.1°$

4. (a) (i) $S = 46$ rt. ∠s
(ii) $S = 4140°$
(b) (i) Int. $\angle = 165.6°$
(ii) Ext. $= 14.4°$

5. (a) (i) $S = 56$ rt. ∠s
(ii) $S = 5040°$
(b) (i) Int. $\angle = 168°$
(ii) Ext. $\angle = 12°$

6. $n = 9$ sides

7. $n = 8$ sides

8. $n = 24$ sides

9. $n = 12$ sides

10. $n = 25$ sides

11. Int. $\angle = 100°$

12. Int. $\angle = 112.5°$

13. Int. $\angle = 115°$

14. Int. $\angle = 119.2°$

15. Int. $\angle = 170°$

16. $n = 8$ sides

17. (a) $D\hat{O}E = 60°$
(b) $DE = 10$ cm
(c) $P = 60$ cm

18. $n = 8$ sides

19. $S = 1260°$ **20.** $n = 12$

21. Ext. $\angle = 30°$. $n = 12$ sides

22. (a) $B\hat{C}D = 120°$
(b) $A\hat{B}F = 30°$

Exercise 9m

1. (a) $\hat{N} = 135°$
(b) $PN = 8.94$ cm

2. (a) $\hat{M} = 135°$
(b) $PN = 10$ cm

3. (a) $\hat{B} = 105°$
(b) $ED = 3$ cm

4. (a) $\hat{E} = 115°$
(b) $BC = 5$cm

5. $h = 4.5$ cm

6. $b = 15$ cm

7. $h = 6$ cm

8. $b = 6.5$ cm

9. $h = 4$ cm

10. $DC = 5.5$ cm

Exercise 9n

1. $A\hat{C}B = 105°$

2. $A\hat{C}B = 48°$

3. $A\hat{O}B = 94°$

4. $A\hat{C}B = 35°$

5. $A\hat{O}B = 216°$

6. $A\hat{C}B = 90°$

7. $BC = 15$ cm

8. $AC = 20$ cm

9. (a) $PR = 25$ cm
(b) $PR = 17$ cm

10. (a) $RQ = 12$ cm
(b) $PQ = 21$ cm

11. $\hat{x} = 37°$ and $\hat{y} = 58°$

12. $A\hat{Q}B = 55°$

13. $\hat{x} = 25°$ and $\hat{y} = 51°$

14. $\hat{x} = 35°$ and $\hat{y} = 28°$

15. $A\hat{P}B = 21°$

16. $A\hat{B}C = 141°$

17. $Q\hat{R}S = 132°$ and $R\hat{S}P = 83°$

18. $\hat{K} = 72°$ and $\hat{N} = 85°$

19. $\hat{Y} = 145°$ and $\hat{Z} = 71°$

20. $\hat{P} = 147°$ and $\hat{S} = 65°$

21. $A\hat{B}C = 93°$

22. $R\hat{S}T = 105°$

23. $X\hat{N}K = 123°$ and $K\hat{N}M = 57°$

24. $A\hat{D}E = 87°$

25. $W\hat{X}Z = 125°$

26. (a) $\hat{x} = 50°$
(b) $\hat{y} = 130°$
(c) $\hat{z} = 60°$
(d) $\hat{p} = 120°$

27. (a) $\hat{a} = 37°$
(b) $\hat{b} = 30°$
(c) $\hat{x} = 48°$ and $\hat{y} = 35°$
(d) $\hat{x} = 55°$ and $\hat{y} = 44°$

28. (a) $\hat{x} = 43°$ and $\hat{y} = 52°$
(b) $\hat{x} = 47°$ and $\hat{y} = 28°$
(c) $\hat{x} = 25°$ and $\hat{y} = 32°$
(d) $\hat{p} = \hat{q} = \hat{r} = \hat{s} = 90°$

29. (a) $\hat{x} = 42.5°$
(b) $\hat{y} = 36°$
(c) $\hat{y} = 90°$
(d) $\hat{y} = 95°$

30. (a) $\hat{p} = 55°$ and $\hat{q} = 72°$
(b) $\hat{r} = 60°$
(c) $\hat{s} = 122.5°$
(d) $\hat{x} = 190°$

31. (a) $\hat{p} = \hat{q} = 50°$
(b) $\hat{a} = 76°$ and $\hat{b} = 152°$
(c) $\hat{r} = 25°$
(d) $\hat{a} = 55°$ and $\hat{b} = 90°$

32. (a) $\hat{p} = \hat{r} = 90°$ and $\hat{q} = 85°$
(b) $\hat{x} = 52°$and $\hat{y} = 128°$
(c) $\hat{x} = 20°$
(d) $\hat{p} = 34°$

33. (a) $\hat{x} = 90°$ and $\hat{y} = 125°$
(b) $\hat{p} = 50°$, $\hat{q} = 90°$,
$\hat{r} = 140°$, $\hat{x} = 25°$,
$\hat{y} = 15°$ and $\hat{z} = 90°$
(c) $\hat{x} = 55°$ and $\hat{y} = 125°$
(d) $\hat{p} = 40°$ and $\hat{q} = 140°$

34. (a) $\hat{x} = 71°$
(b) $\hat{x} = 32°$
(c) $\hat{x} = 100°$
(d) $\hat{x} = 90°$

35. (a) $\hat{y} = 18°$
(b) $\hat{x} = 105°$ and $\hat{y} = 95°$
(c) $\hat{x} = 228°$
(d) $\hat{x} = 118°$

36. (a) $\hat{x} = 90°$
(b) $\hat{x} = \hat{y} = 70°$
(c) $\hat{x} = \hat{y} = 90°$
(d) $\hat{x} = 131°$

37. (a) $\hat{x} = 58°$ and $\hat{y} = 90°$
(b) $\hat{x} = 129°$
(c) $\hat{x} = 70°$ and $\hat{y} = 110°$
(d) $\hat{x} = \hat{y} = 82°$

38. (a) $\hat{x} = \hat{y} = 105°$
(b) $\hat{r} = \hat{s} = 87°$
(c) $\hat{x} = 55°$, $\hat{y} = 110°$ and
$\hat{z} = 105°$

39. (a) $\hat{x} = 35°$, $\hat{y} = 15°$,
$\hat{z} = 90°, \hat{p} = 40°$,
$\hat{q} = 90°$ and $\hat{r} = 130°$
(b) $\hat{x} = 23°$
(c) $\hat{y} = 290°$

40. $B\hat{A}C = 71.5°$

41. $A\hat{E}B = 42°$

42. $A\hat{O}B = 152°$

43. (a) $\hat{x} = 105°$ and $\hat{y} = 93°$
(b) $\hat{z} = 154°$
(c) $\hat{f} = 56°$ and $\hat{g} = 124°$

Exercise 9q

1. (a) (i) 2 windows
(ii) 2 windows
(iii) 2 windows
(iv) 2 windows
(b) 8 windows

2. 3 doors

3. (a) $b = 2$ cm
(b) $b = 1.5$ cm
(c) $b = 1$ cm

4. (a) $l = 6.5$ cm and $b = 6.2$ cm
(b) $l = 4$ cm and $b = 3$ cm
(c) $l = 4$ cm and $b = 3$ cm
(d) $l = 9$ cm and $b =1.5$ cm
(e) $l = b = 1.5$ cm

5. (a) $b = 3$ cm and $b = 4$ cm
(b) $b = 2$ cm each
(c) $b = 1$ cm and $b = 2$ cm
(d) $b = 0.5$ cm each

6. $t = 0.2$ cm

7. 1 cm = 1 m

8. (a) $A = 40.3\ \text{m}^2$
(b) $A= 40.3\ \text{m}^2$
(c) Cost = $1 168.70

9. (a) $A = 12\ \text{m}^2$
(b) $A = 12\ \text{m}^2$
(c) Cost = $450

10. (a) $A = 12\ \text{m}^2$
(b) 192 tiles
(c) Cost = $432

11. (a) $A = 13.5\ \text{m}^2$
(b) 450 tiles
(c) Cost = $427.50

12. (a) $A = 2.25\ \text{m}^2$
(b) 100 tiles
(c) Cost = $160

Exercise 9r

1. $BC = 9$ cm

2. $AC = 4.57$ cm. $D\hat{C}F = 50°$

3. $AC = 8.0$ cm

4. $BC = 5.8$ cm

5. (ii) $DC = 3.2$ cm.
$A\hat{D}C = 98°$

6. (iii) $VM = 20\sqrt{3}$ cm

Exercise 10a

1. Yes **2.** Yes
3. Yes **4.** No
5. No
6. 2 cm horizontally
7. 3.3 cm horizontally
8. 1.5 cm vertically
9. 2.4 cm vertically
10. (a) 1.5 cm horizontally
(b) 1.5 cm vertically
(d) 2.1 cm obliquely

Exercise 10b

6. (a) 2 minor lines
(b) None
(c) 1 mirror line

7. (a) 1 mirror line
(b) 5 mirror lines
(c) 3 mirror lines

8. (a) 6 mirror lines
(b) 4 mirror lines
(c) 4 mirror lines

9. (a) 8 mirror lines
(b) 5 mirror lines
(c) 6 mirror lines

10. (a) 5 mirror lines
(b) 7 mirror lines
(c) 9 mirror lines

Exercise 10c

1. (a) No rotational symmetry
(b) No point symmetry

2. (a) Rotational symmetry of order 2
(b) Point symmetry exists

3. (a) Rotational symmetry of order 2
(b) Point symmetry exists

4. (a) Rotational symmetry of order 2
(b) Point symmetry exists

5. (a) Rotational symmetry of order 2
(b) Point symmetry exists

6. (a) Rotational symmetry of order 2
(b) Point symmetry exists

7. (a) Rotational symmetry of order 4
(b) Point symmetry exists

8. (a) No rotational symmetry
(b) No point symmetry

9. (a) Rotational symmetry of order 5
(b) No point symmetry

10. (a) Rotational symmetry of order 4
(b) Point symmetry exists

11. (a) Rotational symmetry of order 3
(b) No point symmetry

12. (a) Rotational symmetry of order 3
(b) No point symmetry

13. (a) No (b) Yes
(c) No (d) Yes
(e) No (f) Yes
(g) No

14.

Regular polygon	*Number of lines of symmetry*	*Order of rotational symmetry*
Equilateral triangle	3	3
Square	4	4
Pentagon	5	5
Hexagon	6	6
Heptagon	7	7
Octagon	8	8
Nonagon	9	9
Decagon	10	10
Unidecagon	11	11
Duodecagon	12	12

15. It has n lines of symmetry.

16. It has rotational symmetry of order n.

Exercise 10d

1. $A'(4, 6)$, $B'(6, 8)$ and $C'(9, 6)$.

2. $A'(-3, 3)$, $B'(1, 3)$, $C'(0, 5)$ and $D'(-1, 5)$

3. $A'(4, 6)$, $B'(5, 8)$, C'(8, 8) and $D'(7, 6)$

4. $A'(1, 6)$, $B'(7, 4)$ and $C'(7, 6)$
$A''(0, 3)$, $B''(6, 1)$ and $C''(6, 3)$
(a) $\begin{pmatrix}4\\-2\end{pmatrix}$ (b) $\begin{pmatrix}-4\\2\end{pmatrix}$ (c) $\begin{pmatrix}1\\3\end{pmatrix}$

5. $A'(2, 4)$, $B'(6, 4)$ and $C'(6, 7)$
$A''(1, 1)$, $B''(5, 1)$ and $C''(5, 4)$
(a) $\begin{pmatrix}4\\-1\end{pmatrix}$ (b) $\begin{pmatrix}-4\\1\end{pmatrix}$ (c) $\begin{pmatrix}1\\3\end{pmatrix}$

6. $A'(2, 5)$, $B'(6, 4)$ and $C'(5, 6)$
$A''(1, 2)$, $B''(5, 1)$ and $C''(4, 3)$
(a) $\begin{pmatrix}3\\-2\end{pmatrix}$ (b) $\begin{pmatrix}-3\\2\end{pmatrix}$ (c) $\begin{pmatrix}1\\3\end{pmatrix}$

7. $A'(0, 4)$, $B'(4, 4)$ and C' (4, 7)
$A''(-1, 2)$, $B''(3, 2)$ and $C''(3, 5)$
(a) $\begin{pmatrix}2\\0\end{pmatrix}$ (b) $\begin{pmatrix}-2\\0\end{pmatrix}$ (c) $\begin{pmatrix}1\\2\end{pmatrix}$

8. (a) $A'(5, -1)$, $B'(6, 3)$ and C' (4.5, 1)
(b) $T = \begin{pmatrix}2.5\\-1\end{pmatrix}$

9. (a) $A'(-1, 3)$, $B'(1, 7)$ and $C'(-1.5, 6)$
(b) $T = \begin{pmatrix}2\\-3\end{pmatrix}$

10. $A(11, -8)$, $B(13, -6)$ and $C(14, -11)$

11. $K(1, 4), L(2, 2), M(1, 1)$ and $N(4, 3)$

12. $A(-4, -4), B(-3, -1), C(0, -2)$ and $D(-2, -5)$

Exercise 10e

1. $A'(-5, 2), B'(-3, 4)$ and $C'(-1, 2)$

2. $A'(4, -2), B'(1, -2)$ and $C'(1, -4)$

3. $P'(-4, 2), Q'(-1, 2)$ and $R'(-1 , 4)$

4. $P'(-7, 3), Q'(-5, 5)$ and $R'(-3, 2)$

5. $A'(2, -1), B'(6,-1), C'(5,-3)$ and $D'(4 -3)$

6. $A'(5,-2), B'(3, -4)$ and $C'(1, -2)$

7. $A'(-4, 2)$ $B'(-1, 2)$ and $C'(-1, 4)$

8. $P'(4, -2), Q'(1, -2)$ and $R'(1, -4)$

9. $P'(7,-3), W'(5, -5)$ and $R'(3,-2)$

10. $A'(-2, 1), B'(-6, 1)$ $C'(-5, 3)$ and $D'(-4, 3)$

11. (a) $A'(-2, -5), B'(-4, -3)$ and $C'(-2, -1)$
(b) Reflection in the line $y = x$

12. (a) $A'(2, 4), B'(2, 1)$ and $C'(4, 1)$
(b) Reflection in the line $y = x$

13. (a) $P'(-2, -4), Q'(-2, -1)$ and $R'(-4, -1)$
(b) Reflection in the line $y = x$

14. $P'(-3, -7), Q'(-5, -5)$ and $R'(-2, -3)$

15. $A'(1, 2), B'(1, 6), C'(3, 5)$ and $D'(3, 4)$

16. (a) $A'(2, 5), B'(4, 3)$ and $C'(2, 1)$
(b) Reflection in the line $y =- x$

17. $A'(-2, -4), B'(-2, -1)$ and $C'(-4, -1)$

18. (a) $P'(2, 4), Q'(2, 1)$ and $R'(4, 1)$
(b) Reflection in the line $y = -x$

19. (a) $P'(3, 7), Q'(5, 5)$ and $R'(2, 3)$
(b) Reflection in the line $y = -x$

20. $A'(-1, -2), B'(-1,-6), C'(-3, -5)$ and $D'(-3, -4)$

21. $A'(-4, -2), B'(-1, -2)$ and $C'(-1, -4)$

22. $A'(5, 2), B'(3, 4)$ and $C'(1, 2)$

23. $P'(4, 2), Q'(1, 2)$ and $R'(1, 4)$

24. $P'(7, 3), Q'(5, 5)$ and $R'(3, 2)$

25. $A'(-2, -1), B'(-6, -1), C'(-5, -3)$ and $D'(-4, -3)$

26. $P'(4, 3), Q'(0.5, 4.5)$ and $R'(-1, 6)$

27. $P'(-2, 3), Q'(-5.5, 4.5)$ and $R'(-7, 6)$

28. $A'(6, -2), B'(3, -2)$ and $C'(3, -4)$

29. $A'(1, -3), B'(-1, -5)$ and $C'(-3, -2)$

30. $A'(-6, 2), B'(-3, 2)$ and $C'(-3, 4)$

31. $A'(3, -2), B'(1, -4)$ and $C'(-1, -2)$

32. $P'(-9, 1), Q'(-13, 1), R'(-12, 3)$ and $S'(-11, 3)$

33. $P'(0, 1), Q'(3.5, -0.5)$ and $R'(5, -2)$

34. $P'(0, -5), Q'(3.5, -6.5)$ and $R'(5, -8)$

35. $A'(-5, 6), B'(-3, 8)$ and $C'(-1, 6)$

36. $P'(1, -6), Q'(3, -9)$ and $R'(1, -8)$

37. $P'(2, -8), Q'(6, -8), R'(5, -10)$ and $S'(4, -10)$

38. $N\left(2, \frac{1}{2}\right) = N(2, 0.5)$

39. $A(-5, -6), B(-3, -3)$ and $C(0, -1)$

40. $A(3, -2), B(4, -3)$ and $C(5, -1)$

41. $A(5, -3), B(2, -4)$ and $C(1, -5)$

42. $A(5, 2), B(4, 3)$ and $C(3, 6)$

43. $P(3, -4), Q(2, -5)$ and $R(1, -3)$

44. $P(3, -4), Q(5, -5), R(6, -7)$ and $S(7, -9)$

45. $P(3, 0), Q(5, 1), R(6, 2)$ and $S(3, 3)$

Exercise 10f

1. (a) $R_{[0, +225°]}: P \to P'$
(b) $R_{[C, -90°]}: LM \to L'M'$
(c) $R_{[A, -33°]}: \triangle ABC \to \triangle A'B'C'$.

2. (a) Centre of rotation is P
(b) Object = $\triangle XYZ$. Image = $\triangle X'Y'Z'$
(c) Rotation, $\theta = 21°$ clockwise

3. (a) Centre of rotation = C
(b) $PC = P'C$. PC and $P'C$ form a 35° angle
(c) $\triangle PCP'$ is isosceles

4. (a) Rotation, $\theta = 60°$
(b) $L\hat{C}L' = 60°$
(c) $L\hat{L}C' = 60°$

5. $C'\hat{C}X = 85°$. $\theta = 10°$ anti-clockwise

6. $C\hat{P}P' = 67.5°$

7. $X\hat{A}'A = 25°$

8. $C\hat{L}L' = 30°$

9. $X\hat{P}'P = 27.5°$

10. $C\hat{A}C' = 52.5°$

11. $X\hat{P}'P = 43°$

12. $A'(-2, 3), B'(-4, 5)$ and $C'(-7, 4)$

13. $P'(-4, -3), Q'(-3, -5)$ and $R'(-5, -4)$

14. $K'(3, 5), L'(4, 3)$ and $M'(5, 6)$

15. $A'(5, -4), B'(3, -1)$ and $C'(1, -6)$

16. $P'(-2, 3), Q'(-1, 5)$ and $R'(-5, 4)$

17. $L'(-3, -5)$ and $M'(-5, -7)$

18. $A'(-3, -2), B'(-5, -3), C'(-4, -4)$ and $D'(0, -4)$

19. $W'(2, 1)$ $X'(4, 1), Y'(3.5, 4)$ and $Z'(5, 3)$

20. $A'(2, -3), B'(3, -5)$ and $C'(5, -4)$

21. $P'(2, 3), Q'(3, 4)$ and $R'(5, 5)$

22. $K'(3, 5), L'(4, 5)$ and $M'(5, 6)$

23. $A'(5, -3), B'(3, -2)$ and $C'(2, -5)$

24. $P'(1, -3), Q'(2, -4)$ and $R'(5, -4)$

25. $L'(3, 4)$ and $M'(5, 6)$

26. $A'(3, -2), B'(4, -4), C'(5, -3)$ and $D'(0, -3)$

27. $W'(-5, -3.5), X'(-3, -3.5), Y'(-2, -1.5)$ and $Z'(-4, -1.5)$

28. (a) $A'(-1, 2), B'(-1, 6), C'(-3, 5)$ and $D'(-3, 4)$
(b) $A''(2, -5), B''(2, -9), C''(4, -8)$ and $D''(4, -7)$

29. (a) $A'(-2, -1), B'(-3, -5)$ and $C'(0, -4)$
(b) $A''(2, -1), B''(3, -5)$ and $C''(0, -4)$

30. (a) (i) $A'(-2, 3), B'(-5, 3)$ and $C'(-5, 7)$
(ii) $A''(-3, -2), B''(-3, -5)$ and $C''(-7, -5)$

31. (a) $A'(-1, 0), B'(0, 3)$ and $C'(-5, 4)$
(b) $A''(-1, 0), B''(0, -3)$ and $C''(-5, -4)$

32. $A'(3, 2), B'(7, 3)$ and $C'(5, 5)$
$A''(-2, 3), B''(-3, 7)$ and $C''(-5, 5)$

33. $\mathbf{AD} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $\mathbf{BC} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$
$\Rightarrow \mathbf{AD} = \mathbf{BC}$
Or $\mathbf{AB} = \mathbf{DC} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$
$\Rightarrow \mathbf{AB} = \mathbf{DC}$
$A'(-1, -2), B'(-5, -8), C'(-11, -8)$ and $D'(-7, -2)$

34. (b) $A'(4, 5), B'(1, 3), C'(2, 7), A''(4, -1), B''(1, -3)$ and $C''(2, 1)$

35. $A(-2, 3), B(-5, 3)$ and $C(-5, 7)$

36. $P(4, -3), Q(3, -1)$ and $R(1, -5)$

37. $L(-5, -3)$ and $M(-4, -7)$

38. $K(5, -2), L(3, -4)$ and $M(7, -5)$

39. $P(3, -1), Q(5, -1), R(4, -3)$ and $S(1, -6)$

40. $P(1, -3), Q(1, -5)$ and $R(3, -4)$

41. (a) $C(1, 2)$ (b) $\theta = 180°$

42. (a) $P = R_{(0, +90°)}$ = An anti-clockwise rotation of 90° about the origin
(b) $y = -\frac{1}{4}x + \frac{3}{4}$

43. (a) $X(1.5, 0.5)$
(b) $\theta = 180°$

44. (a) $X(-0.5, 2.5)$
(b) $\theta = 270°$ anti-clockwise

45. (b) $BC = B'C$. B and B' are equidistant from C
(c) $C(1, -2)$
(d) $\theta = 270°$ anti-clockwise

46. (b) (i) $AC = A'C$. A and A' are equidistant from C
(ii) $B\hat{M}C = 90°$
(c) $C(0, 4)$. $\theta = A\hat{C}A' = B\hat{C}B' = 90°$ anti-clockwise

Exercise 10g

1. (a) Centre of enlargement = 0
(b) Scale factor, $k = -2$.

2. (a) $E_{(0, 3.5)}: PQ \rightarrow P'Q'$
(b) $E_{\left(X, -\frac{3}{4}\right)}: \triangle ABC \rightarrow A'B'C'$
(c) $E_{\left(c, \frac{1}{4}\right)}$: Quadrilateral $ABCD$ $\rightarrow$ Quadrilateral $A'B'C'D'$

3. (a) $k = 2$
(b) $B'C' = 9$ cm

4. (a) $k = 3$
(b) $Q'R' = 13.5$ cm

5. (a) $k = 2.5$
(b) $AB = 4$ cm

6. (a) $k = 1.75$
(b) $QR = 2.4$ cm

7. (a) $k = -3.5$
(b) $AC = 7$ cm
(c) $B'C' = 17.5$ cm

8. (a) $k = -0.5$
(b) $PQ = 5.4$ cm
(c) $Q'R' = 2.3$ cm

9. (a) $OA' = 7$cm
(b) $OB = 4.5$ cm

10. (a) $OP' = 7.5$ cm
(b) $OQ = 5$ cm

11. $k = 4$

12. $k = 2.8$

13. $OP = 4$ cm

14. $OA = 6.8$ cm

15. (a) $k = 1.5$
(b) $AB = 12$ cm. $A'B' = 18$ cm. $AA' = 10$ cm

16. (a) $k = 5$
(b) $PQ = 6.3$ cm. $P'Q' = 31.5$ cm. $QQ' = 33.6$ cm

17. (a) $Q'R' = 13.5$ cm
(b) $S'R' = 7.5$ cm
(c) $P'R' = 9$ cm

18. (a) $Q'R' = 8.75$ cm
(b) $S'R' = 5.25$ cm
(c) $P'R' = 7$ cm

19. (a) $k^2 = 16$
(b) $k = 4$

20. (a) $k^2 = 25$
(b) $k = 5$

21. (a) $k = 3$
(b) $A_1 = 90$ cm². $A_2 = 810$ m²
(c) $k^2 = 9$

22. (a) $k = 2.5$
(b) $A_1 = 60$ cm². $A_2 = 375$ cm²
(c) $k^2 = 6.25$

27. $A'(2, 4), B'(6, 8)$ and $C'(3, 9)$

28. $P'(-3, 6), Q'(-9, 12)$ and $R'(-13.5, 18)$

29. $A'(2, 4), B'(8.1), C'(10, 6)$ and $D'(4. 6)$

30. $K'(1.5, -1), L'(2.5, -2)$ and $M'(2, -2.5)$

31. $P'\left(\frac{1}{4}, \frac{1}{2}\right), Q'\left(\frac{3}{4}, \frac{1}{2}\right), R'\left(1, 1\frac{1}{4}\right)$ and $S'\left(\frac{1}{2}, 1\frac{1}{2}\right)$

32. (a) $A'(-3, 9), B'(6, 12), C'(9, -6)$ and $D'(-9, -9)$
(b) $A'\left(-\frac{1}{2}, 1\frac{1}{2}\right)$ $B'(1, 2)$, $C'\left(1\frac{1}{2}, -1\right)$ and $D'\left(-1\frac{1}{2}, -1\frac{1}{2}\right)$

33. (a) $A'B' = 120$ cm
(b) $A'\left(1, \frac{1}{2}\right), B'\left(3, \frac{1}{2}\right)$, $C'\left(2\frac{1}{2}, 1\frac{1}{2}\right)$ and $D'\left(2, 1\frac{1}{2}\right)$

34. $A'(4, 2), B'(10, 4)$ and $C'(6, 8)$

35. $A'\left(4\frac{1}{2}, 1\frac{1}{2}\right), B'(9, 3)$ and $C'\left(6, 7\frac{1}{2}\right)$

36. $A'(2, 1), B'(4, 5)$ and $C'(8, 1)$

37. $P'(4, -1), Q'(8, 5)$ and $R'(0, 1)$

38. $K'(-7, 1), L'(1, 1)$ and $M'(-1, 4)$

39. $L'(-7, 8), M'(-3, 8)$ and $N'(-8, 4)$

40. $L'(-7, 8), M'(-3, 8)$ and $N'(-8, 4)$

41. $A'(-2, -2), B'(-6.5, -2)$ and $C'(-8, -5)$

42. $A\left(\frac{1}{2}, 1\right), B\left(1\frac{1}{2}, 2\right)$ and $C(2\frac{1}{2}, 1)$

43. $A\left(-\frac{2}{3}, -1\right), B(-1, -2)$ and $C(-2, -3)$

44. $P\left(-2, \frac{2}{5}\right)$, $Q\left(-2, \frac{4}{3}\right)$ and $R\left(-\frac{2}{5}, 1\frac{1}{5}\right)$

45. $K(-3, -6)$, $L(-9, -12)$ and $M(-6, -9)$

46. $A\left(-1, \frac{1}{2}\right)$, $B\left(-2, 1\frac{1}{2}\right)$ and $C(-3, 2)$

47. $P(-2, 2)$, $Q(-4, 6)$ and $R(-10, 8)$

Exercise 10h

1. $RR' = 5$ cm

2. (c) $CC' = 4.4$ cm
(d) $NC' = NC = 2.2$ cm
NC is the altitude of $\triangle A'B'C'$
(e) Kite.

3. (i) (a) $k = \frac{7}{4}$
(b) $AB = 10$ cm.
$A'B' = 17.5$ cm.

4. (a) $A'(-2, 2)$, $B'(-5, 2)$ and $C'(-2, 6)$
$A''(-2, -2)$, $B''(-2, -5)$ and $C''(-6, -2)$
(b) An anti-clockwise rotation about the origin O, through an angle of 90°.
(c) $AA' = 5.66$ units
(d) $A'\hat{A}A'' = 45°$

5. (a) 20° (b) 70°
(c) An anti-clockwise rotation about O, through an angle of 90°. Or A reflection in the line bisecting k and l.

6. (a) $A'(4, 0)$, $B'(8, 6)$ and $C'(0, 2)$
(b) P is a clockwise rotation about the origin O, through an angle of 90°. Or P is an anti-clockwise rotation about the origin O, through an angle of 270°.
(c) $A'''(-2, 0)$, $B'''(-4, 3)$ and $C'''(0, 1)$.
(d) (i) The area of $\triangle ABC = \frac{1}{4}$ (The area of $\triangle A'B'C'$)
The area of $\triangle ABC$ = The area of $\triangle A''B''C''$
The area of $\triangle ABC$ = The area of $\triangle A'''B'''C'''$

7. (a) (i) $A(2, 1)$, $B(3, 3)$ and $C(5, 1)$
(ii) $A'(2, -1)$, $B'(3, -3)$ and $C'(5, -1)$
(b) (i) $A''(2, 1)$, $B''(4, 5)$ and $C''(8, 1)$
(ii) Area of $\triangle ABC = 3$ cm^2.
Area of $A''B''C'' = 12$ cm

8. (a) (i) $PQ = 13$ km
(ii) 202.6°
(b) (i) $S(5, -1)$ (ii) $R'(4, 3)$

Exercise 11a

1. (a) 0.139 (b) 0.259
(c) 0.629 (d) 0.743

2. (a) 0.799 (b) 0.875
(c) 0.961 (d) 0.998

3. (a) 0.162 (b) 0.299
(c) 0.424 (d) 0.569

4. (a) 0.749 (b) 0.801
(c) 0.906 (d) 0.969

5. (a) 0.927 (b) 0.983
(c) 0.999 (d) 1.000

6. (a) 5° (b) 19°
(c) 28° (d) 31°

7. (a) 41° (b) 53°
(c) 68° (d) 72°

8. (a) 4.5° (b) 13.8°
(c) 29.3° (d) 33.4°

9. (a) 38.1° (b) 49.8°
(c) 54.7° (d) 58.2°

10. (a) 63.8° (b) 74.6°
(c) 75.1° (d) 78.5°

11. (a) $\hat{x} = 9°$ (b) $\hat{y} = 18°$
(c) $\hat{\Theta} = 25°$ (d) $\hat{\Phi} = 34°$

12. (a) $\hat{a} = 37°$ (b) $\hat{b} = 42°$
(c) $\hat{\alpha} = 54°$ (d) $\hat{\beta} = 62°$

13. (a) $A\hat{B}C = 8.7°$
(b) $X\hat{Y}Z = 13.6°$
(c) $P\hat{Q}R = 15.1°$
(d) $S\hat{T}U = 17.4°$

14. (a) $C\hat{A}B = 45.2°$
(b) $B\hat{A}C = 47.8°$
(c) $P\hat{Q}R = 56.1°$
(d) $L\hat{M}N = 59.9°$

15. (a) $\hat{x} = 65.4°$
(b) $\hat{y} = 67.2°$
(c) $\hat{\Theta} = 74.6°$
(d) $\hat{\Phi} = 79.7°$

16. (a) $w = 2.59$ cm
(b) $x = 3.91$ cm
(c) $y = 5.50$ cm
(d) $z = 6.79$ cm

17. (a) $p = 4.99$ cm
(b) $q = 4.50$ cm
(c) $r = 4.24$ cm
(d) $s = 4.02$ cm

18. (a) $a = 2.58$ cm
(b) $b = 3.99$ cm
(c) $c = 4.32$ cm
(d) $d = 5.22$ cm

19. (a) $w = 7.37$ cm
(b) $x = 8.90$ cm
(c) $y = 11.9$ cm
(d) $z = 13.0$ cm

20. (a) $p = 14.4$ cm
(b) $q = 15.6$ cm
(c) $r = 17.6$ cm
(d) $s = 23.0$ cm

21. (a) $w = 24.8$ cm
(b) $x = 22.7$ cm
(c) $y = 18.9$ cm
(d) $z = 17.0$ cm

22. (a) $a = 16.6$ cm
(b) $b = 16.4$ cm
(c) $c = 14.8$ cm
(d) $d = 15.2$ cm

23. (a) $p = 13.4$ cm
(b) $q = 13.5$ cm
(c) $r = 14.6$ cm
(d) $s = 14.6$ cm

24. (a) $k = 14.4$ cm
(b) $l = 14.5$ cm
(c) $m = 15.4$ cm
(d) $n = 16.5$ cm

25. (a) $w = 20.3$ cm
(b) $x = 20.5$ cm
(c) $y = 20.8$ cm
(d) $z = 26.2$ cm

26. (a) $\hat{W} = 30°$
(b) $\hat{X} = 41.8°$
(c) $\hat{Y} = 19.5°$
(d) $\hat{Z} = 14.5°$

27. (a) $\hat{A} = 38.7°$
(b) $\hat{B} = 36.9°$
(c) $\hat{C} = 52.6°$
(d) $\hat{D} = 43.7°$

28. (a) $\hat{P} = 51.0°$
(b) $\hat{Q} = 56.0°$
(c) $\hat{R} = 55.6°$
(d) $\hat{S} = 56.2°$

29. (a) $\hat{K} = 61.2°$ (b) $\hat{L} = 62.9°$ (c) $\hat{M} = 56.0°$ (d) $\hat{N} = 54.5°$

30. (a) $\hat{W} = 33.6°$ (b) $\hat{X} = 35.7°$ (c) $\hat{Y} = 35.7°$ (d) $\hat{Z} = 47.2°$

Exercise 11b

1. (a) 0.996 (b) 0.951 (c) 0.934 (d) 0.875

2. (a) 0.682 (b) 0.602 (c) 0.454 (d) 0.259

3. (a) 0.992 (b) 0.986 (c) 0.884 (d) 0.774

4. (a) 0.706 (b) 0.596 (c) 0.386 (d) 0.245

5. (a) 0.237 (b) 0.153 (c) 0.080 (d) 0.005

6. (a) 0° (b) 22° (c) 31° (d) 43°

7. (a) 53° (b) 62° (c) 75° (d) 89°

8. (a) 31.5° (b) 37.7° (c) 40.6° (d) 45.4°

9. (a) 50.3° (b) 56.9° (c) 62.2° (d) 65.6°

10. (a) 74.2° (b) 78.7° (c) 83.4° (d) 86.3°

11. (a) $\hat{x} = 26°$ (b) $\hat{y} = 34°$ (c) $\hat{\Theta} = 41°$ (d) $\hat{\Phi} = 52°$

12. (a) $\hat{a} = 63°$ (b) $\hat{b} = 75°$ (c) $\hat{\alpha} = 87°$ (d) $\hat{\beta} = 88°$

13. (a) $A\hat{B}C = 20.\ 1°$ (b) $X\hat{Y}Z = 27.9°$ (c) $P\hat{Q}R = 33.2°$ (d) $S\hat{T}U = 40.6°$

14. (a) $C\hat{A}B = 49.1°$ (b) $B\hat{A}C = 54.3°$ (c) $P\hat{Q}R = 56.7°$ (d) $L\hat{M}N = 63.5°$

15. (a) $\hat{x} = 71.2°$ (b) $\hat{y} = 78.4°$ (c) $\hat{\Theta} = 80.9°$ (d) $\hat{\Phi} = 89.7°$

16. (a) $w = 9.56$ cm (b) $x = 11.2$ cm (c) $y = 11.5$ cm (d) $z = 11.9$ cm

17. (a) $p = 8.55$ cm (b) $q = 6.80$ cm (c) $r = 4.82$ cm (d) $s = 3.61$ cm

18. (a) $a = 8.94$ cm (b) $b = 9.91$ cm (c) $c = 9.35$ cm (d) $d = 9.92$ cm

19. (a) $w = 11.1$ cm (b) $x = 11.0$ cm (c) $y = 11.1$ cm (d) $z = 11.3$ cm

20. (a) $w = 11.2$ cm (b) $x = 10.1$ cm (c) $y = 10.6$ cm (d) $z = 6.86$ cm

21. (a) $w = 7.25$ cm (b) $x = 8.37$ cm (c) $y = 10.1$ cm (d) $z = 11.9$ cm

22. (a) $a = 14.2$ cm (b) $b = 17.0$ cm (c) $c = 21.6$ cm (d) $d = 39.1$ cm

23. (a) $p = 13.4$ cm (b) $q = 16.0$ cm (c) $r = 18.5$ cm (d) $s = 23.3$ cm

24. (a) $k = 27.7$ cm (b) $l = 36.0$ cm (c) $m = 34.1$ cm (d) $n = 42.4$ cm

25. (a) $w = 50.7$ cm (b) $x = 71.6$ cm (c) $y = 109$ cm (d) $z = 168.6$ cm

26. (a) $\hat{W} = 60°$ (b) $\hat{X} = 48.2°$ (c) $\hat{Y} = 70.5°$ (d) $\hat{Z} = 75.5°$

27. (a) $\hat{A} = 51.3°$ (b) $\hat{B} = 53.1°$ (c) $\hat{C} = 29.8°$ (d) $\hat{D} = 39.8°$

28. (a) $\hat{P} = 49.5°$ (b) $\hat{Q} = 33.8°$ (c) $\hat{R} = 38.6°$ (d) $\hat{S} = 35.3°$

29. (a) $\hat{K} = 28.8°$ (b) $\hat{L} = 27.1°$ (c) $\hat{M} = 32.9°$ (d) $\hat{N} = 35.4°$

30. (a) $\hat{W} = 56.4°$ (b) $\hat{X} = 48.8°$ (c) $\hat{Y} = 52.8°$ (d) $\hat{Z} = 44.5°$

Exercise 11c

1. (a) 0.141 (b) 0.268 (c) 0.424 (d) 0.532

2. (a) 1 (b) 1.280 (c) 1.804 (d) 3.487

3. (a) 0.142 (b) 0.173 (c) 0.481 (d) 0.716

4. (a) 1.011 (b) 1.299 (c) 2.164 (d) 4.264

5. (a) 5.242 (b) 5.730 (c) 21.20 (d) 63.66

6. (a) 9° (b) 11° (c) 16° (d) 21°

7. (a) 32° (b) 39° (c) 55° (d) 65°

8. (a) 20.2° (b) 28.5° (c) 31.5° (d) 39.6°

9. (a) 40.4° (b) 49.8° (c) 52.5° (d) 59.8°

10. (a) 64.1° (b) 69.2° (c) 72.1° (d) 79.9°

11. (a) $\hat{x} = 2°$ (b) $\hat{y} = 16°$ (c) $\hat{\Theta} = 27°$ (d) $\hat{\Phi} = 36°$

12. (a) $\hat{a} = 41°$ (b) $\hat{b} = 54°$ (c) $\hat{\alpha} = 64°$ (d) $\hat{\beta} = 73°$

13. (a) $A\hat{B}C = 8.5°$ (b) $X\hat{Y}Z = 17.6°$ (c) $P\hat{Q}R = 21.6°$ (d) $S\hat{T}U = 31.5°$

14. (a) $C\hat{A}B = 49.7°$ (b) $B\hat{A}C = 51.8°$ (c) $P\hat{Q}R = 56.4°$ (d) $L\hat{M}N = 63.2°$

15. (a) $\hat{x} = 66.5°$ (b) $\hat{y} = 73.6°$ (c) $\hat{\Theta} = 75.7°$ (d) $\hat{\Phi} = 83.1°$

16. (a) $w = 1.25$ cm (b) $x = 2.54$ cm (c) $y = 4.08$ cm (d) $z = 5.84$ cm

17. (a) $p = 7.81$ cm (b) $q = 9.90$ cm (c) $r = 16.5$ cm (d) $s = 20.0$ cm

18. (a) $a = 2.81$ cm (b) $b = 4.74$ cm (c) $c = 8.53$ cm (d) $d = 9.64$ cm

19. (a) $a = 7.61$ cm
(b) $b = 9.80$ cm
(c) $c = 11.9$ cm
(d) $d = 13.3$ cm
20. (a) $w = 18.1$ cm
(b) $x = 22.1$ cm
(c) $y = 43.6$ cm
(d) $z = 60.7$ cm
21. (a) w = 17.4 cm
(b) $x = 18.2$ cm
(c) $y = 17.2$ cm
(d) $z = 13.3$ cm
22. (a) $a = 15.1$ cm
(b) $b = 11.2$ cm
(c) $c = 8.44$ cm
(d) $d = 8.08$ cm
23. (a) $p = 7.81$cm
(b) $q = 8.14$ cm
(c) $r = 7.78$ cm
(d) $s = 7.69$ cm
24. (a) $k = 6.95$ cm
(b) $l = 5.50$ cm
(c) $m = 5.90$ cm
(d) $n = 5.78$ cm
25. (a) $w = 5.52$ cm
(b) $x = 4.62$ cm
(c) $y = 5.55$ cm
(d) $z = 4.53$ cm
26. (a) $\hat{W} = 26.6°$
(b) $\hat{X} = 18.4°$
(c) $\hat{Y} = 14.0°$
(d) $\hat{Z} = 14.0°$
27. (a) $\hat{A} = 20.6°$
(b) $\hat{B} = 31.0°$
(c) $\hat{C} = 32.0°$
(d) $\hat{D} = 41.2°$
28. (a) $\hat{P} = 33.1°$
(b) $\hat{Q} = 34.3°$
(c) $\hat{R} = 38.1°$
(d) $\hat{S} = 37.9°$
29. (a) $\hat{K} = 42.1°$
(b) $\hat{L} = 37.1°$
(c) $\hat{M} = 39.8°$
(d) $\hat{N} = 39.7°$
30. (a) $\hat{W} = 28.7°$
(b) $\hat{X} = 33.9°$
(c) $\hat{Y} = 31.4°$
(d) $\hat{Z} = 23.7°$

Exercise 11d

1. $AB = 9$ cm
2. $AC = 10$ cm
3. $AB = 4.62$ cm
4. $B\hat{A}C = 14°$
5. $B\hat{A}C = 30°$
6. $B\hat{A}C = 60°$
7. (a) $\alpha = 53.1°$
(b) $OT = 5$ cm
8. (a) $\beta = 22.6°$
(b) $OT = 13$ cm
9. $t = 14.0$ cm
10. $t = 12.3$ cm
11. (a) $RPQ = 45°$
(b) 3.54 m
(c) $PQ = 7.07$ m
12. (a) $QP = 8$ cm
$\triangle MLO \equiv \triangle OPQ$ (R.H.S.)
(b) $R = MO = QO = 10$ cm
(c) $\tan \angle OML = \frac{3}{4}$, $\sin \angle OML = \frac{3}{5}$, $\cos \angle OML = \frac{4}{5}$
13. (a) $D\hat{O}C = 72°$
(b) $DM = 8.82$ cm
(c) $P = 88.2$ cm
14. (a) $R\hat{O}S = 72°$
(b) $RS = 20$ cm
(c) $P = 100$ cm

Exercise 11e

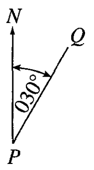
1. cos 63° = 0.454
2. cos 31° = 0.857
3. cos 7° = 0.993
4. cos 54.3° = 0.584
5. cos 41. 7° = 0.747
6. cos 28.6° = 0.878
7. sin 58° = 0.848
8. sin 42° = 0.669
9. sin 23° = 0.391
10. sin 30.7° = 0.511
11. sin 21.5°= 0.367
12. sin 18.2° = 0.312

Exercise 11f

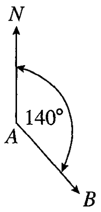
1. 423 m
2. (a) $BC = 10.4$ m
(b) $AC = 10.8$ m
3. 119.2 m **4.** 14.6 m
5. 33.9° **6.** 8.4 m
7. 70 m **8.** 857.8 m
9. 32.2 m **10.** 196.1 m
11. 10.5 m **12.** 13 m
13. 33.3 m **14.** 36.3 m
15. 265.5 m **16.** 74.5 m
17. (a) 58.7 m (b) 7.5 m
18. (a) 77.8 m (b) 22.2 m

Exercise 11g

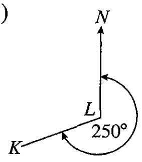
1. (a)



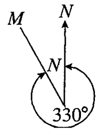
(b)



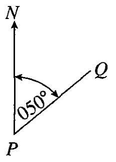
(c)



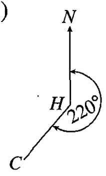
(d)



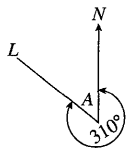
2. (a)



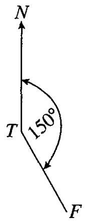
(b)



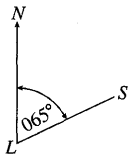
(c)



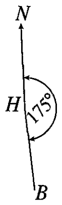
(d)



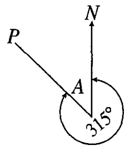
3. (a)



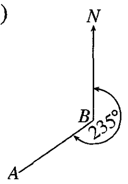
(b)



(c)



(d)



4. 245° **5.** 250° **6.** 264°
7. 315° **8.** 335° **9.** 344°
10. 40° **11.** 70° **12.** 85°
13. 130° **14.** 145° **15.** 159°
16. 64.3 km. 76.6 km
17. 68.9 km. 57.9 km
18. 65.1 km. 54.7 km
19. 32.5 km. 56.3 kg
20. 116.7 km. 116.7 km
21. 88.1 km. 35.6 km
22. 70.7 km. 70.7 km

23. 71.3 km. 49.9 km

24. 116.6° **25.** 205° **26.** 321.3°

27. 36.9° **28.** 68.2° **29.** 287.1°

30. 205.6° **31.** 158.2° **32.** 173.2 m

Exercise 11h

1. (i) $PN = LP = 12$ cm
  (ii) $LM = 13$ cm

2. (a) $XF = 3.5$ m
  (b) 5.1 m

BASIC PROFICIENCY
C.X.C. MODEL EXAMINATIONS
1 to 6 - Paper 1
Examination Numbers

	1	2	3	4	5	6
1.	C	C	B	A	C	D
2.	A	A	A	C	A	B
3.	B	B	C	D	B	C
4.	D	D	D	A	D	D
5.	B	C	D	A	B	A
6.	C	B	B	B	A	D
7.	B	A	C	B	C	C
8.	D	B	D	B	A	A
9.	C	D	B	A	D	C
10.	C	B	A	A	B	C
11.	B	C	D	D	A	B
12.	A	B	C	B	B	D
13.	A	A	B	A	D	A
14.	B	B	D	A	C	D
15.	C	C	B	D	A	B
16.	B	D	B	B	B	B
17.	B	A	B	B	B	C
18.	B	C	D	B	C	B
19.	C	B	A	D	D	C
20.	D	B	A	C	B	C
21.	D	C	B	B	D	D
22.	A	C	D	C	A	C
23.	A	A	C	C	C	D
24.	C	B	A	D	B	A
25.	A	D	D	C	D	B
26.	A	D	C	D	A	D
27.	D	B	A	D	B	B
28.	D	D	C	C	B	C
29.	B	A	C	C	C	D
30.	B	B	A	B	D	A
31.	D	D	D	B	D	D
32.	C	B	B	C	B	A
33.	C	B	B	C	A	A
34.	D	C	D	B	D	D
35.	B	A	D	C	A	B
36.	B	B	B	C	A	C
37.	D	D	C	B	A	D
38.	C	B	C	C	B	B
39.	B	A	D	A	D	D
40.	B	D	B	D	C	B
41.	A	D	B	A	B	C
42.	A	B	C	B	A	D
43.	D	C	A	C	A	B
44.	C	A	B	D	C	A
45.	C	B	D	D	A	A
46.	B	C	C	B	A	B
47.	A	D	B	B	D	D
48.	C	B	D	D	C	C
49.	B	B	C	C	D	B
50.	A	A	D	B	D	D
51.	D	B	B	A	D	C
52.	A	C	A	B	C	B
53.	B	D	A	D	A	B
54.	B	B	C	C	C	C
55.	B	A	B	B	B	B
56.	D	D	D	C	A	A
57.	C	C	D	A	A	B
58.	A	C	C	B	B	A
59.	C	D	D	C	C	D
60.	B	A	B	C	D	A

C.X.C. Basic Proficiency Model Examination 1 - Paper 2

1. (a) 0.71
  (b) (i) $s = 12$ km/h
    (ii) $s = 13\frac{5}{7}$ km/h

2. (a) \$26 804
  (b) \$9 521.60

3. (a) $RQ = 8.07$ cm
  (b) $NN' = 9$ cm

4. (a) Number = 3
  (b) (i) $\{(3, 9), (4, 8), (5, 7), (6, 6), (9, 3), (8, 4), (7, 5)\}$
    (ii) $P(x + y = 7) = \frac{7}{100}$

5. (a) 0.0675 cm
  (b) 1 690 500 m²

6.

x	-3	-1	0	1
y	$\frac{1}{4}$	1	2	4

$x = -2.7$ and $x = 1$

7. (b) Family A had a more equitable distribution of incomes.
  Family A had no family in the lowest income per hour, that is, \$(1–5).
  Family A had more families than Family B in the higher incomes per hour.

8. (a) (i) \$3 525
    (ii) \$2 200
    (iii) \$1 325
  (b) (i) \$3.15
    (ii) 11.55
    (iii) \$32.55
    (iv) \$683.55

9. $\Upsilon = 7.21 \times 10^{-2} = 0.0721$

C.X.C. Basic Proficiency Model Examination 2 - Paper 2

1. (a) 13.9 (b) \$527

2. (a) (i) 3 250 (ii) \$35 600
  (b) \$2 300

3. (a) (i) \$110.31
    (ii) \$3 457.40
  (b) \$3 651.80
  (c) H.P. better. \$194.40

4. (a) $A = 1409.2$ cm²
  (b) $V = 56368$ cm³

5. $m = \$25$. $e = \$15$

6. (a)

x	f
8	12
9	15
10	18
11	20
12	25
13	37
14	32
15	23
16	17
17	13
$n = \Sigma f = 212$	

(b) $\bar{x} = 12.7$ $Q_2 = 13$
Mode = 13
(c) The mode - most popular size
(d) $P(x = 14) = \frac{8}{53}$

7. (a) $\{p: p \leqslant \frac{32}{7}\}$
(b) S.I. = $19943

8. (c) (i) 3.2 min
(ii) $2\frac{2}{3}$ km

9. (a) (ii) $AF = 16$ cm.
$EJ = 10$ cm
(b) (i) $XY = 4$ cm
(ii) $JX = 5$ cm
(iii) $X\hat{J}Y = 38.7°$
$JY = 6.4$ cm

C.X.C. Basic Proficiency Model Examination 3 - Paper 2

1. 2.2
2. $x = 1\frac{1}{2}$
3. (a) $1096. $1781. $2603
(b) 72°. 117°. 171°.
4. (a) $2\frac{1}{2}$ hours
(b) (i) $V = 60$ cm³
(ii) 57.8 cm³
5. (a) (i) TT $4584
(ii) CAN $400
(b) $34992
6. (a) $148.50
(b) $133.65
7. (a) $10c + 4d = 440$ ---- ①
$4c + 3d = 246$ ---- ②
(b) c = $24. d = $50
8. (b) (i) $\bar{x} = 169.2$ cm
(ii) I.Q.R. = 14.5 cm
(iii) $Q_2 = 167$ cm
(c) Modal heights = 160 cm, 167 cm and 178 cm
The sample is tri - modal
(d) $P(x \geqslant 174 \text{ cm}) = \frac{5}{13}$
9. (c) $CC' = 7.5$ cm
(d) $NC = NC'$. NC is the altitude of the triangle formed by NC, the mirror line l and the side CA produced.
(e) $BCCB'$ is and isosceles trapezium
10. (b) $m = \frac{3}{2}$. $c = 20$
(c) $R = \frac{3}{2}v + 20$

C.X.C. Basic Proficiency Model Examination 4 - Paper 2

1. (a) 15.535
(b) 6
(c) $3\frac{5}{8}$
2. (a) −25
(b) $x^2 - 2xy - 3y^2$
(c) $x = 5$
3. (a) 750 km/h
(b) (i) TT $6825
(ii) TT $6006
4. (a) $8m + 6p = 25.00$ ---- ①
$4m + 8p = 30.00$ ---- ②
(b) (i) m = $0.50
(ii) p = $3.50
5. (a) (i) 88 m
(ii) $11498\frac{2}{3}$ m³
(b) (i) 1:200
(ii) 616 cm²
6. (a) 25°. l is the perpendicular bisector of angle LOM.
(b) 65°. k is the perpendicular bisector of angle MON.
(c) $R_{(0, -90°)} = R_{(0, +270°)}$
7. (a) $QR = 7.43$ cm $P\hat{R}Q = 77.4°$
(b) (i) $PM = 15$ cm
(ii) $LN = 19.8$ cm $\triangle LMP$ is isosceles
8. (a) (i) $12150
(ii) $85000
(b) (i) 9657 units
(ii) $1931.40
(iii) $4345.65
(iv) $6277.05
9. (a) $\mathbf{x} = 5.12$ marks
(c) $Q_2 = 5$ marks.
S.I.Q.R. = 2.25 marks
(d) $P(x > 6) = \frac{9}{25}$
10. (a)

x	−2	0	1	4	6
$f(x)$	6	−4	−6	0	14

(d) (i) $x = -1$ and $x = 4$
(ii) (−2, 6) and (5, 6)

C.X.C. Basic Proficiency Model Examination 5 - Paper 2

1. 35.6
2. $x = 3$
3. 240 l
4. (a) $7q + 5j = 27.35$ ----①
$5q + 7j = 25.57$ ----②
(b) (i) q = $2.65
(ii) i = $1.76
5. (a) 55.2 m² (b) 305 m²
(c) 100 m² (d) 300 m²
(e) 915.4 m² (f) 1380 m³
6. (a) $232.75 (b) $6315.80
(c) $49544.20 (d) $1015.50
7. (a) $A'(-2, -3)$, $B'(-4, -5)$ and $C'(-4, -3)$
$A''(-2, 3)$, $B''(-4, 5)$ and $C''(-4, 3)$
(b) M_x
(c) $C'A'' = 6.3$ cm
(d) $A'\hat{A}A'' = 56.3°$
8. (a)

x	f
0	4
1	6
2	3
3	3
4	1
5	5
6	3
$n = \Sigma f = 25$	

(b) $\bar{x} = 2.72$
(c) Modal score = 1
I.Q.R. = 4
(d) $P(x \leqslant 5) = \frac{22}{25}$
9. (a) (i) 330000 students
(ii) 216000 students
(b) (ii) $m = -\frac{1}{3}$

(iii) The line $y = 3x + 1$ and the line PQ are perpendicular. (0, 1). $y = -\frac{1}{3}x + 1$.

10. (a) (i) \$1.85 (ii) \$7.00
  (b) (i) \$1.762
    (ii) \$2643
    (iii) \$1143
    (iv) 2089.08

C.X.C. Basic Proficiency Model
Examination 6 - Paper 2

1. (a) (i) 0.06 (ii) 19.76
  (b) 60

2. (a) $\frac{33}{40}$ (b) 4.5 cm

3. (a) $x = 2$ when $y = 3$
  (b) $\frac{23 - 2x}{12}$

4. (a) (i) $DC = 5.4$ cm
    (ii) 2.1 cm
  (b) 8.82 cm² (c) 3:50 = $1:16\frac{2}{3}$
  (d) (i) 2.7 m
    (ii) 0.245 m²

5. (a) \$1450.80
  (b) (i) \$8094.80
    (ii) \$967.17

6. (a) (i) \$$2x$ (ii) \$$(2x - 9)$
    (iii) \$$(5x - 9)$
  (b) $300 - (5x - 9) = 159$
  (c) \$51

7. (a) \$392 (b) \$224
  (c) 150% (d) \$8500

8. (a) $A'(3, 0)$, $B'(9, 6)$ and $C'(0, 4.5)$
  (b) My
  (c) $A'''(2,0)$, $B'''(6, -4)$ and $C'''(0, -3)$

  (d) (i) Area of $\triangle ABC = \frac{}{2.25}$ (Area of $\triangle A'B'C'$)
    (ii) Area of $\triangle ABC$ = Area of $\triangle A''B''C''$
    (iii) Area of $\triangle ABC$ = Area of $\triangle A'''B'''C'''$

9. (a) (i) $\bar{x} = 142.9$ cm.
      $Q_2 = 143$ cm
    (ii) $P(x < 143 \text{ cm}) = \frac{13}{34}$.
      $P(x \geqslant 143 \text{ cm}) = \frac{21}{34}$
  (b) (i) \$45000. \$20625.
      \$65625

10. (a) (ii) $m = 2$
    (iii) $R(0, -2)$. $y = 2x - 2$
  (b) (i)

x	-2	0	1	3
y	1	-3	-2	6

    (iii) $x = \pm 3$

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